On integrating algebra and geometry in secondary school mathematics

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I want to thank Professors Isabel Ferreirim and José Francisco Rodrigues for their kind invitation, and Larry Francis for his editorial help as usual.

Today I want to share with you some of my thoughts on the teaching of algebra and geometry in secondary school.

These thoughts were originally prompted by my work with teachers from 2000 to 2013. Due to the neglect of the American education establishment, teachers in the U.S. generally have a serious content-knowledge deficit. My goal has always been to eliminate this deficit as best I can.

I did not have any ambitious plans about teaching teachers "higher" mathematics and waiting for them to digest it and use it to elevate their own knowledge of *school mathematics*.

My goal has been much more modest: teach them the mathematics *in the school curriculum*—but in a way that is *mathematically correct*—so that they can directly put it to use in their classrooms.

It was with this mind-set that I started to teach them the algebra and geometry of secondary school. Immediately I came to an impasse because much of the relevant mathematics in almost all the standard textbooks is deeply flawed, in multiple ways.

Let me refer to this body of "knowledge" as Textbook School Mathematics (TSM), for convenience.

One of the most grievous flaws in TSM is the lack of *mathematical coordination* between algebra and geometry. Too often, when a certain geometric fact or a particular geometric point of view is needed to facilitate the algebraic development, that fact or point of view is found to be missing from the curriculum.

There was no way for me to proceed except to devise a usable alternative. Today I would like to talk about this alternative.

My lecture will be divided into two parts:

Part I: A description of the *broken connections* between algebra and geometry in TSM.

Part II: A brief outline of the proposed solution.

I do not expect the Portuguese curriculum to be the same as the American one, but I also believe that certain mathematical issues in the school curriculum transcend national boundaries. I hope you will find some of what I have to say to be relevant.

In addition, although these were my own findings, they have now acquired some legitimacy because the Common Core State Standards for Mathematics (CCSSM) in the U.S., published in 2010, have come to essentially the same conclusion as the proposed solution described in Part II.

PART I: The broken connections

There are at least three basic topics in algebra,

linear equations

quadratic functions

graphs of inverse functions

in which certain geometric connections—*if* established —would bring clarity and understanding to students.

(A) Linear equations. The study of linear equations of two variables is a mainstay of introductory algebra. A main conclusion is that the graph of $ax + by = c$ is a (straight) line. Here are two typical exercises for students:

> What is the equation of the line joining $(3\frac{1}{2}, 5)$ and $(1, -\frac{1}{5})$?
>
> What is the equation of the line with slope $\frac{1}{2}$ and passing through $(-3, 4)$?

It turns out that students have trouble doing these exercises because they were taught the solution method entirely *by rote*.

They also have trouble understanding what slope means. Research shows that "The most difficult problems for students were those requiring identification of the slope of a line from its graph."

Let us look into why this is so. First of all, TSM does not explain why the graph of $ax + by = c$ is a line.

Instead, TSM asks students to plot a few points on the graph of this equation and *observe* that the plotted points seem to lie on a line. This is considered to be sufficient evidence for students to believe that the graph *is* a line.

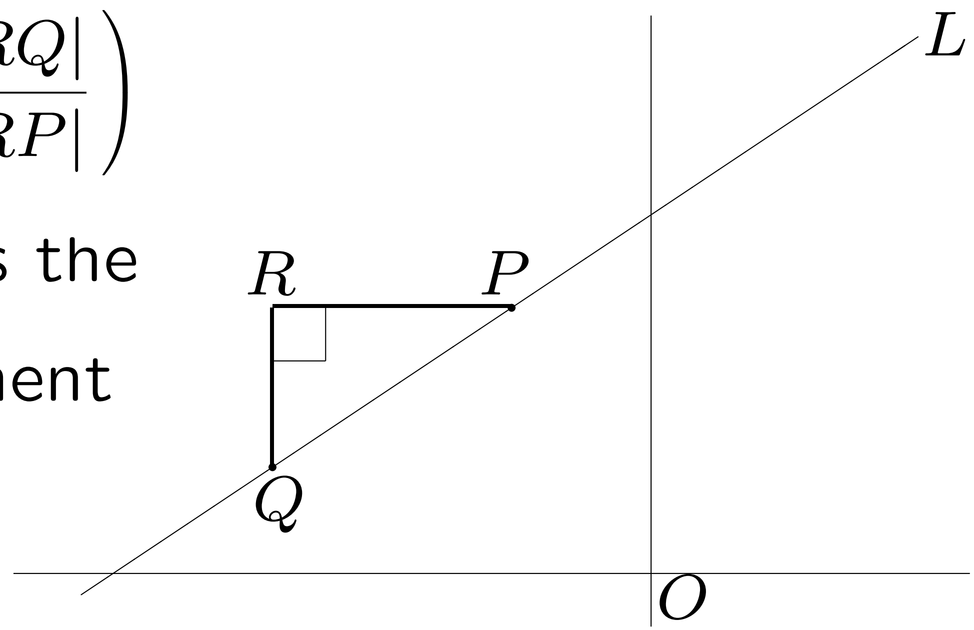
Unfortunately, if students are completely ignorant of the reasoning behind why the graph of $ax+by=c$ is a line, then they have no choice but to memorize by rote how to write down the equation of a line satisfying certain geometric conditions.

The key point of this reasoning is the concept of the **slope** of a line. Here is how TSM introduces this concept:

Let L be a nonvertical line in the coordinate plane and let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be distinct points on L. According to TSM, the definition of the **slope of L** is

$$\frac{p_2 - q_2}{p_1 - q_1} \left(= \frac{|RQ|}{|RP|} \right)$$

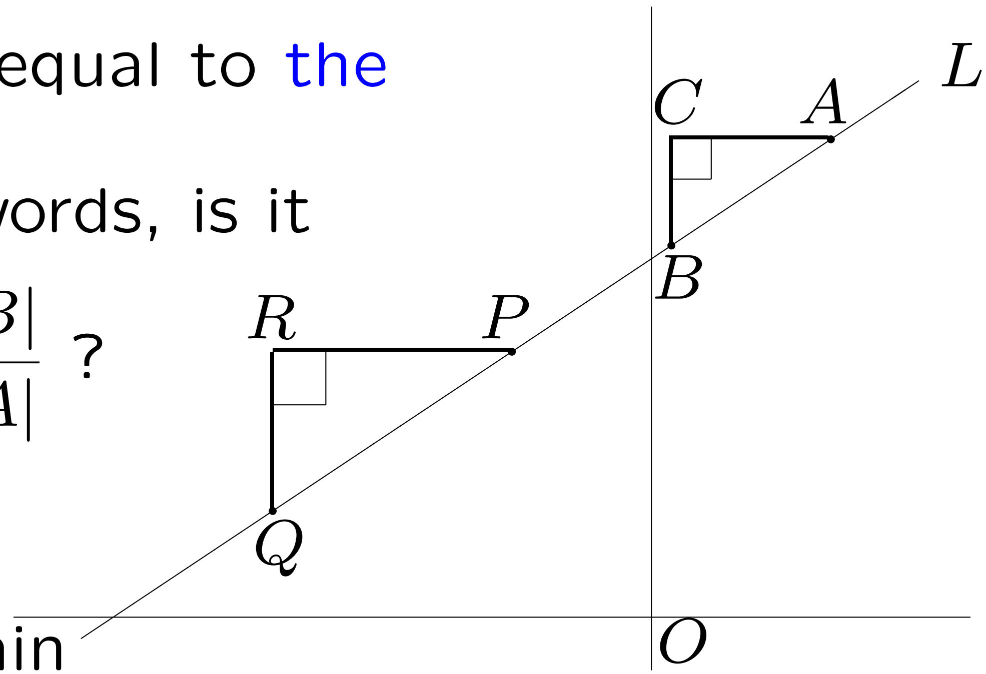
where $|RQ|$ denotes the **length** of the segment RQ, etc.



Suppose two other points A and B on L are chosen.

Then would $\dfrac{|CB|}{|CA|}$ be equal to the slope of L? In other words, is it true that $\dfrac{|RQ|}{|RP|} = \dfrac{|CB|}{|CA|}$?

TSM *does not address this issue.* This is a main reason why **slope** is difficult to understand.



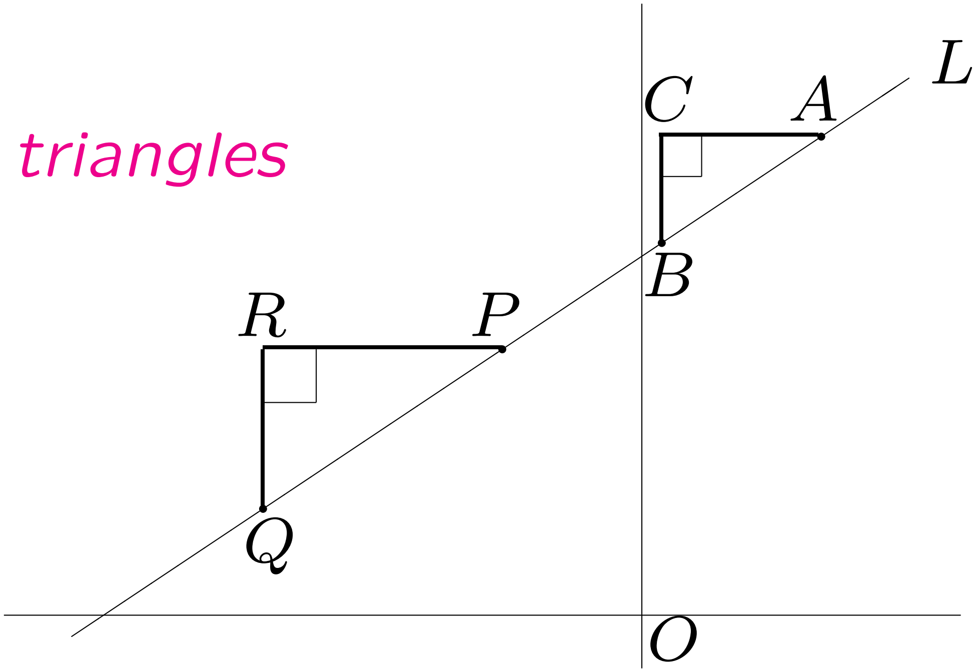
Students do not know that *slope is a single number* attached to L that measures its "slant".

Many believe that *slope* is a pair of numbers—$|RQ|$ and $|RP|$ (so-called *rise-over-run*)—attached to the line L in some mysterious fashion once the two points P and Q on L have been "properly" chosen.

It is therefore not surprising that "The most difficult problems for students were those requiring identification of the slope of a line from its graph."

If students knew about similar triangles, they would know $\triangle ABC \sim \triangle PQR$ (*similar triangles*), which then easily implies that $\dfrac{|RQ|}{|RP|} = \dfrac{|CB|}{|CA|}$.

This is why similar triangles have to be taught in algebra.



At the moment, students either memorize the definition of *slope* using two *fixed points*, or are simply *told*, without explanation, that the slope of a line can be computed using *any* two points on the line.

Without a knowledge of similar triangles, they cannot know the reasoning behind the graph of $ax + by = c$ being a line and, therefore, they have to write down the equation of a line satisfying some geometric conditions by brute force memorization.

In order to understand slope and the algebra of linear equations, student have to be *at ease with the concept of similar triangles and the applications of this concept.* The abstract theory of similarity can wait.

We can therefore introduce them to similarity in an intuitive (but mathematically correct) manner. Schematically, the sequencing of the topics is as follows:

$\{$rotations, reflections, translations$\} \longrightarrow \{$congruence$\}$

$\{$congruence, dilation$\} \longrightarrow \{$similarity$\}$

$\{$similarity$\} \longrightarrow \{$the AA criterion of triangle similarity$\}$

$\longrightarrow \{$correct definition of slope$\}$

We will discuss all these briefly in **PART II**.

(B) Quadratic functions. We will demonstrate that if we can better understand the *graphs* of quadratic functions, then the conceptual simplicity of the *algebra* of quadratic functions will come to light.

This is not at all surprising when we realize that much of our understanding of the *algebra* of linear equations comes from the fact that the graphs of linear equations are lines.

Let us illustrate why knowing the graph of a linear equation being a *line* promotes the understanding of the *equations* themselves.

This fact enables us to visualize the solution of simultaneous linear equations

$$\begin{cases} ax & + & by & = & e \\ cx & + & dy & = & f \end{cases}$$

as the point of intersection of *the two lines of the system*, i.e., the graphs of $ax+by=e$ and $cx+dy=f$.

It also helps us see why the **determinant** $ad - bc$ of the system determines its solvability, as follows:

Assuming $b, d \neq 0$, the slope of the line $ax + by = e$ is $-\frac{a}{b}$ (because $y = -\frac{a}{b}x + \frac{e}{b}$), and the slope of the line $cx + dy = f$ is $-\frac{c}{d}$ (because $y = -\frac{c}{d}x + \frac{f}{d}$).

So the two lines of the linear system are parallel $\iff$ $-\frac{a}{b} = -\frac{c}{d}$, therefore $\iff$ $\frac{a}{b} = \frac{c}{d}$, and therefore $\iff$ $ad = bc$, i.e., $\iff$ $ad - bc = 0$.

So suppose the determinant of the system

$$\begin{cases} ax & + & by & = & e \\ cx & + & dy & = & f \end{cases}$$

is nonzero, i.e., $ad - bc \neq 0$.

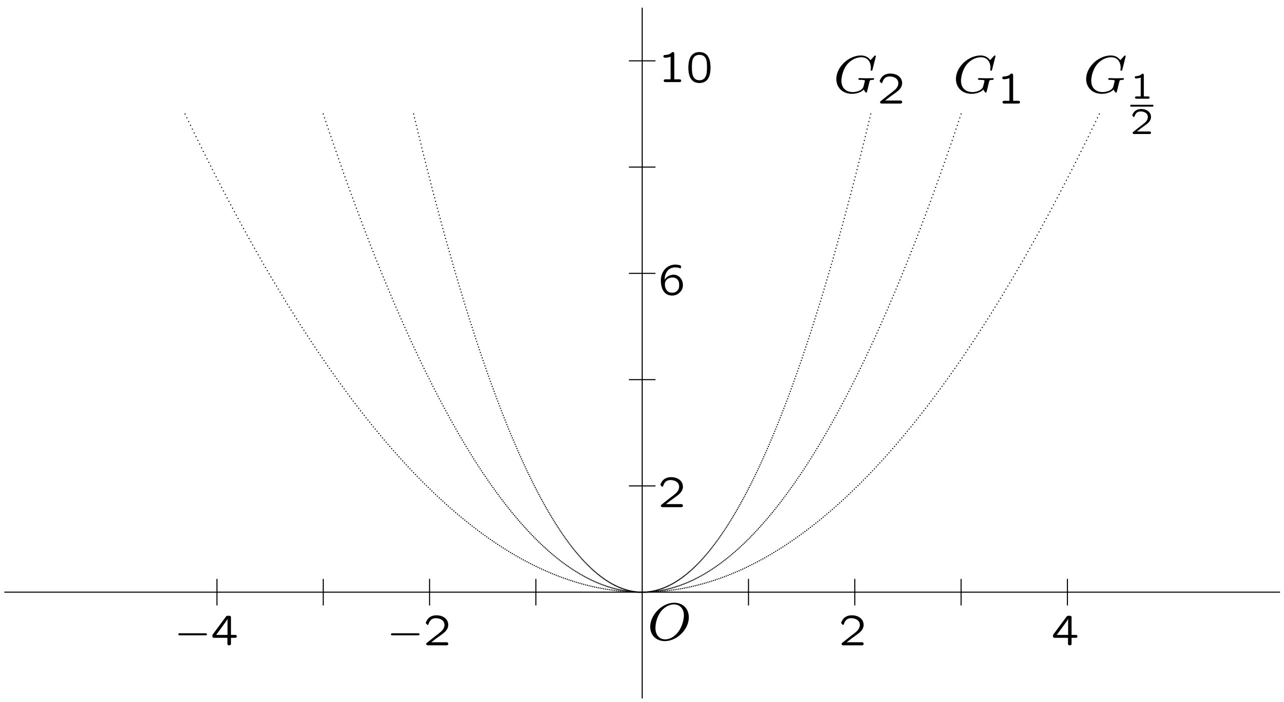
If $b, d \neq 0$, then the two lines of the system are not parallel, $\Rightarrow$ the two lines intersect at a unique point (A, B), $\Rightarrow$ the system has a unique solution $x = A$, $y = B$. (The case of b or $d = 0$ is easy to handle.)

Let us now take up *quadratic functions.*

Since the fact that the *graph* of a linear equation is a line promotes our understanding of the *algebra* of linear equations, we will use the *graphs* of quadratic functions to achieve a similar clarification of the *quadratic functions* themselves.

This is not the way TSM approaches quadratic functions.

The graphs of quadratic functions may seem to be complicated, but actually they are not, as we shall see. Let us concentrate on the graphs G_a of the functions $f_a(x) = ax^2$, where $a > 0$.

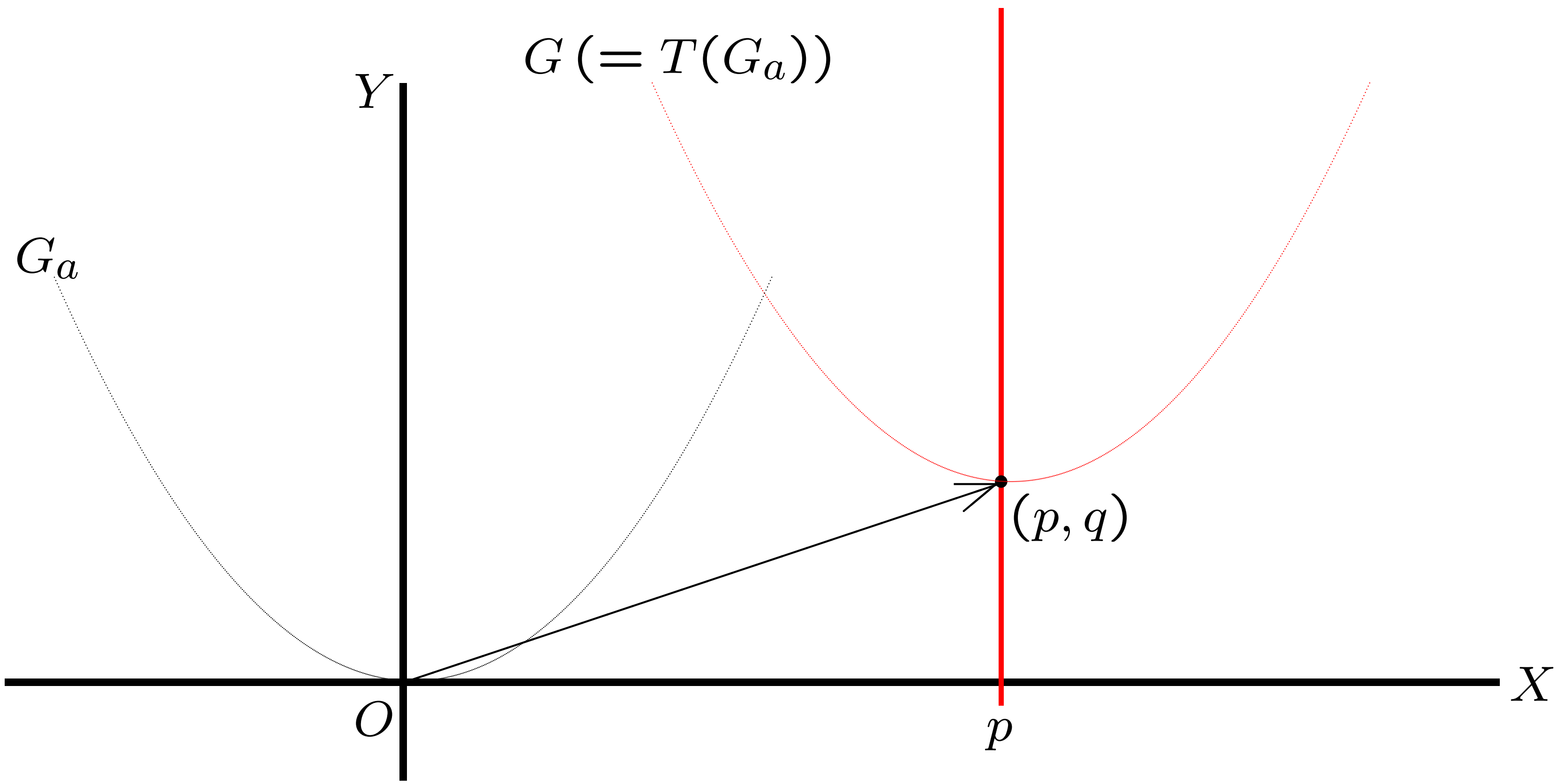


There are two remarkable properties about these graphs G_a of $f_a(x) = ax^2$ in case $\boldsymbol{a > 0}$.

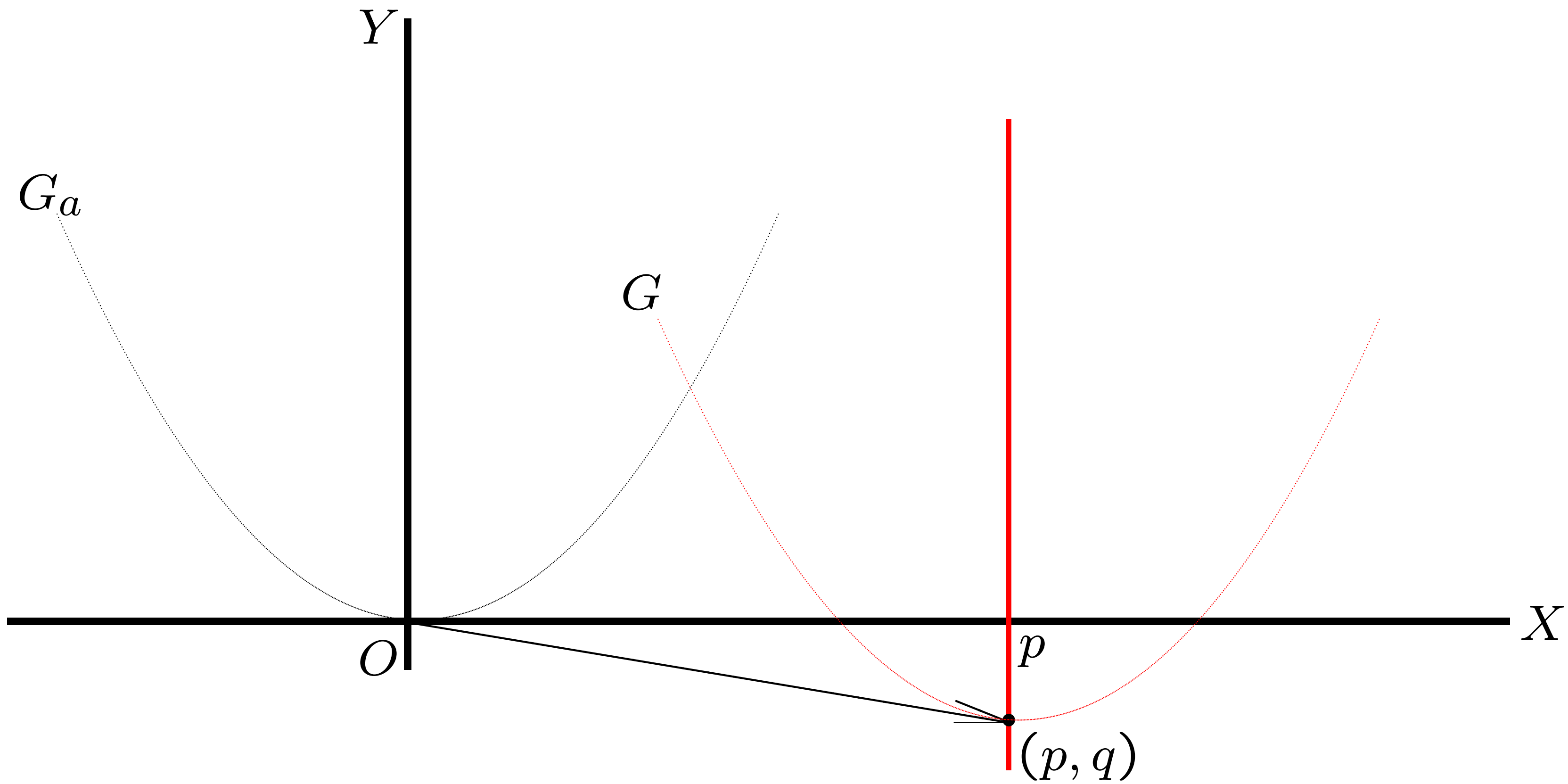
(1) Suppose $a > 0$. Then the graph G of a general quadratic function $f(x) = ax^2 + bx + c$ is *congruent* to G_a, and the congruence is realized by a *translation* T, so that for some fixed (p, q),

$$\begin{aligned} T(x, y) &= (x + p,\, y + q) \\ &\quad \text{for } \textit{all } (x, y) \text{ in the plane} \\ T(G_a) &= G \end{aligned}$$

Observe that $T(0,0) = (p,q)$.



Here is another possible scenario:



You recognize that this (p, q) is nothing other than the so-called **vertex** of G.

With hindsight, we see that the *purpose* of rewriting a quadratic function $f(x) = ax^2 + bx + c$ in its **vertex form** or **normal form**, $\boldsymbol{f(x) = a(x-p)^2+q}$, is precisely to exhibit the congruence of the graph G to the graph G_a.

From the graph G_a, we easily infer the properties of the function $f_a(x) = ax^2$:

It has a minimum at O.

It is decreasing on the interval $(-\infty, 0]$.

It is increasing on the interval $[0, \infty)$.

$f_a(k) = f_a(-k)$ for any k (because the graph G_a is *symmetric with respect to the y-axis*).

The fact that the graph G of $f(x) = ax^2 + bx + c$ is the *translation* of G_a then clearly exhibits the following properties of the function $f(x)$:

It has a minimum at (p, q).

It is decreasing on the interval $(-\infty, p]$.

It is increasing on the interval $[p, \infty)$.

$f(p - k) = f(p + k)$ for all k (because the graph G is *symmetric with respect to the vertical line* $x = p$).

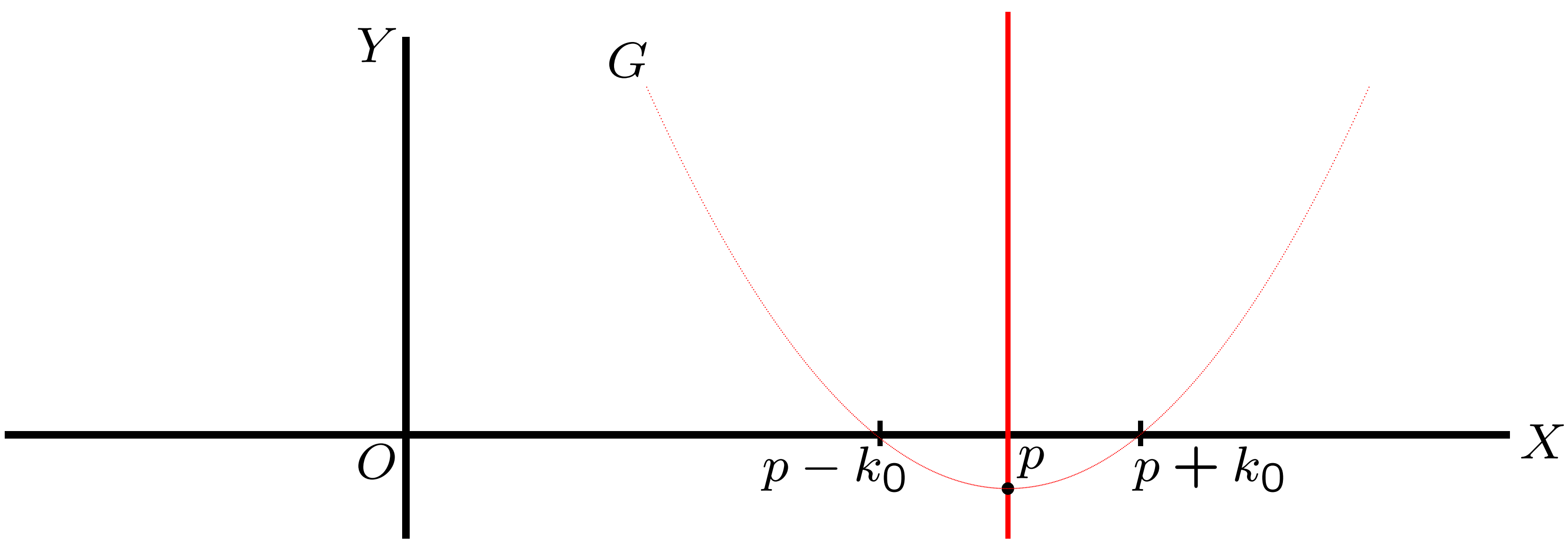
Obviously, these properties go a long way towards helping us understand general quadratic functions.

Less obvious, but no less important, is the fact is that *these properties now make perfect sense because we can now think of them in terms of the graph* G_a *of* $f_a(x) = ax^2$.

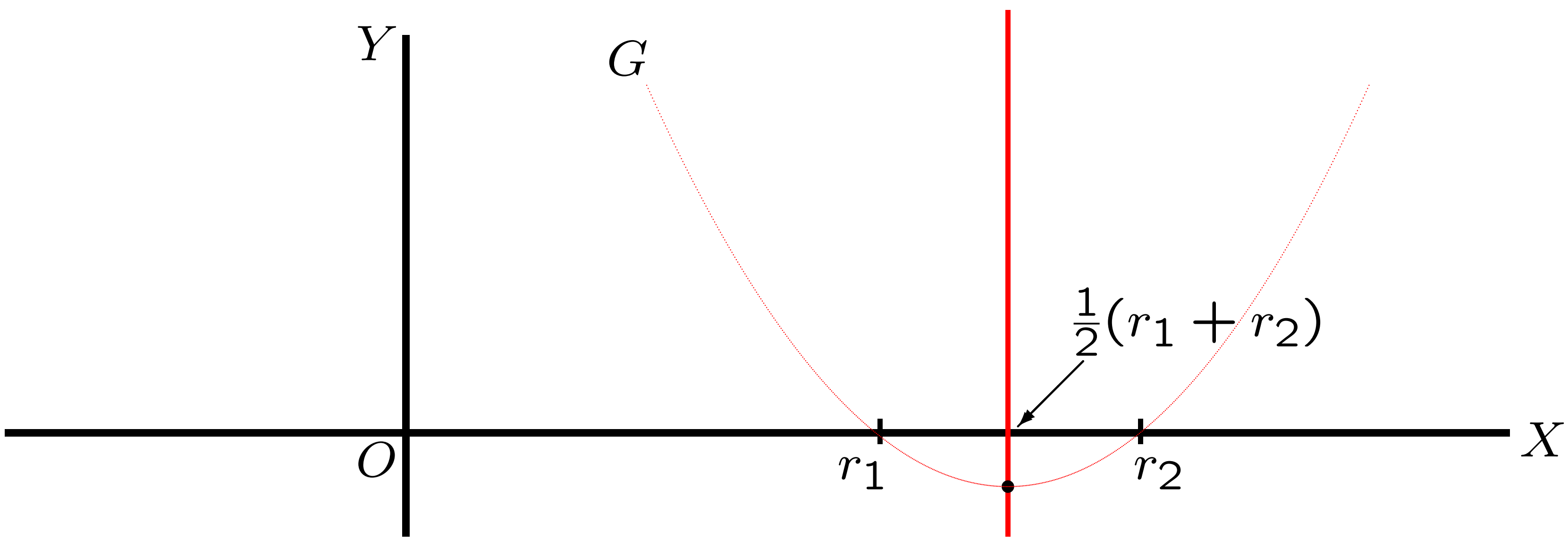
For example, we understand why $ax^2 + bx + c$ must attain a *minimum* somewhere when $a > 0$.

More is true. The *symmetry* of G with respect to the vertical line $x = p$ also explains why the roots (if they exist) of $f(x) = ax^2 + bx + c$ are symmetric with respect to the same vertical line:

$$f(p - k_0) = f(p + k_0) = 0 \text{ for some } k_0.$$



This also explains why, if we know the roots of $f(x) = ax^2 + bx + c$ are r_1 and r_2, then $f(x)$ attains its minimum at $\frac{1}{2}(r_1 + r_2)$ (assuming $a > 0$).



The second property about these $\{G_a\}$ $(a > 0)$ is:

(2) These curves are *similar* to each other. In fact, the *dilation* $(x, y) \longrightarrow (ax, ay)$ sends G_a to G_1 (the graph of $f_1(x) = x^2$).

From these two facts, we can paraphrase the study of quadratic functions $f(x) = ax^2 + bx + c$ (when $a > 0$) by saying that *if we know the function* $f_1(x) = x^2$*, then we know everything about quadratic functions.*

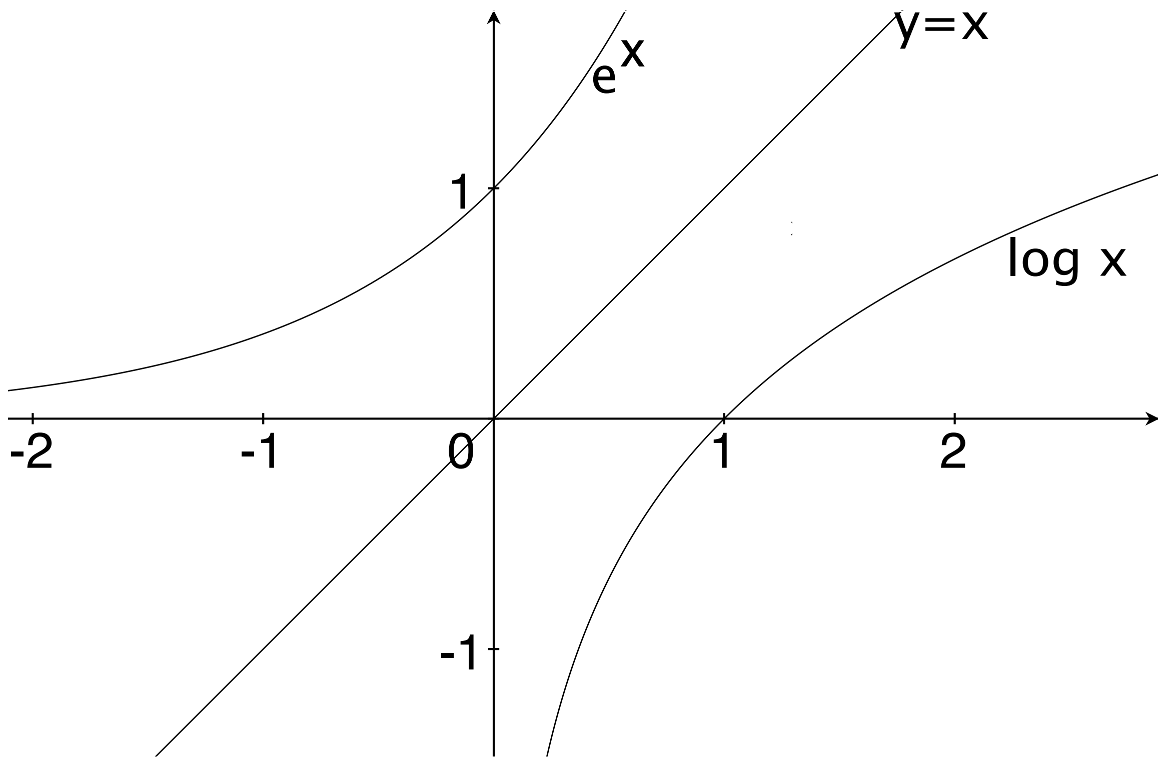
We can quickly dispose of the case $\boldsymbol{a < 0}$ for quadratic functions $F(x) = ax^2 + bx + c$ by observing that the *reflection* Λ across the x-axis clearly reflects the graph of $F(x) = ax^2 + bx + c$ to the graph of $f(x) = \; -ax^2 - bx - c$.

Now since $-a > 0$, everything we know about the graph of $f(x)$ can be transferred to the graph of $F(x)$, and therefore to the function $F(x)$ itself.

I hope you agree that the preceding discussion clarifies the study of quadratic functions.

This discussion would not be possible without the concepts of *translation, congruence, similarity, dilation, and reflection*. So once again, we see the critical need for integrating geometry into algebra if our goal is to facilitate the learning of algebra.

(C) The graphs of inverse functions. It is well-known that graphs of e^x and $\log x$ are *symmetric* with respect to the line $y = x$.

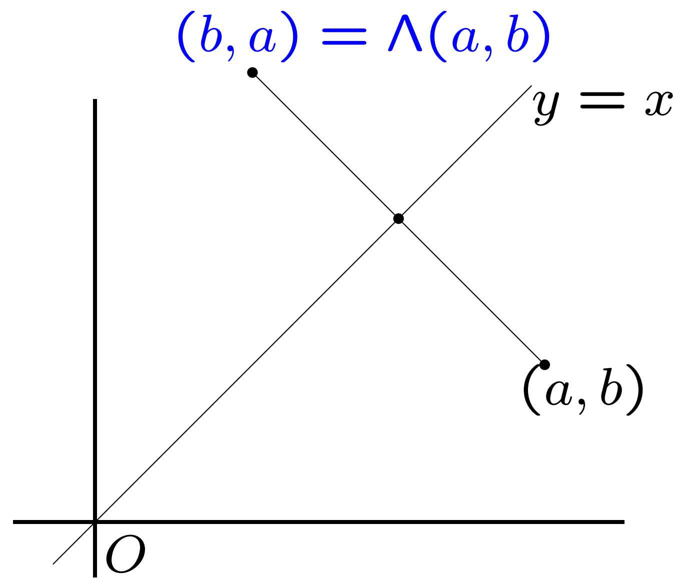


More generally: If a function $f(x)$ has an *inverse function* $g(x)$, then the graphs of $f(x)$ and $g(x)$ are *symmetric* with respect to the line $y = x$.

In TSM, "symmetric" is understood only in the *intuitive* sense. At this stage of students' education, however, they should learn to be precise.

We will make sense of the symmetry by *proving* the following two facts.

(1) If Λ is the reflection across the line $y = x$, then for any point (a, b), $\Lambda(a, b) = (b, a)$.



(2) If a function $f(x)$ has an *inverse function* $g(x)$, then (a, b) is on the graph of $f(x)$ $\Longleftrightarrow$ (b, a) is on the graph of $g(x)$.

Together, we get:

$$\Lambda(\text{graph of } f(x)) \;=\; \text{graph of } g(x)$$

This is the precise meaning of the *symmetry* of the graphs of $f(x)$ and $g(x)$ with respect to the line $y = x$.

Once again, we see the need of the concept of *reflection* to help clarify a basic fact about inverse functions.

In summary: Although we have only touched on truly basic topics in school algebra, we already witnessed the critical role played by geometric concepts related to congruence and similarity in clarifying basic concepts and skills in algebra.

If we get into more advanced topics in school algebra, then we will encounter more examples of the same phenomenon.

For example, the product of complex numbers is best expressed in terms of rotations around the origin of the plane. There is also the relationship between solutions of $x^2 - bx - b^2 = 0$ and the construction of regular pentagons. More profound is the constructibility of regular n-gons in terms of the solutions of $x^n - 1 = 0$.

However, we will limit ourselves to basic topics only.

Major obstacles in changing the geometry curriculum, and the proposed resolutions.

(a) We have seen that a proper treatment of linear equations and the slope of a line requires the concept of *similar triangles*. However, the concept of similarity is sophisticated and is not taught until the latter part of plane geometry, but the teaching of linear equations cannot wait.

Moreover, “similarity” is not just about triangles. We already encountered the *similarity of the graphs of* $f_a(x) = ax^2$ to each other for all $a \neq 0$. So not only should we teach similar triangles, but we must also teach a correct definition of “similarity”.

In college mathematics, a **similarity** is defined to be a transformation F of the plane so that for some positive constant c, $|F(P)\,F(Q)| = c\,|PQ|$ for all P and Q in the plane. *This definition is not usable in school mathematics*, however.

Solution: We can define **congruence** in the plane by using the elementary concepts of rotations, reflections, and translations. We can also define a **dilation** in the plane in an elementary fashion.

Then a **similarity** can be defined as the composition of a dilation followed by a congruence. This definiton is now appropriate for school mathematics.

(b) Even with a usable (and mathematically correct) definition of *similarity*, we cannot wait for a formally correct treatment of similarity before it can be applied to linear equations. We need a shortcut.

We can treat rotations, reflections, and translations in an *intuitive but correct manner*. Likewise for dilations. Then we can get quickly to the **AA criterion** for similar triangles (*two triangles with two pairs of equal angles are similar*). This theorem is sufficient for the applications to linear equations.

A *formal* treatment of congruence and similarity comes later as part of the systematic development of Euclidean geometry.

It retraces the same steps as above, but it gives, at the outset, precise definitions of rotations, reflections, translations, and dilations . Then it gets to similarity as before.

Pedagogically, taking up congruence and similarity twice—an intuitive approach first, to be followed by a precise version—makes sense, because a *precise* treatment of all these concepts requires very careful attention to technical details. The latter can be overwhelming to beginners.

It is better for students to first acquire the needed *intuitive* knowledge.

A comment from the American perspective: The idea of treating congruence and similarity twice—first intuitively and then precisely—is not new; it is in fact the standard practice in the American curriculum.

However, TSM defines congruence *intuitively* as "same size and same shape" and then defines it *precisely* only for polygons:

> Two polygons are said to be **congruent** if their sides and angles are pairwise equal.

This is bad education because it misleads students.

They are led to believe that congruence is a precise mathematical concept only for polygons. For general geometric figures, all one can say is that “congruence” means “same size and same shape” (which is of course unacceptable as mathematics).

Moral: In mathematics education, it is not enough to have a more or less correct idea. *Details matter.*

Essentially the same comments apply to the treatment of *similarity* in the American curriculum. First, similarity means "same shape but not necessarily the same size", and then only *similar polygons* are defined precisely.

We will be careful to avoid this pitfall.

PART II: A different approach to school geometry

We will outline how to teach rotations, reflections, and translations *intuitively*, and on this basis, define congruence.

Then for illustration, we will indicate how to prove the **SAS** criterion for triangle congruence in this setting.

We will next define dilation, and then similarity. We isolate the key fact in any discussion of similarity: *The Fundamental Theorem of Similarity.*

We will also make a few comments on how to teach the same topics *precisely* the second time around.

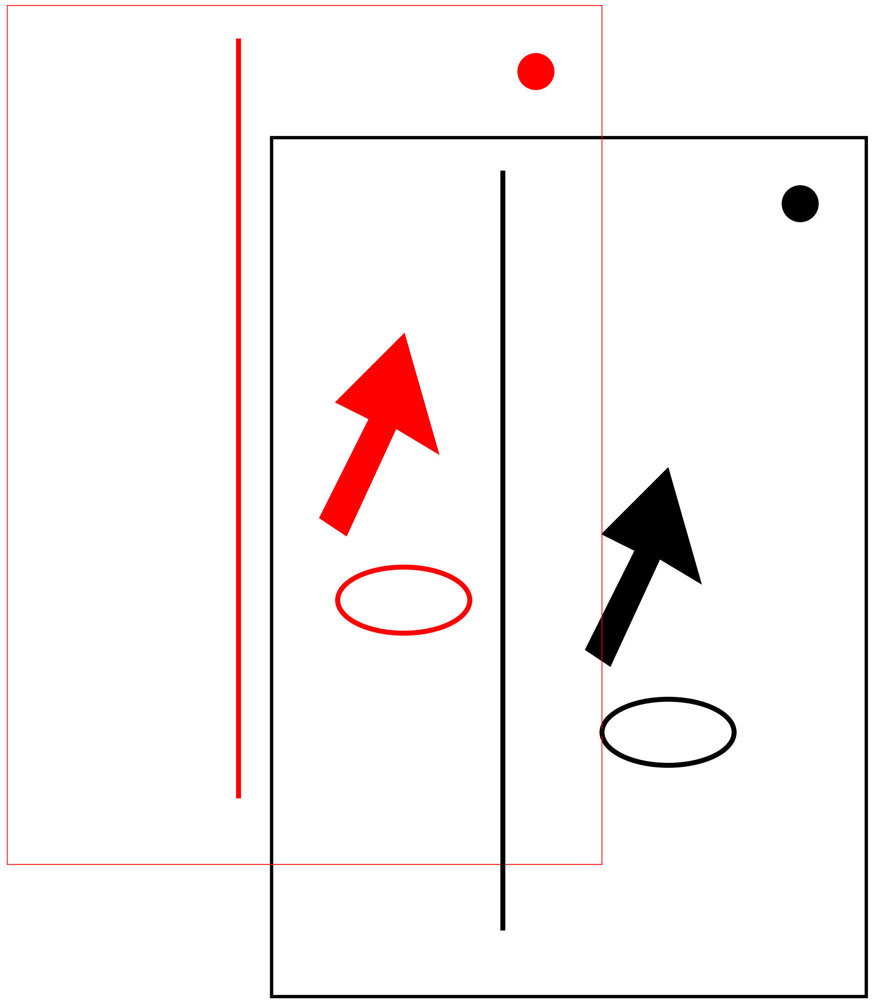
Up to this point, students are used to looking at geometric figures in the plane as *static* objects, in the sense that they don't move. But we are now going to *move every point in the plane* in a rigid manner, in ways to be described.

For starters, we will move the plane in three prescribed ways, to be called **rotations, reflections**, and **translations**. In the classical literature, these three are called *rigid motions*, for good reason!

We will define rotations, reflections, and translations with the help of *overhead projector transparencies*, as follows:

> Draw a geometric figure on a piece of paper in **black** color, copy the figure exactly on a transparency in red. Think of the paper as the plane, and think of moving the transparency as "moving points of the plane".

For example:



Two observations:

(i) Notice that we are not just moving the geometric figure in question, but are also *moving each and every point of the plane.*

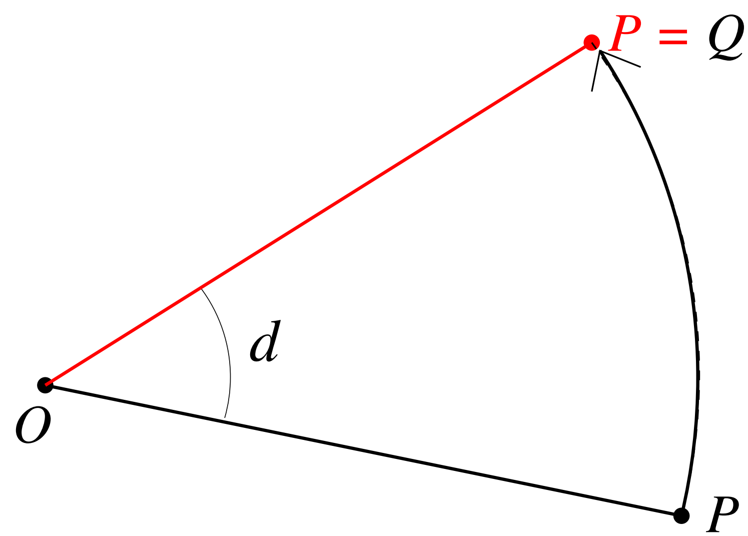
(ii) The following *verbal descriptions* of how to move a transparency will sound exceedingly clumsy. Rest assured that when a teacher demonstrates with transparency and paper, face-to-face with students in a classroom, it will be much easier to understand.

To define **rotation**, we have to choose a point O (the **center** of the rotation) and a number d as the **degree** of the rotation.

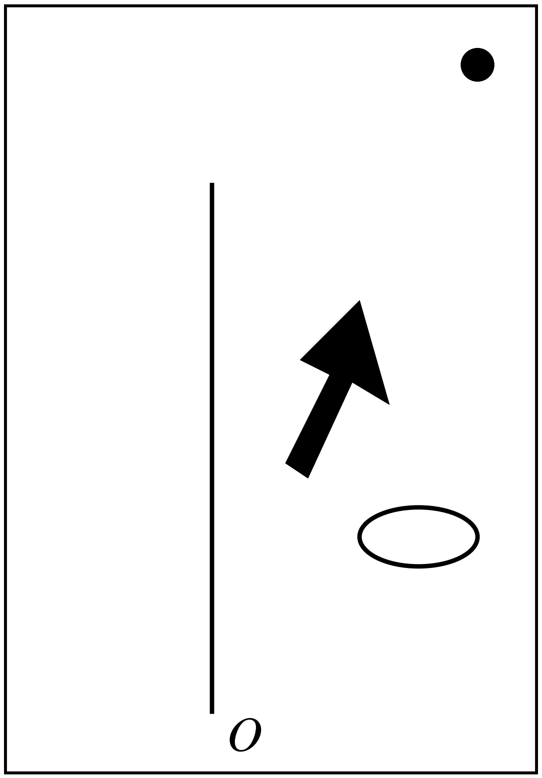
To describe the *counterclockwise rotation around O of d degrees*, all we have to do is describe how it moves a given point P of the plane to another point Q. So we have O and P drawn on the paper, and a transparency on which the points O and P have been copied exactly in *red*.

(1) Pin the transparency to the paper at O.

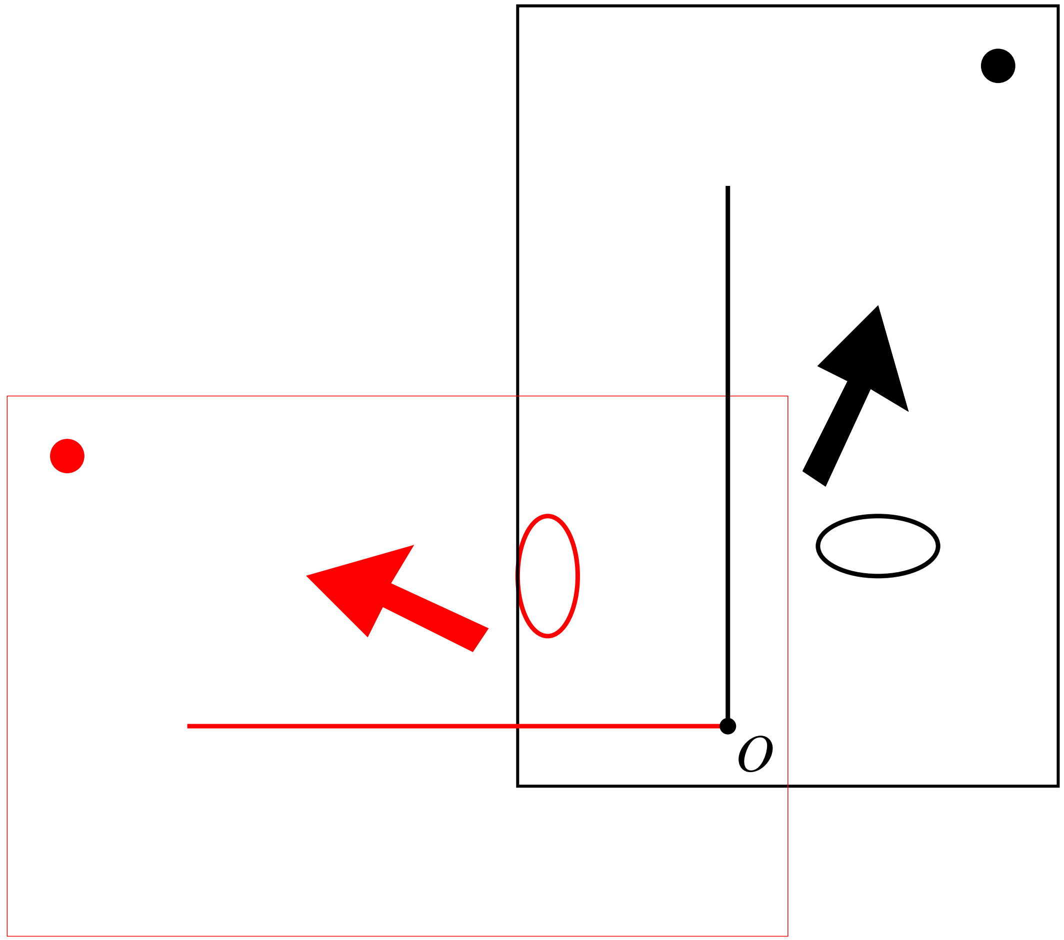
(2) Rotate the transparency d degrees counterclockwise around O. Then the *red* P lands at another point of the paper; this is the point Q.



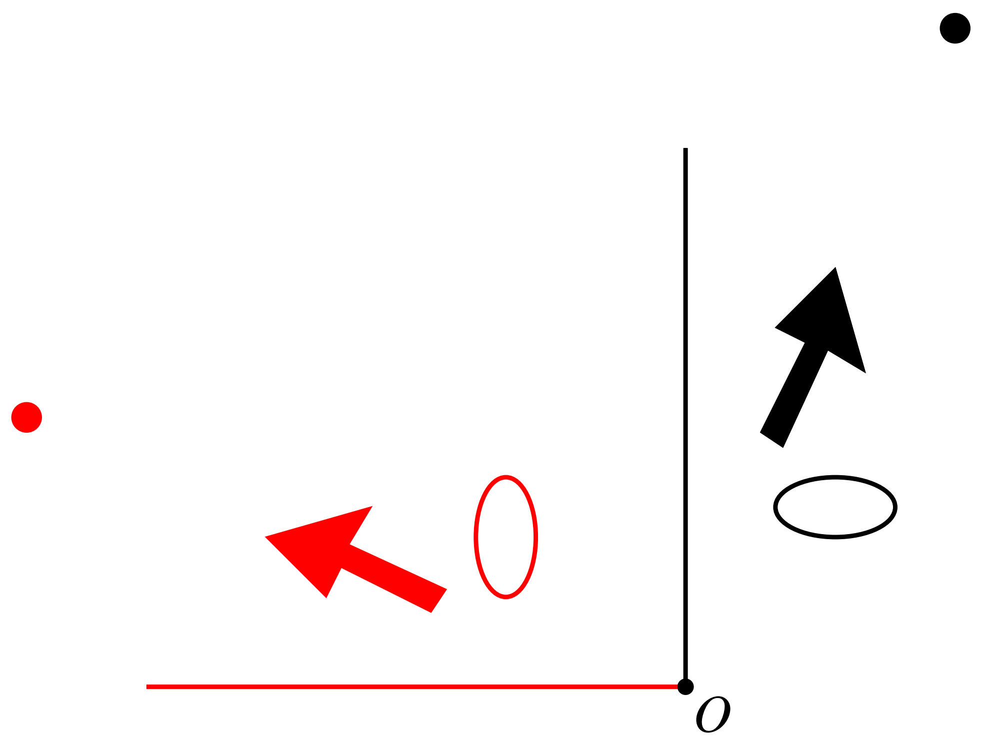
For example, draw the following figure on a piece of paper (the rectangle is the border of the paper) and then copy it in *red* on a transparency:



Here is how a counterclockwise rotation of 90 degrees around O moves the whole figure, point by point:



Now we show this rotation without the border of the paper:

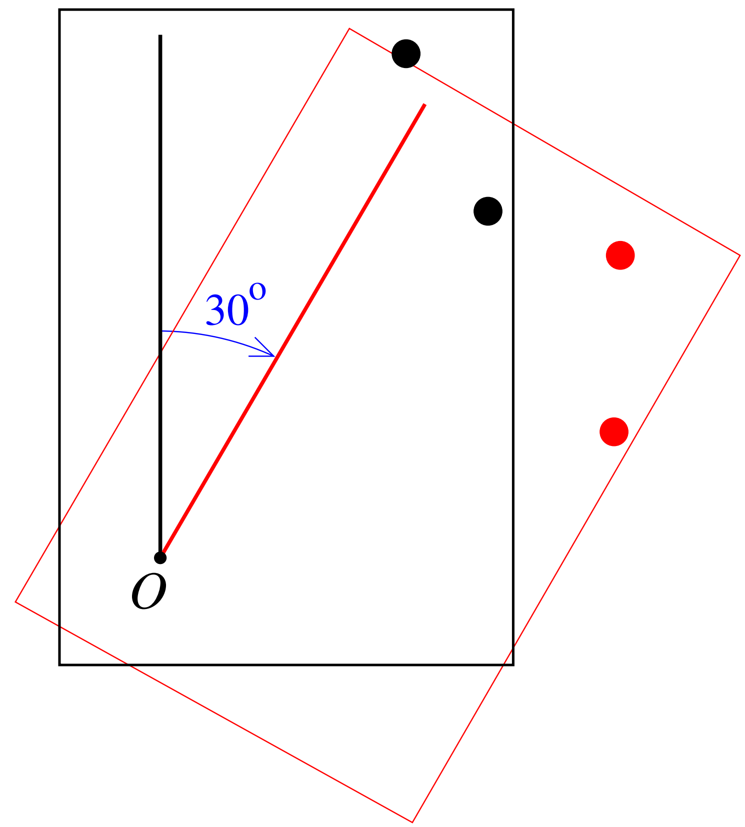


There are some illuminating animations on the definition of rotation by Sunil Koswatta that you should consult:

`http://www.harpercollege.edu/~skoswatt/RigidMotions/rotateccw.html`

`http://www.harpercollege.edu/~skoswatt/RigidMotions/rotatecw.html`

Clockwise rotations are defined likewise. Let students experiment with different choices of the center and the degree of a rotation, using any figure they come up with. For example, here is a 30-degree *clockwise* rotation around O of a figure consisting of a vertical segment and two big dots.

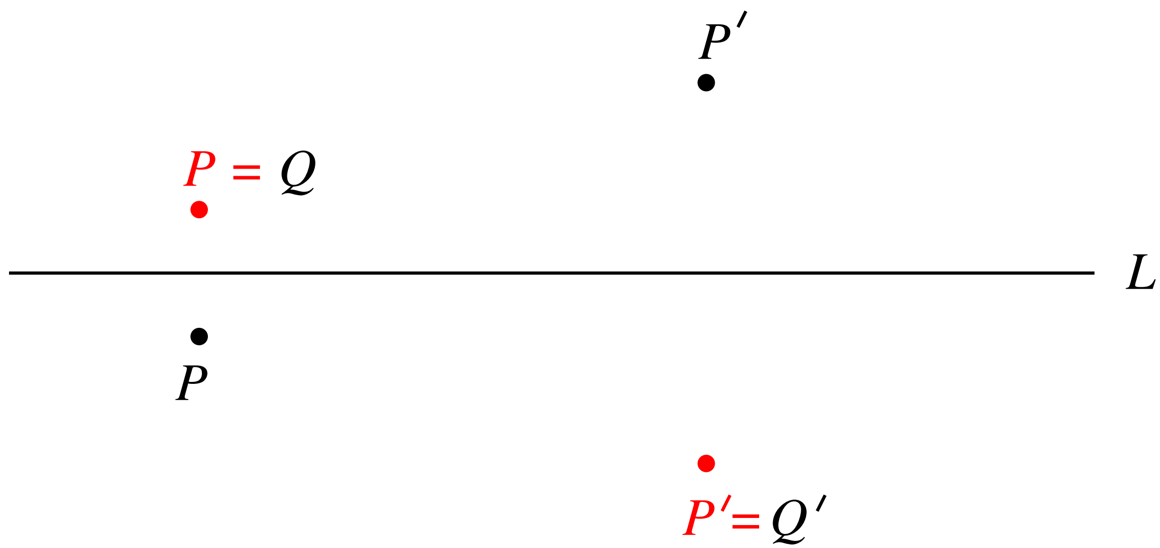


Next, **reflection**. Fix a line L in the plane, and we will describe the *reflection across L*. Given a point P, we will specify how to reflect P across L to a point Q. On the transparency, copy L and P in *red*. Now turn over the transparency across L so that

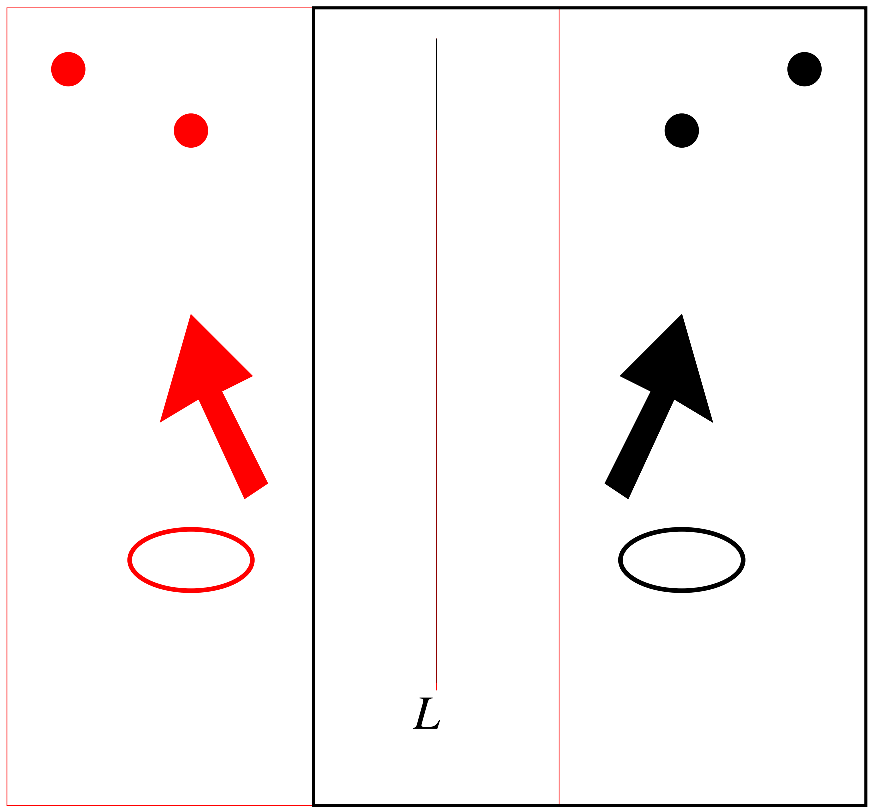
> every point of *red L* falls on itself, and therefore
>
> L also falls on itself, and
>
> the two half-planes of L are interchanged.

Then the point on the paper on which the *red P* lands is the Q we are looking for.

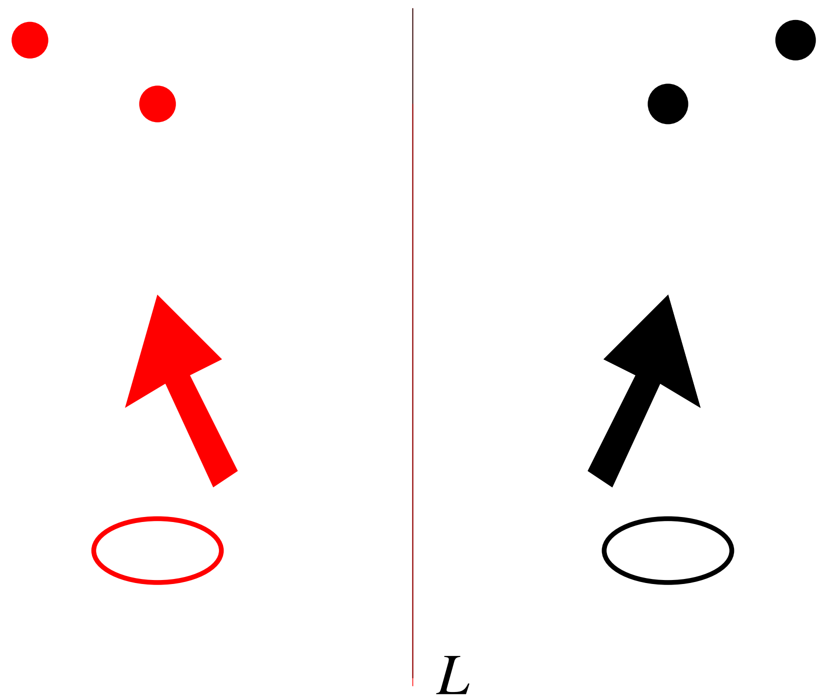
For example, the reflection across the horizontal line L below moves the points P to Q and P' to Q': for example, Q is where the *red* P is, and Q' is where the red P' is.



Here is the reflection across the vertical line L of a black figure consisting of an arrow, an ellipse, and two dots. (Every point on L is reflected to itself.)



Here is the same picture without the border of the paper.



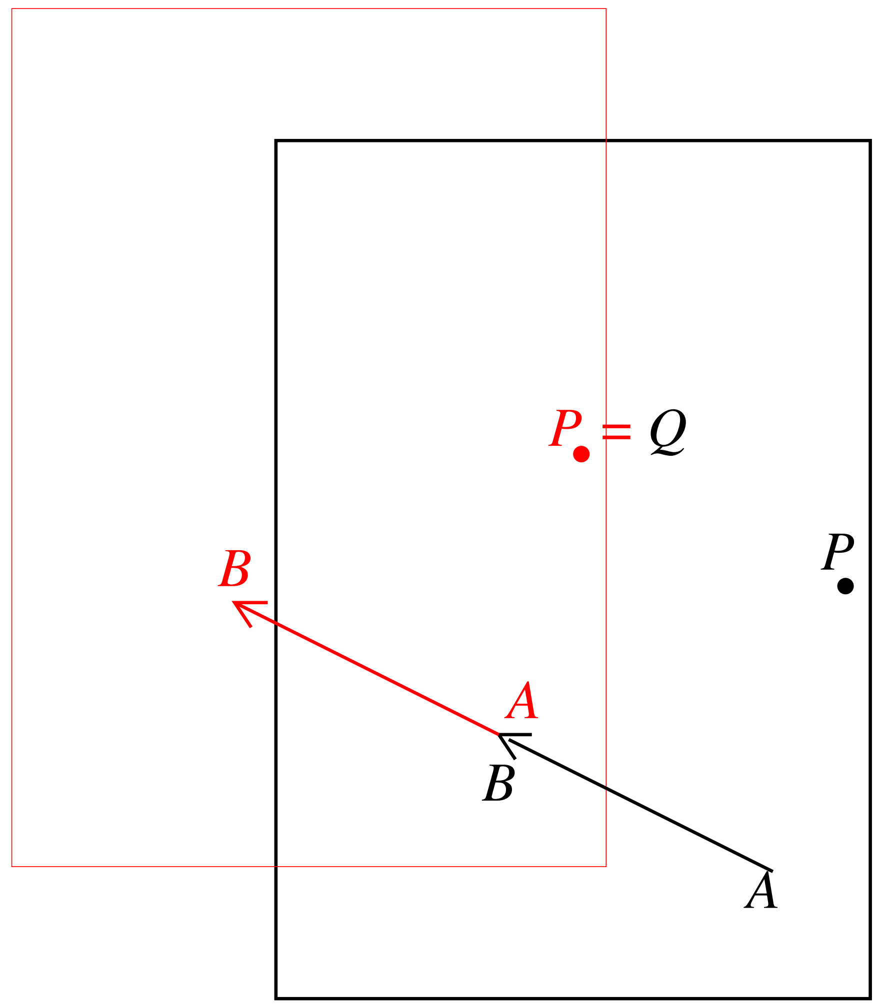
Be sure to consult the animations by Sunil Koswatta on the definition of reflection:

`http://www.harpercollege.edu/~skoswatt/`

`RigidMotions/reflection.html`

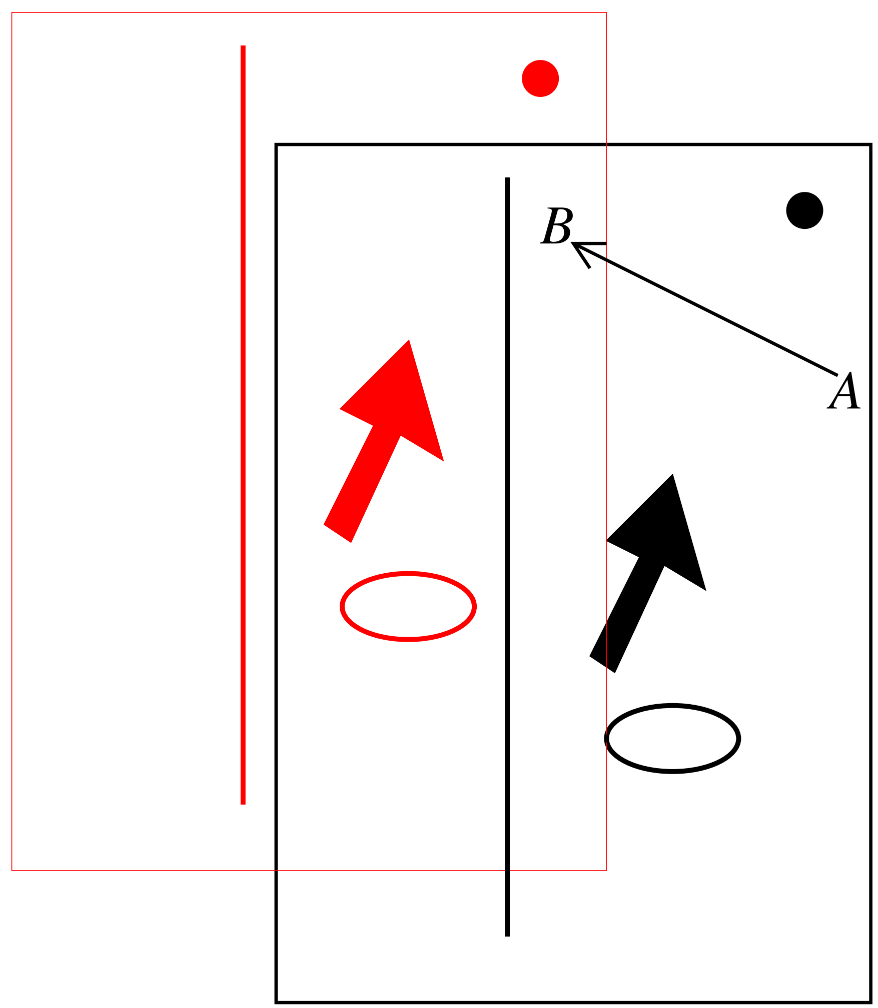
The description of a **translation** requires the choice of a **vector** $\overrightarrow{AB}$ (a segment with a beginning point A and an endpoint B).

Given a point P in the plane, then the *translation along* $\overrightarrow{AB}$ moves P to the point Q, to be described as follows. Copy P and $\overrightarrow{AB}$ in *red* on the transparency as usual. Now slide the transparency so that the *red* $\overrightarrow{AB}$ slides along the *line* L_{AB} (joining A and B) until the *red* A slides to where the black B is. Then Q is where the *red* P rests.

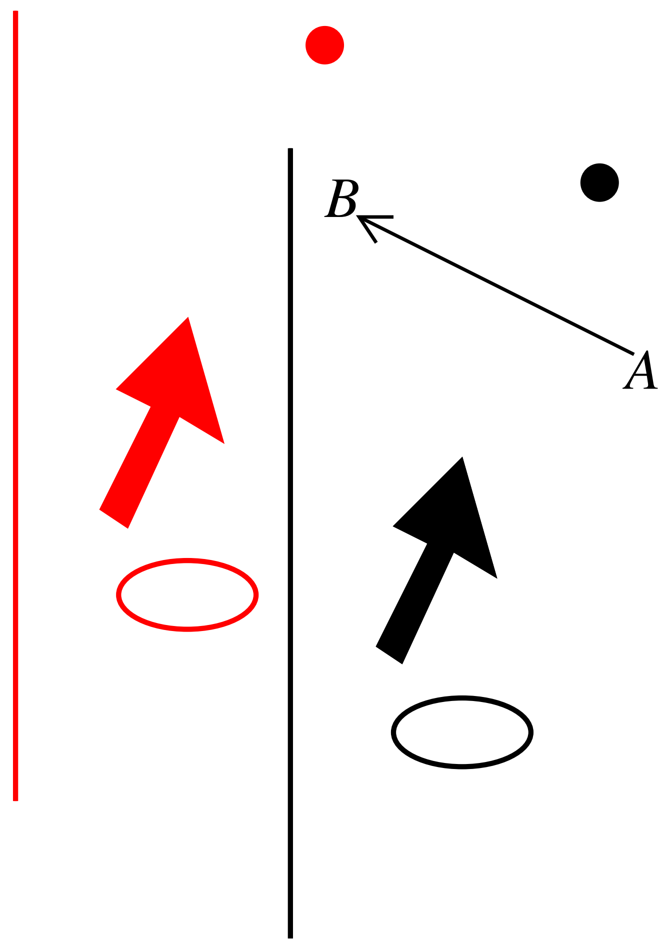


P = Q
P
B
A
B
A

Here is the translation along $\overrightarrow{AB}$ of a by-now familiar figure:



Here is the same picture without the border of the paper.



Again, consult the animations of Sunil Koswatta on the definition of translation:

`http://www.harpercollege.edu/~skoswatt/`

`RigidMotions/translation.html`

We will refer to rotations, reflections, and translations as **basic isometries**.

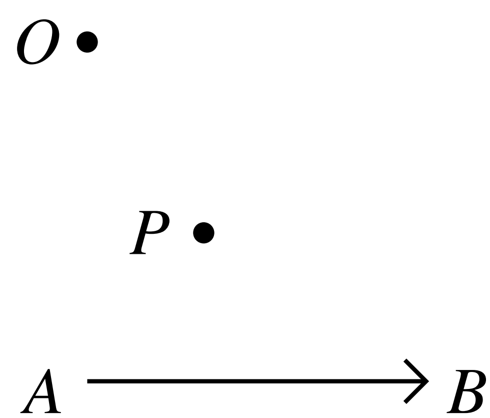
Now that we know the definitions of these basic isometries, we can see from their definitions that:

> **(a)** They move lines to lines, segments to segments, and angles to angles.
> **(b)** They preserve lengths of segments and degrees of angles.

The power of the basic isometries is derived from the two preceding properties together with the ability to "combine" basic isometries, as we now explain.

Suppose F and G are two basic isometries, then we can consider *moving each point of the plane first by F and then by G*. More precisely, if F first moves a point P to Q, then G moves Q to another point R. Altogether, P is moved to the point R. Let us denote this combined motion that moves P to R by $G \circ F$.

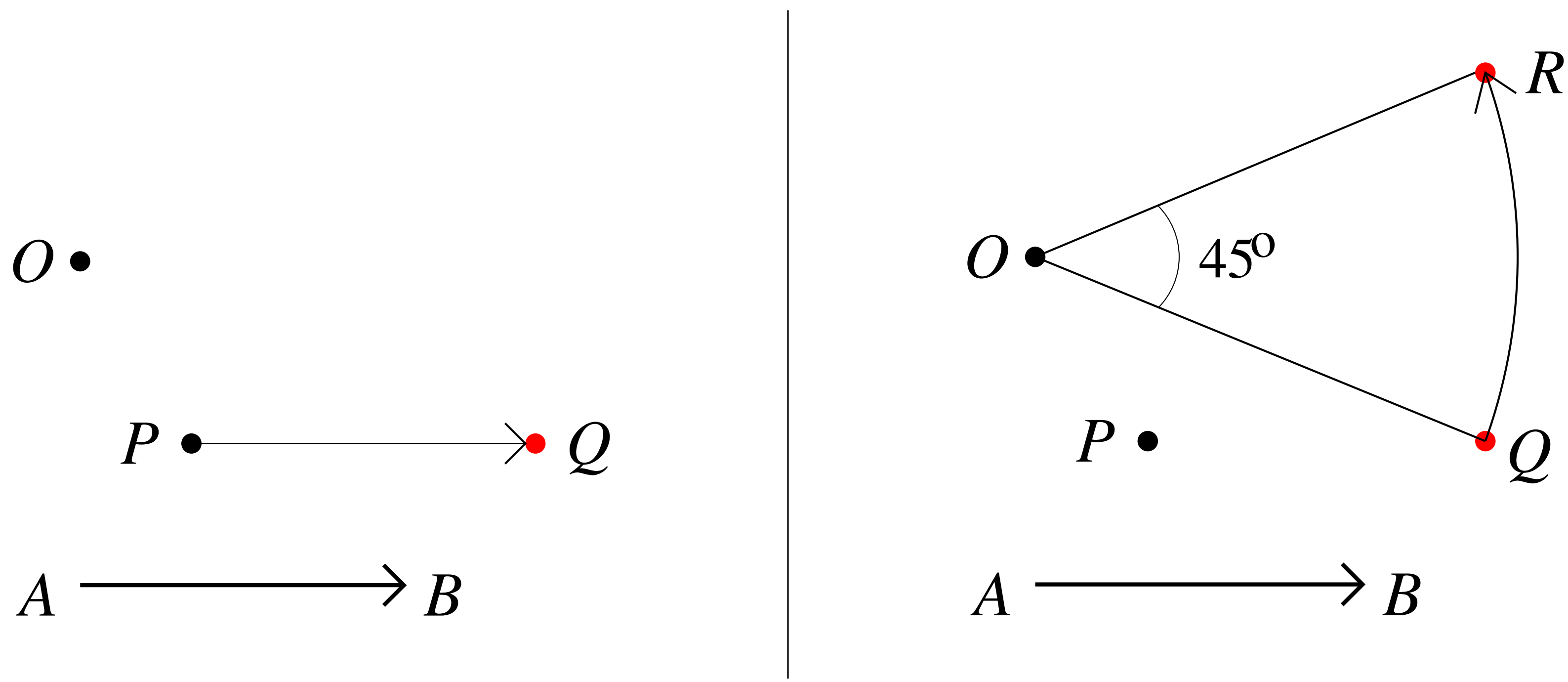
To illustrate, let $\overrightarrow{AB}$ be a given vector and let O be a given point, as shown:



Let F be the translation along $\overrightarrow{AB}$ and let G be the 45° counterclockwise rotation around O.

If $G \circ F$ moves P to R, where is R?

F moves P to Q (see left picture below), and then G moves Q to the point R (see the right picture below).



Once we see how to “combine” two basic isometries, we can iterate the procedure and move points in the plane by “combining” any number of *basic isometries*.

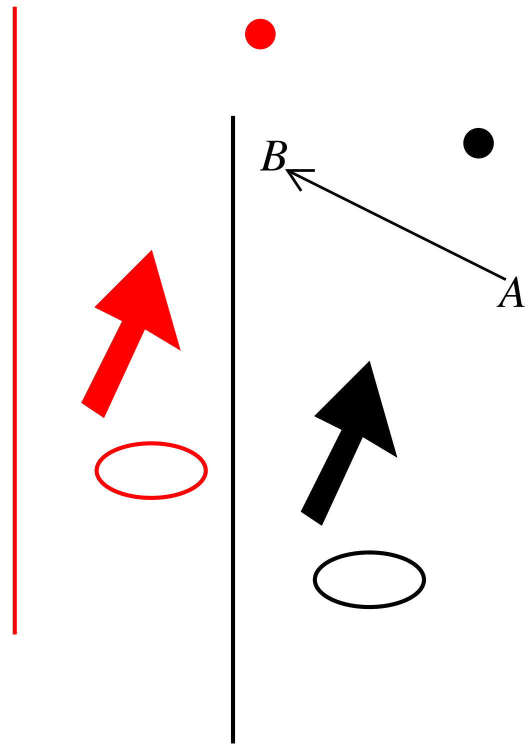
The technical term for “combining basic isometries” is a **composition of basic isometries**.

A **congruence** of the plane is by definition a composition of (any number of) basic isometries.

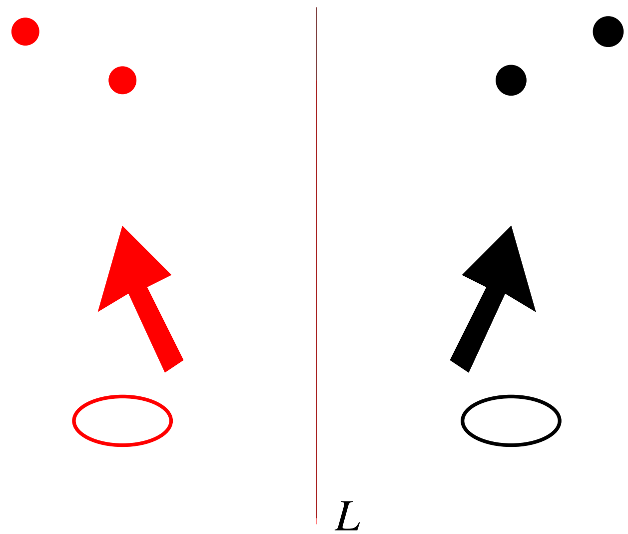
Fortunately, there is a video by Larry Francis that will help clarify the concept of the *composition of basic isometries:*

`http://youtu.be/O2XPy3ZLU7Y`

Remark: We now know what it means even for a curved figure to be congruent to another. For example, the translation along $\overrightarrow{AB}$ shows that the *red ellipse is congruent to the black ellipse*.



Similarly, the following two ellipses are congruent by a reflection.



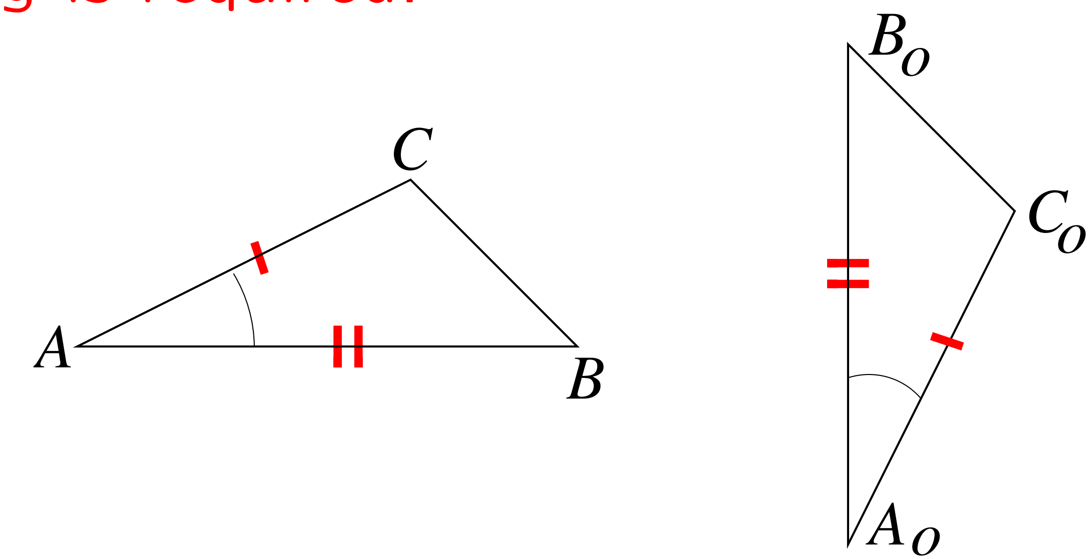
Referring to the earlier discussion of quadratic functions, we now have a new appreciation of the statement that the graph of $y = ax^2$ is *congruent* to the graph of $y = ax^2 + bx + c$ by a translation.

Next, we will use the basic isometries to *prove* the SAS criterion of triangle congruence. The other criteria for triangle congruence can be proved in the same way.

The significance of such a proof lies in its implication on the learning of geometry. Students are in general puzzled by the *abstract* concept of congruence that comes from axiomatic geometry.

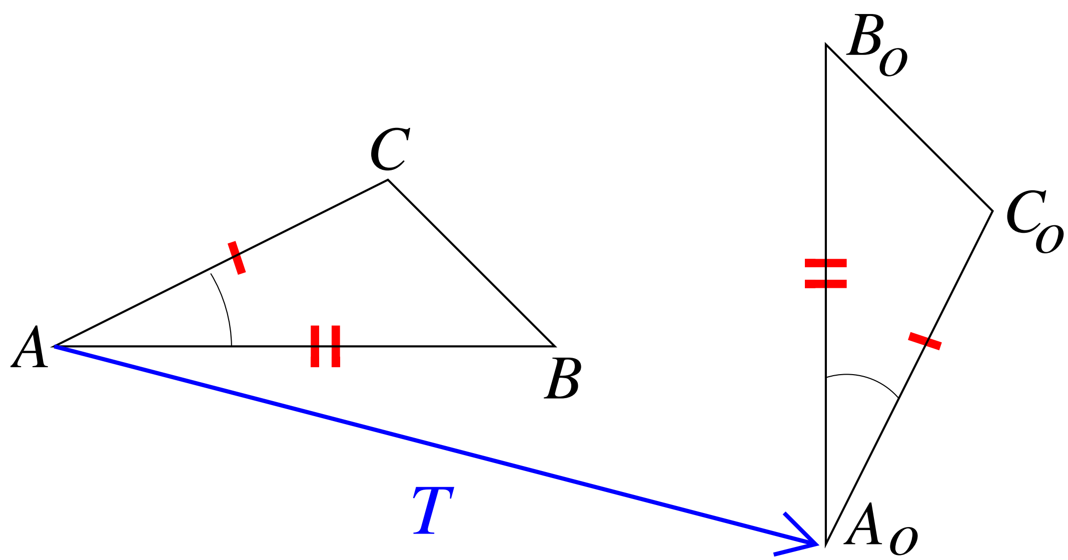
Now congruence becomes a *concrete and tangible* concept that can be realized by hands-on activities.

Proof of SAS: This is a proof meant to be given in the classroom by *moving* (plastic or wooden) *models of triangles on the blackboard or document camera.* No writing is required.

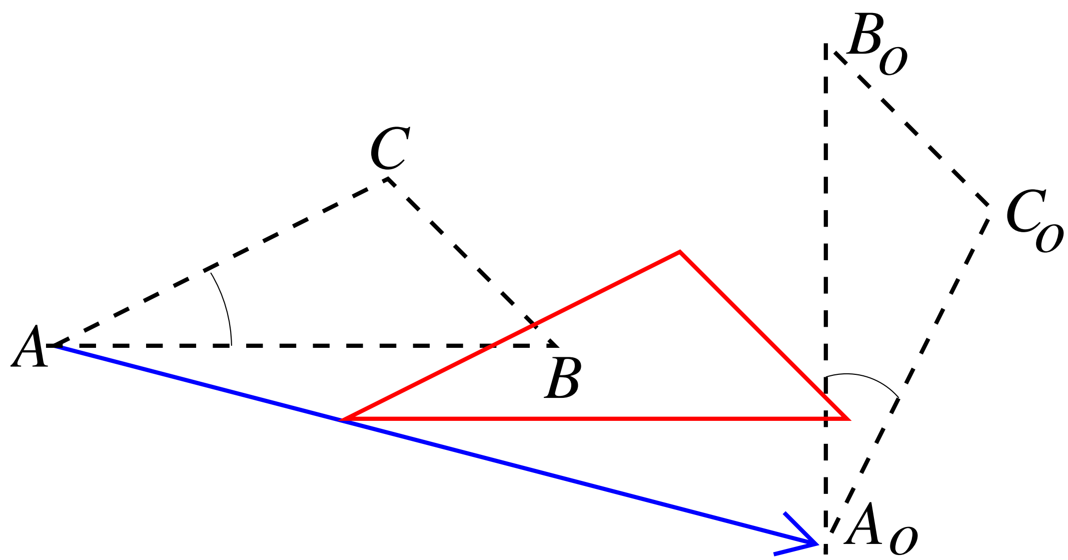


We want to prove that the two triangles as shown are congruent.

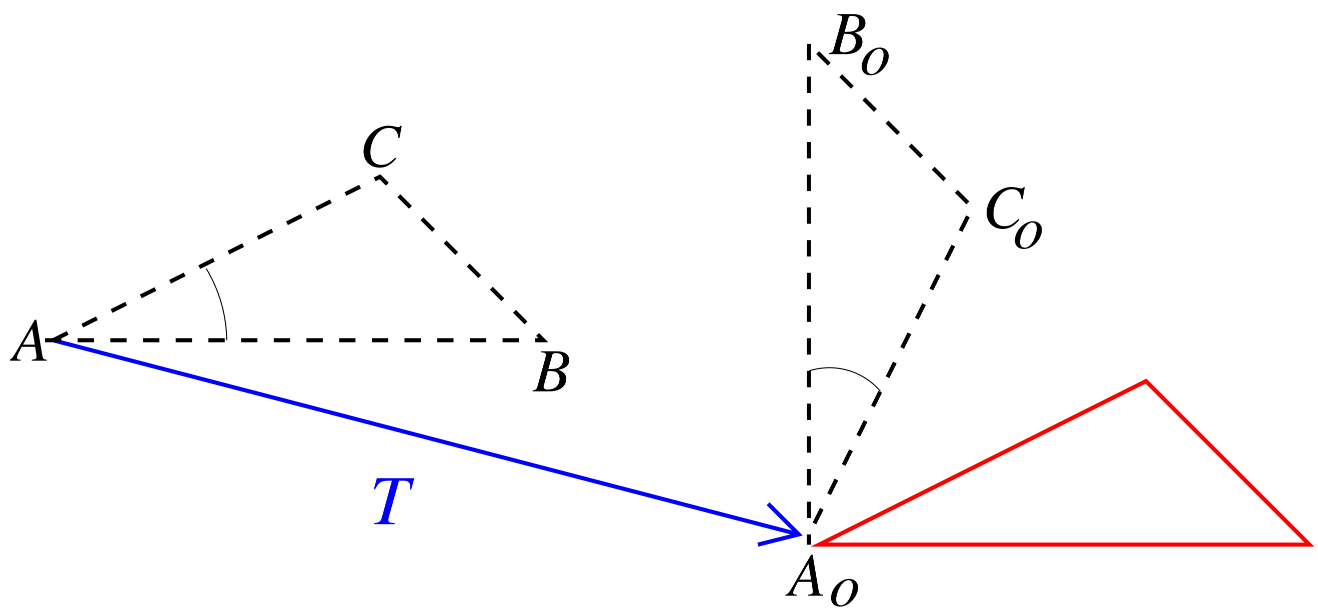
The first step is to bring the vertices of the equal angles together by the translation along $\overrightarrow{AA_0}$. Call the translation T.



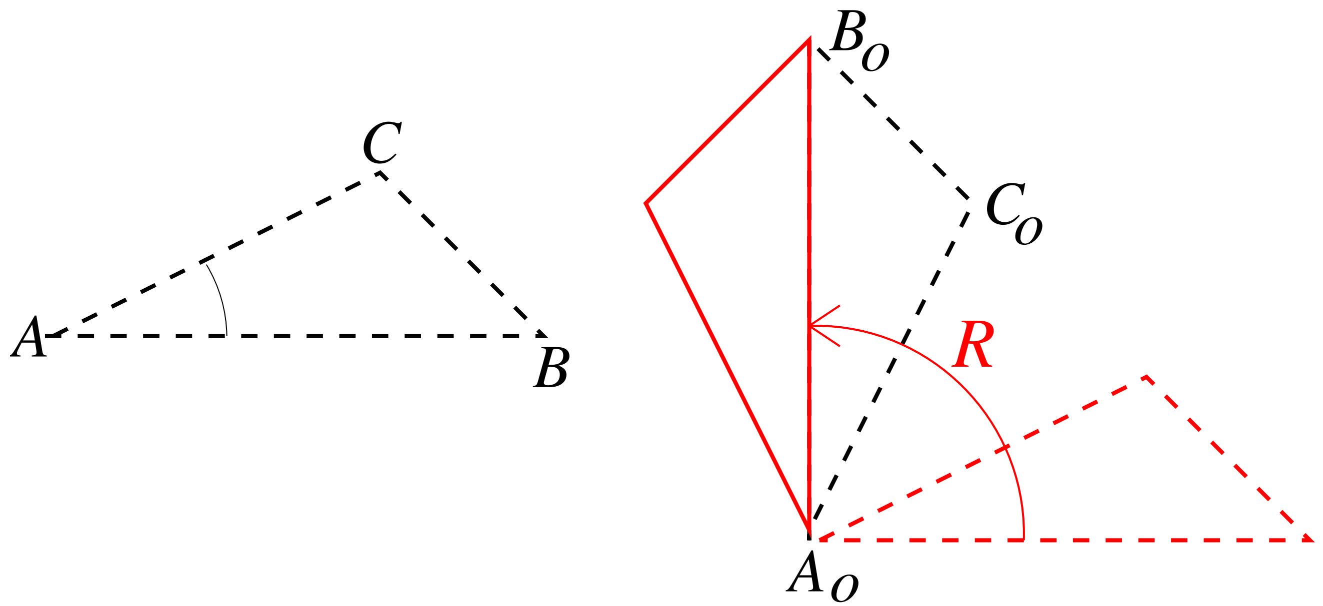
Here is $\triangle ABC$ being translated along $\overrightarrow{AA_0}$. The translation moves *every point in the plane* (including $\triangle A_0B_0C_0$), but we must remember the original positions of the triangles.



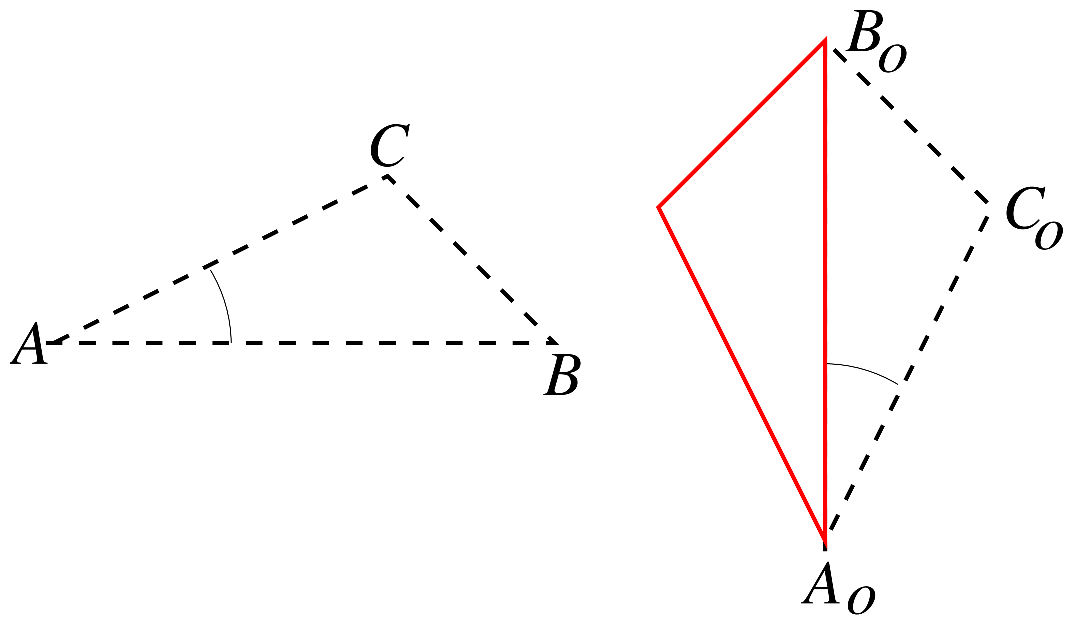
Final result of the translation of $\triangle ABC$, bringing A to A_0.



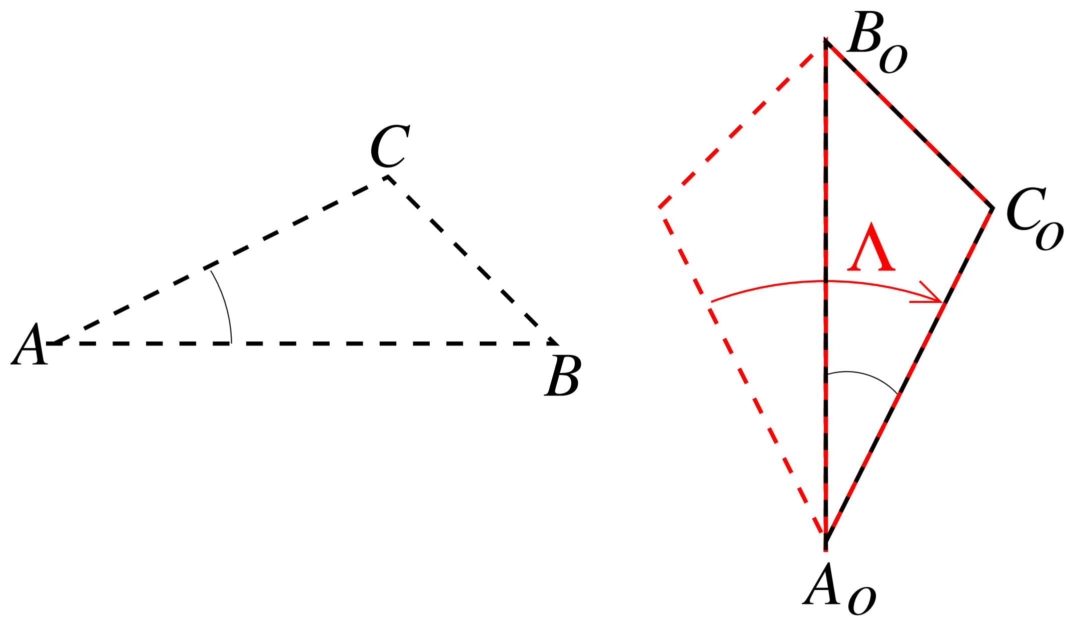
Since it is given that AB and A_0B_0 are equal, a rotation R around A_0 achieves the matching of one side of the red triangle with the *original* side A_0B_0, as shown.



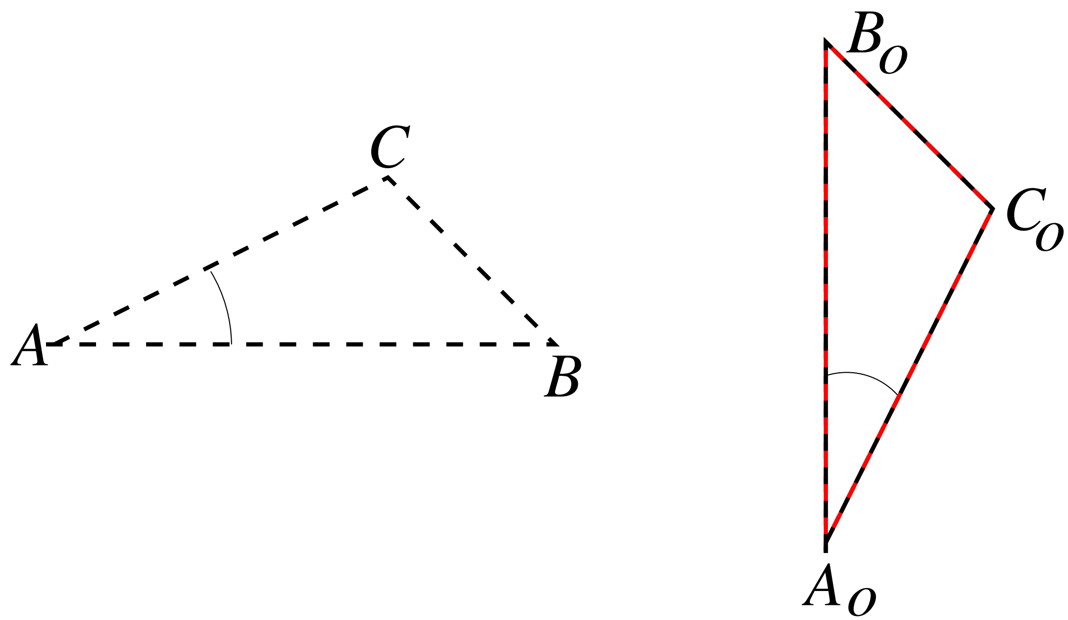
So we have gotten to this stage:



Finally, since $\angle CAB$ and $\angle C_0A_0B_0$ are equal, and also sides AC and A_0C_0 are equal, the reflection Λ across the line $L_{A_0B_0}$ brings the red triangle to match the *original* $\triangle A_0B_0C_0$ exactly.



Thus the congruence $\Lambda \circ R \circ T$ moves $\triangle ABC$ to $\triangle A_0B_0C_0$.



There is an animation by Larry Francis that gives exactly this proof: `http://youtu.be/30dOn3QARVU`

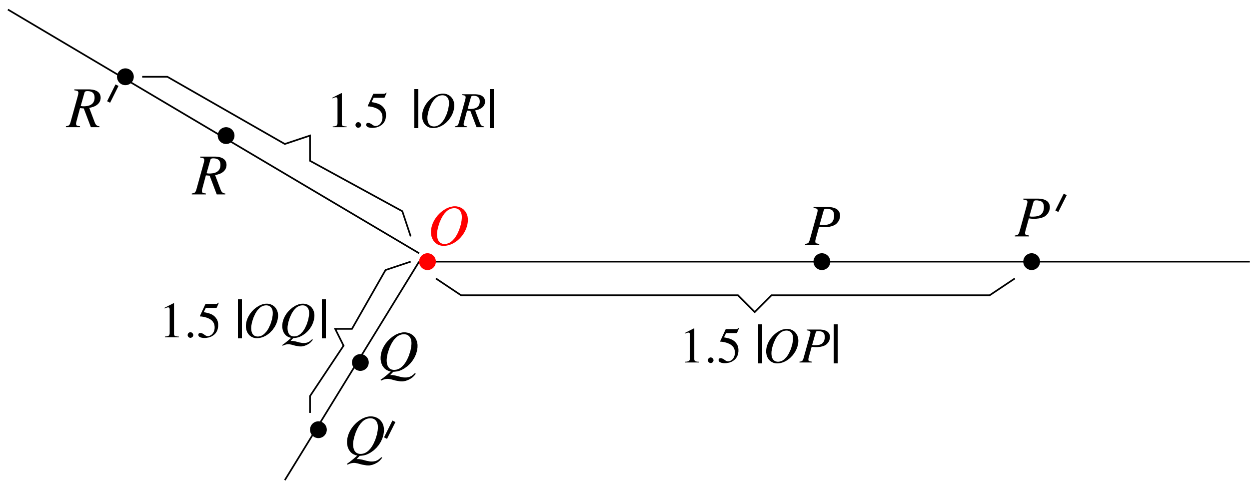
Some brief concluding remarks concerning the teaching of congruence and similarity.

Still on an intuitive level, the next major concept is a *dilation with center O and scalar factor $r > 0$* that moves a point P to a point P' so that:

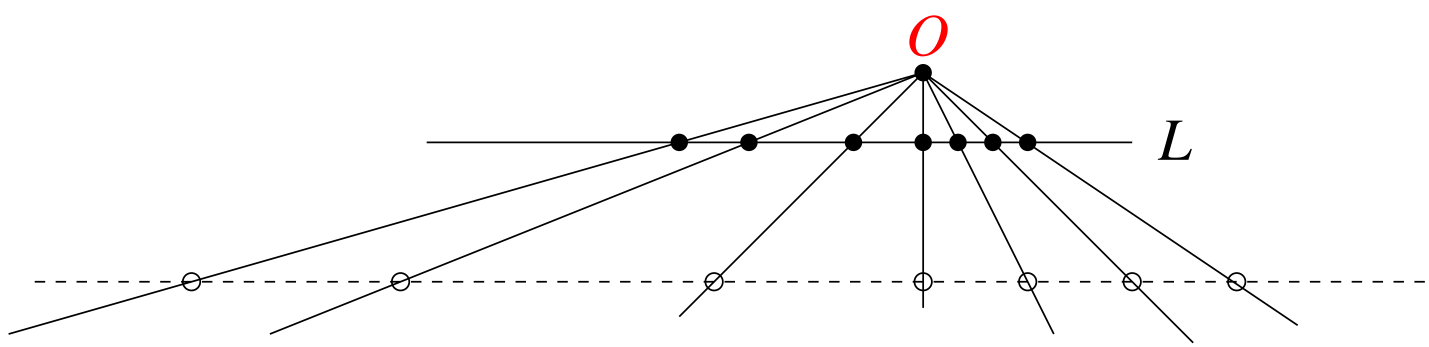
(i) if $P = O$, then $P' = O$, i.e., O stays put.

(ii) if $P \neq O$, then P' lies on the *ray* R_{OP} so that $|OP'| = r \cdot |OP|$.

Here is an example where the scalar factor of the dilation is 1.5 and the dilation moves P, Q, R to P', Q', and R', respectively.

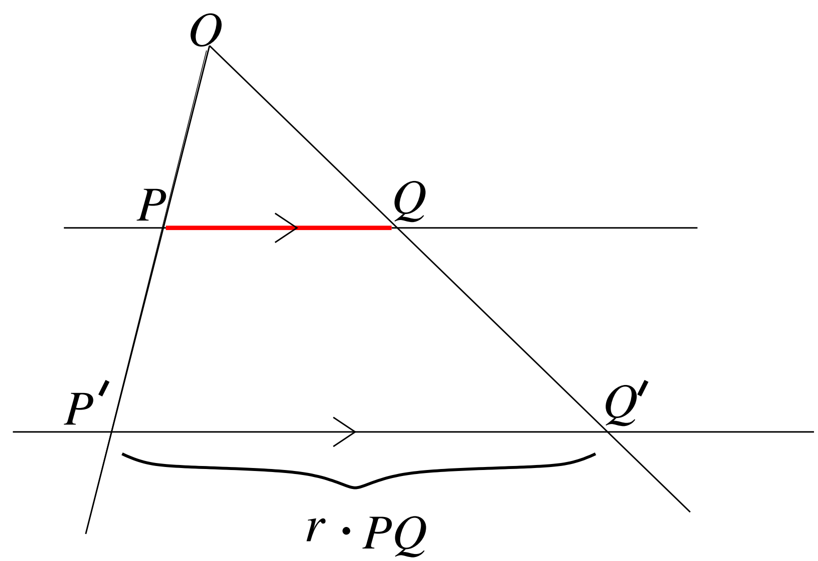


Students can do experiments to verify that if we take a point O not lying on a line L, and dilate the points on L (one at a time) with a fixed scalar factor r, then the dilated points appear to also lie on a line. (In this picture, $r = 3$.)

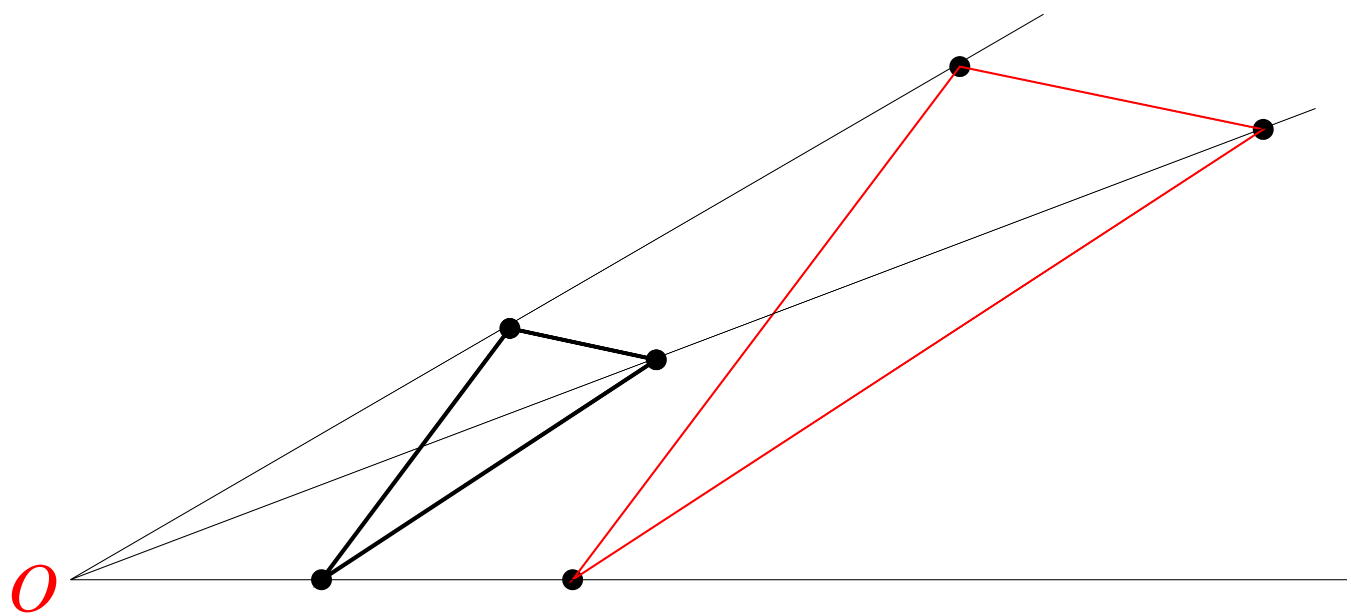


The central fact is this (it will be assumed):

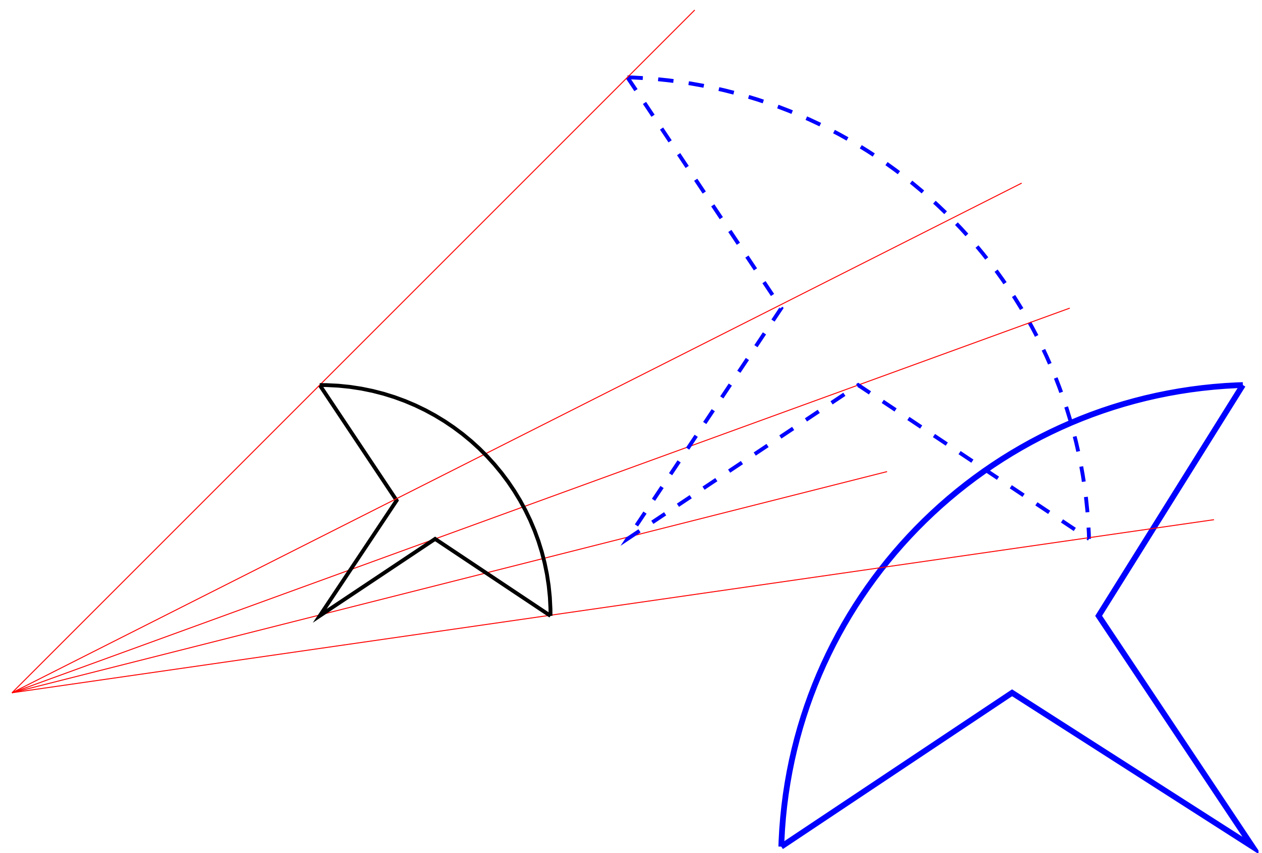
Fundamental Theorem of Similarity. *If a dilation with center O and scale factor r moves two points P, Q in the plane* not collinear with O *to P' and Q', then: (1) the dilation moves the lines L_{PQ} to the line $L_{P'Q'}$ and the segment PQ to the segment $P'Q'$, (2) $L_{PQ} \parallel L_{P'Q'}$, and (3) $|P'Q'| = r \cdot |PQ|$.*



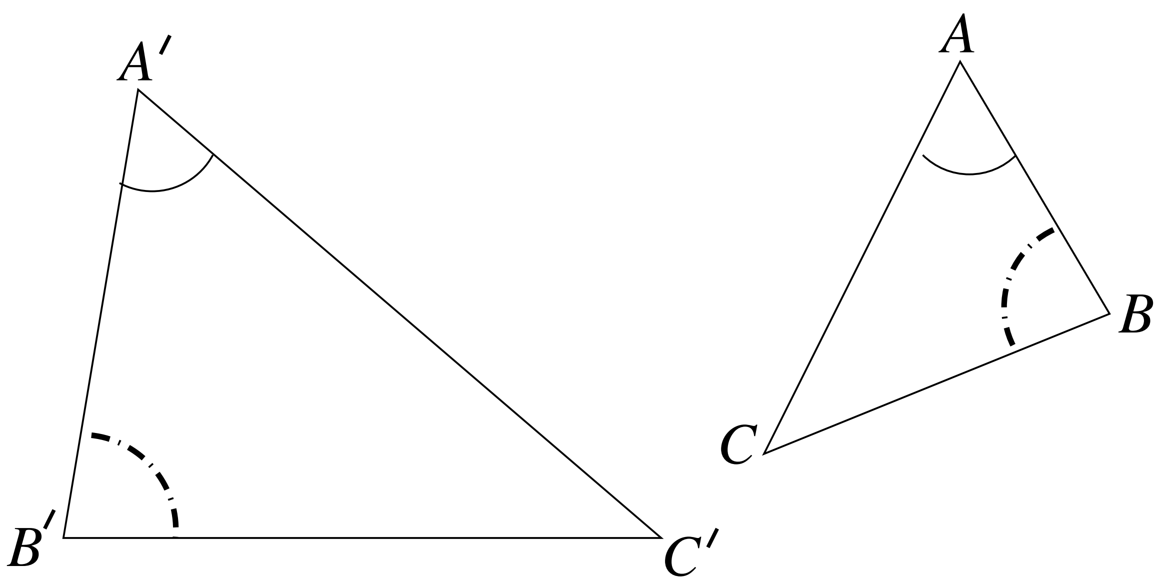
This theorem makes it very easy to draw the dilations of polygons. For example, here is the dilation of the black triangle with a scale factor of 2: notice that the *red* dilated triangle "has the same shape" as the original triangle.



We can now define a *similarity* as the composition of a dilation followed by a congruence. For example, the black figure below is *similar* to the *solid* blue figure:



For completeness, we state the **AA criterion for similarity** without proof: *given two triangles* $\triangle ABC$ *and* $\triangle A'B'C'$, *if* $\angle A$ *and* $\angle A'$ *are equal, and* $\angle B$ *and* $\angle B'$ *are also equal, then* $\triangle ABC \sim \triangle A'B'C'$.

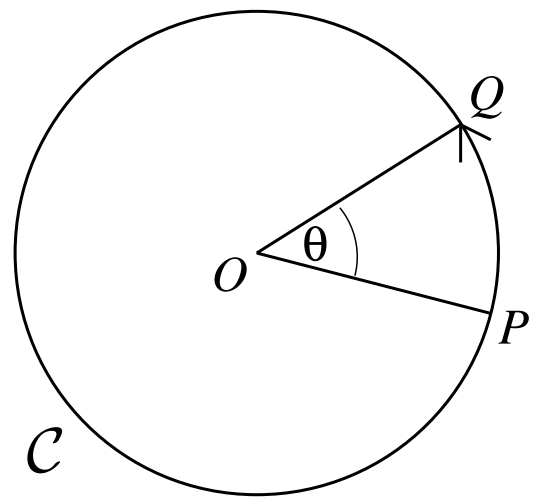


Recall that we said congruence and similarity would each be taught twice in this curriculum: first intuitively, and then as *formal mathematics.*

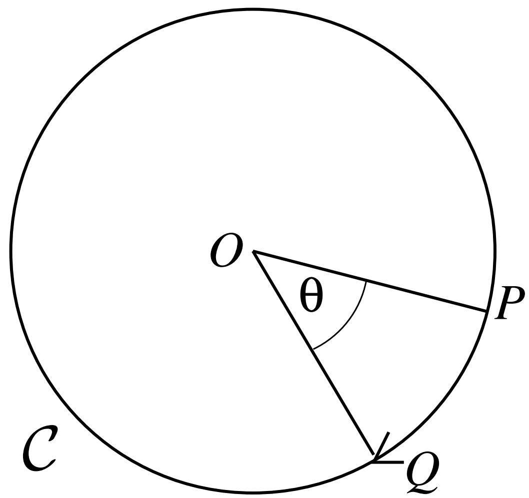
We have briefly outlined the instruction on the intuitive level. It remains to point out that when these topics are taught on a formal level, the definitions of rotations, reflections, and translations can no longer be given by using transparencies. They must be precisely defined.

Here is a definition of rotation: Given a point O and θ so that $-360 \leq \theta \leq 360$, the **rotation of $\boldsymbol{\theta}$ degrees around $\boldsymbol{O}$** is the transformation $\boldsymbol{\varrho_\theta}$ so that $\varrho_\theta(O) = O$, and if $P \neq O$, $\varrho_\theta(P)$ is defined as follows: Let $\mathcal{C}$ be the circle of radius OP centered at O.

If $\theta \geq 0$, $\varrho_\theta(P)$ is the point Q on $\mathcal{C}$ obtained from P by turning P θ degrees in the counter-clock-wise direction along $\mathcal{C}$.



If $\theta < 0$, $\varrho_\theta(P)$ is the point Q on $\mathcal{C}$ obtained from P by turning P $|\theta|$ degrees in the clockwise direction along $\mathcal{C}$.



The two properties **(a)** and **(b)** above concerning basic isometries will now have to be explicitly **assumed** for rotations.

On the basis of these assumptions about rotations, simple theorems such as *opposite sides of parallelograms are equal* will have to proved first before we can give the precise definitions of reflections and translations and show that they are well-defined.

Again properties **(a)** and **(b)** above will be **assumed** for reflections and translations. Together with other natural assumptions such as the Parallel Postulate, we now have the foundation for the usual development of Euclidean geometry in secondary school.

General references:

H. Wu, *Teaching School Mathematics: Pre-Algebra*

H. Wu, *Teaching School Mathematics: Algebra*

Both published by the American Mathematical Society, 2016.

H. Wu, *Mathematics of the Secondary School Curriculum*, I, II, and III (to appear ∼2018)