Mathematical Preparation of Teachers*

MSRI

March 26, 2014

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*This is very slightly expanded version of the presentation I made on March 26, 2014. I wish to thank Larry Francis for his editorial assistance.

An ad by IBM in London's Heathrow Airport (March 2008):

Stop selling what you have.

Start selling what they need.

For the mathematical preparation of teachers:

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Our universities have been too busy selling what they have.

They have forgotten about what pre-service teachers need.

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We have let them down.

The mathematics that has been taught in schools for more or less the past four decades is what we call **TSM**, **T**extbook **S**chool **M**athematics.

TSM is what school textbooks have in common overall: almost no definitions, fragmented presentation of sound bites, blurring the line between a proof and a heuristic argument, and lack of precision.

In other words, *not learnable.*

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Traditional math and reform math are different, but they are both mathematically defective in their own ways.

Some typical consequences of TSM:

1. (2011 TIMSS, 8th grade) $\frac{1}{3} - \frac{1}{4} = ?$ [†]

32% of U.S. students chose $\dfrac{1-1}{4-3}$.

26% chose $\dfrac{1}{4-3}$.

30% got it right. (Taipei: 82%. Finland: 16%.)

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(*Do they try to make sense of anything at all?*)

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2. Many (most?) high school students believe that $\frac{-7}{-3} = \frac{7}{3}$ because:

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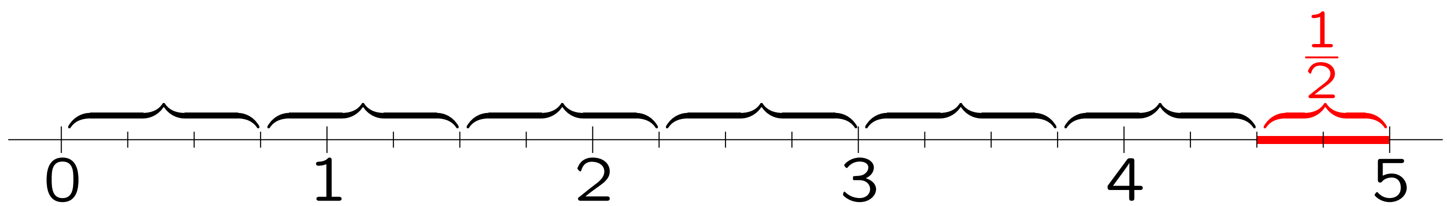
(*Do they try to reason abstractly?*)

3. (Division of fractions) Students are taught $32 \div 5 = 6 \; R \; 2$, therefore $5 \div \frac{3}{4} = 6 \; R \; \frac{1}{2}$;



They *guess* that $\frac{1}{2} = \frac{2}{3} \times \frac{3}{4}$. Therefore $5 \div \frac{3}{4} = 6\frac{2}{3}$.

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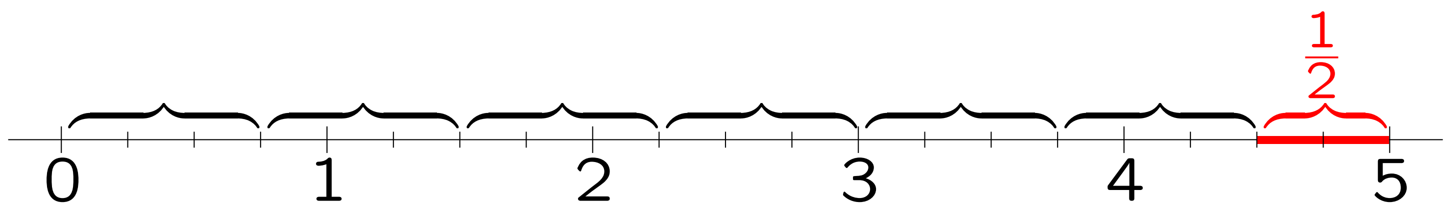

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(*What to do for the division* $\frac{2}{11} \div \frac{81}{29}$ *?*

On what grounds can they critique this reasoning?)

4. The following table gives the

number of miles Helena runs in minutes:

How many miles does she run in 25 min?

min	mi
10	1
20	2
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How many miles does she run in 25 min?

Students learn to model the data by proportional reasoning. The *unit rate* is $\frac{1}{10}$ mi/min. So in 25 minutes she runs $25 \times \frac{1}{10} = 2\frac{1}{2}$ miles.

But it turns out that Helena is an Olympic 400 meter specialist training for a meet. Every 10 minutes, she runs $\frac{1}{2}$ mile in 2 minutes and walks the next $\frac{1}{2}$ mile in 8 minutes. So in 25 minutes, she covers about 2.7 miles.

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(*Why can't proportional reasoning be used to model this situation?*)

5. Students are convinced that for all positive a, b,

$$\sqrt{a}\,\sqrt{b} \;=\; \sqrt{ab},$$

because, on the calculator,

$\sqrt{5}\,\sqrt{7} = \sqrt{35} = 5.9160797831,$

$\sqrt{3}\,\sqrt{6} = \sqrt{18} = 4.24264068712,$ etc.

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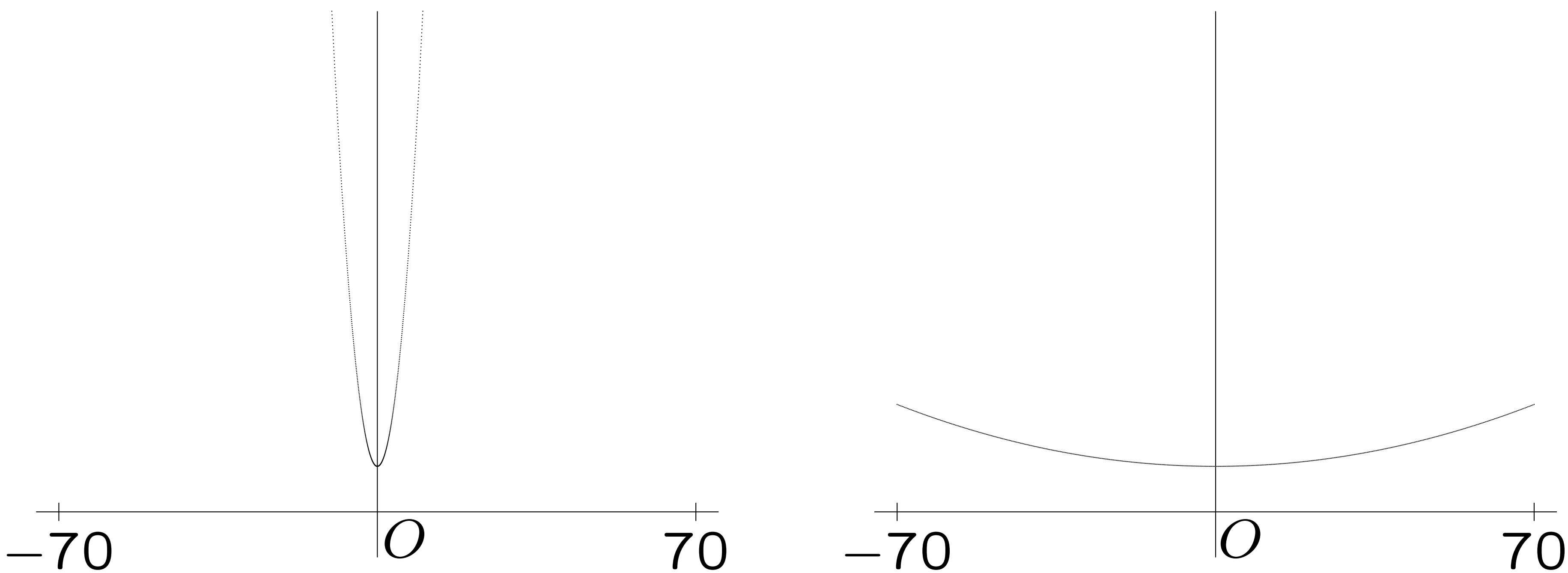
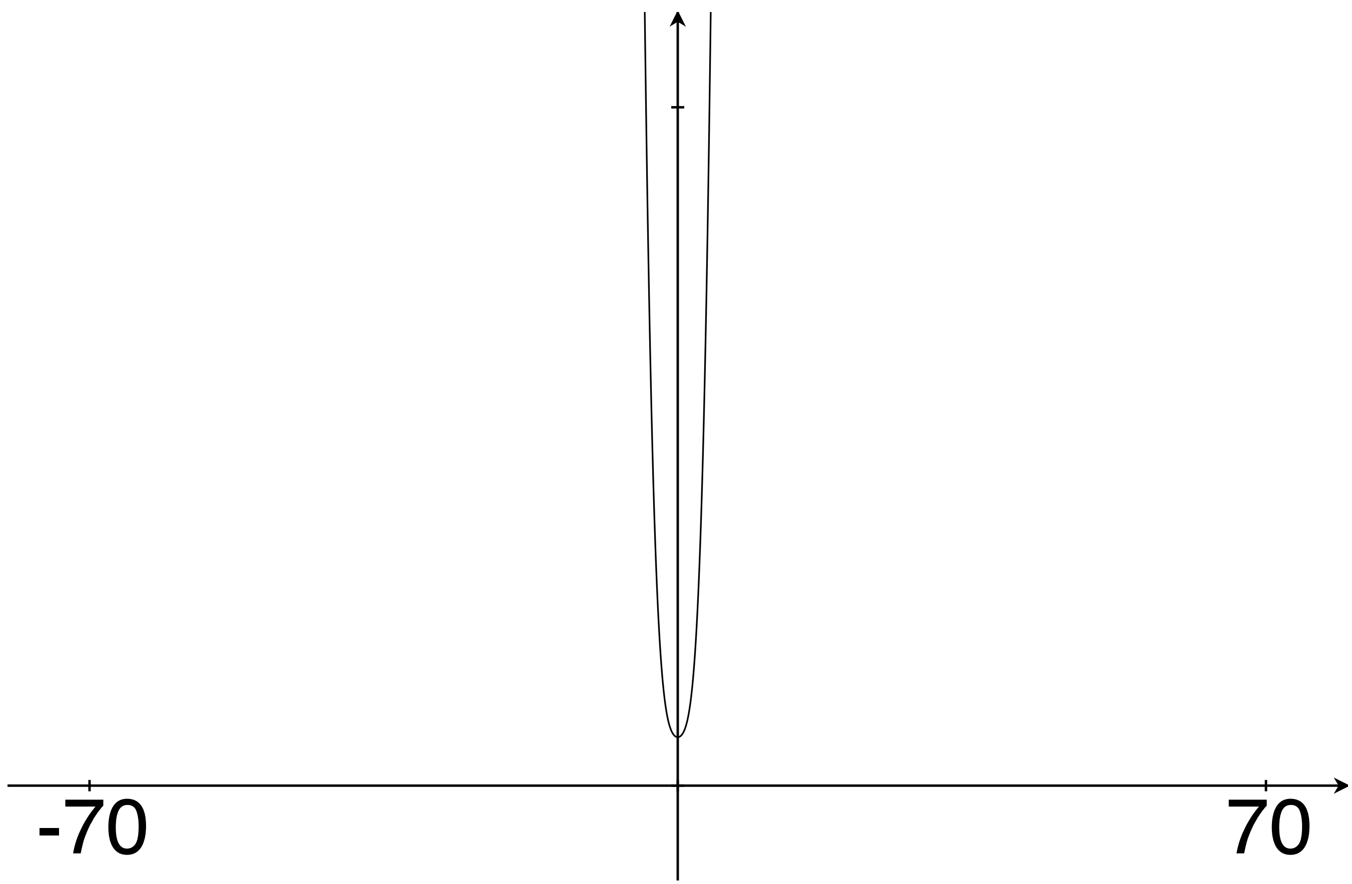
(*Isn't this a strategic use of the calculator?*)

6. Because *similar* means same shape but not necessarily the same size, students believe that the following curves are not similar.



They also believe that the left curve above *is* similar to the following curve:



It turns out the first two curves are graphs of $x^2 + 10$ and $\frac{1}{360}x^2 + 10$, respectively, and are therefore similar.

The third curve is the graph of $\frac{1}{4}x^4 + x^2 + 1$, and is therefore not similar to the first curve.

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The third curve is the graph of $\frac{1}{4}x^4 + x^2 + 1$, and is therefore not similar to the first curve.

(*Perhaps we need a precise definiton of* **similarity***?*)

7. When elementary students take up fractions, the concept of "equivalent fractions" is among the *first* things they encounter.

They learn that $\frac{2}{3} = \frac{8}{12}$, because,

$$\frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{4}{4} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}.$$

The following conversation then takes place:

Carl: You know, I have thought about it, and I don't know why $\frac{2}{3} \times \frac{5}{5} = \frac{2\times 5}{3\times 5}$.

Bryant: Look, you see 2 and 5 on top with $\times$ in between, and you multiply. The same with 3 and 5. You know how it is with whole numbers, right?

Carl: Is that how you do it? So $\frac{2}{3} + \frac{5}{5} = \frac{2+5}{3+5}$?

Diane: Great! Now we can add fractions too!

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(*But is this the right way to make use of structure?*)

8. Students learn about why $(-2) \cdot (-5) = 10$ by observing regularity in repeated reasoning:

$$\begin{aligned} 3 \cdot (-5) &= -15 \\ 2 \cdot (-5) &= -10 \\ 1 \cdot (-5) &= -5 \\ 0 \cdot (-5) &= 0 \\ (-1) \cdot (-5) &= ? \\ (-2) \cdot (-5) &= ? \end{aligned}$$

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(*This is how they will learn algebra?*)

It should not be difficult to see that the preceding eight examples closely parallel the eight **Standards for Mathematical Practice** in the CCSSM.

Consider the typical life-cycles of K–12 math teachers:

In K–12 they learn TSM.

$\longrightarrow$ In college they learn advanced math or more TSM, and strategies to implement what they know about TSM.

$\longrightarrow$ In K–12 they teach by regurgitating TSM.

$\longrightarrow$ Thus they victimize the next generation of teachers by teaching them TSM.

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Equipped only with a knowledge of TSM, teachers can have little hope of implementing the CCSSM.

If a general sends soldiers to the front without any ammunition, he would be court-martialed, at least.

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Yet, universities routinely send prospective teachers to school classrooms *without the content knowledge they need*.

This is not something that math departments—in fact the entire math community—should be proud of.

The time to change is **now.**

Two concluding remarks:

(1) Why not get rid of TSM by writing reasonable *textbooks?* (This will require another workshop.)

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(2) If we in the math departments continue this tradition of inaction by neglecting to teach prospective teachers *correct school mathematics*, we will victimize not only teachers, but math educators as well.