Euclid and High School Geometry

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H. Wu

The teaching of geometry has been in crisis in America for over thirty years.

This is a report on that situation, together with some comments that may be relevant to Portugal.

The American perception of a geometry course in secondary school is that this is the place where students learn about **proofs**.

Not just proofs of some theorems, but **proofs of every theorem starting from axioms.**

This extreme view has to be understood in the context of the usual high school curriculum: outside of geometry, almost nothing is ever proved. In other words, mathematics is largely taught in schools **without** reasoning.

The course on geometry is the only place where reasoning can be found.

The absence of proofs elsewhere adds pressure to the course on geometry to pursue the mythical entity called "proof".

As a result, "proof" in the American school curriculum becomes a **rigid formalism** synonymous with **reasoning from axioms**.

A proof is a sequence of steps going from point A (hypothesis) to point B (conclusion). We want students to learn proofs because they should learn to

(i) recognize clearly the starting point of their logical argument, i.e., what point A is,

(ii) recognize clearly the goal they want to achieve, i.e., what point B is, and

(iii) go from point A to point B by the use of correct reasoning.

For the purpose of learning about proofs, it is immaterial whether point A is at the level of axioms or assumes that the Riemann Hypothesis holds.

The development of a subject from axioms is an organizational issue. Prospective mathematicians should acquire a firsthand experience with such a development in college.

School students should be made aware of it, but there is no compelling reason that they must learn the details.

The idea that developing Euclidean geometry from axioms can be a good introduction to mathematics has a very long tradition. While a few greet it with enthusiasm, such a course has not been a pedagogical success, for at least three reasons.

(A) Such a development is extremely boring at the beginning. A whole month or two devoted to nothing but the proofs of trivial and obvious theorems is not the way to keep the attention of young students.

E.g. On page 177 of a geometry textbook of 567 pages is this theorem:

> If M is a point between points A and C on a line L, then M and A are on the same side of any other line that contains C.

(B) Such a development cannot get to the interesting theorems because the starting point is too low.

(C) Proofs of theorems at the level of axioms, far from being easy, is actually very difficult for beginners. They cannot rely on their intuition, only formal reasoning.

Finally, there is a mathematical reason why **using axioms to teach school geometry** is not good: it promotes the *illusion* that everything is proved.

Twenty-four centuries after Euclid, we have learned that this is not possible without paying a very steep price. It takes something like Hilbert's axioms to really *prove* geometric theorems, but Hilbert's axioms are far too subtle for school use.

For about thirty years, a course on school geometry devoted to proofs-starting-from-axioms has become a farce in America:

teachers don't know what they are doing, and

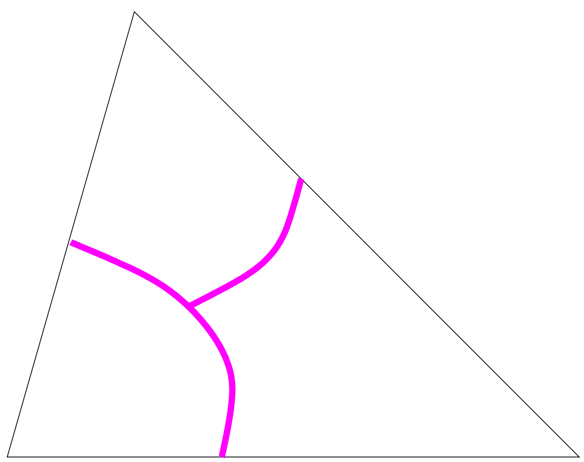
student are reduced to passively following directions.

Learning has ceased to take place in the geometry classroom.

For example, there is a record of a geometry teacher doing construction with ruler and compass, and students were graded not by the validity of their proof of the correctness of the construction, but by the accuracy and neatness of the constructed figures.

To every action there is a reaction. In this case, the reaction is the emergence of geometry courses with **no proofs.**

A textbook for such a course approaches the theorem about the angle sum of a triangle being 180 as follows. It asks students to cut a triangle from a piece of paper, then tear up the triangle into three pieces where each piece contains one angle.



It asks students to re-assemble the three pieces so that the three vertices meet at a point to verify that they add up to 180 degrees.

In conclusion, students are asked to “state a conjecture”, in the following form:

> (**Triangle sum conjecture**) *The sum of the measures* ________.

In an exercise, students are asked to explore this conjecture using geometry software on a computer. Other exercises ask students to use the conjecture to determine unknown angles in a figure.

This book is 834 pages long. The first 732 pages contain no proofs, The last 102 pages are devoted to

> *another type of reasoning, which is called deductive reasoning or proof.*

In these 102 pages one finds a very defective set of axioms and an attempt at explaining what a proof is.

Few teachers make any attempt to get to those 102 pages.

The author's message to the students is that

> This book was designed so that you and your teacher can have fun with geometry . . . and less anxiety.

The forward to the first edition by a math educator says "this is a genuinely exciting book", and the forward to the second edition by the Mathematics Director of a school district says "the second edition is even more exciting".

I imagine that the emphasis on "fun" and "excitement" at the expense of mathematics resonates with you.

How did we get to where we are?

The root cause of our problem with the teaching of secondary school geometry is the overall deterioration of the school curriculum itself.

Because the school mathematics curriculum itself has almost no proofs, those who recognize reasoning as the essence of mathematics see the geometry course as a last chance to teach some mathematics. Therefore they insist on *proving everything*, i.e., start with axioms.

Those who are mesmerized by the "simplicity" of teaching mathematics without proofs naturally insist on teaching geometry without proofs as well. This is how it came to pass that an 834-page geometry textbook with no proofs in its first 732 pages could get published.

In general terms, the solution to the problem of teaching geometry lies in our ability to solve the problem of teaching school mathematics.

If school mathematics is taught in any way resembling *mathematics*, there would be a reasonable amount of reasoning (proofs) to make sense of the disjointed collection of facts. Teaching by rote would be avoided, and many proofs in geometry would be inevitable.

I will not address this universal problem in mathematics education today. But I will narrow my vision and address a specific concern of students in geometry.

Two of the key concepts in geometry are **congruence** and **similarity**. Students cannot come to grips with them because they are told:

> congruence means same size and same shape and similarity means same shape but not necessarily the same size.

These are metaphors. Mathematics needs more than metaphors.

Such key concepts need precise definitions.

There is only enough time to deal with congruence (similarity is slightly more subtle).

We will give a direct, hand-on definition of “congruence”, and more importantly, **use this definition to prove theorems**. In the process, we will try to solve another fundamental problem in the teaching of geometry.

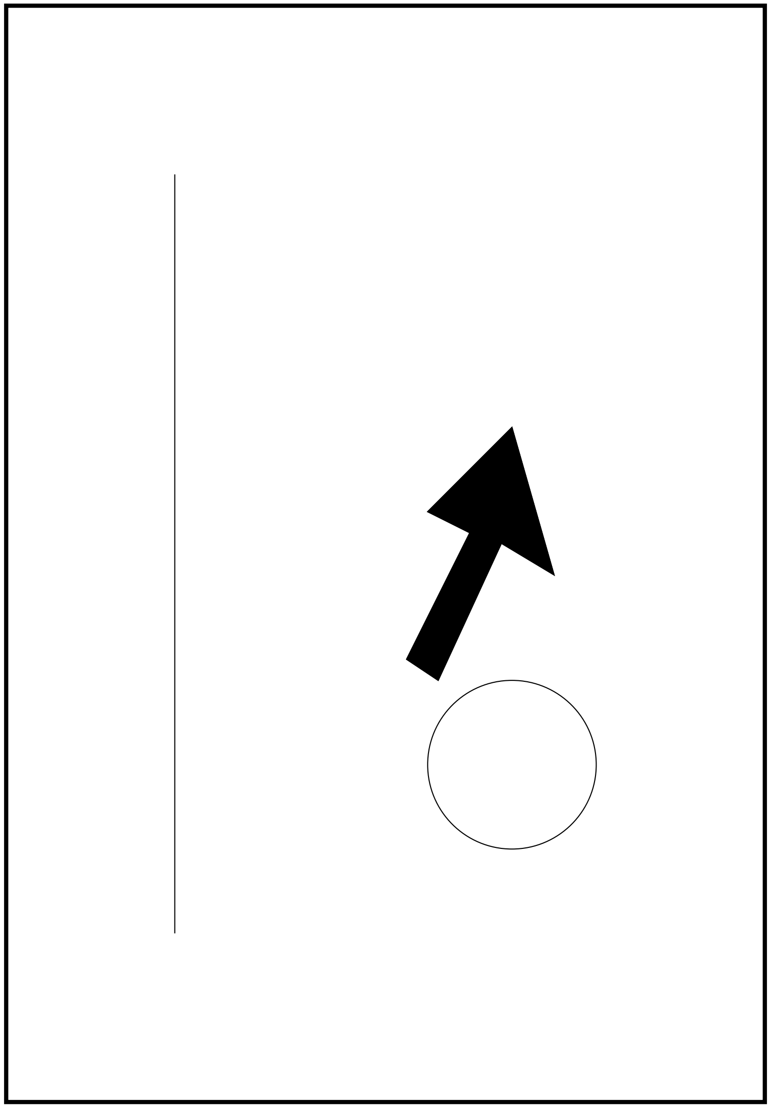
In the plane, we introduce the three **basic isometries**: translations, reflections, and rotations.

The precise definitions of the basic isometries require the concept of a *transformation* of the plane, which is not easy even for university students.

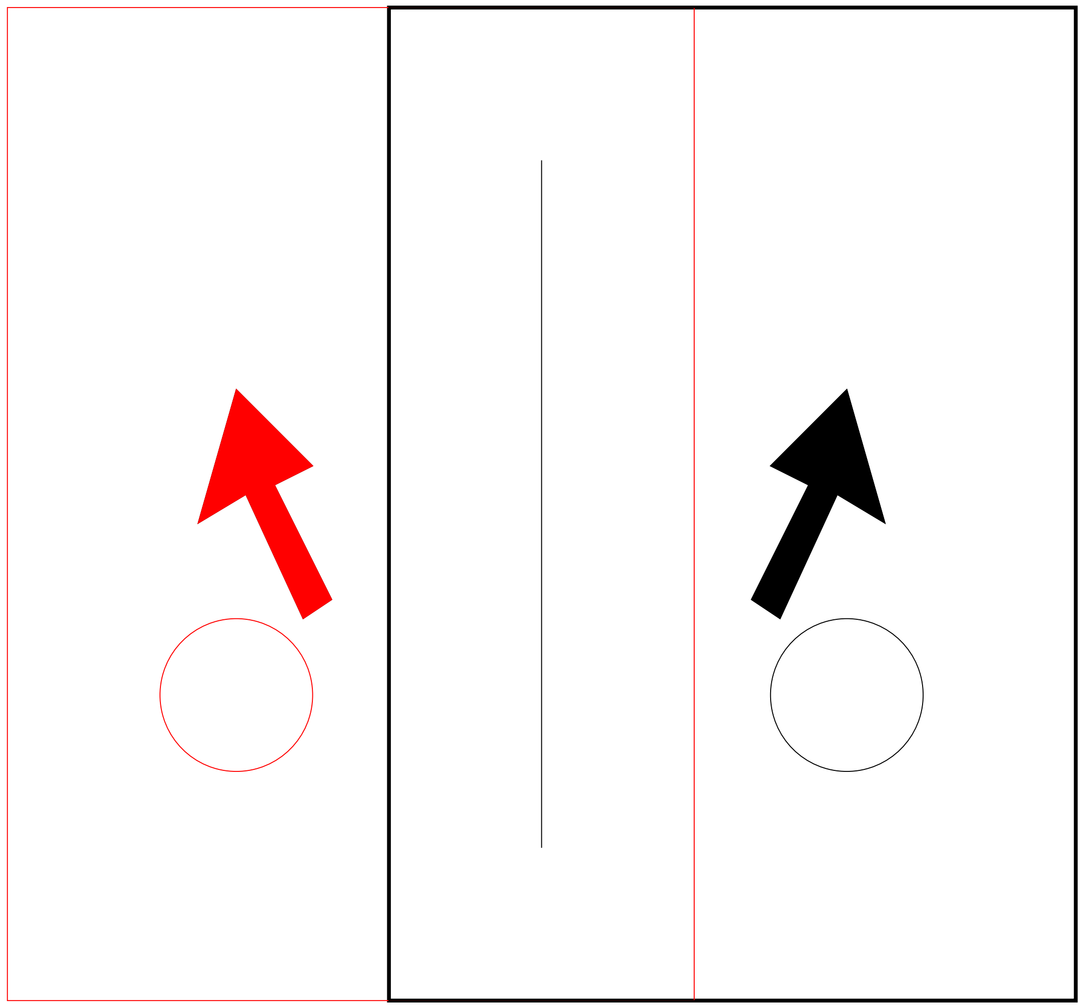
Fortunately, *the basic isometries can be defined with the help of hands-on activities*. We use overhead projector transparencies in the following way:

> draw a geometric figure on one transparency, copy it exactly on another transparency in red; by moving one against the other we make the basic isometries concrete.

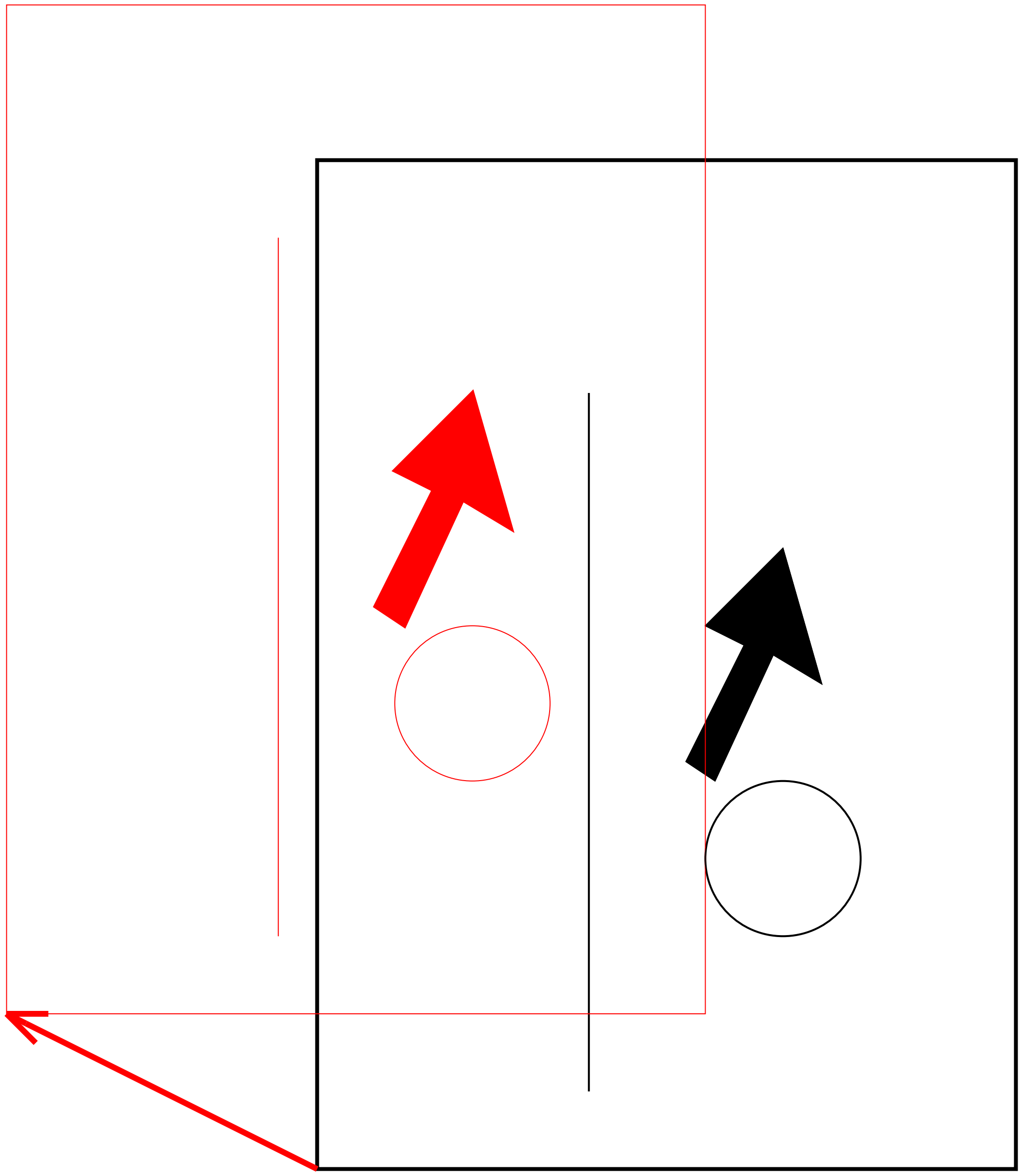
For example, draw the following picture on a transparency:



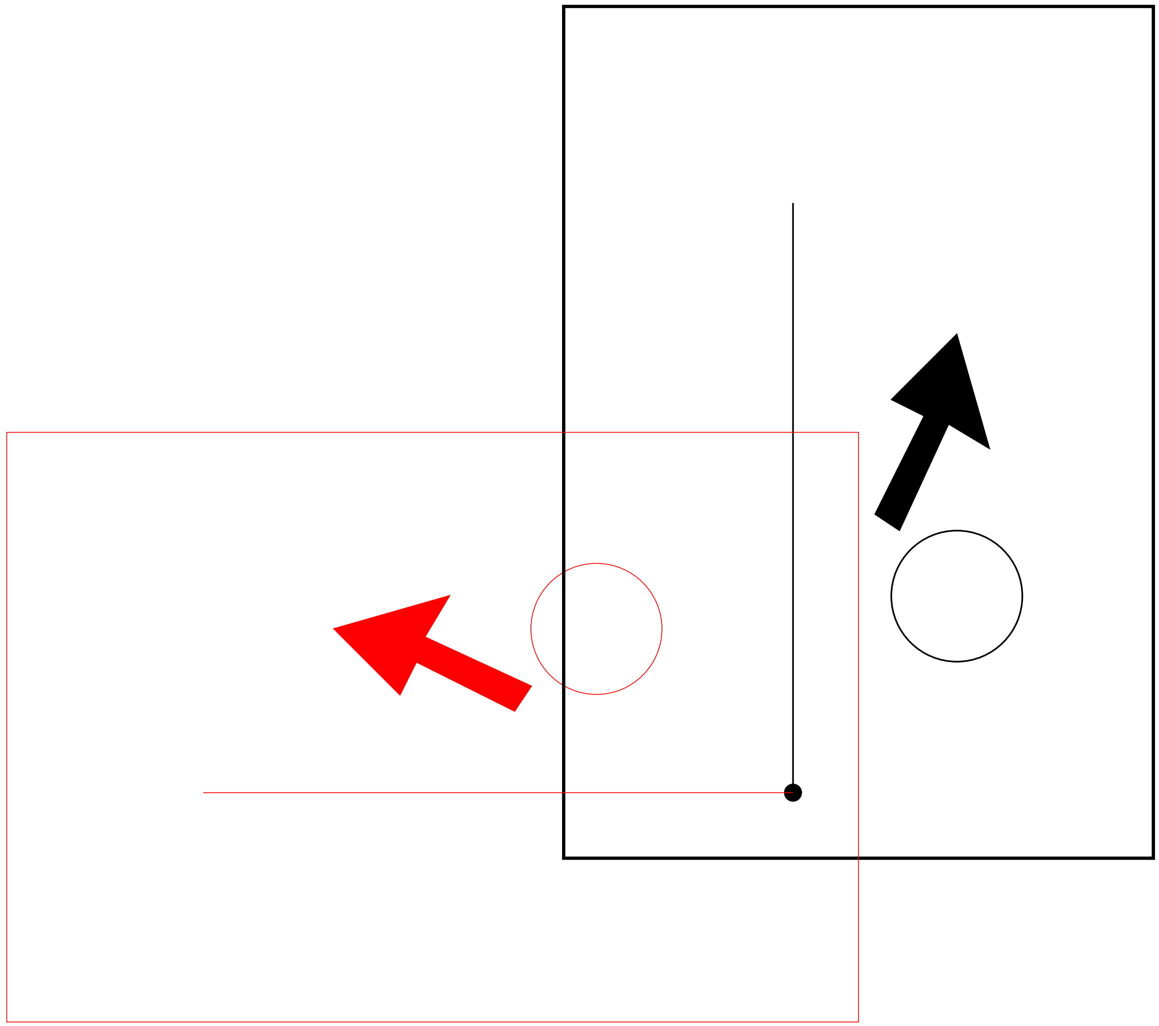
Here is a reflection of the arrow and circle across the line, realized by flipping the red copy across the line:



Here is a translation of the whole picture along the arrow:



Here is a counterclockwise rotation of 90 degrees around the dot:



We will assume that all basic isometries

transform lines to lines, and segments to segments,

preserve lengths of segments,

preserve degrees of angles

This is something all students find it easy to believe after they have gained lots of experience using transparencies.

Two figures $\mathcal{S}$ and $\mathcal{S}'$ are, by definition, **congruent**, if a finite number of isometries carry one onto the other.

Congruence between geometric figures is no longer restricted to triangles. Students can now **direct check** the congruence between ellipses and parabolas, for example.

Nor is congruence a matter of some abstract principles such as SAS, ASA, or SSS. In fact, students can use the basic isometries to **directly check** that SAS, ASA and SSS are correct.

Congruence becomes a concrete and learnable concept.

There is more. The above "fundamental problem in the teaching of geometry" refers to the role played by the basic isometries in school mathematics education.

They are now taught in schools because they are supposed to be important. But students only see them used in discussions of art (Escher prints, tessellations, mosaic art) but nothing about why they are important in mathematics itself.

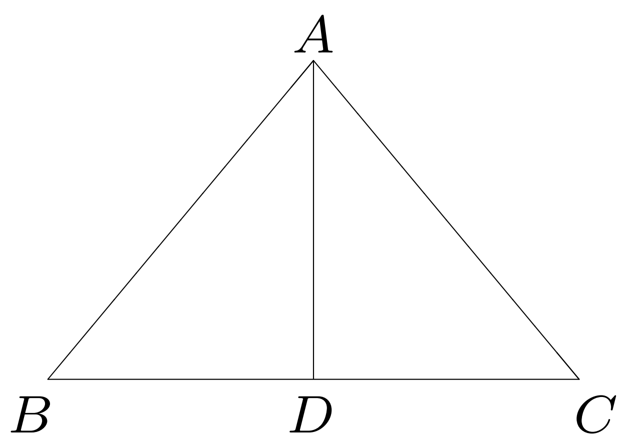
We are going to use them in the proofs of geometric theorems. Their importance in mathematics will no longer be in doubt.

We give two illustrations.

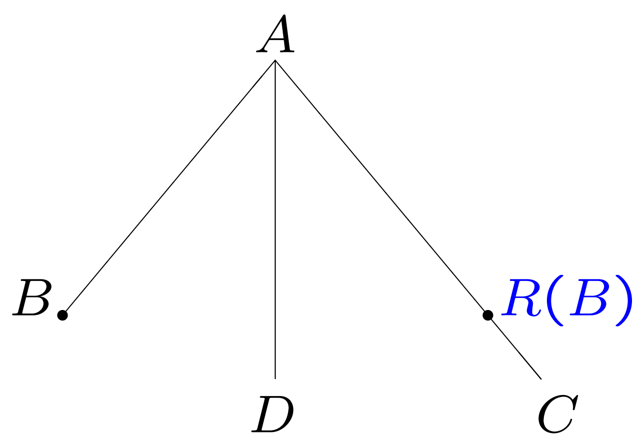
Theorem *Isosceles triangles have equal base angles.*

Proof Let the side AB be equal to AC, and we have to prove that $\angle B$ and $\angle C$ are equal.

Let AD be the angle bisector of the top angle $\angle A$.



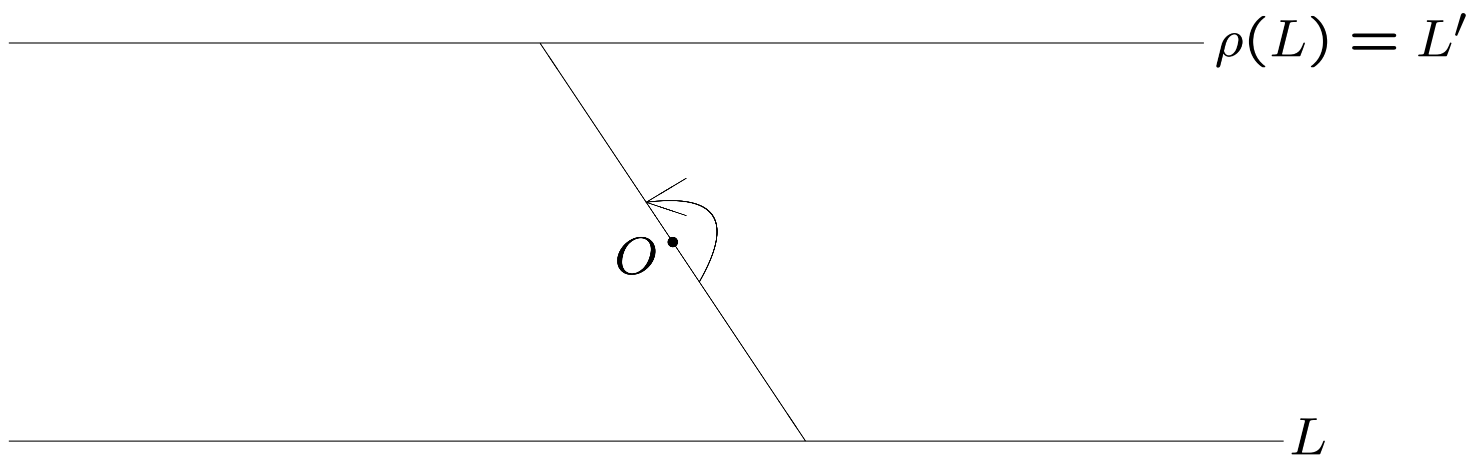
The reflection R across the line AD carries the *half-line* AB onto the *half-line* AC because AD bisects $\angle A$. Without any assumption about $AB = AC$, the point B goes to some point $R(B)$ on the half-line AC.



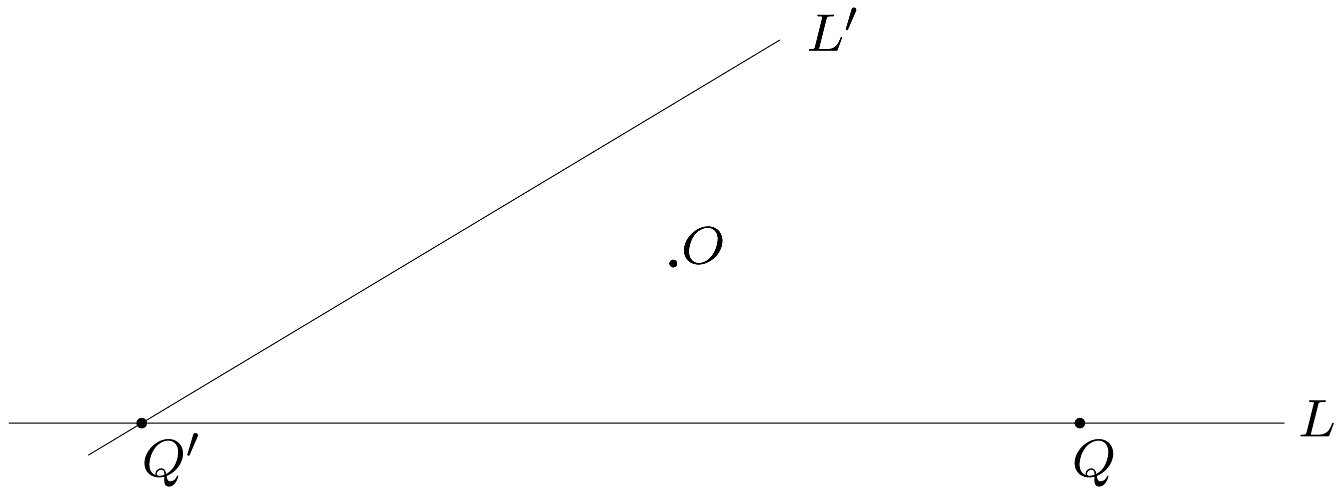
But by assumption, $AB = AC$, so $R(B) = C$. Then R also carries $\angle B$ to $\angle C$, and the angles are equal.

Our next goal is to prove that a **parallelogram** (a quadrilateral whose opposite sides are parallel) must have equal opposite sides. For this we need a lemma.

Lemma *Let O be a point not lying on a line L. The 180 degree rotation ρ around O then carries L to a line parallel to L.*



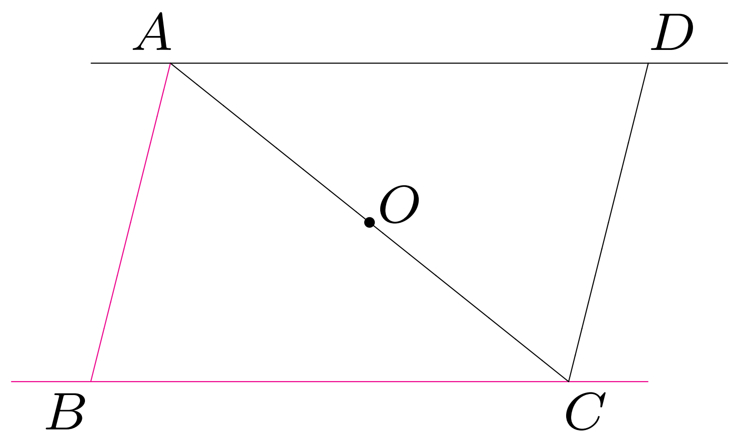
Proof Let $L' = \rho(L)$ and suppose the lemma is false. Let L' intersect L at a point Q'. Consider Q' as a point on L', then there is a point Q on L so that ρ carries Q to Q'.



Because ρ is a 180 degree rotation around O, the three point Q, O, and $\rho(Q) = Q'$ lie on a straight line ℓ. Now ℓ and L are two straight lines which intersect at two distinct points, so $\ell = L$. Then also $O \in L$. Contradiction.

Theorem *Opposite sides of a parallelogram are equal.*

Proof Let O be the midpoint of the diagonal AC of a parallelogram $ABCD$, and let ρ be the 180 degree rotation around O as usual. We claim ρ carries the *line* BC to the *line* AD. This follows from the Parallel Postulate and the Lemma.



Similarly, ρ carries *line* AB to *line* CD. So ρ carries the intersection of line BC and line AB to the intersection of line AD and line CD. Thus ρ carries B to D.

Likewise, ρ also carries C to A. So ρ carries BC to DA. Since ρ preserves length, we have $BC = AD$. The equality $AB = CD$ is proved the same way. The proof is complete.

We repeat: the equality of the segments was not achieved by using some abstract principle such as ASA, SAS, or SSS, but by exhibiting a basic isometry that explicitly carries one segment to the other.

The concept of congruence becomes concrete and tactile, and the basic isometries are seen to be useful in *mathematics itself*.

What is outlined above is the beginning of a complete development of classical Euclidean geometry.

It is built on the basic isometries, and it gets to interesting theorems almost immediately.

It clarifies the fundamental concepts of congruence and similarity for students.

A complete development of plane geometry from the point of view described here will appear in volume II of a three volume set by the author, *Mathematics of the Secondary School Curriculum, I, II, III*, to appear.