Review of three books by Gelfand et al.

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I. M. Gelfand, E. G. Glagoleva and E. E. Shnol, **Functions and Graphs**, *1990. ix + 105 pp.*
I. M. Gelfand, E. G. Glagoleva and A. A. Kirillov, **The Method of Coordinates**, *1990. ix + 73 pp.*
I. M. Gelfand and A. Shen, **Algebra**, *1993. 153 pp.*
—All published by Birkhaüser, Boston-Basel-Berlin.

BACKGROUND

"The most important thing a student can get from the study of mathematics is the attainment of a higher intellectual level." So wrote Gelfand in the common preface to the first two books under review (**FG** and **MC** for short). It may seem truistic in academic context, but anyone who has spent two years surveying the contemporary scene in mathematics education and has had more than his or her share of hyperboles such as "mathematical empowerment of the students" or "political development" in the current reform just might find Gelfand's simple statement refreshing, nay, moving.

A mathematician preoccupied with his own research and his daily duties may not be aware that there is a reform underway in the mathematics education of K-12 (i.e., kindergarten through the 12th grade) in the U.S. In

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the eighties, many educators rightly felt that the traditional mathematics instruction in the schools had degenerated into a ritual, one that no longer had relevance to either mathematics or education. This was easily verifiable through the poor performance of the high school graduates in the high-tech work force. As a result, the business community — among many sectors of society — started to agitate for improvement in the quality of mathematics education. There was also the matter of massive drop-outs and abysmal test scores in mathematics, and the statistics were of course grist for the political mill. For example, when George Bush was campaigning for the presidency in 1988, he saw fit to adopt as a campaign slogan that he, if elected, would be our "education president" and would make the science and mathematics education of this country the first in the world by the year 2000. Subsequently, President Clinton took over this theme and has recently signed the *Goals 2000* legislation.[1] With all these forces at work, attempts at reform became inevitable. In fact, the reform effort has also spread to the teaching of calculus in college. The National Science Foundation has spent millions to sponsor the development of mathematics curricula for both schools and colleges in line with the proposed reform. The prospect is for an exciting era in education.

The publication of the NCTM Standards [N] in 1989 marked the beginning. This document has since become the rallying point in any discussion of the reform. While it may be early to assess the achievements of this effort, there are good reasons to assess the implications of some already recognizable trends. For the purpose of this review, I will limit myself to a brief report, *based on the publications available to me*, on how the reform movement has affected the *content* of the mathematics curriculum *in 9-12 and calculus*. It is to be noted that the reform addresses not just content, but also the method of teaching (e.g., the stress on group learning in the classroom, and the integration of calculators and computers into the instruction) and the method of assessment (e.g., in the current educational jargon, the emphasis on "process" over "product"[2]). These are even more controversial.

[1] In the case of President Clinton, however, one should think twice before ascribing the motive of political expediency to his action, because his track record as governor of Arkansas and his interviews on the subject of education show him to be uncommonly well informed and dedicated to this cause.

[2] One may be forgiven if one hears in this the echo of Tom Lehrer's classic put-down of The New Math: "You take seven from thirteen and that leaves five, well, the answer is actually six, but it is the idea that counts."

The self-imposed restriction to a discussion of the content is nothing more than a sensible decision to get this review written in time.

In general terms, the reform movement has forced many teachers to rethink their day-to-day teaching, to loosen up their previously rigid classroom atmosphere, to jettison some musty topics and ugly technical formulas, and to pay more attention to the student's needs. As with almost anything new, it can be to some a breath of fresh air. The resulting enthusiasm can be seen in the way the NCTM Standards is now embraced by an overwhelming majority of the educators, as well as a sizable portion of the mathematicians connected with education. It may seem churlish, therefore, to say at this point that all is not well with the *content* of the proposed reform and that a more measured response to these innovations is in order. Nevertheless, let me attempt such a response. I shall concentrate on only three areas.

One major emphasis of the current reform is on the "process" in mathematics rather than the "product." Roughly, this means more stress on the general, qualitative reasoning of mathematics at the expense of technical skills and neat formulas. This is clearly a reaction to the mindless and excessive routine computations that characterize many of the elementary texts. The very formal-sounding textbooks still in use in most classrooms are now being replaced by books filled with heuristic arguments, conjectures, and examples. There is no question that this is welcome, but one must ask whether the "process" is backed up by solid mathematics. Qualitative reasoning is important, but so are precise formulas and long computations. Many mathematicians have pointed out that, in response to having "technique" replaced by "technique-without-understanding," the reform movement has now gone to the other extreme of de-emphasizing basic techniques and thereby gutting mathematics. Thus we find in this climate a 9-12 curriculum which allocates the quadratic formula only to the college-bound students in the 12th grade [IMP], a pre-calculus text that does not do the binomial theorem or the geometric series [NC], and a beginning calculus text that does not treat the convergence of infinite series or L'Hospital's rule [HCC].

A second major emphasis of the reform is on the general issue of "relevance," that is to say, the role of mathematics for solving everyday problems. This has to be understood literally. "Applications" in the past used to be synonymous with "applications to the physical sciences," e.g., the deduction of Kepler's three laws from Newton's inverse square law. Now, "applications" means largely statistical phenomena directly related to social issues or tangible problems of our everyday lives, e.g., how to set the "best" speed limit on a given stretch of freeway, or compute the height of a rider on a Ferris wheel as a function of time [NC], [IMP]. The literature of the curriculum development project ARISE (which has been funded by the NSF to develop a complete mathematics curriculum for 9-12) states, for example, that "In ARISE, the mathematics truly arises out of applications. The units are not centered around mathematical topics but rather application areas and themes, with the mathematical topics occurring as strands throughout the unit" [A]. Whether or not this group is proposing to inculcate the idea in high school that "mathematics = industrial and applied mathematics" I leave for the reader to decide.

An obsession with applications can lead to a mathematics curriculum without mathematical cohesion or structure, and to a mathematical education that does violence to mathematics as a branch of knowledge that stands on its own. In addition, one can question the pedagogical value of the current de-emphasis of applications to the physical sciences. Insofar as applications are supposed to demonstrate the power of mathematics, there is no doubt that those arising from the physical sciences do so most convincingly. They are also the ones that carry the most potent mathematical ideas, and have the further advantage that the implication of the mathematical outcome rarely involves any uncertainty of interpretation.

A third emphasis of the reform movement is on minimizing the rôle of proofs in the regular curriculum. Because "proof" is at present a slightly obscene word in mathematics education, an even-handed approach to this issue must begin with the fact that proofs were never accorded their rightful place in the older (traditional) curriculum. Should the reader have any doubts, a casual perusal of almost any algebra text currently in use in the schools would dispel them. What happened in the past was that, when all else failed, one could always count on Euclidean geometry to give the students a modicum of precise logical thinking. Unfortunately, the mathematical training of the average teacher could not (and cannot) be trusted to give adequate instruction in two-column proofs, and the resulting courses in geometry tended to be a travesty.[3] Given this reality, the hostility towards proofs in the education

[3] The problem of teacher qualification, or rather the lack thereof, is a central one in the current "math crisis" in the U.S. It is easy to blame the teachers for this problem until one realizes that it is the mathematicians who train the future teachers and that it is the reward system of our society which indirectly selects them. There is thus enough blame to be shared by all concerned. Unfortunately, we cannot go into this problem here as it would take a full treatise to do it justice.

circle should come as no surprise. One unfortunate example of this extremism in mathematics education is a popular geometry text, highly praised by many teachers, with essentially all the proofs omitted [SE].[4] Another one is a whole curriculum with absolutely no proofs [IMP],[5] which is justified by the statement: "…secondary school is [not] the place for students to learn to write rigorous, formal mathematical proofs. That place is in upper division courses in college" [IW]. Yet another is a calculus text that does not prove a single theorem [HCC]. In all these examples, heuristic arguments are routinely given, and some of them *are* correct proofs. However, since the latter are not clearly separated from others that are logically incomplete or even invalid, students never learn what a proof is. The kind of confusion and abuse such a mathematical education leads to is easy to imagine; for the record, see [W] for examples.

An independent observer may be surprised by the inherent contradiction in the simultaneous emphasis on *process* in mathematics and the de facto banishment of proofs from the curriculum. Perhaps the reform has many concerns, and that of making mathematics accessible to all students overrides all others. Seeing that the art of formulating a correct mathematical argument is not one of universal appeal, some educators were probably persuaded to take the line of least resistance. Be that as it may, the current reform movement is definitely kinder to the students in the lower half than those in the top 20% (say), and the question of how to take proper care of students serious about learning mathematics is left unresolved for now.[6] By coincidence, the Department of Education issued a remarkable document [NA] only six months ago which discusses in depth how the American schools have failed to educate the talented students. It is difficult not to see [NA] as a reproach of this senseless drift in mathematics education, from allowing proofs to be memorized without understanding to essentially denying the students the opportunity to learn about proofs altogether.

One may ask how a well-intentioned document advocating reform such

[4] Discussion of proofs in this 694-page text begins on p.563 (and is badly done), but the students almost never get to p.563.

[5] See the comment below about heuristic arguments

[6] See [RE] for a discussion of this issue. The question why the reform movement would not openly advocate the establishment of a system with built-in choices, whereby the students choose among two or more kinds of courses on the same subject which are differentiated by the amount of technical emphasis in each, is one that has never been answered without social-engineering jargon.

as the NCTM Standards ([N]) could go so wrong. The truthful answer is that the NCTM Standards was written to be all things to all people. On almost every issue, it is open to many interpretations. In fact, one educator was moved to remark that NCTM wants to avoid imposing a curriculum on the teachers so that they would take more initiatives in interpreting and implementing the goals suggested in the Standards. So what are these goals? In general terms, NCTM wants the students to learn to think, and it also wants them to learn "significant mathematics." It is easy to agree with such noble ideals until one realizes that, without precise instructions on the content, pedagogy, and assessment of such a curriculum, *and* especially without a corps of mathematically competent teachers to implement it, they are not achievable goals in the U.S. of A. in 1994. It appears to at least one observer that the tone set by the NCTM Standards has everything to do with the developments detailed in the preceding paragraphs. In its large 256 pages, the Standards makes repeated references to the value of a mathematics education in the schools as a valuable tool to earn a living in the high-tech work force. It also over-emphasizes the applications of mathematics to the social sciences and everyday life, as previously discussed. By contrast, it only mentions in passing (on p.5) the need to learn "to value mathematics" in a cultural and historic context, and that is almost the last time the words "culture" and "history" make an appearance in the book. Given this glaring imbalance, a reader of the Standards is not likely to associate learning mathematics with "the attainment of a higher intellectual level." Such a hard-nosed pragmatic approach to mathematics is bound to spawn anomalous activities.

The deleterious effect of the current reform on the *mathematical* component of elementary mathematics education should be a matter of grave concern to all mathematicians. Yet most are not even aware of the reform, and among those few that are, a majority seem to be enthusiastic about the reform itself as well as about the NCTM Standards [N]. As a consequence, the voice of dissent, so vital to any intellectual enterprise, is thus far largely absent. Although a recent letter [RO] and an article [SA], both extremely critical of [N], did make it to the pages of the *Notices of the American Mathematical Society*, their tendency to overstate their cases and the intemperance of the language give the readers the false impression that only extremists are resisting the reform movement. The mathematical community would be very poorly served indeed if it failed to get the message that a real crisis in mathematical education is looming on the horizon.

THE BOOKS UNDER REVIEW

All three books are a throwback to the bygone era when a student's mathematical achievement was judged by fairly objective standards and the notion of learning for its own sake was still greeted with some approbation. The Foreword to **FG** says, "This book as well as others in this series is intended to to be compatible with computers. However, . . . (the) computer cannot — nor will it ever be able to — think and understand like you can." By and large, these books challenge students to understand mathematics on its own terms. No beautiful computer graphics (in particular, no pictures of fractals), no fancy display of computer power, and no jazzy real-world applications. With this understood, what these books manage to accomplish is to give a surpassing demonstration of the art of mathematical exposition that falls outside the prescriptions of the NCTM Standards.

FG leads the reader through the first steps of graphing simple functions: linear functions, $y = |x|$, quadratic functions, fractional linear functions, and power functions. **MC** gives a leisurely tour of the coordinatization of the line, plane, 3-space, and 4-space. They follow the sound pedagogical principle that if students understand the simple concrete cases thoroughly, the extrapolation to the general situation will not be difficult. The slimness of both volumes gives the correct indication that each has a well-defined and quite limited objective. For this reason, they would most likely be used as supplementary materials in the (American) classroom, and we shall discuss them accordingly. On the other hand, **Algebra** is a more ambitious book and is considered by some as a potential textbook. It traverses a much more extensive mathematical terrain. Of necessity, the review of this book will have a different focus and must be done separately from **FG** and **MC**.

These three books are used as texts in the Gelfand International Mathematics School (a correspondence school[7]). Three other volumes have been promised in the same series (*Calculus, Geometry,* and *Combinatorics*). For the most part, the following will concentrate only on the relation of the three books under review to American high-school education.

[7] One can get further information by writing to: Harriet Schweitzer – AMCS, CMSCE, SERC Building, Room 239, Bush Campus, Piscataway, NJ 08855-1179, or email harriet@@gandalf.rutgers.edu

FG AND **MC**

A sine qua non of a book about mathematics is that it be mathematically correct. While this sounds trivial, the fact remains that many elementary textbooks contain serious errors.[8] Given the stature of the authors in the present case, one can take for granted that these books are mathematically correct. Beyond correctness, a book can impress by its good taste in the selection of materials or the choice of a particular approach to a topic. Or it can impress by the pellucid style of presentation of a subject that easily becomes abstruse in lesser hands. An elementary book can also impress by the inclusion of unexpected insights along a much trodden path. These two books are impressive for all the above reasons, but more is true. To me, their most striking characteristic is the amount of space devoted to a clear and detailed exposition of the inner workings of mathematics: all through both volumes, one finds a careful description of the step-by-step thinking process that leads up to the correct definition of a concept or to an argument that clinches the proof of a theorem. Most if not all beginning students are sorely in need of this material, as are most teachers. We are therefore very fortunate that an account of this caliber has finally made it to the printed page.

Let me give an example: the discussion of the unit cube in $\mathbb{R}^4$ in Part II of **MC**. If my own experience is at all typical, for most of us, our first encounter with n dimensions was tentative and not a little tinged with anxiety. We more or less equated n with 3, as we were told to do on an intuitive level, and we secretly hoped that this oversimplification would not lead us astray. Now enters **MC**, which shows us that this transition from 3 to n in fact can be methodical, smooth and (to use a much abused word) fun. It leads the reader gently to 4 dimensions by first carefully examining the unit cubes in dimensions 1, 2 and 3, and then using all the information so accumulated in the visible world to extrapolate to dimension 4, again carefully and painstakingly. It looks at the situation not only geometrically but also analytically, and makes sure that the reader can correlate the information in these two separate domains. It counts the vertices, edges, and faces of the cubes first

[8] Let me give one example. The usual treatment of sine and cosine in calculus texts assumes that one can compute the length of an arc in the unit circle so that the radian measure of an angle can be defined. But of course, arc-length has not yet been defined up to that point, so that the usual "proof" of the "Theorem" that $d\sin x/dx = \cos x$ purports to prove something about an object not yet properly defined. The use of the word "proof" rather than "heuristic argument" in this context is then an error (although it is easy to fix).

in each of the visible dimensions, so that when it comes to dimension 4, the extension of the counting to the invisible cube by analogy becomes almost effortless. I dare say that anyone who has taken this guided tour will never be intimidated by dimension n ever again. So the only question is: why hasn't any of us thought of writing something like this?

This is the kind of basic mathematical thinking that all high school students and their teachers should be exposed to, independent of the considerations of "real world application,", "technology," "relevance," or what not. Any mathematics education reform should make every effort to insure that the students (and their teachers as well) have access to this kind of writing.

Incidentally, I used the word "painstakingly" above. If this conveys the impression of dullness and pedantry in the exposition of **MC**, then I would like to assure the reader that exactly the opposite is true. The exposition is lively and charming throughout. For other felicitous examples, take the experimental approach to the focus-directrix property of the parabola on p. 42 of **FG**, the motivation for the definition of the tangent of a curve on p. 77 of **FG**, and the discussion of the wings of a butterfly on p. 52 of **MC**.

Every book has its flaws, and these volumes are no exception. There are two that are obvious and pressing: they should have an index, and they should clearly specify the precise knowledge assumed of the reader. (It seems that **MC** should precede **FG**, and that **Algebra** should precede both.)

While the easy-going and conversational tone of the exposition reflects faithfully the smooth progression of the mathematical ideas, there are major discontinuities in three places. Unfortunately, these come without warning. On p. 82 of **FG**, the reader is asked to believe what amounts to

$$\lim_{x \to \pm\infty} \frac{x-1}{x^2+2x+1} = 0.$$

For the intended readers of this volume, some kind of discussion or a more careful argument should be in place to signal the transition to something of greater subtlety. If the decision was not to enter such a discussion, then some disclaimer to this effect would also serve the purpose. Instead, what one gets is "business as usual," and this might just baffle the unsuspecting reader. Next, p. 48 of **MC** gives a beautiful argument to count, asymptotically, the number of lattice points N in a circle of radius $\sqrt{n}$. It reads: "Thus we get the approximate formula $N \approx \pi n$." The problem here is that the symbol "$\approx$" is never defined and there is again no warning about the jump in mathematical sophistication at this juncture. Furthermore, the subsequent argument justifying this formula on p. 49 is on a higher level than the informal tone would seem to indicate. Either it should be greatly expanded, or some cautionary statement to the reader is in order. Finally, Figure 34 on p. 70 of **MC** gives a 3-dimensional projection of the 4-dimensional cube. The paragraph in the lower half of the page purports to explain how that is done, but its terseness is not in line with the very detailed discussion up to that point.

MC has many passages in fine print. It also has some traffic signs (?) next to certain passages. A little explanation of these interesting conventions would be gratefully accepted by the readers, even at the risk of lowering the CQ (Charm Quotient).

High school students (or teachers) reading through these two books would learn an enormous amount of good mathematics. More importantly, they would also get a glimpse of how mathematics is done. This is an example of how the stated goals of the NCTM Standards (learn how to think and learn significant mathematics) can be fulfilled without the use of group learning or technology, and without any extraneous need to make mathematics relevant to daily life. In particular, the detailed presentation of the thinking process in these books allows the readers to discover with the authors something new at each step. There we have the "discovery method" in action without the usual associated educational paraphernalia. True, these books may not be for everyone, but then nothing ever is. If this shows anything at all, it is that there are diverse and equally valid approaches to learning.

ALGEBRA

The topics treated in this volume constitute a good part of what is commonly known as Algebra II in the American schools: review of the basic properties of the ring of integers, including the division algorithm; raising numbers to integer powers; expansion of $(a+b)^n$ and Pascal's triangle; polynomials and rational expressions; arithmetic and geometric progressions; geometric series; quadratic equations; roots and non-integer powers; inequalities and the inequality of arithmetic and geometric means; quadratic means and harmonic means.

The qualities of incisiveness, insight, and impeccable taste that set **FG** and **MC** apart from other books of the same genre also infuse the present volume. Were **Algebra** to be used solely for supplementary reading, it could be wholeheartedly recommended to any high school student or any teacher. (But note the discussion of some flaws below.) In fact, given the long tradition of mistreating algebra as a disjointed collection of techniques in the schools, there should even be some urgency in making this book compulsory reading for anyone interested in learning mathematics. They would discover, perhaps for the first time, that algebra is a logically coherent discipline and at the same time, that its formalism is a product borne of good sense and sensible conventions. Both may come as a surprise.

If one tries to give substance to the preceding general recommendation by specific examples, one is confronted by an embarrassment of riches: where to begin? One might begin by highlighting the down-to-earth character of the whole book. On page 11 and again on pp. 15-17, there are cogent *explanations* of the need for symbols; such a discussion is probably not to be found in any other book of this level. Consider next the perhaps perplexing notation a^{-n} for a positive integer n. It may occur to a beginner to ask why anyone would bother to write this instead of $1/a^n$. Basically, the answer is that once this notational convention is adopted, it would start to "think for us". Look at something different now, the division algorithm for polynomials. Instead of giving the formula and the relevant definitions right away, the book first performs three concrete divisions with low degree polynomials and describes the algorithm in complete sentences, *without* symbols. A final example: before presenting the Pascal triangle and its basic property ($\binom{n-1}{i} + \binom{n-1}{i+1} = \binom{n}{i+1}$), the book examines in detail the expansion of $(a+b)^n$ for $n = 2, 3, 4, 5$, verifies the property in each case, and points out the pattern underlying this property. Then when the property is finally stated in full generality (p. 43), it becomes no more than an afterthought. (It is done without using the notation of the binomial coefficients.)

The next class of examples have to do with a serious mathematical issue: the question of existence and uniqueness. While these concepts undoubtedly lie at the core of mathematics, they would seem to be without any "relevance" whatsoever by the current standard of the reform movement. The book touches on one or the other more than once: for example, the existence and uniqueness of the (positive) square root of a positive number a (p. 98), the uniqueness of the quotient and remainder in the division algorithm for polynomials (p. 66), the existence and uniqueness of the positive n-th root of a (p. 123), and the existence of the maximum of a certain product of numbers (p. 146). Such subtle issues are almost never raised in high-school mathematics. Those who believe that mathematics in the schools should only be strands around real-world applications would no doubt dismiss any discussion of such issues as sterile elitism. I know otherwise, however. My experience with teaching the basic existence and uniqueness theorem of ODE in calculus courses tells me that most of the students never get it, no matter how hard one tries. If the students were given ample exposure to these ideas in high school, would they not be more likely to develop this kind of mental agility?

A final class of examples has to do with the book's effort to present more than one solution to a given problem whenever posible, for example, the solutions to problems on p. 36, p. 68, p. 75, p. 76, p. 84, p. 85, p. 117 and p. 118, three ways to sum a geometric progression, three proofs of the inequality of arithmetic and geometric means, etc. A great deal of the current reform effort seems to be put on convincing students that there is more than one correct answer to a problem, and that there is more than one way to do a problem. The former is clearly a very dangerous position that, in the hands of someone less than completely knowledgeable, can lead to frightful abuses. And it does [W]. The latter is much discussed in the reform movement, but the discussion often lacks substance because the examples used are usually trivial. **Algebra** can serve as a good model to show, correctly for once, that mathematics is indeed "open-middled" and "non-rigid."

With all these good things going for it, would it not be natural to propose **Algebra** as a text in the schools? I believe several changes have to be made before it is suitable as a textbook. Some of them are trivial, others may be less so. Let us go through them systematically:

(1) The book needs an index.

(2) The book needs more exercises. The problems (some of them with solutions) scattered throughout all three volumes are educational, interesting, at times amusing, and always stimulating.[9] However, a textbook needs some easy exercises for the weaker students, and it also needs a few more than what are presently in **Algebra** to keep the stronger students busy. More specifically, a textbook on algebra would need plenty of word problems to force the students to learn to read *and* to translate the verbal information into mathematics. Anyone who has ever taught at the elementary level would

[9] The problem on p. 3 needs some tightening of the language. As it stands, $8+\cdots+8$(125 times) is obviously a solution, but the **Solution** on that page conveys the false impression that the solution is unique.

understand at once that the latter "translation" process is the weak spot in most students' mathematical armor, and that addressing this weakness must be one of the main concerns of an algebra course.

(3) Some topics need to be added. Complex numbers, a thorough discussion of the roots of cubic polynomials with real coefficients, the concept of a function together with an elementary discussion of the exponential and logarithmic functions, are obvious items in an algebra course that are presently missing in **Algebra**. Actually much can be gained in this book if the function concept is introduced and seriously discussed. Take the present treatment of polynomials and rational expressions, for example. §29 on p. 47 tries to say that two polynomials can be equal in two different senses, as members of $\mathbb{R}[x]$ or as functions on $\mathbb{R}$; but it really does not get the point across too well because the function concept is missing. The same remark applies to the equality of rational expression in §34 (pp. 56-58). There the situation is even more critical than the case of polynomials because the need for the concept of the domain of a function becomes acute. There is a good additional reason for introducing the function concept in algebra: when students really get to know polynomial functions, their entry into the world of calculus will be that much smoother.

(4) The informality of the exposition has to be reined in. This recommendation clearly needs a lot of explanation! It has been said that every author of a successful textbook has to be something of a pedant. This is because, for a textbook to be of service to ALL students (so one tries, at least), the i's have to be dotted and the t's crossed. Now **Algebra** maintains the same conversational style as in **FG** and **MC** and is very charming. In fact, the first paragraph of the book says clearly, "This book is about algebra. This is a very old science and its gems lost their charm for us through everyday use. We have tried in this book to refresh them for you." But there comes a time when charm has to give way to official business, and mathematical clarity must precede all other considerations. I will illustrate this point with several examples.

(4a) On p. 10, the book tries to explain that a finite product of integers is independent of the order the multiplication is carried out. Since this comes right after the associative law of multiplication, the book probably expects the reader to know that associativity is involved. Nevertheless, a textbook to be used by all types of students should mention the associative law, and this is not done.

(4b) On p. 29, a^n is defined for $n \in \mathbb{Z}$, but nowhere does it say $a \neq 0$ for

$n = 0$.

(4c) The definition of a polynomial on p. 44 says it is

> an expression containing letters (called *variables*), numbers, addition, subtraction and multiplication. Here are some examples[10]: $a^4 + a^3b + ab^3 + b^4$, $(5-7x)(x-1)(x-3)+11$, 0, $(x-y)^{100}$. These examples contain only addition, subtraction and multiplication, but also positive integer constants as powers. It is legal because they can be considered as shortcuts (e.g., a^4 may be considered as short notation for $a \cdot a \cdot a \cdot a$ which is perfectly legal). But a^{-7} or x^y are *not* polynomials.

My guess is that the book tries to define a polynomial as an element in the ring generated by monomials, but not having the language to do so, it tries to compensate by talking about it in an informal way. In this case, the informality is not a help and a more formal definition would probably fare better. As a matter of fact, when the book comes to fractions of polynomials on p. 63, it feels necessary to add:

> When we say a polynomial must not contain division it does not mean that all its coefficients must be integers; they may be any numbers, including fractions. so for example, $\frac{1}{2}$ is a perfectly legal polynomial of degree zero.

This most likely increases the confusion rather than clarifies it.

(4d) I have already mentioned that the discussion of the equality of two polynomials on. p. 47 is hobbled by the absence of the function concept. As it stands, I am not sure that the informal discussion there conveys the intended message at all.

(4e) On p. 49, the problem is posed:

> Is it possible when multiplying two polynomials that after collecting similar terms all terms vanish (have zero coefficients)?

Then it goes on:

> **Answer**. No.
> **Remark**. Probably this problem seems silly; it is clear that it cannot happen. If it is not clear, please reconsider the problem several years later.

[10] I omit more than half of the examples actually listed in the book.

One can sympathize with the authors for not wishing to open a can of worms (cf. (4d) above), but I doubt that a student would find the preceding passage either informative or edifying.[11]

(4f) On p. 56, §33, a *rational expression* is defined. Again the book probably wishes to say that a rational expression is an element of the quotient field of $\mathbb{R}[x]$, but it chooses to do so informally. The resulting exposition, too long to quote here for a change, is quite a bit less than clear.

(4g) On p. 58, still on the subject of rational expressions, we find:

> Strictly speaking, the cancellation of common factors is not a perfectly legal operation, because sometimes the factor being cancelled may be equal to zero. For example, $(x^3+x^2+x+1)/(x^2-1)$ is undefined when $x = -1$ (but) $(x^2+1)/(x-1)$ is defined. Usually this effect is ignored but sometimes it may become important.

This is charming for a mathematician to read, but may very well be a nightmare for a beginner.

(4h) §61 on pp. 120-121 deals with the solution of a special 4th-degree equation by reducing it to solving a quadratic equation through a clever change of variable. It is very nice, but one can give the students a better perspective if one adds a sentence to this effect: "If one is lucky, one can solve a few equations of high degree by a special substitution."

(4i) On p. 126, the concept of fractional powers of a number a is defined without mentioning that a must be positive.

There are other examples of this type, but I think I have made my point about the need for pedantry. (The curious reader may wish to check: p. 72, lines 9-10; p. 86, lines 7-8; p. 98, last paragraph; p. 100, paragraph above §50; p. 117, first solution to the first problem.)

(5) There should be more applications in the book. By applications, I do not mean exclusively real world applications. Applications to other parts of mathematics which involve enlightening ideas or techniques, such as the counting of lattice points inside a circle on p. 48 of **MC**, would be perfectly legitimate. All the same, I think a textbook in school mathematics should have as many applications as possible, for at least two reasons. First, students need applications to deepen their understanding of both the new concepts and the new techniques. Second, high school students deserve to be given a

[11] It is not important, but note that the answer should not be "no" because one of the factors could be $(x - x)$.

well-rounded view of the subject in order to develop their own interests or to make the correct career decisions. Peter Hilton put it very well when he wrote (Foreward to [CM]):

> We must certainly take into consideration the potential users of mathematics, since the main argument for the importance of mathematics today is precisely the ubiquity of its many applications. ... We also have a responsibility, as teachers of mathematics, to cater to the future citizen, the future adult.

For the case at hand, a good part of algebra was created in response to needs in the other branches of mathematics as well as in the real world, and the students should be informed of this fact, including the connection with the real world, through well-chosen materials on applications. The addition of problems of an applied nature plus a few supplementary sections on applications would bring about a more balanced presentation.

One last suggestion I can make is to flesh out the exposition of **Algebra**. Its Spartan character, everything said only once and not a word to waste, may be intimidating to some students at this level. A few additional reminders or back references would certainly be welcome. In discussing the equality of two rational expressions on p. 72, for example, would it not help the students if a reference were made to p. 47 concerning a similar problem with polynomials? Again, when the existence and uniqueness of the square root of c is discussed on p. 98, a reference to p. 54 where this issue first comes up would refresh the student's memory.

A CONCLUDING THOUGHT

What we have here are three excellent mathematical works from which students and teachers alike would have much to learn. Yet they do not fall within the prescriptions of the prevailing trends in mathematics education reform. I have tried to emphasize the qualities that make these books stand out. To all of us who still have the goal of teaching MATHEMATICS, these books have something special to offer. It is a pity that many in the reform movement choose to close their eyes to the merits of these books, the more so because some have even dismissed them on the ground that they do not conform to the NCTM Standards. The reform is much the worse for that.

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