The Mathematics School Teachers Should Know

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Do school math teachers know enough mathematics to teach?

In America, this question of has been much discussed in recent years. In the opinion of almost all mathematicians, **this is** *the* **overriding issue facing mathematics education today.**

It is worth repeating: The most pressing concern is not whether our teachers know enough pedagogical techniques. It is, rather, whether they know enough mathematics to teach.

In the teaching of mathematics, mathematical content dictates pedagogy in a vast majority of cases.

This explains the great interest in teachers' content knowledge.

There is at present a lack of clarity about *what this content knowledge is*, and *how to enable teachers to acquire this knowledge*.

Today, I will report on the former from an American perspective. It would be surprising if the situation in Portugal is radically different.

It is a nontrivial matter how to demonstrate in a *quantitive* way that "teachers can teach better if they know more mathematics".

In the 1970s, the educator E.G. Begle made a first attempt to demonstrate a connection between teacher content knowledge and student achievement. To his dismay, he found no statistically significant correlation.

Begle measured this knowledge by the *number of mathematics courses taken* and by teachers' *grades* (technically: grade point average). Subsequent research also failed to improve on his findings except for teachers in grades 9-12.

The research methodology of Begle and his followers is flawed.

Why existing university mathematics courses not likely to have significant impact on math teachers' effectiveness in the classroom: the mathematics of these courses

> is *not sufficiently relevant* to the school classroom, and
>
> may be *mathematically deficient.*

Courses of the first kind are generally those given in math departments and taken by secondary math teachers. Those of the second kind are generally taken by elementary teachers.

These conclusions are based on anecdotal evidence, a general knowledge of the undergraduate curriculum, and existing mathematics textbooks written for teachers in America.

These conclusions begin to explain why Begle and others could not obtain any significant correlation between the math courses taken by teachers of K-8 and their students' achievement.

They also make us question the positive correlation between the number of mathematics courses taken by teachers of grades 9-12 and student achievement. There is room for doubt.

I will illustrate with a few examples.

(1) In 1966, I was asked to teach a course on **Number Systems** for teachers. It was about defining the natural numbers $\mathbb{N}$ using the Peano axioms and then

$$\mathbb{Z} \to \mathbb{Q} \to \mathbb{R}$$

Most of the semester was therefore devoted to the difficult construction of $\mathbb{R}$ from $\mathbb{Q}$.

Back in 1966, I was not aware that **school mathematics is all about the rational numbers $\mathbb{Q}$ and not about the real numbers $\mathbb{R}$**.

Why not devise courses that teach teachers about $\mathbb{Q}$, especially about **fractions**, instead of teaching them about $\mathbb{R}$?

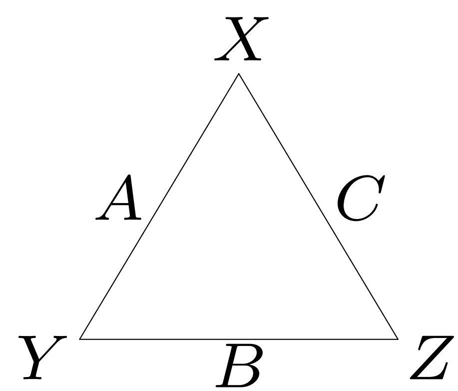
(2) A university mathematician once described to me how he had been presenting "fractions from the *field axioms* point of view" to teachers from grades 6–8.

Thus, beginning with the axioms that the rational numbers $\mathbb{Q}$ satisfy the associative, commutative, and distributive laws with respect to $+$ and $\times$, and that every nonzero number has a multiplicative inverse, he derived all the usual properties of $\mathbb{Q}$.

This knowledge will not help teachers to teach 12-year olds about fractions, because 12-year olds don't care about *axioms of a field*. They have no idea what a field is or what axioms are good for.

(3) A university math educator tried to improve elementary teachers' so-called *number sense* by using the following problem in a content course for elementary teachers. Let X, Y, Z be the number of beans placed on the vertices of a triangle, and let nonzero whole numbers A, B, C be attached to the sides of the same triangle, as shown, so that

$$A = X + Y, \quad B = Y + Z, \quad C = X + Z \qquad (*)$$



The problem: *If whole numbers A, B, C are given, would there be whole numbers X, Y, Z which satisfy the equations in $(*)$?*

He let his teachers use the discovery method to explore how various integer values of X, Y, Z led to different values of A, B, C, and how a solution $\{X, Y, Z\}$ could be obtained by *guess-and-check* given A, B, and C. He also got them to look into replacing the triangle by a quadrilateral.

But he did **not** point out that *there is a unique solution (X, Y, Z) if and only if $A + B + C$ is even and the sum of any two of A, B, C is greater than the third.* This reasoning is accessible to any 6th grade student, and certainly any teacher.

He forgot that **reasoning** is the lifeblood of mathematics, and it is the ability to reason that all teachers must learn.

If we are serious about helping teachers teach effectively, we must provide them with a body of mathematical knowledge that is

> **relevant** to teaching, i.e., not too far from the material they teach in school, and
>
> consistent with the **fundamental principles of mathematics**.

We saw in Examples 1 and 2 the kind of knowledge that is not relevant to teaching, and in Example 3 the kind that runs against the nature of mathematics.

Why **relevant**? Because teachers are not researchers in mathematics. They are not responsible for converting advanced abstract mathematics into something useable in the school classroom. That is a job for professional mathematicians.

Why must teachers know mathematics that respects its **fundamental principles**? Because a teacher cannot afford to misinform students, and because this kind of mathematics is easier for students to learn than *incorrect* mathematics.

We have university courses that satisfy either one of these requirements. **What we need are courses that satisfy both.**

As an example of how the two kinds of courses differ, consider how they teach **fractions** to prospective teachers.

University course on abstract algebra:

Fractions are equivalence classes of ordered pairs of integers under the equivalence relation $(a, b) \sim (c, d) \iff ad = bc$.

Few if any 12-year old can relate to such a sophisticated concept.

What good is something like this for teachers?

Course designed for elementary teachers:

A fraction is a piece of pizza. This *is* simple enough to be used in the elementary classroom. But how to explain, for example, the meaning of $(2/3) \times (11/7)$? How do students learn to multiply two pieces of pizza?

What good is something like this for teachers?

Consider the **division of fractions.** A course on abstract algebra will say that division by B is just multiplication by the multiplicative inverse of B. Thus A/B means AB^{-1}. If $B = \frac{11}{7}$, then $(\frac{11}{7})^{-1} = (\frac{7}{11})$. Therefore,

$$\frac{\frac{2}{3}}{\frac{11}{7}} = \frac{2}{3} \times \left(\frac{11}{7}\right)^{-1} = \frac{2}{3} \times \frac{7}{11}$$

From the mathematical point of view, the **invert-and-multiply** rule is a matter of definition.

12-year olds have no idea why division is just multiplication by the multiplicative inverse.

What good is something like this for teachers?

Content courses specifically designed for elementary teachers try to simplify the concept of fraction division by ignoring the basic principles of mathematics.

A typical approach is this: We are trying to find out what $\frac{\frac{2}{3}}{\frac{11}{7}}$ is. We use equivalent fractions:

$$\frac{\frac{2}{3}}{\frac{11}{7}} = \frac{\frac{2}{3} \times \frac{7}{7}}{\frac{11}{7} \times \frac{3}{3}} = \frac{(2 \times 7)\ \frac{1}{3 \times 7}}{(11 \times 3)\ \frac{1}{7 \times 3}} = \frac{2 \times 7}{11 \times 3},$$

which is the same answer as before.

The trouble is: we do not as yet know what the division $\dfrac{(2\times 7)\,\frac{1}{3\times 7}}{(11\times 3)\,\frac{1}{7\times 3}}$ is, and if not, how to justify canceling the fraction $\frac{1}{3\times 7}$ from the numerator and denominator in order to conclude

$$\frac{(2\times 7)\,\frac{1}{3\times 7}}{(11\times 3)\,\frac{1}{7\times 3}} = \frac{2\times 7}{11\times 3}\ ?$$

What good is something like this for teachers?

The subject of fractions reveals why taking more of the **existing** math courses in universities will not likely lead to more effective teaching in schools.

So long as these courses only teach

> how to work with equivalence classes of ordered pairs of integers, or
>
> how to work with the abstract concept of a field.

they will not enable teachers to teach fractions to school students more effectively.

This is why Begle and his followers wasted their efforts by looking at the usual math courses.

To help teachers improve their teaching of fractions, we have to provide them with a presentation of fractions that

> is suitable for use by 12-year olds, and
>
> does not violate basic mathematical principles.

For the longest time, people working in math education seemed not to be aware that such a presentation is possible.

If we want to produce effective math teachers, the above is the kind of mathematics we should teach them.

To pursue this line of thought, consider now an example in secondary school. A standard problem is to find the point x_0 at which the quadratic function $f(x) = x^2 + 2x - 2$ is a minimum, and also find $f(x_0)$.

At present, few if any university level math courses would discuss such an elementary problem except in calculus. There, the solution is routine: differentiate f and set it to zero, get $2(x+1) = 0$, so $x_0 = -1$ is where f has an extremum. It has to be a global minimum, and $f(-1) = -3$.

But school teachers cannot use calculus to teach the solution of this problem. So what good is something like this for teachers?

At the moment, secondary school students have to **learn by rote** the fact that a quadratic function $f(x) = ax^2 + bx + c$ has a minimum (resp., maximum) at $x = -\frac{b}{2a}$ if $a > 0$ (resp., $a < 0$), and its value at that point is $\frac{4ac-b^2}{4a}$.

In college, the only thing they learn about quadratic functions is that this rote skill can be explained by calculus.

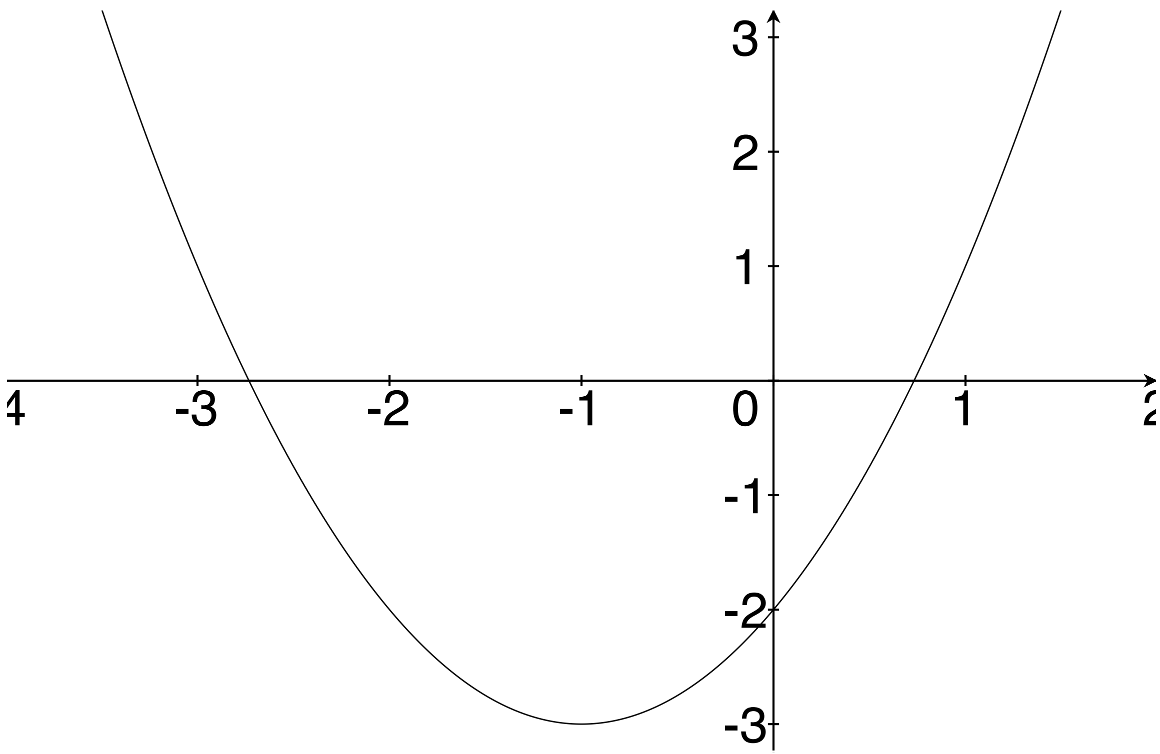
When they become teachers and find themselves having to explain the algebraic solution *without* calculus, they can only fall back on the method they learned as a school student: **teach students by rote**. The vicious cycle continues.

If we want to break the vicious cycle of teaching by rote, we have to teach teachers the **mathematical** knowledge they need in the school classroom, **in a way that they can use it.**

We must give them a coherent presentation of quadratic functions **without** using calculus, so that they can teach this topic to their students, *not by rote*, but *as mathematics.*

For example, they have to know that *completing the square* is not just a technique used to derive the Quadratic Formula. It is rather the key to understanding *everything* about quadratic functions.

They understand that, by completing the square, $x^2 + 2x - 2$ can be put in the form $f(x) = (x+1)^2 - 3$. Now all the information needed to solve the problem about the minimum of $x^2 + 2x - 2$, including the graph of $x^2 + 2x - 2$, can be read off from $f(x) = (x+1)^2 - 3$,



Having looked at several examples, we can understand better an earlier statement: If we are serious about helping teachers teach effectively, we must provide them with a body of mathematical knowledge that is

> **relevant** to teaching, i.e., not too far from the material they teach in school, and
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> consistent with the **fundamental principles of mathematics**.

Thus far, universities have not made a serious effort to do this.

The title of my presentation, *The Mathematics School Teachers should Know*, encompasses a vast terrain. A detailed discussion would include a topic-by-topic analysis of what teachers need to know. This cannot be accomplished in an hour.

On my homepage, there is a 72-page article on *The Mathematics K-12 Teachers Need to know*:

http://math.berkeley.edu/~wu/Schoolmathematics1.pdf

It can be consulted for this purpose. In a real sense though, even 72 pages are not enough. The only way to get the message across is to write books, and this is what I have been doing for several years.

Let me conclude by tying up a loose end. I have been referring to "the fundamental principles of mathematics" all through this presentation. *What are they and why do they matter to a teacher?*

Teachers have to know the *facts* (concepts and skills) as well as their explanations. But beyond facts, they need to know more. Facts are to teachers what words are to novelists: it is how the facts (resp., words) are woven together that matters.

It is the fundamental principles of mathematics that control how the facts are interwoven and give shape to the mathematics in schools.

Precision: Mathematical statements are clear and unambiguous. At any given moment, there is no doubt as what is being asserted to be true, what is known to be true, as well as what is *not* being claimed to be true.

For example, it is mathematically unacceptable to say two geometric figures are **similar** if they have the same shape. No matter how evocative this may sound in everyday life, we reject this in mathematics because "same shape" means different things to different people. It lacks precision.

Definitions: All concepts are precisely defined. They serve as the bedrock of the mathematical structure. They are the platform that launches reasoning. No definitions, no mathematics.

From this perspective, we do not teach mathematics correctly when we teach fractions to elementary school students without defining what a fraction is. Think of the cognitive problem: how can they acquire conceptual understanding about fractions if they don't know what a fraction is?

Could this be a main reason behind the non-learning of fractions?

Reasoning: The lifeblood of mathematics. The engine that drives problem solving. Its absence is the root cause of teaching- and learning-by-rote.

There is no such thing as *mathematics-without-reasoning.* Even in the primary grades, we must teach the simplest mathematical skills, such as the algorithm of adding two whole numbers, with reasoning.

When mathematics is supported by reasoning, it becomes learnable mathematics.

Coherence: Mathematics is a tapestry in which all the concepts and skills are interwoven. It is all of a piece.

Coherent mathematics unfolds naturally like a story. It puts every detail in the right place and it becomes easier to learn.

For example, if student are taught correctly that the arithmetic operations of whole numbers and fractions are *conceptually* all the same, then children's passage from whole numbers to fractions would be greatly eased.

Purposefulness: Mathematics is goal-oriented, and every concept or skill is there for a purpose. Mathematics is not some random fun and games.

Students often find mathematics unattractive because they seem to learn a laundry list of skills for no reason.

For example, rounding a whole number *to the nearest ten, to the nearest hundred*, etc., is taught as a rote skill. If a teacher can explain why a city's population figure of 123,456 should be rounded to the nearest thousand to avoid being nonsense, students would be more motivated to learn about rounding.

Or if students are taught that the *laws of exponents* are needed to make sense of exponential functions, which underlie growth and decay processes, these laws would be more learnable.

In summary, we fervently hope that in years to come, universities will offer courses that teach teachers the kind of mathematics that is

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