What Is School Mathematics?

Lisbon, Portugal
January 30, 2010

H. Wu

*I am grateful to Alexandra Alves-Rodrigues for her many contributions that helped shape this document.

The German conductor Herbert von Karajan once said that there is no vulgar music, only vulgar performance.

I recall this statement by Karajan because mathematicians love to say that school mathematics is *trivial*. Let me therefore paraphrase Karajan by declaring that

> there is no trivial mathematics, only trivial mathematical exposition.

I will argue my case by discussing a few basic topics from school mathematics *from an American perspective*, and let you decide if they are trivial or not.

Let us start with the simple long division of 587 by 3. You know the algorithm:

$$
\begin{array}{cccc|ccc}
 & 5 & 8 & 7 & 3 & & \\ \cline{5-7}
- & 3 & & & 1 & 9 & 5 \\ \cline{1-4}
 & 2 & 8 & & & & \\
- & 2 & 7 & & & & \\ \cline{1-4}
 & & 1 & 7 & & & \\
- & & 1 & 5 & & & \\ \cline{1-4}
 & & & 2 & & &
\end{array}
$$

Two questions:

(1) What does this mean? **(2)** Why is it correct?

The answer to (1) is that if 587 is divided by 3, then the quotient is 195 and the remainder is 2. The answer to (2) will occupy us in the next several minutes.

We cannot answer (2) without a clear understanding of the answer to (1). The usual expression, $\mathbf{587 \div 3 = 195\ \ R\ \ 2}$ carries no information and also does not make sense, because:

> The left side is not a number, and the right side is not a number either.
>
> So what does it mean to say the left side *is equal to* the right side?

The correct meaning of (1) is: we have a division-with-remainder,

$$587 \;=\; (195 \times 3) + 2, \qquad \text{where } 2 < 3.$$

Now we have to prove that the above algorithm *leads to this equation*.

We analyze the long division algorithm **for each digit of the quotient 195**:

$$5 = (1 \times 3) + \mathbf{2}$$

$$\begin{array}{rccc|cc} & 5 & 8 & 7 & 3 & \\ \cline{5-6} - & 3 & & & \mathbf{1} & \\ \cline{1-4} & 2 & & & & \end{array}$$

Then:

$$\mathbf{2}8 = (9 \times 3) + \mathbf{1}$$

$$\begin{array}{rccc|cc} & 5 & 8 & 7 & 3 & \\ \cline{5-6} - & 3 & & & 1 & \mathbf{9} \\ \cline{1-4} & 2 & 8 & & & \\ - & 2 & 7 & & & \\ \cline{1-4} & & 1 & & & \end{array}$$

And finally:

$$17 = (5 \times 3) + 2$$

$$\begin{array}{rrrr|rrr}
 & 5 & 8 & 7 & 3 & & \\ \cline{5-7}
- & 3 & & & 1 & 9 & \mathbf{5} \\ \cline{1-4}
 & 2 & 8 & & & & \\
- & 2 & 7 & & & & \\ \cline{1-4}
 & & 1 & 7 & & & \\
- & & 1 & 5 & & & \\ \cline{1-4}
 & & & 2 & & &
\end{array}$$

Thus the long division algorithm is a *compact* summary of three divisions-with-remainder:

$$\begin{aligned}
5 &= (1 \times 3) + \underline{\mathbf{2}} \\
\underline{\mathbf{2}}8 &= (9 \times 3) + \underline{\mathbf{1}} \\
\underline{\mathbf{1}}7 &= (5 \times 3) + 2
\end{aligned}$$

Here is the proof that $587 = (195 \times 3) + 2$:

Using $5 = (1 \times 3) + \mathbf{2}$, we have:

$$\begin{aligned} 587 &= (\underline{5} \times 100) + (8 \times 10) + 7 \\ &= (((1 \times 3) + \mathbf{2}) \times 100) + (8 \times 10) + 7 \\ &= ((1 \times 100) \times 3) + (\underline{\mathbf{2}8} \times 10) + 7 \end{aligned}$$

Using $\mathbf{2}8 = (9 \times 3) + \mathbf{1}$, we get:

$$\begin{aligned} 587 &= ((1 \times 100) \times 3) + (((9 \times 3) + \mathbf{1}) \times 10) + 7 \\ &= ((1 \times 100) \times 3) + ((9 \times 10) \times 3) + \underline{\mathbf{1}7} \end{aligned}$$

Finally, using $\mathbf{1}7 = (5 \times 3) + 2$, we have:

$$\begin{aligned} 587 &= ((1 \times 100) \times 3) + ((9 \times 10) \times 3) + (5 \times 3) + 2 \\ &= ((1 \times 100) + (9 \times 10) + 5) \times 3 + 2 \\ &= (195 \times 3) + 2 \end{aligned}$$

This proof has the virtue of bringing out the fundamental idea underlying *all* whole number algorithms: each step of the algorithms involves only one digit, when "one digit" is suitably interpreted. So far so good.

Unfortunately, this is sophisticated mathematics. Teachers should know it, of course, but even fifth graders would generally find this too difficult.

To make this proof usable as school mathematics, we have to simplify it as much as possible but *without sacrificing the mathematical validity of the explanation or the central idea that it is one digit at a time*. The following is one possible compromise.

We want whole numbers q and r so that

$$587 \ = \ (q \times 3) + r \quad \text{where } r < 3$$

The number q cannot have 4 digits (because right side would exceed the left side), so it is at most a 3-digit number. Its hundreds digit cannot be ≥ 2 (again because right side would exceed the left side), so its hundreds digit is 1. So $q = 100 + T$, where T is a 2-digit number. Thus

$$587 \ = \ (100 + T) \times 3 + r \ = \ 300 + (T \times 3) + r$$

so that

$$587 - 300 = (T \times 3) + r$$

	5	8	7	3
-	3	0	0	**1**
	2	8	7	

Now $287 = (T \times 3) + r$. The tens digit of T can be as big as 9 without any contradiction (if we want $r < 3$, then we need T to be as large as possible so long as $T \times 3$ does not exceed 287).

So $T = 90 + S$, where S is a single digit number, and therefore

$$287 \;=\; (90 + S) \times 3 + r \;=\; 270 + (S \times 3) + r$$

so that

$$287 - 270 = (S \times 3) + r$$

	5	8	7	3	
-	3	0	0	1	**9**
	2	8	7		
-	2	7	0		
		1	7		

Finally, we have $17 = (S \times 3) + r$. Clearly we should let $S = 5$ and $r = 2$:

$$17 = (5 \times 3) + 2$$

$$\begin{array}{rrrr|rrr} & 5 & 8 & 7 & 3 & & \\ \cline{5-7} - & 3 & 0 & 0 & 1 & 9 & \mathbf{5} \\ \hline & 2 & 8 & 7 & & & \\ - & 2 & 7 & 0 & & & \\ \hline & & 1 & 7 & & & \\ - & & 1 & 5 & & & \\ \hline & & & 2 & & & \end{array}$$

Recall: $587 = (q \times 3) + r$, and we have determined that the hundreds digit, the tens digit, and the ones digit of q are 1, 9, 5 respectively, and $r = 2$. Thus

$$587 = (195 \times 3) + r$$

Everything we have done thus far about long division is part of what we call school mathematics. We observe that:

(a) These considerations do not belong to the university mathematics curriculum. They are too elementary.

(b) The mathematics is not trivial.

(c) Part of this discussion about bringing the mathematics down to the level of fifth graders *goes beyond mathematics per se*.

(d) The discussion in (c) cannot take place without a *complete* understanding of the mathematics underlying the long division algorithm.

The phenomenon exhibited in (a)–(d) is not special to long division, but is shared by *most* topics in school mathematics:

(i) Conversion of a fraction $\frac{m}{n}$ to a decimal by the long division of m by n.

(ii) The concept of a fraction, and everything related to fractions, including ratio and percent.

(iii) The concept of **constant speed** or **constant rate**.

(iv) Axioms and Euclidean geometry.

(v) The concepts of congruence and similarity.

(vi) The concept of length and area, especially the computation of the circumference and area of a circle.

(vii) The concept of a negative number, and everything related to negative numbers.

(viii) Finding the maximum or minimum of a quadratic function.

(ix) The concept of a polynomial and the algebra of polynomials.

Let me give one more illustration of the phenomenon exhibited in (a)–(d) using negative numbers. In America, the Most Frequently Asked question in school mathematics is "Why is **negative** $\times$ **negative** equal to **positive**?"

Mathematicians consider this to be obvious. They prove something more general: For *any* numbers x, y, $(-x)(-y) = xy$.

Proof: We first prove that $(-x)z = -(xz)$ for *any* x and z. Observe that if a number A satisfies $xy + A = 0$, then $A = -(xy)$. But by the distributive law, $xy + \{(-x)y\} = (x + (-x))y = 0 \cdot y = 0$, so $(-x)y = -(xy)$. Now let $z = (-y)$, then we have $(-x)(-y) = -(x(-y))$, which by the commutative law is equal to $-((-y)x) = -(-(yx)) = yx = xy$. So $(-x)(-y) = xy$.

University mathematicians usually do not recognize how sophisticated this simple argument really is. It is not suitable for the consumption of school students.

The basic resistance to accepting **negative** $\times$ **negative** = **positive** is a psychological one. If we can explain this phenomenon for integers (rather than fractions), most of the battle is already won.

We will give a relative simple explanation of why $(-2)(-3) = 2 \times 3$. The key step lies in the proof of

$$(-1)(-1) = 1$$

If we want to show a number is equal to 1, the most desirable way is to get it through a computation, e.g., if

$$A = (12 \times 13) - (6 \times 25) - 5\,,$$

then $A = 1$ because

$$A = 156 - 150 - 5 = 1$$

But sometimes, such a direct computation is not available. Then we have to settle for an indirect method of verification. (Think of dipping a pH strip into a solution to test for acidity.)

So to test if a number A is equal to 1, we ask: is it true that $A + (-1) = 0$? If so, then we are done.

Now let $A = (-1)(-1)$. We have

$$A + (-1) \;=\; (-1)(-1) + (-1) \;=\; (-1)(-1) + 1 \cdot (-1)$$

By the distributive law,

$$(-1)(-1) + 1 \cdot (-1) = \left((-1) + 1\right)(-1) \;=\; 0 \cdot (-1) \;=\; 0$$

So $A + (-1) = 0$, and we conclude that $A = 1$, i.e.,

$$(-1)(-1) \;=\; 1$$

Now we can prove $(-2)(-3) = 2 \times 3$.

We first show $(-1)(-3) = 3$. We have

$$(-1)(-3) = (-1)\left((-1) + (-1) + (-1)\right)$$

which, by the distributive law, is equal to

$$(-1)(-1) + (-1)(-1) + (-1)(-1) = 1 + 1 + 1 = 3$$

Thus $(-1)(-3) = 3$.

Then, $(-2)(-3) = \left((-1)+(-1)\right)(-3) = (-1)(-3)+(-1)(-3)$. By what we just proved, the latter is $3 + 3 = 2 \times 3$. So $(-2)(-3) = 2 \times 3$.

Proof of $(-m)(-n) = mn$ for whole numbers m, n is similar.

So far, I have given you bits and pieces of what School Mathematics is about. Let me go a step further and give you a more comprehensive view.

We know, at least, what it is *not*:

School Mathematics is not University Mathematics.

There is more, however. We have seen that, in order to make a mathematical topic *usable for school students*, we need to take an extra step to ensure that the mathematical substance is not lost. This step involves *more than* mathematical knowledge; it involves a knowledge of the school classroom.

What we are saying is that

> **School Mathematics is the product of Mathematical Engineering.**

What does this mean?

Engineering: The discipline of customizing abstract scientific principles into processes and products that *safely* realize a human objective or function.

Mathematical engineering: The discipline of customizing abstract mathematics into a form that can be *correctly* taught, and learned, in the K–12 classroom.

Mathematical engineering is also known as K–12 mathematics education.

Chemical engineering:

Chemistry $\longrightarrow$ the plexi-glass tanks in aquariums, the gas you pump into your car, shampoo, Lysol, . . .

Electrical engineering:

Electromagnetism $\longrightarrow$ computers, power point, iPod, lighting in this hall, motors, . . .

Mathematical engineering:

Abstract mathematics $\longrightarrow$ school mathematics

The recognition that *school mathematics* is an engineering product lends clarity to the current debate in mathematics education.

There is no controversy in stating that engineering should not attempt to produce anything that caters to human wishes but defies scientific principles, e.g., perpetual motion machines, machines that extract oxygen from water without use of energy.

There is also no controversy in stating that engineering should not waste time producing anything that is irrelevant to human needs no matter how scientifically sound.

Yet school mathematics was once mathematically sound but was decidedly not relevant to the school classroom: the New Math of the 1960's.

Injection of set theory into elementary school.

Over-emphasis of precision (e.g., distinction between "the number *three*" and "the numeral *3* that represents "the number *three*").

Emphasis of abstractions (e.g., modular arithmetic, numbers in arbitrary bases, symbolic logic) at the expense of basic skills (computations in base 10).

Currently, there have been too many attempts in school mathematics to make mathematics easy to learn by ignoring basic mathematical principles.

Consider the subject of fractions in school mathematics. Education researchers and textbook writers $seem$ to believe that no engineering process can bring the mathematics of fractions to the school classroom. Their decision is therefore to teach fractions, not as mathematics, but as a language, which can conceivably be learned

> by osmosis, by listening to stories, by using analogies and metaphors, and by engaging in hands-on activities.

The resulting fear of fractions is there for all to see.

Here are some of the problems in the way fractions is taught in America.

No definition. There is no definition of a fraction (other than as a piece of pizza). The statement "fractions have multiple representations" is meaningless, because if we don't know what it is, what is there to represent? There is also no definition of any of the arithmetic operations on fractions: what does it mean to multiply two pieces of pizza?

No reasoning. Analogies and metaphors replace reasoning. Why use the Least Common Denominator to add fractions? Why not add fractions the same way we multiply fractions: add the numerators and add the denominators? Why compute division by invert-and-multiply?

No coherence. This is perhaps the most serious of the three problems. Fractions are taught as "different numbers" from whole numbers. We are told that "Children must adopt new rules for fractions that often conflict with well-established ideas about whole numbers".

More is true: Decimals are taught as "different numbers" from fractions. There is also no logical connection between various concepts and skills within the subject of fractions.

How can students cope with such fragmented knowledge?

Mathematical engineering, when competently done, **does** produce a presentation of fractions that is

> fully consistent with basic mathematical principles *and*
>
> suitable for students of grades 5–7.

This engineering effort requires a knowledge of the school classroom as well as a deep mathematical knowledge.

Physicists and chemists recognized long ago that real-world applications of their theoretical knowledge are important and should be pursued as a separate discipline. Schools of engineering were born.

However, neither mathematicians nor educators seem to recognize school mathematics as an **engineering product**. Each group end up producing

mathematics not relevant to the school classroom, or

materials usable in the classroom but mathematically flawed.

We need good school mathematics.
We need good mathematical engineering.

For a more thorough discussion of mathematics education as *mathematical engineering*, see

http://math.berkeley.edu/~wu/ICMtalk.pdf