Chapter 2: Fractions (Draft)

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On the Teaching of Fractions

The following is a new approach to the teaching of fractions. It is not new in the sense of introducing new concepts; the subject is too old for that. Rather, it is new in the way the various skills and concepts are introduced and woven together. Whereas it is traditional to ask you to believe that the concept of a fraction is so profound that you have to be willing to accept multiple meanings for it at the outset, we merely ask you to accept one clear-cut definition of a fraction (as a point on the number line), and use reasoning to deduce as logical consequences all other meanings of this concept. In this way, everything that is present in the traditional presentation will make its appearance in this logical and coherent development of the subject. Everything is explained, and nothing is taken for granted.

The logical coherence in a presentation of fractions is critically important. The reason why this subject is the bane of elementary school students has been extensively investigated, and much educational research into the so-called "learning of fractions" has been done. But to our knowledge, no dramatic progress has yet surfaced. If we are to look at this situation without prejudice, we cannot fail to see that a major reason for students' failure to learn fractions is the mystical and mathematically incoherent manner the subject has been presented to them. One consequence is that few students can give a definition of a "fraction" that is at all related to all the manipulations they are made to learn. This being the case, the disjunction between understanding and skills is right there from the beginning. Why then should there be any surprise that students fail to add and divide these mysterious objects with ease or conviction? IN THE FACE OF SUCH MATHEMATICAL DEFECTS IN THE PRESENTATIONS OF FRACTIONS IN SCHOOL TEXTS AND PROFESSIONAL DEVELOPMENT MATERIALS, EDUCATIONAL RESEARCH INTO THE LEARNING OF FRACTIONS BECOMES MORE ABOUT STUDENTS' ABILITY TO OVERCOME THE HANDICAP OF A BAD EDUCATION THAN ABOUT THEIR ABILITY TO LEARN. *If students are not taught correct mathematics, they will not learn correct mathematics. A strong case therefore can be made that one should shore up the mathematical presentation* *of fractions before proceeding with the research.*

In addition to logical coherence, a noteworthy feature of this presentation of fractions is that fractions are treated on the same footing as whole numbers. You will discover that everything you learn from Chapter 1 about whole numbers naturally extends to fractions. There is none of the conceptual discontinuity — so disturbing to a beginning learner — in going from whole numbers to fractions that mars almost all expositions on fractions in school mathematics. Both whole numbers and fractions are just numbers, and they are here treated as such accordingly.

We hope you would find that the present presentation makes sense for a change.[1]

We are going to develop the whole subject of fractions from the beginning, assuming nothing from your previous knowledge of the subject. We will spend times explaining facts (such as equivalent fractions) which you may be already familiar with. It is likely, however, that the explanation itself will be new to you so that it takes time for you to get used to it. For example, we will ask you to remember, again and again, that a fraction is a point on the number line (see §5 of Chapter 1). This will take a little time before it sinks in. Of course, seeing something you think you know but having to rethink it in order to achieve a greater understanding can be a frustrating experience at first, but we think you would find it worthwhile in the long run.

[1] I will not make the usual sales pitch about how it is going to be easy and fun to learn fractions this time around. Let me just say that this presentation will make sense, and it will yield to the normal amount of hard work that must accompany any worthwhile endeavor.

The understanding of this chapter would require a knowledge of at least §§4–5 of Chapter 1 on Whole Numbers. Moreover, it would be to your advantage to first read the Appendix in §13 to get a general idea of where this chapter is headed.

1 Definition of a Fraction

Recall that a number is a point on the number line (§5 of Chapter 1). This chapter deals with a special collection of numbers called fractions, which are usually denoted by $\frac{m}{n}$, where m and n are whole numbers and $n \neq 0$. We begin by defining what fractions are, i.e., specifying which of the points on the number line are fractions. The definition will be both clear and simple. If you find it strange that we are making a point of giving a definition of fractions, it is because this is something thousands (if not hundreds of thousands) of teachers have been trying to get at for a long time. Most school textbooks and professional development materials do not bother to give a definition at all. A few better ones at least try, and typically what you would find is the following:

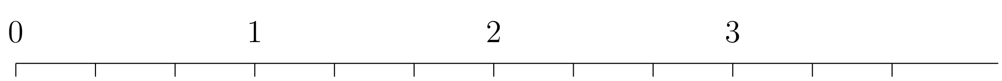
> Three distinct meanings of fractions — part-whole, quotient, and ratio — are found in most elementary mathematics programs.
>
> Part-whole The part-whole interpretation of a fraction such as $\frac{2}{3}$ indicates that a whole has been partitioned into three equal parts and two of those parts are being considered.
>
> Quotient The fraction $\frac{2}{3}$ may also be considered as a quotient, $2 \div 3$. This interpretation also arises from a partitioning situation. Suppose you have some big cookies to give to three people. You could give each person one cookie, then another, and so on until you had distributed the same amount to each. If you have six cookies, then you could represent this process mathematically by $6 \div 3$, and each person would get two cookies. But if you only have two cookies, one way to solve the problem is to divide each cookie into three equal parts and give each person $\frac{1}{3}$ of each cookie so that at the end, each person gets $\frac{1}{3} + \frac{1}{3}$ or $\frac{2}{3}$ cookies. So $2 \div 3 = \frac{2}{3}$.

RATIO The fraction $\frac{2}{3}$ may also represent a ratio situation, such as there are two boys for every three girls.

Such an explanation is unsatisfactory for several reasons. To say that something you try to get to know is three things simultaneously strains one's credulity. For instance, if I tell you I have discovered a substance that is as hard as steel, as light as air, and as transparent as glass, would you believe it? Another reason for objection is that a fraction is being explained in terms of a "ratio", but most people don't know what a ratio is.[2] In addition, while we are used to the idea of a division $a \div b$ where a is a multiple of b (see §3.4 of Chapter 1), we are not sure yet of what $2 \div 3$ means. So to use this to explain the meaning of $\frac{2}{3}$ does not seem to make sense. Finally, we anticipate that fractions would be added, subtracted, multiplied and divided, and it is not clear how one goes about adding, subtracting, multiplying and dividing a part-whole, or a quotient, or a ratio.

This is why we opt for a definition that is both simple and clear.

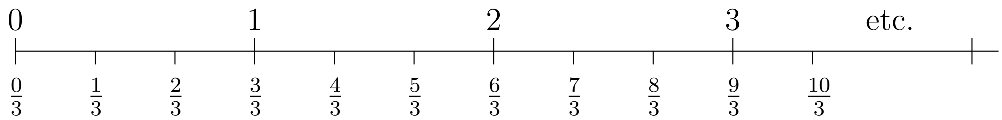
Before giving the general definition of a fraction, let us first consider the special case of all fractions of the form $\frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \ldots$, and try to see what they mean. We have to begin with the concept of two line segments on the number line being of *equal length*: it means that if we slide one of them along the number line until the left endpoints of the segments coincide, then their right endpoints also coincide. Now divide each of the line segments $[0, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, $[4, 5]$, $\ldots$ into three segments of equal length so that each of these segments now acquires two additional division points in addition to its left and right endpoints. The number line now has a new sequence of equally spaced markers superimposed on the original markers corresponding to whole numbers.



By definition, $\frac{1}{3}$ is the first division point to the right of 0, $\frac{2}{3}$ is the second division point to the right, $\frac{3}{3}$ is the third division point, $\frac{4}{3}$ is the fourth

[2] We will precisely define ratio in §11.

division point, etc., and $\frac{m}{3}$ is the m-th division point for any whole number $m > 0$. By convention, we also write 0 for $\frac{0}{3}$. Note that $\frac{3}{3}$ coincides with 1, $\frac{6}{3}$ coincides with 2, $\frac{9}{3}$ coincides with 3, and in general, $\frac{3m}{3}$ coincides with m for any whole number m. Here is the picture:



Each $\frac{n}{3}$ is called a *multiple* of $\frac{1}{3}$. Note that the way we have just introduced the multiples of $\frac{1}{3}$ on the number line is exactly the same way that the multiples of 1 (i.e., the whole numbers) were introduced on the number line in §5 of Chapter 1. In both cases, we start with a fixed unit (in §5 of Chapter 1 it was "1", and here it is "$\frac{1}{3}$"), and then propagate its multiples to the right of 0. One could paraphrase this situation by saying that the multiples of $\frac{1}{3}$ are the analogues of the whole numbers if $\frac{1}{3}$ is used in place of 1 as the unit. From this perspective, we can start with $\frac{1}{n}$ for any nonzero whole number n in place of $\frac{1}{3}$, and then propagate the *multiples* of $\frac{1}{n}$ (i.e., $\frac{m}{n}$ for all whole numbers m) to the right of 0 in exactly the same manner. This then leads to the following general definition:

Definition. Let k, l be whole numbers with $l > 0$. Divide each of the line segments $[0, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, ... into l segments of equal length. These division points together with the whole numbers now form an infinite sequence of equally spaced markers on the number line (in the sense that the lengths of the segments between consecutive markers are equal to each other). The first marker to the right of 0 is by definition $\frac{1}{l}$. The second marker to the right of 0 is by definition $\frac{2}{l}$, the third $\frac{3}{l}$, etc., and the k-th is $\frac{k}{l}$. The collection of these $\frac{k}{l}$'s for all whole numbers k and l, with $l > 0$, is called the *fractions*. The number k is called the *numerator* of the fraction $\frac{k}{l}$, and the number l its *denominator*.

For typographical reasons, a fraction $\frac{k}{l}$ is sometimes written as k/l. We adopt for convenience the **convention** that *the fraction notation* $\frac{k}{l}$ *or* k/l *automatically assumes that* $l > 0$. It is common to call $\frac{k}{l}$ a *proper* fraction if $k < l$, and *improper* if $k \geq l$. Note that by the way we define a fraction,

> *we make no distinction between proper fractions and improper fractions.*

To us, both are just fractions. Note in addition that

> *a whole number is also a fraction.*

This means that a whole number (which is a point on the number line) is among the markers $\frac{k}{l}$ for appropriate choices of k and l. This is true by the very definition of fractions because the markers of the whole numbers are among the markers of the fractions. Of course, we have already pointed out in the discussion of fractions with denominator 3 that m is the same point as $\frac{3m}{3}$ for every whole number m.

We also agree to write 0 for $\frac{0}{n}$ (any $n > 0$).

We can now define the lengths of more line segments than in §3.1. Let $\frac{k}{l}$ be a fraction. We say a *line segment from x to y on the number line, denoted by $[x, y]$, has length* $\frac{k}{l}$ if, after sliding $[x, y]$ to the left until x rests on 0, the right endpoint y rests on $\frac{k}{l}$. It is important to observe that if $l = 1$, then $\frac{k}{l} = k$ and this definition of length coincides with the one given in §3.1. Moreover, it follows from the definition of $\frac{k}{l}$ that:

> $\frac{k}{l}$ *is the length of the concatenation of k segments each of which has length* $\frac{1}{l}$.

For brevity, we shall agree to express the preceding sentence as:

> $\frac{k}{l}$ *is k copies of* $\frac{1}{l}$.

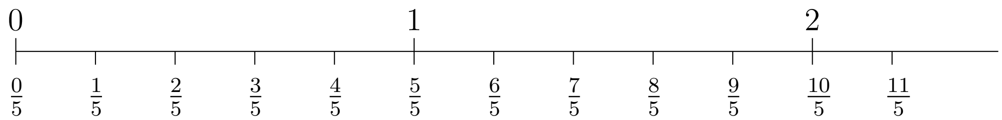
Note that the requirement of $l > 0$ in a fraction $\frac{k}{l}$ is easy to explain from our present standpoint: it is not possible to divide $[0, 1]$ into 0 segments of equal length. Of course, we will eventually show that $\frac{k}{l}$ is the same as k divided by l, so that $l > 0$ guarantees that we are not dividing by 0. (See the discussion after (48) in §4 of Chapter 1.)

It is to be remarked that there is at present some confusion in the meaning of "fraction" in the education literature.

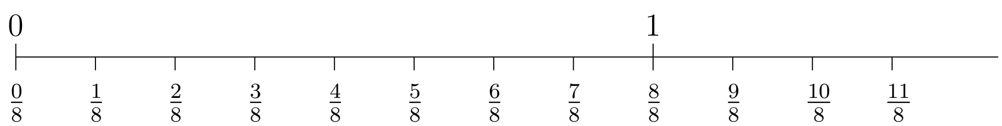
> **In this monograph, a fraction $\frac{a}{b}$ refers only to whole numbers a and b.**

Some writers however define a fraction to be $\frac{x}{y}$, where x and y can be any *real numbers*. Thus $\frac{\sqrt{2}}{5}$ would be a fraction according to the latter, but certainly not according to this monograph. One has to be careful with this conflicting use of the term.

The number line now has many more markers: in addition to the whole numbers, we also have all the fractions $\frac{k}{l}$, where k and l are whole numbers and $l \neq 0$.[3] For example, with $l = 5$, we display the first few markers of the form $\frac{k}{5}$:



The fraction $\frac{7}{5}$ is clearly shown as the 7th marker to the right of 0. Similarly, the number $\frac{11}{8}$ can be precisely located on the number line by dividing each of $[0, 1]$, $[1, 2]$, etc., into 8 segments of equal length and going to the 11th marker to the right of 0. Here is the picture:



You need to be completely at ease with this definition of a fraction. Here is an activity that could help you achieve this goal.

> **Activity:** Using the preceding examples as model, describe in words where each of the fraction is on the number line and also draw a picture to show its location. (a) $\frac{7}{9}$. (b) $\frac{6}{11}$. (c) $\frac{9}{4}$. (d) $\frac{13}{5}$. (e) $\frac{10}{3}$. (f) $\frac{k}{5}$, where k is a whole number between 11 and 14. (g) $\frac{k}{6}$, where k is a whole number between 25 and 29.

[3] It should not be assumed that every number is a fraction. For example, if $[0, c]$ is the line segment with the same length as the diagonal of the unit square, then it will be shown in §5 of Chapter 3 that c is not a fraction. In fact, "most" numbers are not fractions.

In case k is a multiple of l, say $k = nl$, then the fraction $\frac{k}{l}$ is special: it is quite evident (by looking at the numbers $\frac{5}{5}$, $\frac{10}{5}$, and $\frac{8}{8}$ in the two preceding pictures, for instance) that $\frac{l}{l} = 1$, $\frac{2l}{l} = 2$, $\frac{3l}{l} = 3$, $\frac{4l}{l} = 4$, and in general,

$$\frac{nl}{l} = n, \quad \text{for all whole numbers } n,\ l, \text{ where } l > 0 \tag{1}$$

In particular,

$$\frac{n}{1} = n \qquad \text{and} \qquad \frac{n}{n} = 1$$

for any whole number n.

Incidentally, (1) is consistent with the our convention that we write 0 for $\frac{0}{n}$ for any $n > 0$.

In common language, we say $\frac{k}{\ell}$ is "k-ℓ ths of 1", e.g., $\frac{2}{7}$ is two-sevenths of 1 and $\frac{4}{5}$ is four-fifths of 1.

Certain issues that arise in the preceding discussion are quite subtle and deserve to be examined at some length. First and foremost: it is imperative for our purpose that we have an *explicit definition* of a fraction. In this monograph, every mathematical statement is supported by a logical explanation. In the case of fractions, this definition furnishes the starting point of all logical explanations we have to offer about fractions. The way mathematics works is to start with one clearly stated meaning (i.e., a precise definition) of a given concept, and on the basis of this meaning we explain everything that is supposed to be true of this concept (including all other meanings and interpretations) using logical reasoning. Such an approach puts a premium on your reasoning skill as well as your knowledge of precise definitions. So please go back to study the definition of a fraction and get used to it (cf. §5 of Chapter 1). Some of the comments below may help you achieve this goal.

A second point is *the important role of a unit disguised as the number 1 in the definition of a fraction.* Compare the discussion in §4 of Chapter 1, especially the discussion surrounding (45). We have so far limited ourselves to looking at the unit 1 abstractly as a point on the number line, but suppose we use as unit the *weight* of a piece of ham which weighs three pounds. So the number 1 stands for the weight of this piece of ham. The number 4 is then the weight of four pieces of ham of the same weight (therefore twelve pounds). Now what would $\frac{1}{3}$ be? According to our definition, we divide our unit (three pounds of ham) into 3 parts of equal weight (each part therefore weighs one pound), and one of these parts is $\frac{1}{3}$.[4] In this context, $\frac{1}{3}$ is one pound of ham. In ordinary language, $\frac{1}{3}$ in the present setting is exactly what we mean by "a third of the piece of ham by weight". More generally, the same reasoning tells us that, if we decide on using the weight of an object X as our unit, then $\frac{5}{7}$ (say) would mean 5 parts of X after it has been partitioned into 7 parts of equal weight. Therefore in this setting, $\frac{5}{7}$ is what we usually refer to as "five-sevenths of X by weight". (compare §10 below.) Thus depending on what the unit 1 is, a fraction can have many interpretations. A fraction $\frac{5}{7}$ could be the volume of five-sevenths of a bucket of water, the volume of five-sevenths of a pie,[5] five-sevenths in dollars of you life-savings, etc.

In terms of a fixed unit, we can *paraphrase* the definition of a fraction as follows:

> Let k, l be whole numbers with $l > 0$. Then $\frac{1}{l}$ is by definition one part when the unit is divided into l equal parts, and $\frac{k}{l}$ is by definition k of these parts.

It is well to keep in mind that this does not give the precise meaning of a fraction because "one part" and "divided into l equal parts" are vague statements.[6] If the unit is the length of a certain segment, then "one part" would be a segment of a fixed length and "equal parts" would mean "segments of the same length". If a unit is the volume of a fixed set, then "one part" would be a subset of a fixed volume and "equal parts" would mean "subsets of equal volume", and so on. In the case the unit in question is the length of a segment, then the perceding paraphrase gives essentially the original

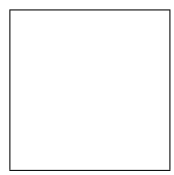
[4] Notice that we did not say "divide our unit into 3 equal parts", but said instead *3 parts of equal weight*. The reason is that the former statement does not make clear "equal" in what respect. Same shape? Same volume?

[5] In textbooks, this is often left vague, thereby leading to unfortunate misconceptions, see Exercise 1.7. The usual statement is to just say "cut the pie into 7 equal parts". Because the pie is represented by a circle, textbooks then show 7 circular sectors with the same central angle. Students should know that "equality" here means equality of the *volumes* of the parts of the pie. Because the circular pie is represented by a cross-section (which is a circle) and we may always assume that cutting equal areas of the circular cross-section leads to cutting equal volumes of the pie, it is legitimate to settle for cutting "regions of equal area in the circle". If however these considerations are never made explicit to students once and for all, there will be misconceptions about "equal division" and therefore about fractions.

[6] See preceding footnote.

definition of a fraction.[7] In the classroom, this way of introducing a fraction in terms of a fixed unit may be more acceptable to some students, but if it is used, do not forget to constantly remind the students about the presence of the unit.

Other than the length of a unit interval, the most interesting and the most common unit is the area of a unit square. Recall from §4 of Chapter 1 that "unit square" refers to a square with each side of length 1 (a certain unit being assumed to have been chosen on the number line).



Because we will have to use the concept of *area* in a more elaborate fashion, let us first give a more detailed discussion of the basic properties of area. The basic facts we need are summarized in the following:

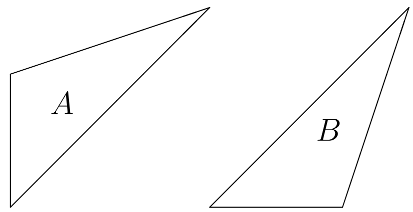
(a) The area of the unit square is by definition 1.

(b) If two regions are *congruent*, then their areas are equal.

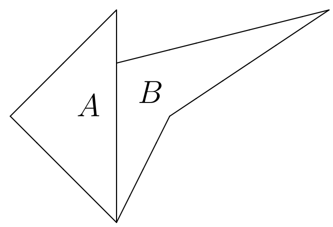
(c) If two regions have at most (part of) their boundaries in common, then the area of their union is the sum of their individual areas.

We shall not define "congruent regions" precisely except to use the intuitive meaning that congruent regions have the same shape and same size, or that one can check congruence by sliding, rotating, and reflecting one region to see if it can be made to coincide completely with the other. Thus the following two regions A and B are congruent:

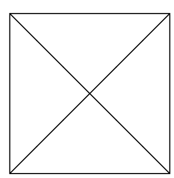
[7] What is missing from this paraphrase is the clarity of a fraction as a definite point on the number line. Here, a "part" must be left to the imagination.



The meaning of (C) is illustrated by the following: the area of the union of A and B below is the sum of the areas of A and B.

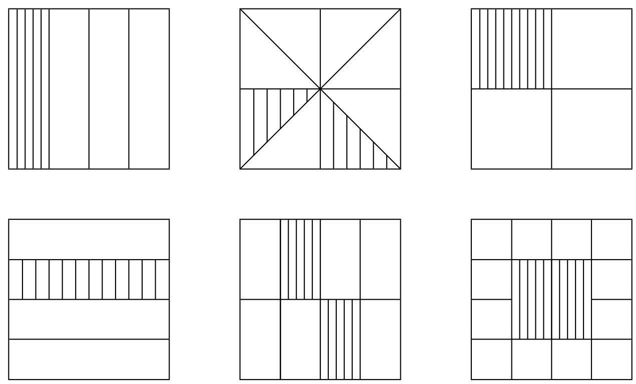


Let us show, for example, that each of the following four triangular regions in the unit square has area $\frac{1}{4}$:



By (a) above, the area of the unit square is 1, which we now regard as the unit of the number line. Each of these triangular regions is congruent to the others, and by (b), these four regions have equal area. Because the union of the four triangular regions is the unit square, the sum of these four areas is therefore equal to 1, by (c). Therefore, *area-wise*, these four regions give a division of the unit 1 into four equal parts. By the definition of $\frac{1}{4}$ in the above definition of fractions, the area of any of these triangular regions is thus $\frac{1}{4}$.

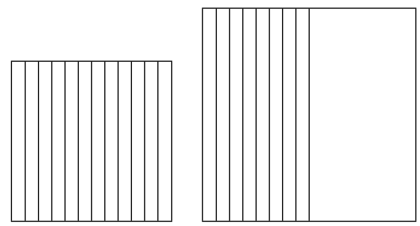
This kind of argument can be carried out in like manner in similar situations, and $\frac{1}{4}$ can be seen to have many pictorial representations. It can be a part of any division of the unit square into four parts of equal area. Each of the following shaded area is an example:



If it helps you to think of a fraction as some kind of a pictorial object, — part of a pie, part of a square (such as above), or a collection of dots, — by all means do so. *In mathematics, do whatever it takes to help you learn something, provided you do not lose sight of what you are supposed to learn.* In the case of fractions, it means you may use any pictorial image you want to process your thoughts on fractions, but *at the end, you should be able to formulate logical arguments in terms of the original definition of a fraction as a point on the number line.* This precise definition of a fraction is our starting point, and is therefore the *only* reference point at our disposal for logical arguments. For this reason, we will try to present such a number-line argument for every assertion we make about fractions in addition to all other forms of picture-based reasoning.

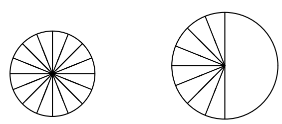
One word of caution about the use of pictures: even in informal reasoning, we should try not to damage students' intuitive grasp of the basic mathematics. Therefore if the area of a square or a circle is used as a unit, what we should be careful about is to keep the size of the unit the same (or as much as hand-drawing allows!) under all circumstances. Here are some examples of how **not** to do it.

EXAMPLE 1. Tell students that the following shaded area represents $\frac{3}{2}$:



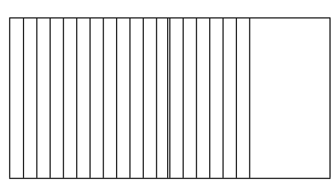
What is wrong is that, if the area of the left square is implicitly taken as the unit, then the area of the right square (which is visibly bigger) would be bigger than 1 and consequently the total shaded area would represent more than $\frac{3}{2}$. Or, if the area of the right square is taken as the unit 1, then the total shaded area would be smaller than $\frac{3}{2}$.

Example 2. Tell students that the following shaded area represents $\frac{3}{2}$:



This is exactly the same visual misrepresentation of a fraction as the preceding example, with only a square replaced by a pie. If the area of the left pie is taken as a unit, then the right pie represents a number bigger than 1, so that the total shaded area would be more than $\frac{3}{2}$. Because the pie representation of a fraction is so popular, it is hoped that by bringing this problematic issue to the forefront, such misleading representations of fractions will disappear from the classroom. An additional remark is that many teachers manage to avoid this misrepresentation because they only work with proper fractions, in which case only one pie would be needed at all times. *We would like to suggest that such a practice is not pedagogically sound because students must get used to seeing all kinds of fractions, proper or improper.*

Example 3. Ask students what fraction is represented by the following shaded area:



The problem here is that the unit is not clearly specified, so that it would be perfectly legitimate to assume that the area of the whole rectangle is the unit 1, in which case the shaded area would be $\frac{3}{4}$ instead of $\frac{3}{2}$. It would be a good idea to avoid this kind of ambiguity right from the beginning by emphasizing the important role of a unit.

A third point is that there is supposed to be some unease among students concerning the strange notation $\frac{k}{l}$ for a fraction; they wonder why it takes *two* whole numbers k and l and a bar between to denote a single object. We believe this notation is strange only when the concept of a fraction is not clearly defined and therefore remains a mystery, so that *any* notation used to denote a myterious object would seem strange. But we know better now: fractions are a definite collection of points on the number line, most of which lie *between* whole numbers. To denote $\frac{5}{3}$, for example, ask your students if they believe there is any way to specify the location of $\frac{5}{3}$ between 1 and 2 by using only *one* whole number. Obviously not. Furthermore, explain to them that the bar between 5 and 3 is strictly for clarity and nothing else. When such explanations are supplied, the notation $\frac{5}{3}$ would look much less strange.

A fourth point is that you may have some concern about the idea of dividing anything into 7 or 11 or 9 equal parts. To be specific, our definition of a fraction freely assumes the possibility of dividing the unit segment into 11 equal parts, for example, in order to place the fractions with denominator equal to 11 on the number line. In practical terms, this may seem far-fetched as most of us have trouble dividing anything into equal parts other than halves or fourths. However, we are talking here not about *practical* means of doing this but the *theoretical* possibility. Should you have a real psychological block against such considerations, §2 following describes how such theoretical equi-division of segments can be easily achieved.

A final point that merits discussion is this: *what does the equality of two fractions mean?* This is a continuation of the discussion of the equality of two whole numbers begun in §2 of Chapter 1. For example, we claimed in (1) that $\frac{nl}{l} = n$. For whole numbers, it was a matter of counting both sides to see if the two whole numbers come out to the the same number. For fractions, counting is out of the question. But we now see a great advantage in defining a fraction as a point on the number line, because it allows us to define:

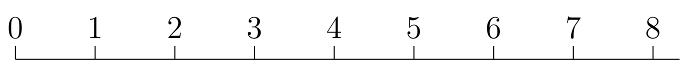
> *the* equality $\frac{a}{b} = \frac{k}{l}$ *means that the two points* $\frac{a}{b}$ *and* $\frac{k}{l}$ *are precisely the same point on the number line.*

In other words, the two points represented by $\frac{a}{b}$ and $\frac{k}{l}$ *coincide*. It follows immediately from this definition that

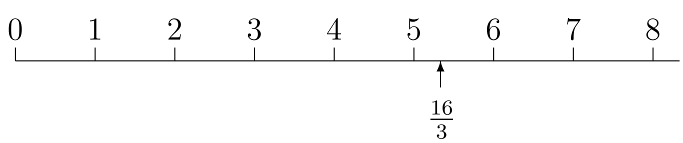
> *if* $k \neq 0$, *then* $\frac{k}{l} \neq 0$.

The meaning of the equality of two fractions will assume increasing importance in subsequent sections as we will often be called upon to verify that two fractions are equal. This definition tells us that the only way to do so is to show that the two fractions represent the same point on the number line. Incidentally, it is time to formally point out the general phenomenon that *the same point on the number line can be denoted by different symbols*. In §3 below, we will enter into a fuller discussion of this phenomenon.

We now give some examples on how to locate fractions, approximately, on the number line. For example, on the following line, where should the fraction $\frac{16}{3}$ be placed?



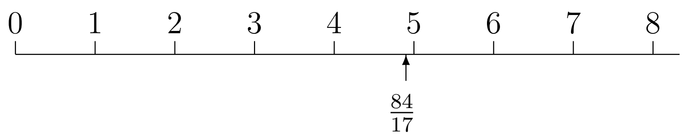
For this simple case, we can do it by a simple mental calculation. We know $15 = 5 \times 3$ and $18 = 6 \times 3$. Therefore $\frac{15}{3} = 5$ and $\frac{18}{3} = 6$, by virtue of (1), so that inasmuch as $15 < 16 < 18$, $\frac{16}{3}$ must be somewhere between 5 and 6, and closer to 5 than to 6, as shown:



In general, when simple mental calculation does not come as easily, it would be necessary to use the division-with-remainder in §3.4. To illustrate, consider the problem of where to put $\frac{84}{17}$ on the same number line. By the division-with-remainder, $84 = 4 \times \boxed{17} + \boxed{16}$. So

$$\frac{84}{17} = \frac{(4 \times 17) + 16}{17}.$$

and it should be the point on the number line which is the $(4 \times 17 + 16)$-th multiple of $\frac{1}{17}$. Of course, the (4×17)-th multiple of $\frac{1}{17}$ is exactly 4, by the observation in (1). So after that we need to go further to the right of 4 another 16 steps, each step having length $\frac{1}{17}$. If we go 17 more steps, we would get to 5. Therefore $\frac{84}{17}$ should be quite near 5, as shown:



In general, if $\frac{m}{n}$ is a fraction and

$$\frac{m}{n} = \frac{qn+k}{n}\,, \qquad \text{where } 0 \le k < n,$$

then the position of $\frac{m}{n}$ on the number line would be between q $(= \frac{qn}{n})$ and $q+1$ $(= \frac{qn+n}{n})$. In this case, while it is not necessary to do so, the common practice is to write

$$\frac{qn+k}{n} \qquad \text{as} \qquad q\,\frac{k}{n} \qquad (0 \le k < n)$$

as a clearer indication of the position of $\frac{qn+k}{n}$ on the number line, namely, that it comes after q but before $q+1$. The new notation $q\frac{k}{n}$ is called a *mixed fraction*, or sometimes a *mixed number*. We emphasize that this is no more than a notational convention.

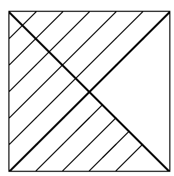
> *In other words,* $q\frac{k}{n}$ *is an alternative notation for* $\frac{qn+k}{n}$*, no more and no less.*

For example, the fractions $\frac{16}{3}$ and $\frac{84}{17}$ above could be written as $5\frac{1}{3}$ and $4\frac{16}{17}$, respectively. We will come to a better understanding of this notation after we introduce the concept of the addition of fractions. See (11) in §6 and the ensuing discussion.

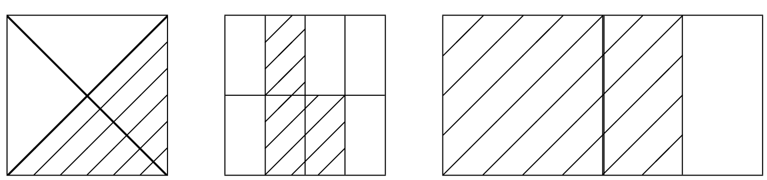
A final remark on the definition of a fraction is that, from a mathematical standpoint as well as in our present approach to fractions, there is no difference between "big" fractions and "small" fractions (in the sense of the numerators or denominators being big numbers), because on the number line, all numbers look alike. The preceding examples as well as some of the following exercises should make this fact very clear. You will never need to favor fractions with special denominators and numerators, which was exactly what a document from Minnesota once did. It advocated that in teaching fractions, one should avoid the use of fractions whose denominators are not 2, 3, 4, 5, 8, 10, 16. If you want to say you understand fractions, let us make sure you don't mean you "only understand fractions with certain denominators and numerators", because if you do so, it would be the same as saying you don't understand fractions at all.

Exercise 1.1 With the area of a unit square as the unit 1, draw two distinct pictorial representations of each of: (a) $\frac{5}{6}$. (b) $\frac{7}{4}$. (c) $\frac{9}{4}$.

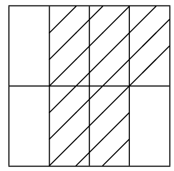
Exercise 1.2 Suppose the unit 1 on the number line is the area of the following shaded region in a division of the given square into four equal parts:



Write down the fraction representing the shaded area of each of the following, and *give a brief explanation of your answer*:



Exercise 1.3 Repeat the preceding exercise if the unit 1 is now the area of the following shaded region:



Exercise 1.4 Indicate the approximate position of each of the following on the number line, and also write it as a mixed number. (a) $\frac{67}{4}$. (b) $\frac{205}{11}$. (c) $\frac{459}{23}$. (d) $\frac{1502}{24}$.

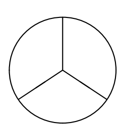
Exercise 1.5 (a) After driving 352 miles, we have done only two-thirds of the driving for the day. How many miles did we plan to drive for the day? (b) After reading 168 pages, I am exactly four-fifths of the way through. How many pages are in the book? (c) Helena was three quarters of the way to school after having walked $2\frac{2}{5}$ miles from home. How far is her home from school?

Exercise 1.6 (a) I have a friend who earns two dollars for every three times she walks her parents' dog. She knows that this week she she will walk the dog twelve times. How much will she earn? (b) Suppose your friend tells you that he taught his fifth grade class to do the problem in part (a) by using fractions and setting up proportions,

$$\frac{2}{3} = \frac{?}{12}$$

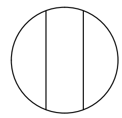
and he wonders why his class didn't "get it". How would you straighten him out to help him?

Exercise 1.7 A text on professional development claims that students' conception of "equal parts" is fragile and is prone to errors. As an example, it says that when a circle is presented this way to students



they have no trouble shading $\frac{2}{3}$, but when these same students

> are asked to construct their own picture of $\frac{2}{3}$, we often see them create pictures with unequal pieces:



(a) What kind of faulty mathematical instruction might have promoted this kind of misunderstanding on the part of students? (Hint: Note the phrase "unequal pieces" above, and see footnotes 4 and 5.) (b) What would you do to correct this kind of mistakes by students?

2 Equal Division of a Segment by Ruler and Compass

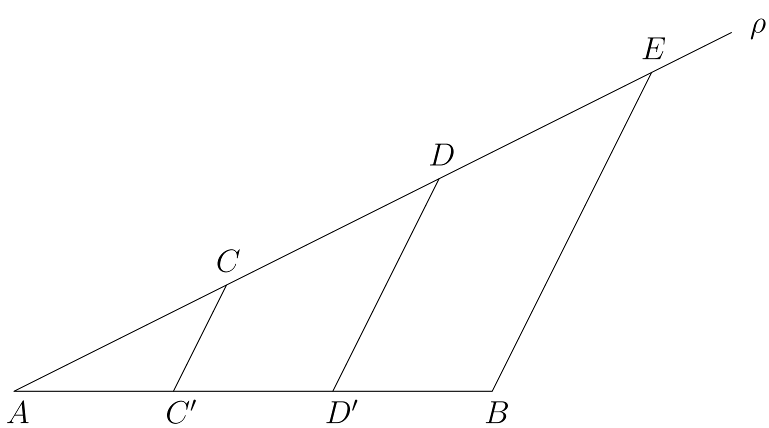
[**This section may be skipped on first reading.**]

From the preceding section, you see that the fundamental idea underlying the concept of a fraction is the division of a segment of length 1 into segments of equal length. Without fear of confusion, we will henceforth — in the context of line segments — refer to "segments of equal length" as "equal parts". While people are comfortable with dividing anything into two equal halves, and therefore four equal parts, eight equal parts, etc.,[8] there is a psychological barrier to dividing into, say, 7 equal parts or 11 equal parts. In fact, this may be one of the reasons why the state of Minnesota at one point advised teachers to use only fractions with denominators 2, 3, 4, 5, 8, 9, 10, and 16, as mentioned at the end of the last section. It could be that the people who wrote that document thought students would feel uneasy with equal divisions into 7 or 19 parts. With this in mind, we introduce a standard geometric hands-on method of dividing a given segment into *any* number of equal parts using only compass and straightedge (a ruler without markings). This is a fun activity that most people enjoy and it helps to put

[8] There is a good reason why people are comfortable with such divisions: there is a practical method to divide say a segment into two equal halves, and threrefore into four, eight, sixteen, etc. equal parts. Indeed if the end points of a segment are A and B, one simply folds the paper so that the point A falls on the point B. Where the crease of the fold intersects the segment then gives the correct point of equal division of the segment AB.

you psychologically at ease with the idea of arbitrary equi-divisions of a given segment. The validity of the method can be easily made believable through empirical verification, but we note that one could prove the validity by using standard arguments in Euclidean geometry.

Suppose we have to divide a given segment AB into 3 equal parts. We draw an arbitrary ray ρ issuing from A and, using a compass, mark off three points C, D, and E in succession on ρ so that $AC = CD = DE$ — the precise length of AC being irrelevant. Since you are most likely using a ruler to do this construction, the markings on the ruler would even make the use of a compass superfluous in getting the points C, D and E. For example, you can make each of AC, CD and DE an inch long, or 2 cm long, or whatever. Join BE, and through C and D draw lines parallel to BE which intersect AB at C' and D'. The points C' and D' are then the desired divisions points of AB, *i.e.,* $AC' = C'D' = D'B$.



It should be added that while there is a standard Euclidean construction with ruler and compass to draw a line through a given point parallel to a given line, the practical (and quick) way to draw such lines in the event that you have a plastic triangle is the following. Position such a triangle so that one side is exactly over the points B and E. *Keeping the triangle fixed in this position*, place a ruler snugly along another side of the triangle. (There are two other sides of the triangle in this situation, but practice and common sense will tell you which of these two sides to use as well as how to strategically position the ruler and the triangle.) Next, *hold the ruler* but glide the triangle along the ruler so that the side of the triangle that was originally on top of B and E now passes over the points D as the triangle slides along the ruler in the direction of the point A. With this side of the triangle over the point D, press down tightly on the triangle and draw a line along this side through D; this line is then parallel to BE. Do the same same for the point C.

There is nothing special about the number 3 in the preceding construction. We could have asked for a division of AB into 7 equal parts, in which case we would mark off 7 points $A_1, A_2, \ldots, A_7$ in succession on the ray ρ so that $AA_1 = A_1A_2 = \cdots = A_6A_7$. Join A_7B, and then draw lines through $A_1, \ldots, A_6$ parallel to A_7B as before. The intersections of these parallel lines with AB then furnish the desired division points of AB.

In teaching fractions in the school classroom, allow for ample class time to do this construction, say for divisions into 3, 5, 6 or 7 equal parts. Again, remember that the purpose of this activity is to make students feel comfortable with the idea of arbitrary division of a line segment into equal parts.

3 Equivalent Fractions (Cancellation Law)

The main purpose of this section is to show that if k, l, m are whole numbers with $m \neq 0$,

$$\frac{km}{lm} = \frac{k}{l} \tag{2}$$

First recall from the end of §1 that this equality between two fractions means: we have to show the two markers $\frac{km}{lm}$ and $\frac{k}{l}$ on the number line coincide. By tradition, the two fractions $\frac{km}{lm}$ and $\frac{k}{l}$ in (2) are said to be *equivalent fractions*, although to us, they are *equal*, period. But there is no point fighting against tradition unless it is absolutely necessary. In this case, "equivalent" is harmless enough, so we just take note of the terminology and go on. The equality (2) is also called the *cancellation law* of fractions (you "cancel" the m from both the top and bottom of $\frac{km}{lm}$). In particular, by changing the number m, we see that there are an infinite number of names for the same point $\frac{k}{l}$ no matter what $\frac{k}{l}$ may be.

If $l = 1$ in (2), then the resulting equality $\frac{km}{m} = k$ is just (1) in §1. Now if we write (2) as

$$\frac{km}{m}\left(\frac{1}{l}\right) = k\left(\frac{1}{l}\right),$$

we may think of (2) intuitively as equation (1) when the unit is changed from 1 to $\frac{1}{l}$. This is a good way to think of the cancellation law (2), although we shall prove it by a more formal argument.

The cancellation law is of great importance in any discussion of fractions. Before giving the explanation of why (2) is true, let us first describe one of the ways it is used in practice. Given $\frac{10}{15}$, say, suppose we notice that $5|10$ and $5|15$ (recall from §3.4 of Chapter 1, for whole numbers a and b, $b|a$ means b divides a, or a is a multiple of b). In fact, $15 = 3 \times 5$ and $10 = 2 \times 5$. So we apply (2) and get

$$\frac{10}{15} = \frac{2 \times 5}{3 \times 5} = \frac{2}{3}.$$

The same idea gives also

$$\frac{38}{57} = \frac{2 \times 19}{3 \times 19} = \frac{2}{3}.$$

These and similar facts show why (2) is important on a practical level. After all, wouldn't you rather deal with $\frac{2}{3}$ than $\frac{10}{15}$ or $\frac{38}{57}$?

The cancellation law is sometimes given in a different form which may be more convenient for the purpose of performing a cancellation: if a whole number p divides both k and l, then

$$\frac{k}{l} = \frac{k \div p}{l \div p}\ .$$

In other words, if (2) is true, then so is this claim. We can verify this claim for $k = 38$ and $l = 57$ easily enough (assuming (2)). Indeed, we see that $19|38$ and $19|57$ so that

$$\frac{38}{57} = \frac{38 \div 19}{57 \div 19} = \frac{2}{3}.$$

The general proof of the claim on the basis of (2) is no different. If p divides k, let $k = mp$ and if p divides l, let $l = np$ for some whole numbers m and n. So assuming (2) is true, then

$$\frac{k}{l} = \frac{mp}{np} = \frac{m}{n}.$$

But by definition, $m = k \div p$ and $n = l \div p$. So

$$\frac{k}{l} = \frac{m}{n} = \frac{k \div p}{l \div p},$$

as desired.

The passage from $\frac{k}{l}$ to $\frac{k \div p}{l \div p}$ is called *reducing the fraction* $\frac{k}{l}$ *to* $\frac{k \div p}{l \div p}$. Here is another example of reducing a fraction:

$$\frac{105}{49} = \frac{105 \div 7}{49 \div 7} = \frac{15}{7}.$$

You will note that when the cancellation law is used for the purpose of reducing a fraction, its effectiveness is entirely dependent on one's ability to find a whole number (different from 1) that divides both numerator and denominator. Sometimes this is not so easy. It is not obvious that $171/285$ is equal to $\frac{3}{5}$, for instance, because 57 does not present itself as a number that divides both 171 and 285. Of course one can produce other striking examples at will, such as the following: $1651/762 = \frac{13}{6}$

Activity: Prove the last assertion.

A fraction $\frac{a}{b}$ is said to be *reduced*, or *in lowest terms*, if there is no whole number $c > 1$ so that c divides both a and b. It is a fact that every fraction is equal to one and only one fraction in reduced form. For example, $\frac{2}{3}$ is the reduced form of $\frac{18}{27}$ and $\frac{13}{6}$ is the reduced form of $1651/762$. This fact is plausible, but its proof (to be given in §5 of Chapter 3) is not so trivial as it requires something like the Fundamental Theorem of Arithmetic or the Euclidean algorithm. However, it is good to keep this fact in mind because it justifies our occasional reference to *the* reduced form of a given fraction.

Pedagogical Comments: It seems to be a tradition in school mathematics to regard non-reduced fractions as something "illegitimate", and students usually get points deducted if they give an answer in terms of non-reduced fractions. This is perhaps the right place to give this issue some perspective. You know that a fraction is a fraction is a fraction. See the definition in §1. Nowhere does it say that some fractions are "better' than others. So from a *mathematical* standpoint, all fraction are on the same footing. For example, there is nothing wrong with $\frac{5}{10}$. You may *like* to see

$\frac{1}{2}$ instead, but we must remember that whether something is mathematically correct or not has nothing to do our likings. Moreover, it is sometimes difficult to justify what we like even under the most generous conditions. If we insist that every fraction should be in lowest terms, should we accept $\frac{38}{57}$ from a fifth grader? After all, few adults would recognize that it is not in lowest terms. And what about an extreme case like $1333/2279$? (It is actually equal to $\frac{31}{53}$.) So a pedantic insistence on having everything in lowest terms can be difficult to defend.

On the other side of the ledger, it does get annoying if students get into the habit of never simplifying fractions such as $\frac{4}{2}$ or $\frac{9}{3}$. Some common sense is thus called for in this situation. One suggestion is to lead by example: at the board, the teacher should always simplify the obviously simplifiable fractions and also tell students that reducing fractions to lowest terms is a skill they need to acquire for certain needs. On exams, explicitly call for certain answers to be reduced, but otherwise make allowance for nonreduced fractions. **End of Pedagogical Comments.**

We now turn to the proof of (2). First, let us see how it is usually handled in school textbooks. A typical explanation is the following:

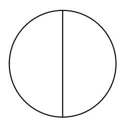
> Different fractions which name the same amount are called EQUIVALENT FRACTIONS.
>
> We will practice a method for making equivalent fractions. If we multiply a number by a fraction that is equivalent to 1, the answer will be a different name for the same number. Thus
>
> $$\frac{1}{2} \times 1 = \frac{1}{2} \times \frac{2}{2} = \frac{2}{4}, \qquad \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{3}{3} = \frac{3}{6}$$
>
> The fractions $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{3}{6}$ are equivalent fractions.

Let us assume that students understand what is meant by "name the same amount". The most worrisome feature of an explanation of this kind is that the concept of equivalent fractions is made to depend on the concept of fraction multiplication. As we shall see in §7 below, the latter needs the kind of careful definition that is usually missing in school texts. Moreover, such an explanation of equivalent fractions obscures the simple intuitive idea behind (2), which we now proceed to describe.

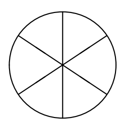
We will first prove (2) for the special case of

$$\frac{3}{6}\ \left(=\frac{1\times 3}{2\times 3}\right)=\frac{1}{2}\ .$$

This example is actually *too* simple to help with the proof in the general case. However, it does give a good introduction to the basic ideas involved. The reasoning is a good illustration of the importance of having precise definitions. According to the definition of a fraction, $\frac{3}{6}$ is 3 copies of $\frac{1}{6}$'s,[9] and we want to know why this is equal to $\frac{1}{2}$. Let us first look at this from the intuitive point of view of cutting pies. Of course the pies are represented two-dimensionally by circles and we try to cut the circles into congruent circular sectors. So $\frac{1}{2}$ is represented by half of a circle that represents 1:



Now we cut each half into three congruent pieces, thereby obtaining a division of the pie into 6 congruent smaller pieces. Each of these smaller pieces is therfore $\frac{1}{6}$.



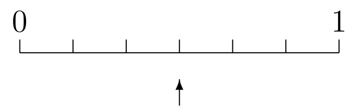
Look at say the left half of the circle, which is $\frac{1}{2}$. It is now divided into 3 congruent pieces, each being $\frac{1}{6}$. This is then the statement that $\frac{1}{2}$ is 3 copies of $\frac{1}{6}$, i.e., $\frac{1}{2}$ is $\frac{3}{6}$.

The intuitive idea of the prceding argument can be easily translated into a formal argument using the number line. In the interest of brevity of expression,

> we shall henceforth write **equal parts** for **segments of equal length** in the context of the number line.

[9] Recall from §1: this means 3/6 is the length of the concatenation of 3 segments each of length 1/6.

Now divide $[0,1]$ into two equal parts, the point of division (i.e., $\frac{1}{2}$) being indicated by the vertical arrow below the line segment:

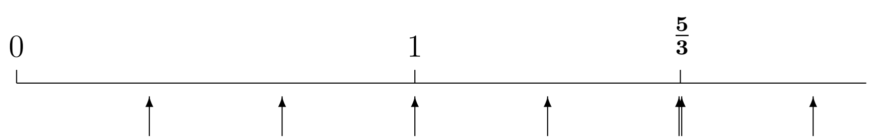


Now divide each of the segments of length $\frac{1}{2}$ into three equal parts, then $[0,1]$ is now divided into 6 equal parts, and each of these parts therefore has length $\frac{1}{6}$. The picture makes it clear that $\frac{1}{2}$ is 3 copies of $\frac{1}{6}$. Knowing that 3 copies of $\frac{1}{6}$ is equal to $\frac{3}{6}$, we see that $\frac{1}{2}$ is $\frac{3}{6}$.

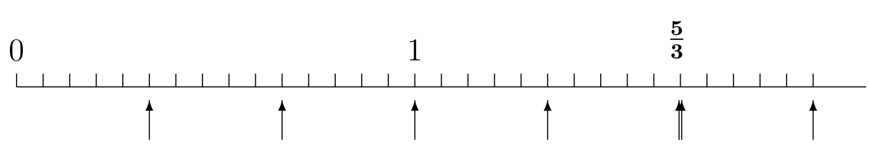
Let us look at another example:

$$\frac{25}{15} = \frac{5}{3}\,.$$

The argument in this example fully illustrates the complexities of the general situation, but observe how this argument elaborates on — rather than breaks away from — the simple ideas of the preceding number line argument for $\frac{1}{2} = \frac{3}{6}$. We begin with the right side: $\frac{5}{3}$ is 5 copies of $\frac{1}{3}$, and we want to know why it is also 25 copies of $\frac{1}{15}$. On the number line, consider all multiples of $\frac{1}{3}$. The segments between consecutive multiples are all of length $\frac{1}{3}$. These are the segments between the vertical arrows below the number line:



Now divide each of these segments of length $\frac{1}{3}$ into 5 equal parts; call these parts *small segments*. Here is a picture of these *small segments*:



What is the length of each *small segments*? The unit interval $[0, 1]$ is the concatenation of 3 copies of $\frac{1}{3}$'s, and each of the latter is the concatenation of 5 *small segments*. Thus $[0, 1]$ is the concatenation of $3 \times 5 = 15$ *small segments*. Thus each small segment has length $\frac{1}{15}$, and since each segment of length $\frac{1}{3}$ is the concatenation of 5 *small segments*, we see that $\frac{1}{3}$ is 5 copies of $\frac{1}{15}$. We therefore have:

> Since $\frac{5}{3}$ is 5 copies of $\frac{1}{3}$ and $\frac{1}{3}$ is 5 copies of $\frac{1}{15}$, it follows that $\frac{5}{3}$ is $5 \times 5 = 25$ copies of $\frac{1}{15}$.

In other words, $\frac{5}{3}$ is $\frac{25}{15}$.

The general argument for (2) should be clear now. We want to prove

$$\frac{km}{lm} = \frac{k}{l}.$$

We know that $\frac{k}{l}$ is k copies of $\frac{1}{l}$, and we want to prove that it is also km copies of $\frac{1}{lm}$. On the number line, consider all multiples of $\frac{1}{l}$. The segments between consecutive multiples are all of length $\frac{1}{l}$. Now divide each of these segments of length $\frac{1}{l}$ into m equal parts; call these parts *small segments*. We first compute the length of each *small segments*. The unit interval $[0, 1]$ is the concatenation of l copies of $\frac{1}{l}$'s, and each of the latter is the concatenation of m *small segments*. Thus $[0, 1]$ is the concatenation of lm *small segments*. Thus each small segment has length $\frac{1}{lm}$, and since each segment of length $\frac{1}{l}$ is the concatenation of m *small segments*, we see that $\frac{1}{l}$ is m copies of $\frac{1}{lm}$. So:

> Since $\frac{k}{l}$ is k copies of $\frac{1}{l}$ and $\frac{1}{l}$ is m copies of $\frac{1}{lm}$, it follows that $\frac{k}{l}$ is km copies of $\frac{1}{lm}$.

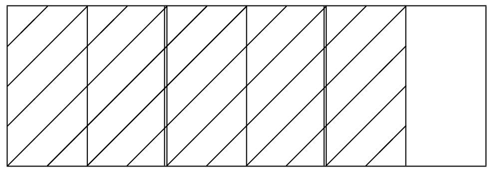
But $\frac{km}{lm}$ is also km copies of $\frac{1}{lm}$, we have proved that $\frac{k}{l}$ is equal to $\frac{km}{lm}$.

There is a different way to present the proof of equality (2) which may be more intuitive to some people. The idea is to use the area of a unit square as the unit 1. We can illustrate this idea with a concrete example. Let us see why

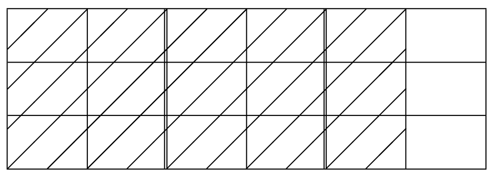
$$\frac{15}{6} = \frac{5}{2}$$

If 1 is the area of a unit square, and $\frac{5}{2} = 2\frac{1}{2}$, $\frac{5}{2}$ is the area of two and a half unit squares. The following is the concatenation of three unit squares, and

$\frac{5}{2}$ is represented by the area of 5 of the vertical half squares, as indicated by the following shaded area.



Prompted by the fact that $15 = 5 \times 3$ and $6 = 2 \times 3$, we divide each unit square *horizontally* into equal thirds to get:



Now we see that this new shaded area consists of $3 \times 5 = 15$ small rectangles, all congruent to each other. But 6 of these rectangles pave a unit square, so the usual argument using the basic properties of area (see (a)–(c) in §1 and subsequent discussion) shows that the area of a small rectangle is $\frac{1}{6}$, so that the area of the shaded region is $\frac{15}{6}$. Therfore $\frac{15}{6} = \frac{5}{2}$.

Exercise 3.1 Reduce the following fractions to lowest terms. (You may use a four-function calculator to test the divisibility of the given numbers by various whole numbers.)

$$\frac{27}{126}, \quad \frac{72}{48}, \quad \frac{42}{91}, \quad \frac{52}{195}, \quad \frac{204}{85}, \quad \frac{414}{529}, \quad \frac{1197}{1273}.$$

Exercise 3.2 Explain each of the following directly, without using (2):

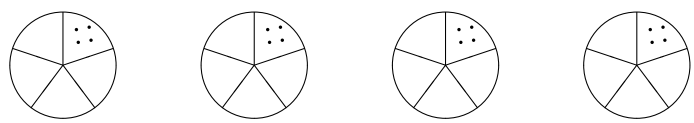
$$\frac{6}{14} = \frac{3}{7}, \quad \frac{28}{24} = \frac{7}{6}, \quad \frac{30}{12} = \frac{5}{2}, \quad \text{and} \quad \frac{12}{27} = \frac{4}{9}.$$

Exercise 3.3 Use the representation of 1 as the area of a unit square to give a proof of why $\frac{6}{14} = \frac{3}{7}$ and $\frac{30}{12} = \frac{5}{2}$.

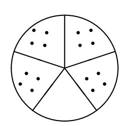
Exercise 3.4 Discuss in depth the similarities between (1) of §1 and (2) of this section, both in terms of the formal similarity and the similarity in the reasoning that underlies both.

4 Fraction as Division

Suppose we have four pizzas to be shared equally among five people, how shall we do it? First of all, we will represent pizzas by circles of equal radius and will also interpret "equal sharing" to mean dividing the circles into regions of equal areas.[10] Thus we have four circles of a fixed radius and the problem is how to divide them into five regions of the same area. One idea that comes to mind immediately is to divide each circle into five congruent circular sectors (so that they have the same area) and then let each person take one sector from each circle. For example, one person could take the dotted sectors:



How much pizza did each person get? If we put the four dotted sectors together into the same circle, then they take up four of the five circular sectors and therefore comprise four-fifths of the pizza. Thus in this sharing scheme, each person gets $\frac{4}{5}$ of a pizza:



This is a very surprising result! Maybe not to you if you think of this as something you do almost everyday, but look at it this way. The sharing was done by looking at *all four pizza together* in order to device a method of equal sharing, yet the net result that each person gets $\frac{4}{5}$ of a pizza could have been obtained *by looking at a single pizza alone*. By the latter, we mean: let the unit on the number line be the area of the pizza (circle), then the area of four parts in a division of the circle into five parts of equal area would be exactly $\frac{4}{5}$. Thus we have, *area-wise*:

$$\frac{4}{5} \text{ of a pizza} \quad = \quad \text{one part in a division of 4 pizzas into 5 equal parts.}$$

[10] Compare the discussion in footnote 5 of §1.

As you know, this is a special case of a general phenomenon which shows that a fraction as defined in §1 can be interpreted as a kind of "equal division". Precisely, we shall give several *proofs* of the following:

$$\frac{k}{l} \;=\; \text{the length of one part when a segment of of length } k \text{ is divided into } l \text{ equal parts,} \tag{3}$$

(Recall from §3: *equal parts* in the context of the number line means *segments of equal length.*) This gives a completely different way of computing the number $\frac{k}{l}$, i.e., a completely different way of locating the point $\frac{k}{l}$ on the number line. Let us make sure that you understand what this means. In the original definition, you can get hold of $\frac{k}{l}$ by looking exclusively at $[0,1]$: divide this segment into l equal parts, then the length of the concatenation of k of these parts is $\frac{k}{l}$. On the other hand, the prescription in (3) for computing $\frac{k}{l}$ involves looking at the whole segment $[0,k]$ from 0 to k from the very beginning and dividing this (presumably long) segment into l equal parts. There is no a priori reason why these two operations are related, much less the fact that they produce the same point on the number line.

Incidentally, there is no difference between (3) and the definition of a fraction $\frac{k}{l}$ in case $k = 1$.

To more firmly anchor these ideas, we apply them to something other than pizzas. Let us choose our unit 1 to be a collection of 15 pencils. At the risk of stating the obvious, the number 2 now represents two collections, and therefore 30 pencils; the number 3 represents three collections, and therfore 45 pencils; and so on. What is $\frac{4}{5}$? By the definition in §1, we first divide the unit (corresponding to $[0,1]$ on the number line) into 5 equal parts (i.e., equal number of pencils), so each part consists of 3 pencils. Now we aggregate 4 of these parts (corresponding to concatenating 4 segments each of length $\frac{1}{5}$), thereby obtaining 12 pencils. So far we have only made use of the definition of the fraction $\frac{4}{5}$. Now we look at what (3) says about another way to compute $\frac{4}{5}$. We put together 4 collections of these pencils (corresponding to $[0,4]$) and divide the totality (of $4 \times 15 = 60$ pencils) into 5 equal parts. Then the size of a part in terms of the number of pencils, which is $60 \div 5 = 12$ pencils, is what is represented by the fraction $\frac{4}{5}$. You see that the two numbers at the end of these two processes are the same, but the intermediate steps of the processes look completely different.

There is of course nothing special about the unit of "15 pencils" in the preceding reasoning. It could have been any other kind of unit. In general then, (3) may be *paraphrased* as follows. Starting with a fixed unit, define $\frac{k}{l}$ to be the magnitude of k of the parts when this unit is divided into l equal parts. Then:

$$\frac{k}{l} \;=\; \text{the magnitude of a part when the whole consisting of } k \text{ units is divided into } l \text{ equal parts, } l > 0$$

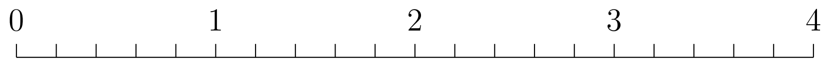
The drawback of this paraphrase is of course the fact that the meanings of "magnitude" and "part" are left vague.

The interpretation (3) of a fraction as an equal division is a key result in the subject of fractions. Unfortunately, the reason why this interpretation is valid is, at best, slurred over in textbooks. The difference in the treatment of (3) between what is done in this monograph and what is done elsewhere deserves a closer look. The most obvious one is that every word in (3) has been carefully defined and (3) makes sense. On the other hand, when it is generally claimed that (for example) "the fraction $\frac{2}{3}$ is also a division $2 \div 3$", this sentence *has no meaning* because the meaning of $2 \div 3$ is generally not given. A division of a number by another is supposed to yield a number, but, apart from the ambiguity of the meaning of a "number" in school mathematics, there is no explanation of what number would result from 2 divided by 3, much less why this number should be equal to a "part of a whole" which is $\frac{2}{3}$.

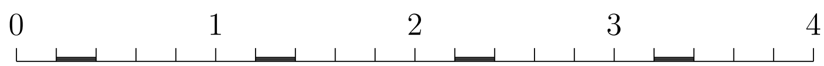
We will eventually give more than one explanation of (3) in recognition of its importance. Let us first look at a proof of the special case of $\frac{4}{5}$. Thus we have to prove:

$$\frac{4}{5} \;=\; \text{the length of a part when the segment } [0,4] \text{ is divided into 5 equal parts}$$

The reasoning will be particularly instructive because it follows closely the reasoning in the pizza example. Imagine each pizza to be a segment of length 1, so that $[0,1]$, $[1,2]$, $[2,3]$, $[3,4]$ represent four pizzas. The segment $[0,4]$ then represent the whole collection of four pizzas. Divide each of $[0,1]$, $[1,2]$, $[2,3]$, and $[3,4]$ into 5 equal parts; we call each of these parts a "short" segment.



There are $4 \times 5 = 20$ short segments in $[0, 4]$. Now pick the first[11] short segment from each of $[0, 1]$, $[1, 2]$, $[2, 3]$, and $[3, 4]$ — this corresponds to picking the dotted sector from each pizza — and let A_1 be the segment obtained by concatenating these 4 short segments. We claim that the length of A_1 is $\frac{4}{5}$. This is intutuively obvious, but it would be nicer if we can prove it decisively because we want to be sure that it is ok to make a similar claim even when $\frac{4}{5}$ is replaced by an arbitrary fraction. The way to do this is to similarly pick the second short segment from each of $[0, 1]$, $[1, 2]$, $[2, 3]$, and $[3, 4]$ and let A_2 be the segment obtained by concatenating these 4 short segments. Segments A_3, A_4, and A_5 are likewise defined. For example, in the figure below, A_2 is the concatenation of the thickened segments:



Clearly all of A_1, A_2, A_3, A_4, A_5 have the same length, and together they form a division of $[0, 4]$ into 5 equal parts. Therefore the length of A_1 (say) is the length of a part when $[0, 4]$ is divided into 5 equal parts. On the other hand, the length of A_1 must be $\frac{4}{5}$ because the length of each short segment is clearly $\frac{1}{5}$ and there are 4 such short segments in A_1, i.e., A_1 is four copies of $\frac{1}{5}$ (in the language introduced in §1). Therefore $\frac{4}{5}$ is the length of a part when $[0, 4]$ is divided into 5 equal parts.

The proof of (3) in general is not at different from the preceding special case. So we want to show that $\frac{k}{l}$ is the length of a part when $[0, k]$ is divided into l equal parts. We denote by $\mathbb{S}$ the collection of k segments

$$\{[0, 1], [1, 2], [2, 3], \ldots [k-1, k]\}.$$

Divide each of these k segments in $\mathbb{S}$ into ℓ equal parts, and call each of these parts a "short" segment. (Note that each short segment then has length $\frac{1}{\ell}$.) Let us order these short segments by counting from left to right. For each

[11] We may always count from the left so that the first short segment is the same as the leftmost short segment.

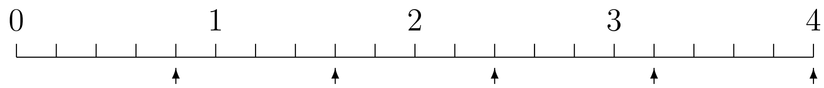
$i = 1, 2, \ldots \ell$, denote by A_i the segment obtained by concatenating the i-th short segment in each of the k segments in $\mathbb{S}$. These A_1, A_2, $\ldots A_\ell$ then partition $[0, k]$ into ℓ equal parts. The length of A_1, in particular, is then the length of a part when $[0, k]$ is divided into ℓ equal parts. But A_1 is itself partitioned into k short segments, and each short segment (as noted) has length $\frac{1}{\ell}$. By the definition of $\frac{k}{\ell}$ in §1, the length of A_1 is therefore also equal to $\frac{k}{\ell}$. Thus $\frac{k}{\ell}$ is equal to the length of a part when $[0, k]$ is divided into ℓ equal parts. This proves (3).

The proof we have just given of (3) (which corresponds to the way we divided four pizzas into five equal portions) is not the only way to think about equal division. For example, suppose five people run a *relay race* of four miles and each is supposed to *run the same distance*. You understand how relay races are run: one doesn't run a short distance, hands the baton to the next person, and then comes back in to run some more. One runs a fixed distance and quits. Therefore, we have to divide up the four-mile course into five *unbroken* segments of equal length (distance). We do so by first dividing four miles into $4 \times 5 = 20$ segments of the same distance. One way to do this is of course to divide each mile into 4 parts of equal distance. But now we give the first 4 of the 20 equal parts to the first runner, the second 4 of the 20 parts to the second runner, the third 4 of the 20 parts to the third runner, and so on. Then we also conclude that the distance run by each runner is $\frac{4}{5}$ of a mile, because each of these 20 parts is one part of the division of a mile into 5 equal parts, so that each part if $\frac{1}{5}$ of a mile long. Four of these parts them make up $\frac{4}{5}$ of a mile.

We can formalize this arguemnt in terms of the number line. Again we shall prove that

$$\frac{4}{5} \;=\; \begin{array}{l}\text{the length of a part when the segment } [0,4] \\ \text{is divided into 5 equal parts}\end{array}$$

Referring to the segment $[0, 4]$, we divide $[0, 4]$ into 20 equal parts, 20 being the product of (numerator of $\frac{4}{5}$) $\times$ (denominator of $\frac{4}{5}$). Thus there are 20 equally spaced markers to the right of 0, and the 20th marker is at 4. The 4th, 8th, 12th, 16th and 20th markers then furnish a division of $[0, 4]$ into 5 equal parts. Our job is to show that the 4th marker is in fact at $\frac{4}{5}$.



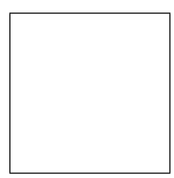
We begin by finding out where the first marker is located. Now the 5th, 10th, 15th and 20th markers divide $[0, 4]$ into four equal parts. But 1, 2, 3 and 4 already divide $[0, 4]$ into four equal parts, so we conclude that, in fact, the 5th, 10th, 15th and 20th markers are exactly at 1, 2, 3 and 4. In particular, the 5th marker is at 1, so that the first five markers give a division of the unit segement $[0, 1]$ into five equal parts. Thus the first marker is at $\frac{1}{5}$, and therefore the 4th marker is at $\frac{4}{5}$ by the definition of a fraction.

This idea can be used to give another proof of (3). We leave further explorations to the exercises.

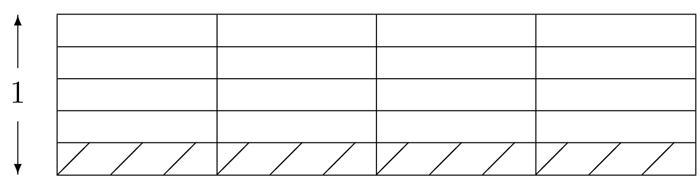
The first proof of (3) can be rephrased using the area of a unit square as the unit 1. Again, we illustrate with the concrete example of

$$\frac{4}{5} = \text{the length of a part when the segment } [0,4] \text{ is divided into 5 equal parts}$$

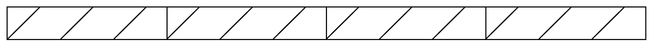
Let the number 1 be the area of the unit square:



We have to show that $\frac{4}{5}$ is the area of a part when the whole consisting of 4 unit squares is divided into 5 equal parts. To this end, we divide 4 unit squares into 5 equal parts in the following manner:



The shaded area is one of the parts, and we have to show that its area is $\frac{4}{5}$. Now it consists of 4 horizontal strips each of which is contained in a unit square.



By the way we divided the 4 unit squares, each horizontal strip is $\frac{1}{5}$ of the unit square. Since there are 4 such horizontal strips in the shaded area, the latter is $\frac{4}{5}$, by the definition of $\frac{4}{5}$.

The proof of (3) in general can also be given using this choice of the unit 1 on the number line. You will get some practice with this approach in the exercizes.

The intepretation (3) of a fraction as equal division allows us to make an important observation. Let k and l be whole numbers. Suppose we are given k objects, and k is a multiple of l, with l always assumed to be nonzero in this discussion. Then according to the partitive interpretation of $k \div l$ (see §3.4 of Chapter 1):

> $k \div l$ is the number of objects in each group when the collection of k objects is partitioned into l equal groups.

Assuming k is a multiple of l, then we have according to (3):

$$\frac{k}{l} = k \div l.$$

Now we take note of the fact that if k and l are arbitrary whole numbers (in particular, *k may no longer be a multiple of l*), the notation "$k \div l$" has no meaning up to this point. This is because division has so far been considered only in the context of whole numbers, and (for example) $4 \div 5$ is meaningless according to the partitive interpretation of division. One cannot divide 4 objects into 5 equal groups of objects. But if we put ourselves in the context of fractions, so that instead of 4 dots or 4 books, we consider a segment of length 4, then a partition of this segment into 5 equal parts makes perfect sense. This prompts us to *extend the meaning of division among ALL whole numbers* in the following way:

Definition. For whole numbers k and l with $l > 0$, $k \div l$ is by definition the length of a part when a segment of length k is divided into l equal parts.

With this definition in place, the full significance of (3) can now be displayed in one symbolic statement: for any whole numbers k and l with $l > 0$,

$$\frac{k}{l} = k \div l.$$

For this reason, we write from now on $\frac{k}{l}$ in place of $k \div l$ for all whole number k and l, $l > 0$. **We shall henceforth retire the symbol $\div$ from all symbolic computations and instead use fractions to denote division.**

The preceding idea of extending the meaning of a concept to make it more inclusive (e.g., in this case, giving meaning to k *divided by* l regardless of whether k is a multiple of l or not) will be a recurrent one in this monograph.

Incidentally, this discussion puts in clear evidence the advantage of using the number line to define fractions. Imagine, for example, interpreting whole numbers only in terms of discrete objects. It would be awkward indeed to interpret $\frac{5}{7}$ in terms of (3) if we start with 5 chairs (say) and try to divide them into 7 parts of equal weight or equal volume or whatever, or for that matter, divide 5 dots into 7 equal parts. (This is the reason why in §4 of Chapter 1, we stated that we preferred the area model for multiplication rather than the dot model.) On the other hand, the concept of partitioning any segment $[0, k]$ into any number of equal parts is so natural that the extension of whole number arithmetic to fractions becomes seamless.

To summarize what we have done thus far: we gave a clear definition of a fraction as a point on the number line in terms of parts-whole (cf. §1), and demonstrated why on the basis of this definition alone, $\frac{k}{l}$ is equal to $k \div l$ when k is a multiple of l. When k is not a multiple of l, we also proved in (3) that $\frac{k}{l}$ has the *formal* property of a division in the partitive sense. Then modifying the partitive interpretation of division, we *defined* "k divided by l" in general. We concluded that, on account of (3), a fraction $\frac{k}{l}$ as defined in §1 furnishes the correct notion of "k divided by l" for arbitrary whole numbers k and l.

Exercise 4.1 (a) Suppose we try to put 2710 pieces of candy into 21 bags with an equal number in each bag. What is the maximum number of candy we can put in each bag, and how many are left over? (b) Suppose we try to divide a (straight) path of 2710 feet into 21 parts of equal length, how many feet are in each part?

Exercise 4.2 Assume that you can cut a pie into any number of equal portions (in the sense of cutting circle into congruent circular sectors). (a) How would you cut 7 pies in order to give equal portions to 11 kids? (b) Find two different ways to cut 11 pies to give equal portions to 7 kids.

Exercise 4.3 Without appealing to (3), show directly that $\frac{4}{7}$ is the length of a part when a segment of length 4 is divided into 7 equal parts. Do the same for $\frac{7}{3}$ and $\frac{5}{4}$. In each case, do it in at least two different ways.

Exercise 4.4 [The following problem is a fifth grade problem. You are asked to do it without the use of proportions. You are also asked to explain the solution clearly to the class.] Ballpoint pens are sold in bundles of 4. Lee bought 20 pens for 12 dollars. How much would 28 pens cost?

Exercise 4.5 Explain to a fifth grader in what sense $\frac{15}{4} = 15 \div 4$.

Exercise 4.6 (a) After driving 113 miles, we are only four-fifth of the way to our destination. How much further do we have to drive? (b) Helena walked to school from home but quit after having walked $2\frac{1}{2}$ of a mile. She was $\frac{5}{8}$ of the way to school. If x is the number of miles from her home to school, and x is known to be a whole number, what is x?

Exercise 4.7 James gave a riddle to his friends: "I was on a hiking trail, and after walking $\frac{7}{12}$ of a mile, I was $\frac{5}{9}$ of the way to the end. How long is the trail?" Help his friends solve the riddle. [You do not need to know anything more about fractions before you do this problem, but we shall revisit this problem after we know something about fraction multiplications.]

Exercise 4.8 Use the "relay race" method (the second method described in this section) of equal division to give another proof of (3).

5 Ordering Fractions (the Cross-Multiplication Algorithm)

Before we approach the addition of fractions, we first consider the more elementary concept of *order*, i.e., comparing two fractions to see if one is bigger than or equal to the other (recall that equality in this case means they are the same point on the number line). Given two fractions A and B, *we say* $A < B$ *if* A *is to the left of* B *as points on the number line.* This is the same as saying that the segment $[0, B]$ is longer than the segment $[0, A]$. According to the discussion of order among whole numbers in §2 in Chapter 1, this definition of order is *consistent with the same concept among whole numbers.*

We emphasize once again the need to put fractions and whole numbers on the same footing. It would have been preposterous to define order among fractions in a way that is different from the definition of order among whole numbers. Observe also the *ease* with which we define order among fractions when the number line is at our disposal.

The main objective of this section is to show that a comparison of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ can be made by inspecting their "cross products" ad and bc. This so-called cross-multiplication algorithm has gotten a bad name in recent years because it is supposed to be part of learning-by-rote, and the reason for *that* is because many textbooks just write it down and use it without any explanation. As a reaction, the curricula of recent years have a tendency of not even mentioning this algorithm. Using the algorithm without explanation and not mentioning the algorithm at all represent the two extremes of mathematics education. Neither is good education, because this algorithm is a useful tool which can be simply explained.

Consider the following example. Which is the bigger of the two: $\frac{4}{7}$ or $\frac{3}{5}$? In terms of segments, this should be rephrased as: which of $[0, \frac{4}{7}]$ and $[0, \frac{3}{5}]$ is longer? Now by definition:

$\frac{4}{7}$ is 4 copies of $\frac{1}{7}$
$\frac{3}{5}$ is 3 copies of $\frac{1}{5}$

This comparison is difficult because the two fractions are expressed in terms of different "units": $\frac{1}{7}$ and $\frac{1}{5}$. However, imagine for a moment that the following statements were actually true for some whole number c :

$\frac{4}{7}$ is 20 copies of $\dfrac{1}{c}$

$\frac{3}{5}$ is 21 copies of $\dfrac{1}{c}$

Then we would be able to immediately conclude that $\frac{3}{5}$ is the bigger of the two because it includes one more segment (of the same length) than $\frac{4}{7}$. This suggests that the way to achieve the desired comparison is to express both $\frac{1}{7}$ and $\frac{1}{5}$ in terms of a common "unit". Before embarking on the search of this hypothetical unit, we may be able to better explain the underlying idea by considering a more mundane problem: which is longer, 3500 yards or 3.2 km? In this case, everybody knows that we need to reduce both *yard* and *km* to a common unit, say, meter. One finds that 1 yard = 0.9144 meter and 1 km = 1000 m., so that

3500 yards = 3200.4 m
3.2 km = 3200 m

Conclusion: 3500 yards > 3.2 km.

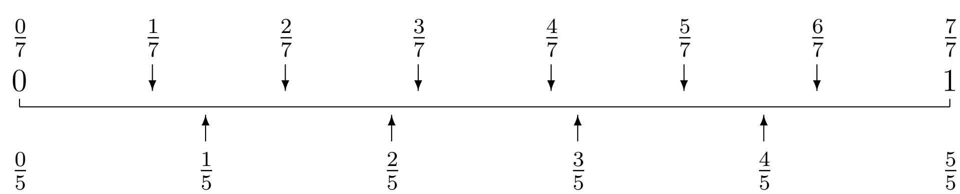
In order to imitate this procedure, we have to decide on a common unit for $\frac{1}{7}$ and $\frac{1}{5}$. The cancellation law (2) suggests the use of $\frac{1}{35}$ because

$$\frac{1}{7} = \frac{5 \times 1}{5 \times 7} = \frac{5}{35} = 5 \text{ copies of } \frac{1}{35}$$
$$\frac{1}{5} = \frac{7 \times 1}{7 \times 5} = \frac{7}{35} = 7 \text{ copies of } \frac{1}{35}$$

More generally then,

$$\frac{4}{7} = \frac{5 \times 4}{5 \times 7} = \frac{20}{35} = 20 \text{ copies of } \frac{1}{35}$$
$$\frac{3}{5} = \frac{7 \times 3}{7 \times 5} = \frac{21}{35} = 21 \text{ copies of } \frac{1}{35}$$

Conclusion: $\dfrac{4}{7} < \dfrac{3}{5}$. A closer look of the preceding also reveals that this conclusion is based on the inequality $5 \times 4 < 7 \times 3$ between the numerators and denominators of the two fractions. We are thus witnessing the cross-multiplication algorithm in a special case. In picture:



Before proceeding further, doing an activity on the above strategy may help you understand the situation better:

> **Activity:** Compare the following pairs of fractions using the preceding idea: $\frac{5}{6}$ and $\frac{4}{5}$; $\frac{5}{6}$ and $\frac{3}{4}$.

In general, suppose we are to compare $\frac{k}{l}$ and $\frac{m}{n}$. The reasoning of the preceding example tells us that we should use the cancellation law (2) to rewrite them as:

$$\begin{aligned} \frac{k}{l} &= \frac{kn}{ln} \\ \frac{m}{n} &= \frac{lm}{ln} \end{aligned}$$

We now show that these equalities lead immediately to a pair of statements:

$$\begin{aligned} \frac{k}{l} < \frac{m}{n} \qquad &\text{exactly when} \qquad kn < lm \\ \frac{k}{l} = \frac{m}{n} \qquad &\text{exactly when} \qquad kn = lm \end{aligned} \tag{4}$$

Either of the two is referred to as the *cross-multiplication algorithm*. The reason for this terminology is obvious by looking at the positions of k, n, l and m in $\frac{k}{l} = \frac{m}{n}$.

Before proving (4), we pause to give an explanation of the phrase *exactly when*: it means the same thing as *is the same as* which we explained in §2 of Chapter 1. Recall that this means: the mathematical statements on both sides of this phrase (e.g., "$\frac{k}{l} < \frac{m}{n}$" and "$kn < lm$" in the first assertion of (4)) imply each other. In greater detail, the first assertion of (4) is actually the composite of the following two statements:

(4a) if $\frac{k}{l} < \frac{m}{n}$, then $kn < lm$,

and also

(4b) if $kn < lm$, then $\frac{k}{l} < \frac{m}{n}$.

Let us prove (4a) and (4b). For (4a), suppose $\frac{k}{l} < \frac{m}{n}$, then we know this may be rewritten as $\frac{kn}{ln} < \frac{lm}{ln}$. By the definition of a fraction in §1, this inequality means that in terms of the markers on the number line which are $\frac{1}{ln}$ apart, the kn-th marker is to the left of the lm-th marker. This shows that $kn < lm$. Next we take up (4b). Suppose $kn < lm$. Then $\frac{kn}{ln} < \frac{lm}{ln}$. By (2), this says $\frac{k}{l} < \frac{m}{n}$. Thus (4b) is proved.

The same reasoning also proves the second assertion in (4) (simply replace the inequality symbol "$<$" everywhere by the equality symbol "$=$" in the preceding argument).

The cross-multiplication algorithm is a powerful tool for deciding whether or not two fractions are equal in situations where visual inspection may reveal nothing, as the following examples amply demonstrate. The following situation is also common in applications: we are given

$$\frac{a}{b} = \frac{c}{d},$$

where a, b, c and d are whole numbers, and we would like to conclude that

$$\frac{a}{c} = \frac{b}{d}.$$

We may do so because, by (4), both equalities are the same as the equality $ad = bc$. So you can easily convince yourself that the cross-multiplication algorithm is worth knowing, and knowing well. Note also that in applications, one must be careful *not* to conclude from $kn < lm$ that $\frac{m}{n} < \frac{k}{l}$. (The correct conclusion is rather $\frac{k}{l} < \frac{m}{n}$. One way may be to always start at the "upper left corner", which is k.)

Example 1. Let us first reprise the example of $\frac{4}{7}$ and $\frac{3}{5}$ at the beginning of this section, making direct use of the cross-multiplication algorithm. Because $4 \times 5 < 3 \times 7$, (4b) implies $\frac{4}{7} < \frac{3}{5}$.

Example 2. Which is bigger: $\frac{14}{21}$ or $\frac{38}{57}$? Because $14 \times 57 = 21 \times 38 (= 798)$, the fractions are actually equal by virtue of the second assertion in (4). (Both equal $\frac{2}{3}$.)

EXAMPLE 3. $\frac{84}{119}$ and $\frac{228}{323}$ are equal, because $84 \times 323 = 27132 = 119 \times 228$. In fact, both fractions turn out to be equal to $\frac{12}{17}$. It would be fair to say that without the cross-multiplication algorithm, it would be difficult to do this problem.

EXAMPLE 4. I have a whole number x with the property that when 39 is divided by x (see the definition in §4), it equals $\frac{63}{105}$. What is x ?

Solution: We are given $\frac{39}{x} = \frac{63}{105}$. By the second assertion of (5), $39 \times 105 = 63x$, so that $63x = 4095$. By (48) of §4 in Chapter 1, $x = \frac{4095}{63} = 65$.

EXAMPLE 5. Which is greater: $\frac{23}{24}$ or $\frac{24}{25}$? Do it both with and without computation.

Using the cross-multiplication algorithm, we see that $\frac{23}{24} < \frac{24}{25}$ because $23 \times 25 = 575 < 576 = 24 \times 24$. However, this result could also be done by inspection: Both fractions are points on the unit segment $[0, 1]$:

It suffices therefore to decide which of the two is the longer segment: the one between $\frac{23}{24}$ and 1, or the one between $\frac{24}{25}$ and 1. But the first segment has length $\frac{1}{24}$ while the latter has length $\frac{1}{25}$; clearly the latter is shorter. So again $\frac{23}{24} < \frac{24}{25}$.

EXAMPLE 6. Prove that

$$\text{if} \quad \frac{a}{b} < \frac{c}{d}, \text{ then } \quad \frac{b}{a} > \frac{d}{c},$$

assuming that a, b, c, and d are all nonzero.

This must be done carefully. By (4a), $\frac{a}{b} < \frac{c}{d}$ implies $ad < bc$, which may be rewritten as $da < cb$. By (4b), the latter implies $\frac{d}{c} < \frac{b}{a}$, which is the same as $\frac{b}{a} > \frac{d}{c}$.

Remark. Given a fraction $\frac{a}{b}$ (both a and b nonzero), the fraction $\frac{b}{a}$ is called its *reciprocal*. What Example 6 says is that taking reciprocals reverses an inequality between fractions. This would be more intuitive if we look at the simplest case of whole numbers. Start with $3 < 5$. Then because $3 = \frac{3}{1}$ (by (1)), the reciprocal of 3 is $\frac{1}{3}$. Similarly, the reciprocal of 5 is $\frac{1}{5}$. But clearly $\frac{1}{3} > \frac{1}{5}$ (if this is not "clear" to you, please review the definition of a fraction in §1), so inequality-reversal is understandable in this case. You may also observe that $\frac{1}{7} < \frac{1}{3}$ implies $7 > 3$. And so on.

Example 7. We want to make some red liquid. One proposal is to mix 18 fluid ounces of liquid red dye in a pail of 230 fluid ounces of water, and the other proposal is to mix 12 fluid ounces of red dye in a smaller pail of 160 fluid ounces of water. The question: which would produce a redder liquid?

In the first method, we have 18 parts of red dye out of $230 + 18 = 248$ parts of liquid. In the second method, we have 12 parts of red dye out of $160 + 12 = 172$ fluid ounces of liquid. Which of $\frac{18}{248}$ and $\frac{12}{172}$ is greater should produce — if common sense prevails — the redder liquid. Now $18 \times 172 = 3096$ and $12 \times 248 = 2976$. Thus $12 \times 248 < 18 \times 172$, and by (4b), $\frac{12}{172} < \frac{18}{248}$. So we know that the first method gives a redder liquid.

We can also think about the problem in a different way. In the first method, we distribute 18 parts of red dye among 230 parts of water. By the partitive interpretation of division, each part of water gets $\frac{18}{230}$ parts of red dye. Similarly, in the second method, each part of water gets $\frac{12}{160}$ parts of red dye. We now compare $\frac{18}{230}$ and $\frac{12}{160}$. We have $12 \times 230 = 2760 < 2880 = 18 \times 160$. By (4b) again, $\frac{12}{160} < \frac{18}{230}$, and so by common sense we expect the first method to provide a redder liquid.

We pause to note that both ways of doing the problem end up with the same conclusion. Is this just luck, or is something deeper involved here? How are the two inequalities

$$\frac{12}{172} < \frac{18}{248} \quad \text{and} \quad \frac{12}{160} < \frac{18}{230}$$

related? Two obvious relationships stand out:

$$\frac{12}{172} = \frac{12}{160 + 12} \quad \text{and} \quad \frac{18}{248} = \frac{18}{230 + 18}.$$

The following theorem then shows that the two ways of thinking about this problem are mathematically the same. (So no luck was involved!)

Theorem The following say the same thing about any four whole numbers a, b, c, and d, with $b \neq 0$ and $d \neq 0$:

(a) $\dfrac{a}{b} < \dfrac{c}{d}$.

(b) $\dfrac{a}{a+b} < \dfrac{c}{c+d}$.

(c) $\dfrac{a+b}{b} < \dfrac{c+d}{d}$.

Proof Because we are mainly interested in (a) and (b) being the same, we will only prove this part. The rest of the proof we leave to an exercise.

Why (a) implies (b): If (a) is true, then by (4a), $ad < bc$. Adding ac to both sides gives $ac + ad < ac + bc$ (by (11) in §2 of Chapter 1), which is the same as $a(c+d) < (a+b)c$. By (5b), this implies $\frac{a}{a+b} < \frac{c}{c+d}$.

Why (b) implies (a): If (b) is true, then by (4a), $a(c+d) < c(a+b)$, which is the same as $ac+ad < ac+bc$, which is the same as $ad < bc$ (by (11) of §2 in Chapter 1). Now (4b) says $\frac{a}{b} < \frac{c}{d}$. Q.E.D.

Exercise 5.1 Compare the following pairs of fractions.

$$\frac{4}{9} \text{ and } \frac{3}{7}, \quad \frac{9}{29} \text{ and } \frac{4}{13}, \quad \frac{13}{17} \text{ and } \frac{19}{25}, \quad \frac{12}{23} \text{ and } \frac{53}{102}\,.$$

(You may use a calculator to do the multiplications of the last item.)

Exercise 5.2 (a) Compare the fractions $\frac{94}{95}$ and $\frac{311}{314}$ both ways, with and without using the cross-multiplication algorithm. (b) Do the same for $\frac{83}{119}$ and $\frac{227}{328}$ (compare Example 3).

Exercise 5.3 Use calculator to do the whole number computations (and *only* the whole number computations) if necessary to see which is greater: $\frac{112}{234}$ and $\frac{213}{435}$, $\frac{577}{267}$ and $\frac{863}{403}$

Exercise 5.4 How would you explain to a student that the reason the inequality $\frac{4}{9} > \frac{3}{7}$ is true is because $4 \times 7 > 3 \times 9$?

Exercise 5.5 Write down a fraction that is between $\frac{31}{63}$ and $\frac{32}{63}$, and one between $\frac{5}{8}$ and $\frac{8}{13}$.

Exercise 5.6 If a and b are nonzero whole numbers such that $a < b$, is it true that $\frac{1}{a} < \frac{1}{b}$? If so, explain. If not, what is the right conclusion, and why?

Exercise 5.7 Let $\frac{a}{b}$ be a nonzero fraction. Compare successively $\frac{a+1}{b+1}$, $\frac{a+2}{b+2}$, $\frac{a+3}{b+3}$, ... with $\frac{a}{b}$. Generalize. (Hint: try $\frac{a}{b} = \frac{2}{3}$, and then try $\frac{a}{b} = \frac{3}{2}$.)

Exercise 5.8 Show that (a) and (c) of the preceding theorem are the same.

Exercise 5.9 Formulate and prove the analogue of the preceding theorem with "<" replaced by "=".

Exercise 5.10 (a) Which is the better buy: 3 pencils for 59 cents or 10 pencils for $1.99 ? (b) Which is the better buy: 12 candles for $1.75 or 3 candles for 45 cents?

Exercise 5.11 An alcohol solution mixes 5 parts water with 23 parts alcohol. Then 3 parts water and 14 parts alcohol are added to the solution. Which has a higher concentration of alcohol, the old solution or the new?

6 Addition and Subtraction of Fractions

We are now in a position to deal with the addition of fractions. For whole numbers, addition is calculated by combining two groups of objects and just count. For fractions, we do not have that luxury. It is not possible to combine two segments, one of length $\frac{11}{13}$ and another of length $\frac{4}{7}$ and "count", or combine $\frac{7}{8}$ of a bucket of water and another $\frac{5}{11}$ of a bucket of water and "count". However, (43) in §4 of Chapter 1 gives the geometric formulation of the addition of two whole numbers which makes sense verbatim for the addition of any two fractions. For example, we can concatenate two segments of lengths $\frac{7}{8}$ and $\frac{5}{11}$ and then measure how long the resulting segment is. This is what we shall use as the definition of addition for fractions. (All the work in §§4–5 will finally pay off, as we shall see.)

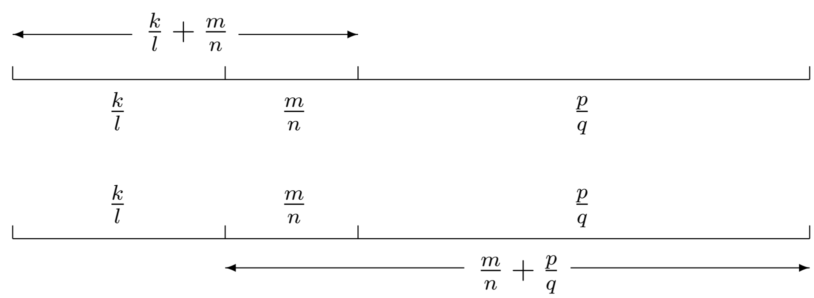
Definition. Given fractions $\frac{k}{l}$ and $\frac{m}{n}$, we define

$$\begin{aligned} \frac{k}{l} + \frac{m}{n} \;&=\; \text{the length of two concatenated segments,} \\ &\quad\;\text{one of length } \frac{k}{l}\text{, followed by one of length } \frac{m}{n} \end{aligned} \tag{5}$$

It follows directly from this definition that

$$\left(\frac{k}{l} + \frac{m}{n}\right) + \frac{p}{q} = \frac{k}{l} + \left(\frac{m}{n} + \frac{p}{q}\right)$$

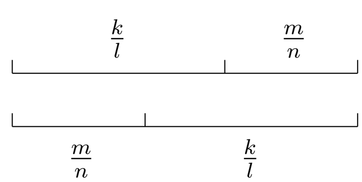
for any fractions $\frac{k}{l}$, $\frac{m}{n}$ and $\frac{p}{q}$, as the following picture shows:



This is the associative law of addition for fractions. Comparing with the discussion after (5) and (6) in §2 of Chapter 1, we see that there is no conceptual difference between whole numbers and fractions as far as the associative law is concerned. Similarly, we also have the commutative law for the addition of fractions:

$$\frac{k}{l} + \frac{m}{n} = \frac{m}{n} + \frac{k}{l}\,.$$

The corresponding picture is:



Exactly the same reasoning as in §2 of Chapter 1 leads to the conclusion that the sum

$$\frac{k}{l} + \frac{m}{n} + \frac{p}{q}$$

makes sense without the use of parenthesis, and that the same holds for an arbitrary sum of fractions, e.g.,

$$\frac{a}{b} + \frac{c}{d} + \cdots + \frac{y}{z}\,.$$

This said, we now observe that it follows from the definition of addition in (5) that

$$\frac{k}{l} = \frac{1}{l} + \frac{1}{l} + \cdots + \frac{1}{l} \quad (k \text{ times}) \tag{6}$$

(the point being that the right side makes sense without parentheses). Also immediate from the definition (5) is the fact that

$$\frac{m}{l} + \frac{k}{l} = \frac{m+k}{l}$$

because, using (6), both sides are equal to the length of $m+k$ concatenated segments each of length $\frac{1}{l}$.

More generally, we have

$$\frac{k_1}{l} + \frac{k_2}{l} + \cdots + \frac{k_n}{l} = \frac{k_1 + k_2 + \cdots + k_n}{l} \tag{7}$$

for any whole numbers k_1, k_2, $\ldots$, k_n, l ($l > 0$), as both sides are equal to the length of $k_1 + k_2 + \cdots + k_n$ concatenated segments each of length $\frac{1}{l}$.

We need a formula that explicitly expresses the sum in (5) in terms of k, l, m, n. This is the counterpart of the addition algorithm for the sum of two whole numbers. The previous consideration of comparing fractions prepares us well for this task. First of all, note that in the case of equal denominators (i.e., the case of $l = n$ in (5)), such a formula is already contained in (7). However, we want a general formula regardless of whether the denominators are equal or not. The problem we face can be seen in a concrete case: we cannot add $\frac{2}{5} + \frac{1}{3}$ directly, in the same way that we cannot add the sum (1 m. + 1 ft.) until we express both meter and foot in terms of a common unit. One can, for example, express them in terms of centimeters to obtain: (1 m. + 1 ft.) $=$ 100 cm + (12×2.54) cm $=$ 130.48 cm. For the fractions themselves, we follow the reasoning given above (4) of §5 to express both fractions in terms of the "new unit" $\frac{1}{3\times 5}$ to arrive at

$$\frac{2}{5} + \frac{1}{3} = \frac{3\times 2}{3\times 5} + \frac{5\times 1}{5\times 3} = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}\,,$$

by virtue of (2) and (7). The general case is just more of the same: given $\frac{k}{l}$ and $\frac{m}{n}$, we use the cancellation law (2) to rewrite

$$\frac{k}{l} + \frac{m}{n} = \frac{kn}{ln} + \frac{lm}{ln} = \frac{kn + lm}{ln},$$

where the last step uses (7). Thus we have the general formula:

$$\frac{k}{l} + \frac{m}{n} = \frac{kn + lm}{ln} \tag{8}$$

This formula is different from the usual formula given in textbooks involving the lcm (least common multiple[12]) of the denominators l and n, and we shall comment on the difference below.

We call special attention to the fact that formula (8) was obtained by a deductive process that is conceptually simple and entirely natural. It should go a long way towards explaining why the addition of fractions does *not* take the form of

$$\frac{k}{l} + \frac{m}{n} = \frac{k+m}{l+n}.$$

(Try $k = m = 1$ and $l = n = 2$, and compare the result with (8).)

The special case of (8) when $l = 1$ should be singled out: since $\frac{k}{1} = k$ (by (1)),

$$k + \frac{m}{n} = \frac{kn+m}{n} \tag{9}$$

Thus, $5 + \frac{3}{2} = \frac{13}{2}$ and $7 + \frac{5}{6} = \frac{47}{6}$. Using the notation of a mixed number introduced at the end of §1, we see that

$$k\frac{m}{n} = k + \frac{m}{n} \qquad \text{in case} \quad m < n. \tag{10}$$

In terms of the addition of fractions, the notation of a mixed number makes sense: $3\frac{1}{4}$, for example, means the length of the concatenation of a segment of length 3 and a segment of length $\frac{1}{4}$, and in general, $q\frac{a}{b}$ means the length of the concatenation of a segment of length q and one of length $\frac{a}{b}$. Operations with mixed fractions are generally regarded with a sizable amount of trepidation. The equality (10) serves as a reminder that a mixed fraction is merely a shorthand notation to denote the sum of a whole number and a fraction. Since a whole number k is also a fraction $\frac{k}{1}$ (see (1)), a mixed fraction should be handled in the same routine manner as any other fraction.

We note in this connection that if a and b are whole numbers and q is the quotient and r the remainder of $a \div b$, then

$$\frac{a}{b} = q\frac{r}{b}$$

[12] A knowledge of the lcm of two whole numbers is not needed here. Any discussion involving lcm, such as later in this section, is only peripheral to our purpose. For this reason, we do not even define what lcm means. For the definition and a detailed discussion, see Chapter 3.

because $a = qb + r$ so that $\frac{a}{b} = \frac{qb+r}{b} = \frac{qb}{b} + \frac{r}{b} = q + \frac{r}{b} = q\frac{r}{b}$. In schools, students are taught as part of the long division algorithm that "if a divided by b has quotient q and remainder r, then you can write $a \div b$ as $q\frac{r}{b}$". This is the reason why.

Although the tradition in school mathematics is to insist that every improper fraction be automatically converted to a mixed fraction, *there is no mathematical reason why this must be so.* This tradition seems to be closely related to the one which insists that every answer in fractions must be in reduced form. See the discussion in §3 about reduced fractions.

Formulas (9) and (10) combined give the usual conversion of a mixed number to an improper fraction. Conversely, the conversion of an improper fraction to a mixed fraction is done by appealing to the division-with-remainder. Let $\frac{k}{l}$ be given, where $k > l$. By the division-with-remainder, $k = ql + r$ for whole numbers q and r so that $0 \le r < l$. Thus,

$$\frac{k}{l} = \frac{ql + r}{l} = \frac{ql}{l} + \frac{r}{l} = q + \frac{r}{l} = q\frac{r}{l},$$

and $\frac{r}{l}$ is a proper fraction because $r < l$.

> **Activity:** Convert each of the following improper fraction to a mixed number, and vice versa: $7\frac{5}{6}$, $4\frac{5}{7}$, $6\frac{1}{7}$, $13\frac{4}{5}$, $\frac{32}{7}$, $\frac{148}{9}$, $\frac{166}{15}$.

Example. Compute $2\frac{5}{9} + \frac{7}{8}$ and $15\frac{4}{17} + 16\frac{12}{13}$.

Using the associative and commutative laws of addition without comment, we have:

$$2\frac{5}{9} + \frac{7}{8} = 2 + (\frac{5}{9} + \frac{7}{8}) = 2 + \frac{103}{72} = 2 + 1 + \frac{31}{72} = 3\frac{31}{72}.$$

$$15\frac{4}{17} + 16\frac{12}{13} = (15 + 16) + \frac{256}{221} = 31 + 1 + \frac{35}{221} = 32\frac{35}{221}.$$

However, we could have carried out the additions differently:

$$2\frac{5}{9} + \frac{7}{8} = \frac{18 + 5}{9} + \frac{7}{8} = \frac{23}{9} + \frac{7}{8} = \frac{23 \times 8 + 7 \times 9}{9 \times 8} = \frac{247}{72}$$

$$
\begin{aligned}
15\frac{4}{17} + 16\frac{12}{13} &= \frac{15 \times 17 + 4}{17} + \frac{16 \times 13 + 12}{13} \\
&= \frac{259}{17} + \frac{220}{13} = \frac{259 \times 13 + 220 \times 17}{17 \times 13} \\
&= \frac{7107}{221}
\end{aligned}
$$

The fact that both answers are equally acceptable in each case is something we already had occasion to emphasize. The fact that the two answers are actually equal to each other in each case is something for you to verify!

> **Activity:** Verify that the two answers above are indeed the same.

We should now address some of the fine points of formula (8). The salient feature of this formula is its simplicity, not only its *formal* simplicity, but also the simplicity of the reasoning behind its derivation. Another noteworthy feature is its generality: it is valid under all circumstances. As a rule, however, its generality also works against it in special situations where there are usually cute tricks to provide shortcuts. For example, in the case of fractions with equal denominators, (7) clearly supersedes (8) because the latter would give $\frac{m}{l} + \frac{k}{l} = \frac{kl+ml}{l^2}$, which is more clumsy than (7). It is well to note that, clumsiness notwithstanding, this answer is *correct*, because $\frac{kl+ml}{l^2} = \frac{(k+m)l}{ll} = \frac{k+m}{l}$, where the last step uses the cancellation law (2). This in fact illustrates the value of (8): it provides an easy-to-use formula for the addition of fractions for all occasions. This is a "security blanket" that is invaluable to many students.

Another example is when one denominator is a multiple of another. In this case, instead of appealing to (8), we should just use the bigger denominator:

$$\frac{2}{9} + \frac{5}{36} = \frac{8}{36} + \frac{5}{36} = \frac{13}{36}.$$

More generally,

$$\frac{m}{nl} + \frac{k}{l} = \frac{m}{nl} + \frac{nk}{nl} = \frac{m+nk}{nl},$$

whereas by comparison, (8) would give us

$$\frac{m}{nl} + \frac{k}{l} = \frac{ml+nkl}{nl^2}.$$

We point out again that this answer is correct, because

$$\frac{ml + nkl}{nl^2} = \frac{(m + nk)l}{nl^2} = \frac{m + nk}{nl}.$$

The next special case of (8) is worthy of a more elaborate discussion. Consider the following example.

EXAMPLE Compute $\frac{3}{4} + \frac{5}{6}$.

$$\frac{3}{4} + \frac{5}{6} = \frac{18 + 20}{24} = \frac{38}{24} = \frac{19}{12}$$

which could also be written as $1\frac{7}{12}$. However, in this case one sees that it is not necessary to go to $\frac{1}{24}$ as common unit of measurement of the fractions $\frac{1}{4}$ and $\frac{1}{6}$, because we could use $\frac{1}{12}$ instead: $\frac{1}{4} = \frac{3}{12}$ and $\frac{1}{6} = \frac{2}{12}$. Hence we could have computed this way:

$$\frac{3}{4} + \frac{5}{6} = \frac{3 \times 3}{12} + \frac{2 \times 5}{12} = \frac{19}{12} = 1\frac{7}{12},$$

as before.

This example brings us to the consideration of the usual formula for adding fractions. So suppose $\frac{k}{l}$ and $\frac{m}{n}$ are given. Suppose we know that there is a whole number A which could be different from ln but which is nevertheless a multiple of both n and l. (In terms of the preceding example, if we let $\frac{k}{l} = \frac{3}{4}$ and $\frac{m}{n} = \frac{5}{6}$, then $l = 4$, $n = 6$ and we may let $A = 12$. Note that $A \neq 4 \times 6 = ln$.) Then we have $A = nN = lL$ for some whole numbers L and N. Letting $A = nl$ would always work, of course. Another candidate for A is the lcm[13] of n and l, which is generally different from nl. For example, the lcm of 4 and 6 is 12, not 24. In any case, given such an A, $\frac{k}{l} = \frac{kL}{lL} = \frac{kL}{A}$ and $\frac{m}{n} = \frac{mN}{nN} = \frac{mN}{A}$, so that

$$\frac{k}{l} + \frac{m}{n} = \frac{kL + mN}{A} \quad \text{where} \quad A = nN = lL \tag{11}$$

As remarked earlier, if A is taken to be the lcm of n and l, then this is the formula used in most textbooks to teach students how to add fractions.

[13] This concept will be defined in §5 of Chapter 3, but we merely use it here for illustration.

Formula (11) is a useful skill that all students of fractions should learn, as we shall see presently. We should add a strong word of caution, however, against its abuse in the school curriculum. The worst possible abuse is the use of (11) — with A as the lcm of n and l — as the *definition* of the addition of the two fractions $\frac{k}{l}$ and $\frac{m}{n}$. There are reasons from advanced mathematics as to why this is the *wrong* way to define the addition of fractions[14], but for our purpose, it is enough to point out that formula (11) is a pedagogical disaster when used as the definition of adding fractions. Often (11) is offered as the definition *without any explanation*. But even when an explanation is given, one would have to start with the concept of the lcm of two numbers, which is usually preceded by the concept of the gcd[15] (greatest common divisor) of two numbers. Students tend to confuse gcd with lcm, unfortunately, so an unnecessary roadblock is inserted in their learning path. Moreover, the consideration of lcm renders the simple concept of adding fractions too complicated to understand and too clumsy to use. The difficulty task of teaching fractions is therefore made even more difficult. There is also one more argument against the use of lcm to define the addition of fractions, which is how to prove that the distributive law holds ofr fractions? (See (15) of §7.) Of course a proof can be given, but it is made unnecessarily difficult by having to consider lcm.

All these objections have to be understood in the context of the alternative, namely, the definition of addition using (5) or (8). We have seen how simple and natural it is to explain (see also the proof of (15) in §7), and especially how easy it is to use. Therefore there is no contest: one should never teach fractions using (11) as the definition of addition.

We now give an example to illustrate the difference between (9) and (12) as definitions.

Example. Compute $\dfrac{2}{323} + \dfrac{3}{493}$.

Solution. According to (8):

$$\frac{2}{323} + \frac{3}{493} = \frac{(2 \times 493) + (3 \times 323)}{323 \times 493} = \frac{1955}{159239}$$

Now suppose a sixth grader is taught to add fractions only by using (11) with A as the lcm of l and n, she would have a difficult time finding the lcm of 323

[14] This would imply that addition cannot be defined in the quotient field of an integral domain unless the domain is an UFD.

[15] See §3 of Chapter 3.

and 493 and may therefore give up doing the problem altogether. But we have just seen that there is *nothing* at all difficult with such a routine problem: 1955/159239 is the answer (with the help of a four-function calculator).

We may add that the lcm of 323 and 493 is in fact $17 \times 19 \times 29$. Moreover, $323 = 17 \times 19$ and $493 = 17 \times 29$, so that according to (11),

$$\frac{2}{323} + \frac{3}{493} = \frac{2 \times 29 + 3 \times 19}{17 \times 19 \times 29} = \frac{115}{9367}.$$

It also goes without saying that $1955/159239 = 115/9367$. In both practical and theoretical terms, however, there is little advantage in having 115/9367 as the answer instead of 1955/159239.

We mentioned that as a special skill, (11) is important. Here is one reason.

Example. If n is a whole number, we define $n!$ (read: *n factorial*) to be the product of all the whole numbers from 1 through n. Thus $5! = 1 \times 2 \times 3 \times 4 \times 5$. We also define the so-called *binomial coefficients* $\begin{pmatrix} n \\ k \end{pmatrix}$ for any whole number k satisfying $0 < k < n$ as

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!\; k!}$$

Then a very useful formula says:

$$\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} n-1 \\ k \end{pmatrix} + \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$$

We now use (11) to prove this formula. We shall start from the right side and show that the addition of these two fractions gives the left side. Thus

$$\begin{aligned} \begin{pmatrix} n-1 \\ k \end{pmatrix} + \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} &= \frac{(n-1)!}{(n-1-k)!\; k!} + \frac{(n-1)!}{((n-1)-(k-1))!\; (k-1)!} \\ &= \frac{(n-1)!}{(n-k-1)!\; k!} + \frac{(n-1)!}{(n-k)!\; (k-1)!} \end{aligned}$$

Observe now:

$$\begin{aligned} (n-k)!\; k! &= [(n-k-1)!\; k!] \cdot (n-k) \\ (n-k)!\; k! &= [(n-k)!\; (k-1)!] \cdot k \end{aligned}$$

So using (11) with A as $(n-k)!\; k!$, we have:

$$\frac{(n-1)!}{(n-k-1)!\; k!} + \frac{(n-1)!}{(n-k)!\; (k-1)!} = \frac{(n-1)!\; (n-k) + (n-1)!\; k}{(n-k)!\; k!}$$

By the distributive law, the numerator can be simplified as follows:

$$(n-1)!\,(n-k) + (n-1)!\,k = (n-1)!\,\{(n-k)+k\} = (n-1)!\,n = n!$$

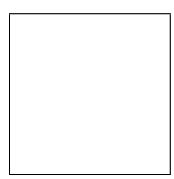
Thus,

$$\frac{(n-1)!}{(n-k-1)!\,k!} + \frac{(n-1)!}{(n-k)!\,(k-1)!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

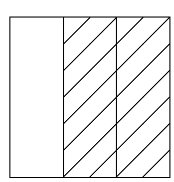
So the formula is proved.

Pedagogical Comments: Because of the important role of the addition of fractions in the context of elementary mathematics education, we now make an excursion into pedagogy. The point is that in an average fifth grade classroom, adding fractions by concatenating line segments — while desirable — may not be the most popular move one can make. Is there perhaps an alternative that may be more palatable to students? We wish to make a suggestion. The key idea behind the use of segments — and indeed the use of the number line — is really that in any discussion of fractions, one must refer to a fixed unit. This suggests that, if necessary, we may forego the formality of the number line and simply fix the unit 1 as the area of a given unit square and go on from there (not forgetting, of course, to remind the students periodically that they must refer to this fixed unit square anytime they use a fraction). Instead of concatenating squares, we now combine parts of unit squares. Let us illustrate with the addition of $\frac{2}{3} + \frac{1}{2}$.

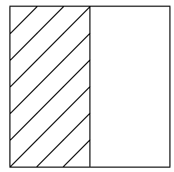
We begin by fixing a unit square whose area is our reference unit 1:



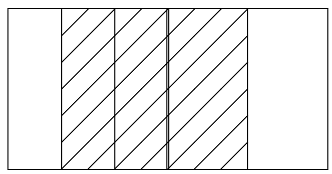
Then the following shaded area represents $\frac{2}{3}$:



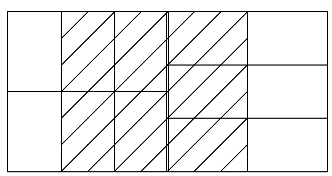
and the following shaded area represents $\frac{1}{2}$:



To compute $\frac{2}{3} + \frac{1}{2}$, one has to compute the area of the combined shaded area:



However, suppose we divide both squares into 6 ($= 2 \times 3$) equal parts in the following way: introduce a horizontal division of the left square into two equal halves, and introduce a horizontal division of the right square into three equal thirds:



The net effect of these divisions is that each of the two squares is now divided into 6 (mutually congruent) "small rectangles", and we may now use one of these small rectangles as "yardsticks" to measure the combined shaded area. The shaded area of the left square is paved by four small rectangles, and that of the right square is paved by three of them. The combined shaded area therefore contains $3 + 4 = 7$ small rectangles. But each small rectangle is $\frac{1}{6}$ of the unit square, and the unit square has area equal to 1. Thus the area of the small rectangle is also $\frac{1}{6}$, and by the definition of a fraction (see §1), the combined shaded area has area $\frac{7}{6}$.

We now express this symbolically. The fact that $\frac{2}{3} = \frac{2\times 2}{2\times 3}$ expresses the further division of the left unit square into two horizontal halves so that the original 3 vertical thirds are now split into 2×3 small rectangles, and the original 2 vertical thirds of the shaded area are now split into 2×2 small rectangles. Similarly, for the right unit square, $\frac{1}{2} = \frac{3\times 1}{3\times 2}$. Hence,

$$\frac{2}{3} + \frac{1}{2} = \frac{2 \times 2}{2 \times 3} + \frac{3 \times 1}{3 \times 2} = \frac{2 \times 2 + 3 \times 2}{3 \times 2},$$

which is the special case of (9) for $\frac{k}{l} = \frac{2}{3}$ and $\frac{m}{n} = \frac{1}{2}$.

By working out a few examples like this, students would get the main idea of the reasoning behind (8). **End of Pedagogical Comments.**

Now we show how to use the addition of fractions to re-prove the interpretation (3) of a fraction as a division of whole numbers (in the sense of the definition in the latter part of §4). **The idea of this proof is an important one and will be used again on several other occasions.** We begin with a general observation . Suppose A is a fraction such that

$$\underbrace{A + A + \cdots + A}_{l} = k$$

for some whole number k and l. By the definition of the addition of fractions in (5), we know that by concatenating l segments each of length A, we obtain a segment of length k. Looked at another way, this says that the segment $[0, k]$ can be divided into l equal parts and each part has length A. By the extended definition of division between whole numbers in §4, we have that $k \div l = A$.

Now suppose $\frac{k}{l}$ is given. Let us consider the sum

$$\underbrace{\frac{k}{l} + \frac{k}{l} + \cdots + \frac{k}{l}}_{l}.$$

By (7), this equals $\dfrac{\overbrace{k + k + \cdots + k}^{l}}{l} = \frac{lk}{l} = k$, where we have used (1). Thus

$$\underbrace{\frac{k}{l} + \frac{k}{l} + \cdots + \frac{k}{l}}_{l} = k.$$

The previous observation then implies that $k \div l = \frac{k}{l}$, as desired.

We now say a few words about the subtraction of fractions; the brevity of our comments is warranted by the similarity of subtraction to addition. Suppose as usual that $\frac{k}{l}$ and $\frac{m}{n}$ are given and that $\frac{k}{l} \geq \frac{m}{n}$. Imitating the case of whole numbers in (44) in §4 of Chapter 1, we define the difference $\frac{k}{l} - \frac{m}{n}$ as

$$\begin{aligned}\frac{k}{l} - \frac{m}{n} \quad &= \quad \text{the length of the remaining segment when} \\ &\quad\ \text{a segment of length } \frac{m}{n} \text{ is removed from one end of} \\ &\quad\ \text{a segment of length } \frac{k}{l}\end{aligned}$$

Now $\frac{k}{l} \geq \frac{m}{n}$ means, by virtue of (4), that $kn \geq lm$ so that one can subtract lm from kn. Hence the following makes sense:

$$\begin{aligned}\frac{k}{l} - \frac{m}{n} \quad &= \quad \frac{kn}{ln} - \frac{lm}{ln} \\ &= \quad \text{the length of the remaining segment when } lm \text{ copies} \\ &\quad\ \text{of } \frac{1}{ln} \text{ are removed from } kn \text{ copies of } \frac{1}{ln} \\ &= \quad \text{the length of } \ (kn - lm) \ \text{ copies of } \frac{1}{ln} \\ &= \quad \frac{kn - lm}{ln},\end{aligned}$$

where the last equally is by definition of the fraction $\frac{kn-lm}{ln}$. This yields the formula:

$$\frac{k}{l} - \frac{m}{n} = \frac{kn - lm}{ln} \tag{12}$$

when $\frac{k}{l} \geq \frac{m}{n}$.

The subtraction formula (12) is in particular applicable to mixed fractions and, in that contenxt, brings out a special feature which is not particularly important but which is interesting nonetheless. As usual, instead of explaining this feature using symbolic notation, we shall illustrate it with an example. Consider the subtraction of $17\frac{2}{5} - 7\frac{3}{4}$. There is an obvious way to make (12) directly applicable to this case regardless of what the relevant numbers may be, which is to convert the mixed numbers into improper fractions:

$$17\frac{2}{5} - 7\frac{3}{4} = \frac{85+2}{5} - \frac{28+3}{4} = \frac{87}{5} - \frac{31}{4} = \frac{87 \times 4 - 31 \times 5}{5 \times 4} = \frac{193}{20}.$$

(We emphasize again that there is no need to convert this back to a mixed fraction unless of course there is an explicit instruction to do so.) However, there is another way to do the computation:

$$17\frac{2}{5} - 7\frac{3}{4} = (17 + \frac{2}{5}) - (7 + \frac{3}{4}).$$

Now we use the analog of identity (23) in §3.2 of Chapter 1 for fractions. As explained in §3.2 of Chapter 1, we shall give a full explanation of this identity for fractions in Chapter 5. Such being the case, we get:

$$17\frac{2}{5} - 7\frac{3}{4} = (17 - 7) + (\frac{2}{5} - \frac{3}{4}) = 10 + (\frac{2}{5} - \frac{3}{4}),$$

and we note that (12) is not applicable as it stands to $\frac{2}{5} - \frac{3}{4}$ because $\frac{2}{5} < \frac{3}{4}$. (Can you *prove* this inequality?) We now recall the subtraction algorithm in §3.2: if in doing the subtraction (for example) $82 - 57$ we find that the subtraction in the ones digit (i.e., $2 - 7$) cannot be done using whole numbers, then we trade a 1 from the tens digit to the ones digit to make it work. Similarly, we are going to trade a 1 from the whole number 17 in this situation. Formally, what we are doing is to *appeal to the associative law of the addition of fractions*, so that

$$\begin{aligned} 17\frac{2}{5} - 7\frac{3}{4} &= (16 + 1\frac{2}{5} - (7 + \frac{3}{4}) \\ &= (16 + \frac{7}{5}) - (7 + \frac{3}{4}) \\ &= (16 - 7) + (\frac{7}{5} - \frac{3}{4}) \\ &= 9 + (\frac{28 - 15}{20}) \\ &= 9 + \frac{13}{20} = 9\frac{13}{20} \end{aligned}$$

The whole computation looks longer than it actually is because we interrupted it with explanations. Normally, we would have done it the following way:

$$17\frac{2}{5} - 7\frac{3}{4} = (16 + \frac{7}{5}) - (7 + \frac{3}{4}) = (16 - 7) + (\frac{7}{5} - \frac{3}{4}) = 9\frac{13}{20}.$$

Incidentally, $9\frac{13}{20} = \frac{193}{20}$, exactly the same as before.

We conclude this section with some remarks on inequalities among fractions. With the avaailbility of the concept of addition among fractions, the direct extensions of some of the assertions in §2 of Chapter 1 are straightforward: let A, B, C, D be fractions, then

> (A) the statement $A < B$ is the same as $A + C = B$ for a nonzero fraction C;
>
> (B) the statement $A < B$ is the same as $A + C < B + C$ for some C;
>
> (C) $A < B$ and $C < D$ imply $A + C < B + D$.

Because in terms of the number line, there is no conceptual difference between the addition of whole numbers and the addition of fractions, we can safely leave the proofs of these statements to the exercises.

Pedagogical Comments on the Use of Calculators in the Learning of Fractions: The increasing use of large numbers in both the main text as well as the exercises is because we consider it important to get everyone out of the habit of working only with single-digit numerators and denominators. This habit seems to be linked to several bad practices which obstruct both the teaching and learning of fractions:

> (1) The over-reliance on drawing pictures of pies in every phase of the learning of fractions. Students see no need to acquire a more abstract understanding of what a fraction is, thereby retarding their acquisition of the basic disposition towards algebra. If however fractions such as $\frac{159}{68}$ and $\frac{21}{825}$ appear often, then the pressing need of coming to grips with the fraction concept and all its associated operations would be self-evident.
> (2) The exclusive reliance on (11) as a way of adding fractions. The minute large numbers are used, the silliness of (11) as the definition of the addition of fractions is exposed.
> (3) The failure to acquire the needed computational fluency with fractions. So long as fractions are equated with fractions-with-one-digit-numbers, there is no need to remember — or indeed, to understand — the formulas for adding, subtracting, and dividing fractions. For these simple fractions, it is a common practice to draw pictures of pies to do fraction computations, and this

> practice unfortunately has even been accorded the dignified status of being the bearer of "increased conceptual understanding". To us, it is rather a symptom of the breakdown in mathematics education.

This monograph therfore strongly advocates the frequent use of a four-function calculator to alleviate the tedium of computing with LARGE **numbers.**

There is an additional pragmatic issue to address. When calculator use is explictly allowed, the teacher should make sure that all intermediate steps of a computation are clearly displayed so that the calculator may not short-circuit students' need to remember the computational algorithms. We recall an earlier calculation as an example:

$$\frac{2}{323} + \frac{3}{493} = \frac{(2 \times 493) + (3 \times 323)}{323 \times 493} = \frac{1955}{159239}$$

In this instance, the calculator enters only in the last step, but in a way that is invisible. There is no way of telling whether the arithmetic computations are done by hand or the calculator. At the level of grades 4-6, it may be a good rule of thumb that if the presence of the calculator is invisible in students' work, then the calculator is not a distraction in students' learning. **End of Pedagogical Comments.**

Exercise 6.1 $\frac{17}{50} - \frac{1}{3} = ?$ $\quad 4 - \frac{2}{7} = ?$ $\quad \frac{6}{17} + \frac{4}{3} = ?$ $\quad \frac{8}{5} + \frac{16}{11} = ?$ $\quad 3\frac{1}{5} - 2\frac{7}{8} = ?$

Exercise 6.2 Large numbers are used in (a) and (b) below on purpose. You may use a four-function calculator to do the calculations with whole numbers (and *only* for that purpose.) (a) $81\frac{25}{311} + 145\frac{11}{102} = ?$ (b) $310\frac{22}{117} - 167\frac{3}{181} = ?$ (c) $78\frac{3}{54} - \frac{67}{14} = ?$

Exercise 6.3 (a) Find a fraction A so that $17\frac{5}{62} = A + 8\frac{4}{25}$. (b) Find a fraction B so that $4\frac{2}{5} - B = 1\frac{3}{4}$.

Exercise 6.4 Without computing the exact answer, estimate which of the following is bigger: $(\frac{91}{624} + \frac{8}{9})$ and 1. Explain how you did it.

Exercise 6.5 Explain to a sixth grader why every fraction can be expressed as a mixed number. (Don't forget: 0 is a whole number.)

Exercise 6.6 Let A and B be two fractions such that $A < B$. Show that there is always a fraction C so that $A < C < B$. (After finding what you think is a good candidate for C, don't forget to actually prove that $A < C < B$.)

Exercise 6.7 Prove assertions (A)–(C) about inequalities among fractions.

Exercise 6.8 Let A, B, C be fractions. (a) Prove that $A + B < C$ is the same as $A < C - B$. (b) Suppose $C < A$ and $C < B$. Prove that $A < B$ is the same as $A - C < B - C$. (Cf. Exercise 3.14 in Chapter 1.)

7 Multiplication of Fractions

Before we can discuss the multiplication of fractions, we must first understand what it means to multiply two fractions. The usual meaning of multiplication as repeated addition among whole numbers (e.g., $3 \times 5 = 5 + 5 + 5$) cannot be used for fractions because $\frac{2}{5} \times \frac{1}{4}$ is neither adding $\frac{1}{4}$ to itself $\frac{2}{5}$ times, nor adding $\frac{2}{5}$ to itself $\frac{1}{4}$ times. Most textbooks simply duck the issue of what it means to multiply two fractions but assume instead students are ready and willing to mutiply everything in sight with no questions asked. The following is a typical introduction to fraction multiplication in grade 5.

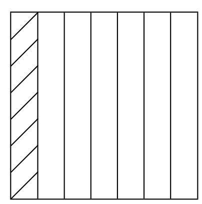
> In this lesson we will multiply fractions. Consider this multiplication problem: How much is one half of one half? Using our fraction manipulatives, we show one half of a circle. To find one half of one half, we divide the half in half. We see that the answer is one fourth. Written out, the problem looks like this.
>
> $$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
>
> WE FIND THE ANSWER TO A FRACTION MULTIPLICATION PROBLEM BY MULTIPLYING THE NUMERATORS TO GET THE NEW NUMERATOR AND MULTIPLYING THE DENOMINATORS TO GET THE NEW DENOMINATOR.

So the attitude here is pretty much that students already know that they must learn to multiply fractions, and are ready to just follow the rule of multiplying the numerators and denominators without asking why. The reminder of the text goes on to give numerous examples of how to use fraction multiplication to obtain answers.

We have to approach this topic differently. There are at least three ways to do this, the first is geometric and the other two are algebraic. We do the geometric one first.

We saw in §2 of Chapter 1 that there is another interpretation of the multiplication of whole numbers in terms of area: $m \times n$ is the area of a rectangle of sides m and n (more precisely, of "vertical" side m and "horizontal" side n), where m and n are whole numbers. Of course the area of a rectangle *does* make sense even when its sides are fractions. Defining the unit 1 to be the area of the unit square (recall: this is the square whose side has length 1), the argument in §1 using the basic properties (a)–(c) of area then shows that any rectangle contained in the unit square with sides of length 1 and $\frac{1}{n}$ will have area $\frac{1}{n}$. This is because n such rectangles provide a division of the unit (i.e., the area of the unit square in this case) into n equal parts.

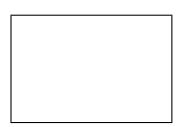
$n = 7$



It follows that the rectangle contained in the unit square with sides 1 and $\frac{m}{n}$ will have area $\frac{m}{n}$, because this rectangle is paved by m congruent rectangles with sides 1 and $\frac{1}{n}$ (whose area is $\frac{1}{n}$, as we have just seen). This prompts the following *extension* of the meaning of multiplication: we *define* for any two fractions:

$$\frac{k}{l} \times \frac{m}{n} \quad = \quad \text{the area of a rectangle with sides } \frac{k}{l} \text{ and } \frac{m}{n}$$

$\frac{k}{l}$



$\frac{m}{n}$

If $l = n = 1$, then this coincides with the interpretation given in §2 of Chapter 1 of the product of the whole numbers k and m. (Recall that we already did this kind of extension with regard to the concept of division at the end of §4 in this chapter.)

We are treating fractions conceptually the same way as we treat whole numbers, so it is important to see the similarity between the definition of the multiplication of whole numbers and that of fractions.

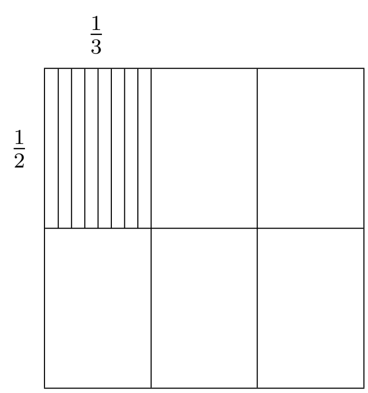
In subsequent discussions of multiplication, the unit 1 will be understood to be the area of unit square. You may wish to review the relevant discussions in §§2 and 4 of Chapter 1 and §1 of this chapter.

7.1 Formula for the product, and first consequences

As in the case of adding fractions, we want a formula that expresses $\frac{k}{l} \times \frac{m}{n}$ directly in terms k, l, m, n. We first establish the formula in a special case, but this case will turn out to be the kernel of the main argument.

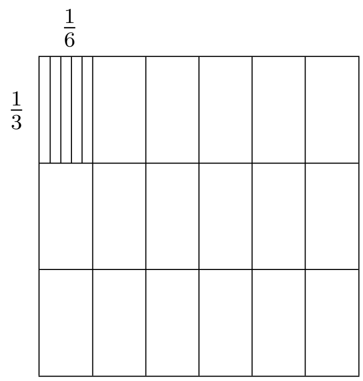
$$\frac{1}{l} \times \frac{1}{n} = \frac{1}{ln} \tag{13}$$

Before explaining (13) in general, let us begin by looking at a concrete example: why is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$? Take a unit square and divide one side into 2 equal parts and the other into 3 equal parts. Joining corresponding points of the division then partitions the square into 6 identical rectangles:



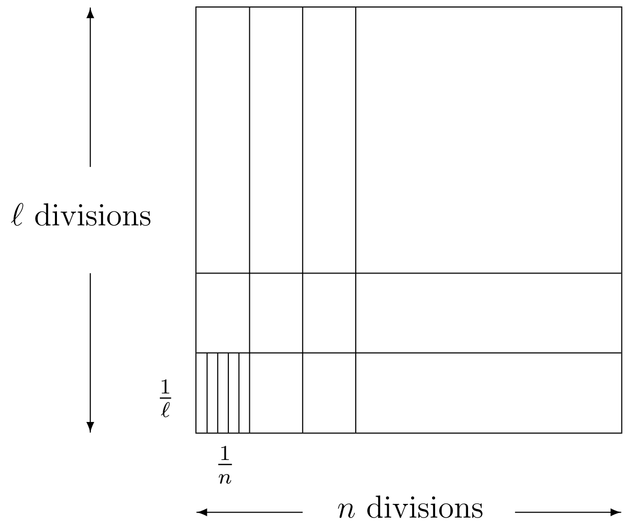
By construction, each of the 2×3 $(= 6)$ rectangles has sides $\frac{1}{2}$ and $\frac{1}{3}$ and its area is by definition $\frac{1}{2} \times \frac{1}{3}$. However, the total area of these 6 rectangle is the area of the unit square, which is 1, so the shaded rectangle is one part in a partition of the unit square into 2×3 parts of equal area. Therefore the area of the shaded rectangle is $\frac{1}{2\times 3}$ by the definition of a fraction in §1. Thus $\frac{1}{2} \times \frac{1}{3} = \frac{1}{2\times 3}$.

Let us look at another concrete example. We will show: $\frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$. Again we divide one side of a unit square into 3 equal parts and the other side 6 equal parts. Joining the corresponding points leads to a partition of the unit square into 3×6 $(= 18)$ congruent rectangles:



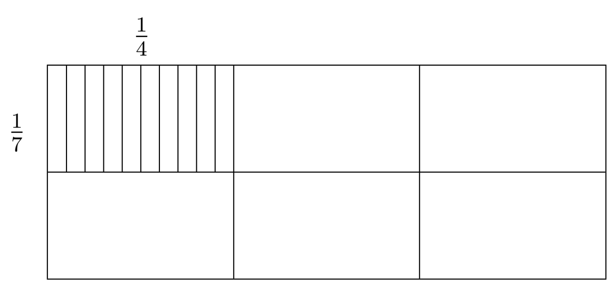
By construction, each rectangle has sides of lengths $\frac{1}{3}$ and $\frac{1}{6}$, so its area is $\frac{1}{3} \times \frac{1}{6}$, by definition. By since these 18 rectangles are identical and they partition the unit square which has area equal to 1, the area of each rectangles is $\frac{1}{18}$.

The general case in (13) can be handled in a similar way. Divide the two sides of a unit square into ℓ equal parts and n equal parts, respectively. Joining the corresponding division points creates a partition of the unit square into ℓn identical rectangles.



Because each of these rectangles has sides $\frac{1}{\ell}$ and $\frac{1}{n}$ by construction, its area is $\frac{1}{\ell} \times \frac{1}{n}$ by definition. Moreover, these ℓn congruent rectangles partition a square of area equal to 1, so each of them has area $\frac{1}{\ell n}$, in the same way that the length of a part when the unit segment is divided into ℓn equal parts is $\frac{1}{\ell n}$ (see §1). Thus $\frac{1}{\ell} \times \frac{1}{n} = \frac{1}{\ell n}$, which proves (13).

Before attacking the general case of $\frac{k}{l} \times \frac{m}{n}$, let us again consider a concrete example: $\frac{2}{7} \times \frac{3}{4}$. This is by definition the area of a rectangle with the width $\frac{2}{7}$ and length $\frac{3}{4}$. By definition of $\frac{2}{7}$, the width consists of two concatenated segments each of length $\frac{1}{7}$. Similarly, the length consists of three concatenated segments each of length $\frac{1}{4}$. Joining the obvious corresponding points on opposite sides yields a partition of the original rectangle into 2×3 identical small rectangles.

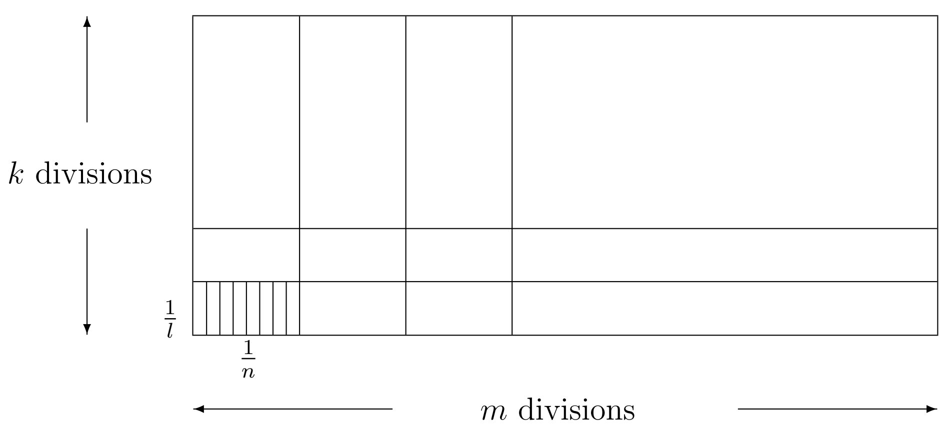


Now each of the small rectangles has sides $\frac{1}{7}$ and $\frac{1}{4}$ and therefore, by (13), has area $\frac{1}{7\times 4}$. Since the big rectangle contains exactly 2×3 such identical rectangles, its area (as a fraction) in terms of the unit area 1 is $\frac{2\times 3}{7\times 4}$, by the definition given in §1 of the fraction $\frac{2\times 3}{7\times 4}$. So at least in this case, we have $\frac{2}{7} \times \frac{3}{4} = \frac{2\times 3}{7\times 4}$.

Finally we prove in general:

$$\frac{k}{l} \times \frac{m}{n} = \frac{km}{ln} \tag{14}$$

We construct a rectangle with width $\frac{k}{l}$ and length $\frac{m}{n}$, so that its area is $\frac{k}{l} \times \frac{m}{n}$, by definition. Our task is to show that its area is also equal to $\frac{km}{ln}$, so that we would have $\frac{k}{l} \times \frac{m}{n} = \frac{km}{ln}$. By definition, its width consists of k concatenated segments each of length $\frac{1}{l}$ and its length m concatenated segments each of length $\frac{1}{n}$. Joining corresponding division points on opposite sides leads to a partition of the big rectangle into km small rectangles.



Since each of these small rectangles has sides equal to $\frac{1}{l}$ and $\frac{1}{n}$, its area is $\frac{1}{ln}$ by virtue of (13). But the original rectangle is paved by exactly km such small rectangles, so its area is $\frac{km}{ln}$, thereby proving (14).

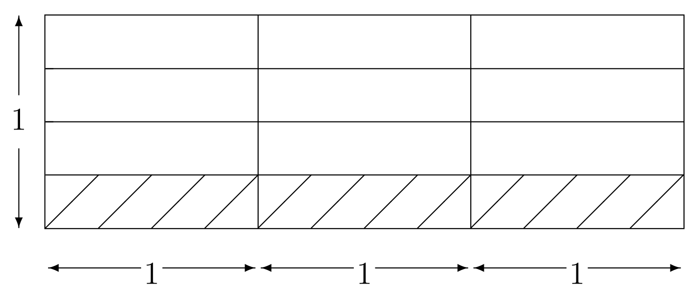
Observe that if we let $l = 1$ in (14), we would have

$$k \times \frac{m}{n} = \frac{k}{1} \times \frac{m}{n} = \frac{km}{n} = (\frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n}) \quad (k \text{ times})$$

where the last is because of (7). In particular,

$$1 \times \frac{m}{n} = \frac{m}{n}.$$

We therfore see that our definition of the multiplication of fractions is consistent with the usual intuitive understanding of what "multiplication by a whole number" means, namely, repeated addition. It is instructive to see a direct explanation of this fact. We present it for the special case where $k = 3$ and $\frac{m}{n} = \frac{1}{4}$. In that case, $3 \times \frac{1}{4}$ is the area of the shaded rectangle below.



The part of the shaded rectangle in each unit square has area $\frac{1}{4}$. So the area of the original $\frac{1}{4}$ by 3 shaded rectangle is $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, by the definition of the addition of fractions. So $3 \times \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, as desired. The reasoning clearly applies in general.

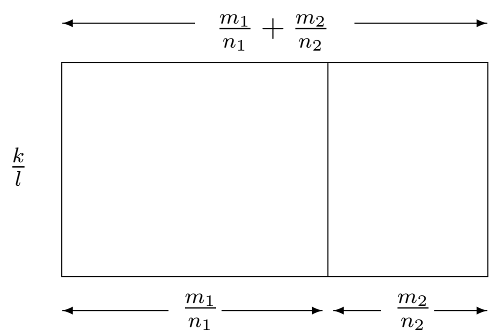
This is the right place to note that *whereas in the context of whole numbers, "multiply by a number" always results in magnification, in the context of fractions this is no longer true*. For example, if we start with 15, then mutiplying it by $\frac{1}{75}$ gets $\frac{1}{5}$, which is far smaller than 15.

We noted in §6 the validity of the *commutative law* and the *associative law* for the addition of fractions. The same laws for multiplication are straightforward consequences of (14). However, we should single out the *distributive law* for discussion because it has nontrivial computational consequences. It states that for whole numbers k, l, m_1, m_2, n_1, n_2, with l, n_1, and n_2 being nonzero:

$$\frac{k}{l} \times \left(\frac{m_1}{n_1} \pm \frac{m_2}{n_2}\right) = \left(\frac{k}{l} \times \frac{m_1}{n_1}\right) \pm \left(\frac{k}{l} \times \frac{m_2}{n_2}\right) \tag{15}$$

This can be proved in two ways: algebraically and geometrically. The algebraic proof consists of a straightforward computation using (8) and (14), and we will leave that as an exercise:

We will give the geometric proof of the case of "+" in (15); the "−" case can be handled similarly. Consider a rectangle with one side equal to $\frac{k}{l}$, and such that the other side consists of two concatenated segments of lengths $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$, respectively. This gives rise to a partition of the rectangle into two smaller rectangles as shown.



Then the distributive law (15) is merely the statement that the area of the big rectangle (left side of (15)) is equal to the sum of the areas of the two smaller rectangles (right side of (15)).

As an application of the distributive law, consider $181\frac{1}{6} \times \frac{3}{7}$. It is equal to

$$(181 + \frac{1}{6}) \times \frac{3}{7} = \frac{543}{7} + \frac{1}{14} = \frac{1086 + 1}{14} = 77\frac{9}{14}$$

Of course one may also do this calculation without appealing to the distributive law for fractions:

$$181\frac{1}{6} \times \frac{3}{7} = \frac{1087}{6} \times \frac{3}{7} = \frac{1087}{14} = 77\frac{9}{14}$$

At this point, we can bring closure to some of the discussions in §§3 and 4. With formula (14) at our disposal, the proof of the cancellation law (2) for fractions is now immediate:

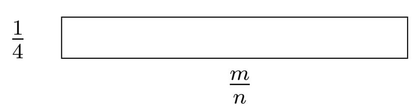
$$\frac{km}{lm} = \frac{k}{l} \times \frac{m}{m} \overset{(1)}{=} \frac{k}{l} \times 1 = \frac{k}{l}.$$

Again, we emphasize that while many textbooks employ this argument to "prove" the equivalence of fractions, there is a marked difference between what is done there and what we have done here. In those textbooks, the concept of multiplying fractions is taken as known even when the concept of a fraction is never defined, and the formula (14) is also just decreed as true without explanation. Thus their "proof" is based on unproven facts and undefined concepts. Such "proofs" are likely to corrupt students' conception of what mathematical reasoning is about. By contrast, we devoted a great deal of effort in §§2 and 4 of Chapter 1 and in §1 of this chapter to lay the necessary groundwork for the precise statement and proof of (14). The superficial similarity does not tell the whole story.

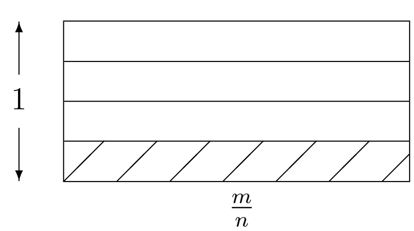
Next we want to give a different interpretation of the multiplication of fractions that will prove to be useful in many contexts. We begin with a special case. If k, m, n are nonzero whole numbers, then

$$\frac{1}{k} \times \frac{m}{n} \;=\; \begin{array}{l}\text{the length of a part when a segment of length} \\ \frac{m}{n} \text{ is divided into } k \text{ equal parts}\end{array} \qquad (16)$$

We give two proofs of (16). First, a geometric one. For the ease of drawing figures below, it would be best if we let k be a concrete number, say 4. It will be seen that the general reasoning is exactly the same. We begin by letting the unit 1 be the area of the unit square. Such being the case, $\frac{1}{4} \times \frac{m}{n}$ is the area of the rectangle:



Now we stack 4 of these rectangles vertically, obtaining the following:



The shaded rectangle is the original one whose area is $\frac{1}{4} \times \frac{m}{n}$. Moreover, the shaded area is also a part of the division of the big rectangle into 4 parts of equal area. Because the area of the big rectangle is $1 \times \frac{m}{n} = \frac{m}{n}$, this shows that $\frac{1}{4} \times \frac{m}{n}$ is the area of a part when $\frac{m}{n}$ is divided into 4 parts of equal area. This proves (16).

The second proof is algebraic, and it makes use of an idea already used in §6. Let $A = \frac{1}{k} \times \frac{m}{n}$. We are going to add A to itself k times and apply the distributive law for fractions (15). But first a general observation: Suppose A is any fraction and for some fraction $\frac{m}{n}$, we have

$$A + A = \frac{m}{n},$$

then by the definition of the addition of fractions, A is the length of a part when (a segment of length) $\frac{m}{n}$ is divided into two equal parts. Next, suppose

$$A + A + A = \frac{m}{n},$$

then A is the length of a part when $\frac{m}{n}$ is divided into three equal parts. More generally, if

$$\underbrace{A + A + \cdots + A}_{k} = \frac{m}{n}$$

for some whole number k, then A is the length of a part when $\frac{m}{n}$ is divided into k equal parts.

Now we return to the case at hand, where $A = \frac{1}{k} \times \frac{m}{n}$. By the distributive law (15):

$$\begin{aligned}
\underbrace{A + A + \cdots + A}_{k} &= \frac{1}{k} \times \underbrace{\left(\frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n}\right)}_{k} \\
&\overset{(7)}{=} \frac{1}{k} \times \frac{km}{n} \\
&\overset{(14)}{=} \frac{1}{k} \times \left(k \times \frac{m}{n}\right) \\
&= (\frac{1}{k} \times k) \times \frac{m}{n} = 1 \times \frac{m}{n} \\
&= \frac{m}{n}
\end{aligned}$$

This exhibits $\frac{m}{n}$ as k copies of A, so that A is length of a part when a segment of length $\frac{m}{n}$ is divided into k equal parts. But by definition, $A = \frac{1}{k} \times \frac{m}{n}$, so (16) is proved.

> We remark once again that although the associative law for fraction multiplication was not explicitly mentioned in the preceding proof, it was nevertheless used in the step: $\frac{1}{k} \times (k \times \frac{m}{n}) = (\frac{1}{k} \times k) \times \frac{m}{n}$. It should be taken for granted by now that the basic laws of operations that we discussed in §2 of Chapter 1 indeed play a crucial role in almost all our computations, whether concrete or symbolic. This role will be in fact dominant in the next subsection, §7.2.

Finally, we point out a connection between (3) and fraction multiplication. We have:

$$\frac{l}{k} = \frac{1}{k} \times l$$

By (16), the right side is the length of a part when $[0, l]$ is divided into k equal parts. Thus this equality says precissely the same thing as (3).

A noteworthy consequence of (16) is the fact that if B is a fraction (and not just a whole number), then $\frac{1}{k} \times B$ is — in the partitive sense of division described in §3.4 — what "B divided by k" *ought* to mean. In other words, what (16) says is that

> "divide by a whole number k" in the intuitive sense of division is correctly expressed by "multiply by $\frac{1}{k}$".

We now give an interpretation of fraction multiplication that will be useful in §7.3. Let k, ℓ, m, n by nonzero whole numbers. Then by (16),

$$\begin{aligned} \frac{k}{\ell} \times \frac{m}{n} &= k \times \left(\frac{1}{\ell} \times \frac{m}{n}\right) \\ &= \text{the length of } k \text{ concatenated segments} \\ &\quad \text{when } [0, \frac{m}{n}] \text{ is divided into } \ell \text{ segments} \\ &\quad \text{of equal length} \end{aligned} \tag{17}$$

We can go further. Suppose we use $\frac{m}{n}$ as a unit on the number line. Then we get a new number line where the unit 1 is $\frac{m}{n}$, 2 is $\frac{m}{n} + \frac{m}{n} = \frac{2m}{n}$, 3 is $\frac{m}{n} + \frac{m}{n} + \frac{m}{n} = \frac{3m}{n}$, etc. To avoid confusion, we shall refer to this new 1 as the *new unit*. In terms of the new unit, $\frac{1}{\ell}$ is the length of a part when a segment of length $\frac{m}{n}$ is divided into ℓ equal parts. By (16), such a part has length $\frac{1}{\ell} \times \frac{m}{n}$. Consequently, $\frac{k}{\ell}$ in terms of the new unit is $\frac{k}{\ell} \times \frac{m}{n}$, by (17). We therefore have:

Given $\frac{m}{n}$, let a new unit be chosen to be $\frac{m}{n}$. Then with respect to the new unit, a fraction $\frac{k}{\ell}$ is exactly the length of $\frac{k}{\ell} \times \frac{m}{n}$

We conclude with the fraction analogues of the inequalities in (11) at the end of §2 in Chapter 1. First, recall that if A and B are fractions, then $A < B$ means A is to the left of B on the number line (see §5).

But this means the line segment $[0, B]$ is the concatenation of the segment $[0, A]$ and the segment $[A, B]$ from A to B. In terms of the definition of the addition of fractions, this means $A + C = B$, where C is the length of $[A, B]$. Therefore, for two fractions A and B,

$A < B$ *is the same as* $B = A + C$ *for some nonzero fraction* C.

Now let the numbering be a continuation of that in §6. Again, let A, B, C, D be fractions.

(D) If $A \neq 0$, $AB < AC$ is the same as $B < C$.

(E) $A < B$ and $C < D$ imply $AC < BD$.

The proofs are left as exercises.

Exercise 7.1 Verify (15) directly by expanding both sides using (8) and (14).

Exercise 7.2 Use a calculator to do the whole number computations if necessary (and *only* for that purpose), compute: (a) $4\frac{2}{9} \times 6\frac{11}{13} = ?$ (b) $15\frac{4}{17} \times 23\frac{9}{25} - 16\frac{8}{19} \times 15\frac{4}{17} = ?$ (c) $2\frac{7}{8} \times 14\frac{4}{5} \times 3\frac{1}{6} = ?$

Exercise 7.3 Prove the inequalities (D) and (E) for fractions.

Exercise 7.4 Without using (14), explain directly to a sixth grader why $\frac{3}{7} \times \frac{4}{5} = \frac{12}{35}$.

Exercise 7.5 A small rectangle with sides $1\frac{2}{3}$ and $2\frac{1}{7}$ is contained in a larger rectangle with sides $12\frac{1}{3}$ and $6\frac{2}{5}$. Find the area of the region between these rectangles. (Use a four-function calculator.)

Exercise 7.6 (a) A rectangle has area 6 and a side of length $\frac{1}{3}$. What is the length of the other side? (b) A rectangle has area $3\frac{1}{3}$ and one side of length $\frac{2}{3}$. What is the length of the other side? (c) A rectangle has area $\frac{7}{8}$ and one side of length $1\frac{1}{3}$. What is the length of the other side?

Exercise 7.7 Prove the inequalities in (D) and (E) above Exercise 7.1.

Exercise 7.8 [This is Exercise 4.7 in §4. Now do it again using the concept of fraction multiplication.] James gave a riddle to his friends: "I was on a hiking trail, and after walking $\frac{7}{12}$ of a mile, I was $\frac{5}{9}$ of the way to the end. How long is the trail?" Help his friends solve the riddle.

7.2 The first alternative approach

[The content of this section will not be needed until Chapter 5.]

We will now sketch a purely *algebraic* approach to the multiplication of fractions. In reading this subsection, you are asked to pretend that you have never heard of what is in §7.1, and you would start all over again assuming only what we have done in §§1–6.

Let us first reflect on what we have done in regard to the multiplication of fractions: we have given a precise definition of multiplying two fractions by extending, in a *geometric* setting, what we have come to understand about the multiplication of two whole numbers. Then we derived formula (14) on the basis of this definition. What we propose to do next is to imitate this procedure, except that we replace the geometry with algebra. We look at what we usually do in the context of the multiplication of whole numbers, and ask what would happen if we insist that the laws of operations in §2 of Chapter 1 continue to hold not only for whole numbers but also for fractions. It turns out that by pursuing this line of thinking, we manage to deduce formula (14). This then gives us assurance that, far from a random congregation of symbols, formula (14) is in fact dictated *a priori* by the general reasoning of algebra. So we adopt it as the definition of fraction multiplication.

Let k be a whole number and let $\frac{m}{n}$ be a fraction. We first consider the question of what a reasonable definition of $k \times \frac{m}{n}$ should be. We know that if p is a whole number, then

$$kp = p + p + \cdots + p \qquad (k \text{ times})$$

i.e., multiplication of p by k is just repeated addition k times. *Suppose* we believe that this should be true even when p is a fraction, then we would *define*:

$$k \times \frac{m}{n} = \frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n} \quad (k \text{ times}).$$

Since

$$\frac{m + m + \cdots + m}{n} = \frac{km}{n}.$$

we have:

$$k \times \frac{m}{n} = \frac{km}{n} \tag{18}$$

Note in particular that (18) implies

$$1 \times \frac{m}{n} = \frac{m}{n}.$$

Next, consider what should be a reasonable definition of $\frac{1}{\ell} \times \frac{m}{n}$ for a nonzero whole number ℓ. In order for this discussion to go forward, it will be necessary to assume that $\frac{1}{\ell} \times \frac{m}{n}$ is also a fraction, and furthermore that, just as with multiplication among whole numbers,

MULTIPLICATION OF FRACTIONS OBEYS THE ASSOCIATIVE LAW.

There should be no misunderstanding of what we are doing: we don't know what it means to multiply fractions yet, but as a matter of faith and habit, we are going to *assume* that it must be associative. This is one reason why we saw fit to discuss the associative law back in §2 of Chapter 1, because at this point, this law dictates how our mathematical thinking should proceed. In any case, we can now assert that

$$\ell \times (\frac{1}{\ell} \times \frac{m}{n}) = (\ell \times \frac{1}{\ell}) \times \frac{m}{n} = \frac{\ell}{\ell} \times \frac{m}{n} = 1 \times \frac{m}{n} = \frac{m}{n}.$$

We have made use of (18) to conclude that $\ell \times \frac{1}{\ell} = \frac{\ell}{\ell}$. Therefore we have:

$$\ell \times \left\{ \frac{1}{\ell} \times \frac{m}{n} \right\} = \frac{m}{n} \tag{19}$$

for any nonzero whole number ℓ and for any fraction $\frac{m}{n}$. On the other hand, by (18),

$$\ell \times \frac{m}{\ell n} = \frac{\ell m}{\ell n} = \frac{m}{n} \ ,$$

by the cancellation law of fractions (2). Thus, we have:

$$\ell \times \left\{ \frac{m}{\ell n} \right\} = \frac{m}{n} \tag{20}$$

again for any nonzero whole number ℓ and for any fraction $\frac{m}{n}$.

Comparing (19) and (20), we conclude that the only reasonable way to define $\frac{1}{\ell} \times \frac{m}{n}$ in general is to say

$$\frac{1}{\ell} \times \frac{m}{n} = \frac{m}{\ell n} \tag{21}$$

We now make one more assumption:

MULTIPLICATION OF FRACTIONS OBEYS THE COMMUTATIVE LAW.

As with associativity, this assumption is also purely an article of faith. Then (18) can now be rewritten as:

$$\frac{m}{n} \times k = \frac{mk}{n} \tag{22}$$

for any whole number k and for any fraction $\frac{m}{n}$.

We can now put everything together and deduce how fractions should be multiplied in general:

$$\begin{aligned}
\frac{k}{\ell} \times \frac{m}{n} &= (\frac{1}{\ell} \times k) \times \frac{m}{n} && \text{(by (22))} \\
&= \frac{1}{\ell} \times (k \times \frac{m}{n}) && \text{(associcative law)} \\
&= \frac{1}{\ell} \times \frac{km}{n} && \text{(by (18))} \\
&= \frac{km}{\ell n} && \text{(by (21))}
\end{aligned}$$

That is:

$$\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n} .$$

To summarize, what we have shown is that *if we believe* that

> (i) multiplication of a fraction by a whole number k should behave as in the case of whole numbers, i.e., adding the fraction to itself k times, and
> (ii) mutiplication of fractions is associative, and
> (iii) mutiplication of fractions is commutative,

then fractions $\frac{k}{\ell}$ and $\frac{m}{n}$ can only be multiplied in the following way:

$$\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n} ,$$

and we are back to (14) again.

From our experience with whole numbers, there is no reason not to believe that (i)–(iii) *above are totally reasonable.* On this basis, we adopt this formula (i.e., (14)) as our *definition* of fraction multiplication.

So the grand conclusion is that there are good reasons to adopt (14) as our definition of the multiplication of fractions. Where then do we stand at this point? We must start from the beginning and use (14) to rederive all the usual properties of multiplication: commutativity, associativity, and ditributivity. All this is entirely mechanical and will be left as an exercise.

We further note that the following obviously holds for all fractions $\frac{m}{n}$ and all whole numbers k and ℓ:

$$\begin{aligned} 1 \times \frac{m}{n} &= \frac{m}{n} \\ k \times \frac{m}{n} &= \frac{km}{n} = \left(\frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n}\right) \qquad (k \text{ times}) \\ \frac{1}{\ell} \times m &= \frac{m}{\ell} \end{aligned}$$

The mathematical discussion of the multiplication of fractions can now proceed as before.

To recapitulate: We have arrived at the usual formula of fraction multiplication by a fairly elaborate algebraic process. It is well to repeat that one should never use (14) as a definition of fraction multiplication without some prior discussion of what fraction multiplication means. Either of the preceding two approaches, or the next one in §7.3, is acceptable. They all make it abundantly clear that what we do in fractions is nothing but a natural extension of what we do in whole numbers. One cannot exaggerate the importance of explaining to students that, *conceptually*, there is no difference between whole numbers and fractions.

Exercise 7.9 Using (14) as the definition of multiplying fractions, show that fraction multiplication is commutative, associative, and distributive.

Exercise 7.10 Use (14) to derive the preceding formulas for $1 \times \frac{m}{n}$, $k \times \frac{m}{n}$, and $\frac{1}{\ell} \times m$.

7.3 The second alternative approach

This section may be skipped on first reading.

What we are going to do is to make use of the interpretation (17) at the end of §7.1 as a starting point to introduce the definition of fraction mulriplication. We are trying to make sense of fraction multiplication. Fix a fraction $\frac{m}{n}$, what could $\frac{k}{l} \times \frac{m}{n}$ mean for any fraction $\frac{k}{l}$? Conceptually, what we are going to do is to introduce a new unit $\overline{1}$ on the number line so that

$$\overline{1} = \frac{m}{n}.$$

Denoting the corresponding numbers on the number line by a bar, e.g., $\overline{\frac{k}{l}}$, then $\frac{k}{l} \times \frac{m}{n}$ will turn out to be exactly $\overline{\frac{k}{l}}$. See the discussion in §4 of chapter 1 above (48). However, a presentation along this line turns out to be a notational nightmare. So we will adopt a more down-to-earth approach instead and just stay with the original number line. So we define:

$$\frac{k}{l} \times \frac{m}{n} \;=\; \text{the length of } k \text{ parts of a division when } [0, \tfrac{m}{n}] \text{ is divided into } l \text{ equal parts.} \qquad (23)$$

We reiterate that this definition is completely modeled on (17). Let us denote the length of a part of the division of $[0, \frac{m}{n}]$ into l equal parts by $\frac{p}{q}$. Then by definition,

$$\frac{k}{l} \times \frac{m}{n} = \frac{kp}{q}. \tag{24}$$

By the definition of $\frac{p}{q}$, we have

$$\underbrace{\frac{p}{q} + \cdots + \frac{p}{q}}_{l} = \frac{m}{n}.$$

But the sum on the left side is just lp/q. So we have

$$\frac{lp}{q} = \frac{m}{n}.$$

But we can see by inspection that letting $p = m$ and $q = ln$, we would have

$$\frac{lp}{q} = \frac{lm}{ln} = \frac{m}{n},$$

by virtue of the cancellation law for fractions. Therefore

$$\frac{p}{q} = \frac{m}{ln}.$$

It follows from (24) that

$$\frac{k}{l} \times \frac{m}{n} = \frac{km}{ln}.$$

We have therefore derived formula (14) from the definition of what the product of fractions should be as given in (23).

The one advantage of this approach is that the interpretation (23) of $\frac{k}{\ell} \times \frac{m}{n}$ is built into the definition. This meaning of a product of fractions corresponds directly to the usual meaning of the usuage of “a fraction of”, as in “two thirds of the audience booed, while the other third cheered wildly,” or “two-fifths of a class of n students”. For this reason, some people would find this definition to be the most appealing approach among the three presented here.

8 Division of Fractions

In order to understand the division of fractions, we have to understand an important feature which distinguishes fractions from whole numbers. Having put them on the same conceptual footing, we are now looking for differences. In terms of addition, subtraction, and multiplication, we have observed no differences between the two, at least operationally. But of course they are different, and we are going to showcase the critical difference by exhibiting a multiplication problem which is always solvable in fractions but rarely in whole numbers. It is the following

Given any two fractions A and B, $B \neq 0$, then there is a unique fractions C so that $A = BC$. (25)

The truth of (25) is easy to demonstrate: If $A = \frac{k}{l}$ and $B = \frac{m}{n}$, we can simply write down a C that satisfies $A = BC$, namely, $C = \frac{nk}{ml}$. In so doing, we have made use of the assumption that $B \neq 0$ because $B \neq 0$ implies $m \neq 0$ so that the denominator of C is indeed nonzero. As to the uniqueness of C, we have to show that if D is another fraction that satisfies $A = BD$, then necessarily $D = C$. But from $A = BD$ we get $\frac{k}{l} = \frac{m}{n} \times D$. Multiplying both sides by $\frac{n}{m}$ shows immediately that $\frac{nk}{ml} = \frac{n}{m} \times \frac{k}{l} = \frac{n}{m} \times (\frac{m}{n} \times D) = (\frac{n}{m} \times \frac{m}{n}) \times D = D$ so that $D = \frac{nk}{ml} = C$. (Notice that we made use of $B \neq 0$ a second time to be able to write the fraction $\frac{n}{m}$.)

A passing remark about the uniqueness part of (25) may be appropriate. In view of the cancellation law of fractions (§3), it must be understood that the assertion of the equality of C and D in the preceding paragraph means only that C and D are the same point on the number line, but not that they have equal numerators and denominators. For example, if $C = \frac{2}{5}$ and $D = \frac{36}{90}$, then $C = D$.

A few additional comments may also shed some light on both (25) and its solution. The fact that we can write down the fraction $C = \frac{nk}{ml}$ is due to the assumption of $B \neq 0$, as already noted above. It is standard terminlogy to call $\frac{n}{m}$ the (*multiplicative*) *inverse* of $B = \frac{m}{n}$, and denote it by B^{-1}. Thus $BB^{-1} = B^{-1}B = 1$ and

the solution C of $A = BC$ in (25) is given by $C = AB^{-1}$.

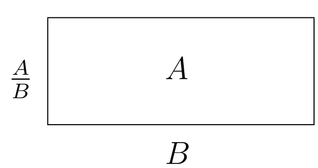
(In §5, we also introduced the terminolgy that $\frac{n}{m}$ is the reciprocal of $\frac{m}{n}$. Thus for a fraction, its inverse happens to be its reciprocal.) Second, (25) is false if $B = 0$. For if we chose a nonzero A, then we would get a contradiction because $A = BC = 0 \times C = 0$. Finally, and this is the most important, the anlogue of (25) for whole numbers is almost always false. Let us begin by giving a precise statement of this analogue:

$$\text{Given any two whole numbers } A \text{ and } B,\ B \neq 0, \text{ then there is a unique whole number } C \text{ so that } A = BC. \tag{26}$$

This would be a true statement if A happens to be a multiple of B, but false otherwise. To see the latter, take for instance $A = 5$ and $B = 2$, or $A = 27$ and $B = 4$.

One interpretation of the failure of (26) is that it poses the following mathematical problem: *Given any two whole numbers A and B, $B \neq 0$, when is there a "number" C so that $A = BC$?* From this point of view, fractions are the "numbers" we must add to whole numbers in order to secure an affirmative answer to this question. Indeed, we can get all fractions whose denominators are 2 by letting $B = 2$ and letting A be successively 1, 2, 3, $\ldots$. Because then the requisite C's are exactly $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, $\ldots$ Next, if $B = 3$ and A is successively 1, 2, 3, $\ldots$ then we get $C = \frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$ $\ldots$ In this way we get all the fractions.

Now we return to the consideration of the division of two fractions. Recall from §4 (especially (49)) of Chapter 1 that if A, B are two whole numbers, $B \neq 0$, then $\frac{A}{B}$ is the other side of a rectangle whose area is A and one of whose sides is B.[16]



Notice that we have now allowed $\frac{A}{B}$ to be a fraction, so that it is no longer necessary to require, as we did in §4 of Chapter 1, that the whole number A divides the whole numnber B. In any case, in our quest to put whole numbers and fractions on an equal footing, we are naturally led to define the division of fractions in exactly the same manner:

[16] Recall that we agreed henceforth to denote $A \div B$ by A/B.

> GIVEN TWO FRACTIONS A AND B, WITH $B \neq 0$, THE *quotient* $\frac{A}{B}$ (OR A/B) IS THE LENGTH OF THE OTHER SIDE OF A RECTANGLE WHOSE AREA IS A AND ONE OF WHOSE SIDES IS B.

Because the area of a rectangle is the product of its sides, we see that by definition, $\frac{A}{B}$ satisfies:

$$A = \frac{A}{B} \times B$$

There is a subtle point here: We have defined $\frac{A}{B}$ as *the* fraction with the preceding property. How do we know that there *is* such a fraction, and why is it unique? Fortunately, (25) answers both questions simultaneously in the affirmative, so we know the division $\frac{A}{B}$ is always possible provided $B \neq 0$.

We can be more explicit about the division of fractions: Given $A = \frac{k}{l}$ and $B = \frac{m}{n}$ with $m \neq 0$, the reasoning leading to (25) tells us that $\frac{A}{B} = \frac{nk}{ml}$. Hence, our definition of the division of fractions may be presented in an essentially equivalent formulation:

$$\frac{\frac{k}{l}}{\frac{m}{n}} = \frac{kn}{lm} \tag{27}$$

The right side is more easily remembered as $\frac{k}{l} \times (\frac{m}{n})^{-1}$. This is of course the famous "invert and multiply" rule, but to us, this is not a rule we adopt blindly, but rather one that is logically deduced from understanding what "division" means.

Note an interesting side-light: the notation $A = B \times \frac{A}{B}$ "suggests" that we obtain A by "cancelling" B on the right side. It is a virtue of the notation that it would suggest the correct answer, but let it be noted that no cancellation ever took place because the only cancellation we know about at this point applies only to the case where A and B are *whole numbers* (see §§3 and 7). Here, A and B are fractions, and the cancellation law applicable to this situation has yet to be formally proved (see (b) of §9 below).

The special case of (27) where $k = l = m = 1$ is of particular interest: it says

$$\frac{1}{1/n} = n$$

It can be interpreted as follows: Suppose we have a water tank with capacity T gallons and a bucket with capacity b gallons. Then naturally the number of buckets of water needed to fill the tank is T/b. Now suppose the tank capacity is 1 gallon, *i.e.,* $T = 1$, and the bucket capacity is $\frac{1}{n}$ of a gallon, *i.e.,* $b = \frac{1}{n}$. Then the number of buckets of water needed to fill the tank is $\dfrac{1}{1/n}$. However, this number can be computed directly: if $n = 2$ (bucket holds half a gallon of water), it takes 2 buckets to fill the tank, if $n = 3$ (bucket holds a third of a gallon of water), it takes 3 buckets to fill the tank, etc,. and for a general n the same reasoning shows that it takes n buckets to fill the tank. Thus, $\dfrac{1}{1/n} = n$, exactly as predicted by our elaborate definition of the division of fractions.

Note that the preceding is a very limited interpretation of the division of fractions in the measurement sense (see §3.4 of Chapter 1). We now proceed to broaden this discussion. But first, let us take note of the fact that currently there is a fascination with the drawing of pictures to "make sense" of the measurement interpretation of the division of fractions. The following is a typical example:

> A rod $7\frac{5}{6}$ meters long is cut into pieces which are $\frac{4}{3}$ meters long.
> How many short pieces are there?

The denominators are usually rigged so that they can be easily brought to be the same, in this case, $7\frac{5}{6} = \frac{47}{6}$ while $\frac{4}{3} = \frac{8}{6}$, then pictures can be drawn to show directly that because $47 = 5 \times 8 + 7$, there are 5 short pieces with $\frac{7}{8}$ of a piece left over. Now, given all the work we have done, how do we handle problems of this nature? Indeed, while pictures are always useful, we can do much harder problems of this type quite effortlessly. For example:

> A rod $15\frac{5}{7}$ meters long is cut into pieces which are $2\frac{1}{8}$ meters long.
> How many short pieces are there?

We recognize roughly that this is a division problem, because if $15\frac{5}{7}$ and $2\frac{1}{8}$ are replaced by whole numbers 15 and 3 (say), then this would indeed be a division problem in the measurement sense of division. We shall presently explain *precisely* why division gives the correct answer. We have

$$\frac{15\frac{5}{7}}{2\frac{1}{8}} = \frac{\frac{110}{7}}{\frac{17}{8}} \overset{(27)}{=} \frac{880}{119} = 7\frac{47}{119}\ .$$

Our answer is that there are 7 short pieces with $\frac{47}{119}$ of a short piece left over. The reason is simple. We understand the meaning of division: the fact that

$$\frac{15\frac{5}{7}}{2\frac{1}{8}} = 7\frac{47}{119}$$

means exactly that

$$\begin{aligned} 15\frac{5}{7} &= 7\frac{47}{119} \times 2\frac{1}{8} \\ &= \left(7 + \frac{47}{119} \times 2\frac{1}{8}\right) \\ &= \left(7 \times 2\frac{1}{8}\right) + \left(\frac{47}{119} \times 2\frac{1}{8}\right) \ . \end{aligned}$$

That is,

$$15\frac{5}{7} = \left(7 \times 2\frac{1}{8}\right) + \left(\frac{47}{119} \times 2\frac{1}{8}\right) \ .$$

But this is precisely the statement that the rod $(15\frac{5}{7})$ is the sum (concatenation) of 7 copies of the short piece (i.e., $7 \times 2\frac{1}{8}$), and the remaining piece which is only $\frac{47}{119}$ of the short piece (i.e., the last term $\frac{47}{119} \times 2\frac{1}{8}$).

Remark: As in the case of multiplications, it is good to remind students that *dividing one fraction by another does not necessarily make the first fraction smaller*, e.g., $2/\frac{1}{5} = 10$.

In §3.4 of Chapter 1, we discussed the partitive and measurement interpretations of the division of whole numbers. Let us recast that discussion in the broader context of the division of fractions. The preceding example concerning rods illustrates very well how the division of fractions can be interpreted in the measurement sense. In general, the division of fractions, as defined here, can also be given a partitive interpretation. To see this, let us revisit the motion problem first discussed at the end of §3.4 of Chapter 1.

Recall that for simplicity, we assume the constancy of speed throughout the subsequent discussion. So suppose we embark on a full-day hike to the beach starting from some park headquarters. The Bear Valley Trail is $12\frac{1}{3}$ miles. When we start off in the morning, we count on the ability to maintain a brisk pace of $3\frac{1}{2}$ miles an hour, and we want to compute how long it would take us to get to the beach. As discussed in §3.4 of Chapter 1, this is a measurement division problem because we want to know how many $3\frac{1}{2}$'s there are in $12\frac{1}{3}$. What we claim is that the way the division of fraction is defined here automatically gives us a measurement interpretation. To see this, we invert and multiply according to (27) to get:

$$\frac{12\frac{1}{3}}{3\frac{1}{2}} = \frac{\frac{37}{3}}{\frac{7}{2}} = \frac{37}{3} \times \frac{2}{7} = 3\frac{11}{12}.$$

Now, recall that this division fact is by definition equivalent to

$$12\frac{1}{3} = 3\frac{11}{21} \times 3\frac{1}{2},$$

which may be rewritten as:

$$12\frac{1}{3} = 3\frac{11}{21} \times 3\frac{1}{2} = \left(3 + \frac{11}{21}\right) \times 3\frac{1}{2} = 3 \times \left(3\frac{1}{2}\right) \; + \; \frac{11}{21} \times \left(3\frac{1}{2}\right).$$

This is an explicit statement that $12\frac{1}{3}$ contains 3 of $3\frac{1}{2}$'s plus a leftover of $\frac{11}{21}$ of $3\frac{1}{2}$. Therefore the hike to the beach will take 3 and $\frac{11}{21}$ of an hour, or roughly 3 hours and 31 minutes.

On the way back from the beach to the park headquarters, we take it easy. We leave at 4 pm and get back at 8:45 pm. What is our speed? So we want the speed of a hike that takes $4\frac{3}{4}$ hours to cover $12\frac{1}{3}$ miles, i.e., the number of miles covered in any one hour interval. This is a partitive division problem because: if the speed is M miles per hour, then in 4 hours we cover $4M$ miles, and in $\frac{3}{4}$ hours we cover an additional $\frac{3}{4}M$ miles, and $4M$ and $\frac{3}{4}M$ fit into $12\frac{1}{3}$ in the sense that $12\frac{1}{3} = 4M + \frac{3}{4}M$. However, this implies $12\frac{1}{3} = (4 + \frac{3}{4})M = 4\frac{3}{4} \times M$. By the definition of fraction division,

$$M = \frac{12\frac{1}{3}}{4\frac{3}{4}}.$$

So again we can obtain the speed M by invert-and-multiply:

$$M = \frac{37}{3} \times \frac{4}{19} = 2\frac{34}{57}.$$

The speed of hike back is therefore $2\frac{34}{57}$ miles an hour, which is not quite 3 miles an hour.

For this particular case, there is another way to look at the partitive division that is equally interesting. Observe that $4\frac{3}{4}$ hours is the same as 19 quarter-hours (15 minutes). Thus we may think of the speed of walking $12\frac{1}{3}$ miles in $4\frac{3}{4}$ hours as that of walking $12\frac{1}{3}$ miles in 19 quarter-hours. Assuming the constancy of the speed, we want to know how many miles are covered per quarter-hour, i.e., what is $12\frac{1}{3}/19$? Thus we have a straightforward partitive division problem of dividing $12\frac{1}{3}$ into 19 equal parts. By invert-and-multiply,

$$\frac{12\frac{1}{3}}{19} = \frac{\frac{37}{3}}{19} = \frac{37}{57}.$$

Thus we walked a steady $\frac{37}{57}$ miles per quarter-hour. The speed per hour is therefore

$$4 \times \frac{37}{57} = \frac{148}{57} = 2\frac{34}{57}.$$

It is of some interest to look at another approach to the division of fractions that is commonly offered. It goes as follows. Given $\frac{k/l}{m/n}$, we use the equivalence of fractions to get

$$\frac{\frac{k}{l}}{\frac{m}{n}} = \frac{\frac{k}{l} \times ln}{\frac{m}{n} \times ln} = \frac{\frac{kln}{l}}{\frac{mln}{n}} = \frac{kn}{lm}\,,$$

and therefore (so the argument goes) it is a valid mathematical fact that

$$\frac{\frac{k}{l}}{\frac{m}{n}} = \frac{kn}{lm}\,.$$

Now the conclusion is superficially consistent with (27) as both conclude that the invert-and-multiply rule is correct, but the flaws in this approach are subtle. Consider the first step:

$$\frac{\frac{k}{l}}{\frac{m}{n}} = \frac{\frac{k}{l} \times ln}{\frac{m}{n} \times ln} \tag{28}$$

Does it make sense? Not at all, because recall that *we are trying to find out what $\frac{k/l}{m/n}$ means*. If we don't know what it is, how can we compute with it? By the same token, we also do not know what $\frac{k/l \times ln}{m/n \times ln}$ means. Therefore, to assert the equality of two things we know not the meaning of is fantasy and not mathematics. This is the mathematical equivalent of the statement that "we saw two angels, one in a red garb and the other in blue, and we measured their heights and found them to be the same". Moreover, the reason for the equality in (28) is usually given as "using equivalent fractions", but all we know about equivalent fractions in §3 is that $\frac{K}{L} = \frac{KM}{LM}$ where K, L, M are *whole numbers*, not fractions. Therefore this approach heaps logical difficulty upon logical difficulty. It is by virtue of ignoring these difficulties that the preceding reasoning succeeds in presenting (27) as a mathematically proven fact rather than as a definition.

The preceding approach can be amended to make it valid. One should not present the preceding reasoning as a sequence of logical deductions based on known facts, but rather (along the line of reasoning of the algebraic approach to the multiplication of fractions in §7.2) as speculations based on certain hypotheses. For example, one of these hypotheses must be that the cancellation law $\frac{K}{L} = \frac{KM}{LM}$ has meaning and is valid even when K, L, and M are fractions. The hypothesis is plausible. Thus the argument should be rephrased as one that shows that, under reasonable assumptions on the behavior of fractions, (27) is the only possible *definition* to use for the division of fractions.

Of course, once the division of fractions has been precisely defined, then (b) of the next section (§9) would give an after-the-fact justification of this way of thinking about division.

There is a variant of the preceding approach, and it goes as follows:

$$\frac{\frac{k}{l}}{\frac{m}{n}} = \frac{\frac{k}{l} \times 1}{\frac{m}{n} \times 1} = \frac{\frac{k}{l} \times \frac{n}{n}}{\frac{m}{n} \times \frac{l}{l}} = \frac{\frac{kn}{ln}}{\frac{lm}{ln}} = \frac{kn \times \frac{1}{ln}}{lm \times \frac{1}{ln}},$$

and thus the division $\frac{k/l}{m/n}$ becomes the division of kn *new units* (where the new unit is $\frac{1}{ln}$) divided by lm of the same new unit. Naturally, the result is $\frac{kn}{lm}$. Therefore,

$$\frac{\frac{k}{l}}{\frac{m}{n}} = \frac{kn}{lm}.$$

Exercise 8.1 Let a, b be whole numbers, and let q and r be the quotient and remainder of $a \div b$. Let also Q be the fraction so that $a = Qb$. Determine the relationship among Q, q, and r.

Exercise 8.2 Analyze the preceding approach to the definition of the division of fractions, describe what the difficulties are, and make suggestions on how to make it into a a valid plausibility argument. (Keep in mind §7.2.)

Exercise 8.3 Recall that whole numbers are fractions. Now express the following as ordinary fractions for nonzero whole numbers a, b, and c: $a/\frac{b}{c}$, $\frac{a}{b}/c$, $12/\frac{4}{5}$, $\frac{4}{5}/8$, $5/\frac{1}{7}$, $\frac{1}{7}/5$.

Exercise 8.4 Shawna used to spend $\frac{2}{3}$ of an hour driving to work. Now that her firm has moved 12 miles farther from her home, she spends $\frac{5}{6}$ of an hour driving to work at the same speed. How far is her firm from her home?

Exercise 8.5 I drove from Town A to Town B at a constant speed of 50 mph, and I drove back from Town B to Town A at a faster speed of 60 mph. The roundtrip took $14\frac{2}{3}$ hours. How far apart are the towns.

Exercise 8.6 Find a fraction q so that $28\frac{1}{2} = q \times 5\frac{3}{4}$. Do the same for $218\frac{1}{7} = q \times 20\frac{1}{2}$. Make up a word problem for each situation.

Exercise 8.7 It takes 2 tablespoonfuls of a chemical to de-chlorinate 120 gallons of water. Given that 3 *tea*spoonfuls make up a tablespoonful, how many *tea*spoonfuls of this chemical are needed to de-chorinate 43 gallons?

9 Complex Fractions

We now come to a central topic in the study of fractions: the arithmetic operations with *complex fractions*, i.e., quotients of the form $\frac{a}{b}/\frac{c}{d}$, where $\frac{a}{b}$ and $\frac{c}{d}$ are fractions with $\frac{c}{d} \neq 0$. If truth be told, this topic is generally not regarded as being central in the study of fractions. In fact, most school textbooks on fractions hardly mention complex fractions, and even professional development materials hardly pause to make sense of them. But if you suspend your disbelief for a few moments and read through this section first, you will be able to verify for yourself that, indeed, complex fractions are of central importance.

Writing A for $\frac{a}{b}$ and B for $\frac{c}{d}$, we can abbreviate the complex fraction $\frac{a}{b}/\frac{c}{d}$ to $\frac{A}{B}$. There is a reason for this notation. First of all, let a and b be nonzero whole numbers and let A and B be the fractions $A = \frac{a}{1}$ and $B = \frac{b}{1}$. Because we know that $A = a$ and $B = b$ as fractions (cf. (1)), we expect the complex fraction $\frac{A}{B}$ to be equal to the ordinary fraction $\frac{a}{b}$. *And indeed it is*, according to the invert-and-multiply rule (27). So the notation is at least consistent with the existing notation. Moreover, recall that at the beginning of this chapter, our objective was to give meaning to the symbol $\frac{A}{B}$ when A and B were whole numbers. Now we have progressed to the point of being able to *extend* the meaning of the symbol to the case where A and B are not just whole numbers but fractions. Naturally, we wish to know how much of the knowledge accumulated for $\frac{A}{B}$ when A and B are whole numbers carries over to the case where A and B are fractions. The answer: *virtually everything*. This will therefore be a stunning display of the power of the symbolic notation, in that it makes possible a tremendous saving of mental energy by encoding two parallel developments using only one set of formualas. Make no mistake about it: this is good mathematics.

Now we proceed to list the basic rules concerning complex fractions adumbrated in the preceding paragraph. Let A, B, $\ldots F$ be fractions (which will be assumed to be nonzero in the event any of them appears in the denominator). In the following, we shall omit the multiplication symbol "$\times$" between letters, as usual. Thus $A \times B$ will be simply written as AB. With this understood, then the following are valid:

(a) $A \times \dfrac{B}{C} = \dfrac{AB}{C}$.

(b) *Cancellation law*: if $C \neq 0$, then

$$\frac{AC}{BC} = \frac{A}{B}$$

(c) $\dfrac{A}{B} > \dfrac{C}{D}$ (resp.,$\dfrac{A}{B} = \dfrac{C}{D}$) exactly when $AD > BC$ (resp., $AD = BC$).

(d) $\dfrac{A}{B} \pm \dfrac{C}{D} = \dfrac{(AD) \pm (BC)}{BD}$

(e) $\dfrac{A}{B} \times \dfrac{C}{D} = \dfrac{AC}{BD}$

(f) *Distributive law* :

$$\frac{A}{B} \times \left(\frac{C}{D} \pm \frac{E}{F}\right) = \left(\frac{A}{B} \times \frac{C}{D}\right) \pm \left(\frac{A}{B} \times \frac{E}{F}\right)$$

Here is the correspondence between the items on this list and their cognate facts in ordinary fractions:

$$\begin{array}{rll} (a) \leftrightarrow (14), & (b) \leftrightarrow (2), & (c) \leftrightarrow (4) \\ (d) \leftrightarrow (8) \text{ and } (12), & (e) \leftrightarrow (14), & (f) \leftrightarrow (15). \end{array}$$

The algebraic proofs of (a)–(f) are entirely mechanical and somewhat tedious, and are based on (14), (2), (4), (8), (12), and (15) (which will be used without comment in the following). For this reason, only the proofs of (a), (d) and (f) will be given for the purpose of illustration, and the proofs of the rest will be left as exercises. In a sixth grade classroom, one or two such proofs ought to be presented, but perhaps no more than that.

Proof of (a). Let $A = \frac{k}{l}$, $B = \frac{m}{n}$, and $C = \frac{p}{q}$. Then $\frac{B}{C} = \frac{mq}{np}$, so that

$$A \times \frac{B}{C} = \frac{kmq}{lnp}.$$

But

$$\frac{AB}{C} = \frac{\frac{km}{ln}}{\frac{p}{q}} = \frac{kmq}{lnp} = A \times \frac{B}{C},$$

so (a) is proved.

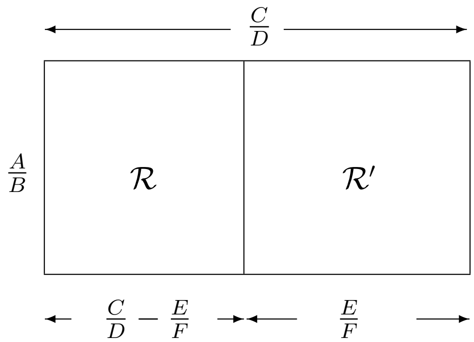
Proof of (d). With A, B, C as above and $D = \frac{r}{s}$, we have:

$$\frac{A}{B} \pm \frac{C}{D} = \frac{\frac{k}{l}}{\frac{m}{n}} \pm \frac{\frac{p}{q}}{\frac{r}{s}} = \frac{kn}{lm} \pm \frac{ps}{qr} = \frac{knqr \pm lmps}{lmqr},$$

and

$$\begin{aligned} \frac{(AD) \pm (BC)}{BD} &= \frac{\frac{kr}{ls} \pm \frac{mp}{nq}}{\frac{mr}{ns}} = \frac{\frac{(krnq) \pm (lsmp)}{lsnq}}{\frac{mr}{ns}} \\ &= \frac{ns[(krnq) \pm (lsmp)]}{mr(lsnq)} = \frac{krnq \pm lsmp}{mrlq} = \frac{A}{B} \pm \frac{C}{D}. \end{aligned}$$

Proof of (f). An algebraic proof can be given in a routine fashion as with (15), but it will be left as an exercise. However, a geometric proof is far more enlightening. For a change, we will handle the "−" case of (f) and leave the "+" case to the reader. Let $\mathcal{R}$ be a rectangle with one side equal to $\frac{A}{B}$ and the other side $\frac{C}{D} - \frac{E}{F}$. From the definition of $\frac{C}{D} - \frac{E}{F}$ (given above (12)), the other side of $\mathcal{R}$ is the remaining segment when a segment of length $\frac{E}{F}$ is removed from one of length $\frac{C}{D}$. See the figure below.



Let $\mathcal{R}_1$ be the rectangle with one side $\frac{A}{B}$ and the other side $\frac{C}{D}$ (the big rectangle), and $\mathcal{R}'$ the rectangle with one side $\frac{A}{B}$ and the other side $\frac{E}{F}$ (the smaller rectangle on the right). Then

$$\begin{aligned}\frac{A}{B} \times \left(\frac{C}{D} - \frac{E}{F}\right) &= \text{area of } \mathcal{R} = \text{area of } \mathcal{R}_1 - \text{ area of } \mathcal{R}' \\ &= \left(\frac{A}{B} \times \frac{C}{D}\right) - \left(\frac{A}{B} \times \frac{E}{F}\right) .\end{aligned}$$

EXAMPLE 1. If A, B, C are fractions and $B \neq 0$, then

$$\frac{A}{B} + \frac{C}{B} = \frac{A+C}{B}$$

(Compare (7) of §6.)

We can see this directly: let $A = \frac{k}{l}$, $B = \frac{m}{n}$, and $C = \frac{p}{q}$. Then,

$$\frac{A}{B} + \frac{C}{B} = \frac{kn}{lm} + \frac{pn}{qm} = \frac{(kn)q + (pn)l}{lqm} = \frac{n(kq+lp)}{lqm} ,$$

and

$$\frac{A+C}{B} = \frac{\frac{kq+lp}{lq}}{\frac{m}{n}} = \frac{n(kq+lp)}{lqm} = \frac{A}{B} + \frac{C}{B} .$$

We can also make use of (a) and (f):

$$\begin{aligned}\tfrac{A}{B} + \tfrac{C}{B} &\overset{(a)}{=} (\tfrac{1}{B} \times A) + (\tfrac{1}{B} \times C) \\ &\overset{(f)}{=} \tfrac{1}{B} \times (A+C) \overset{(a)}{=} \tfrac{A+C}{B}\end{aligned}$$

Or, we can use (d) and (b):

$$\frac{A}{B} + \frac{C}{B} \overset{(d)}{=} \frac{AB + CB}{BB} = \frac{(A+C)B}{BB} \overset{(b)}{=} \frac{A+C}{B}.$$

EXAMPLE 2. Let A and B be fractions and $B \neq 0$. Then

$$\underbrace{\frac{A}{B} + \cdots + \frac{A}{B}}_{j} = \frac{jA}{B}.$$

We can prove this directly, of course, but we choose to use Example 1 repeatedly:

$$\begin{aligned}\underbrace{\frac{A}{B} + \cdots + \frac{A}{B}}_{j} &= \frac{2A}{B} + \underbrace{\frac{A}{B} + \cdots + \frac{A}{B}}_{j-2} \\ &= \frac{3A}{B} + \underbrace{\frac{A}{B} + \cdots + \frac{A}{B}}_{j-3} = \cdots = \frac{jA}{B}.\end{aligned}$$

The importance of the algebraic operations of complex fractions has not been properly recognized in the K–12 mathematics curriculum. Let us illustrate in a simple way why the statements (a)–(f) are useful by considering the approximate size of $67\frac{1}{2}\%$ (here we anticipate the discussion of the next section on percentage; if neceesary, read the next section before reading the remainder of the this section). We recognize that $\frac{2}{3}$ is $66\frac{2}{3}\%$, and therefore *by common sense*, $67\frac{1}{2}\%$ should be roughly $\frac{2}{3}$. What we want to do is to replace "common sense" by correct reasoning to render our conclusion beyond reproach. We have

$$67\frac{1}{2} = 66\frac{2}{3} + \left(67\frac{1}{2} - 66\frac{2}{3}\right) = 66\frac{2}{3} + \frac{5}{6}.$$

Therefore,

$$\begin{aligned} 67\frac{1}{2}\% &= \frac{67\frac{1}{2}}{100} \\ &= \frac{66\frac{2}{3} + \frac{5}{6}}{100} \\ &= \frac{66\frac{2}{3}}{100} + \frac{\frac{5}{6}}{100} \qquad \text{(by (d))} \\ &= \frac{2}{3} + \frac{\frac{5}{6}}{100} \end{aligned}$$

Moreover, $\frac{5}{6} < 1$. By (c),

$$\frac{\frac{5}{6}}{100} < \frac{1}{100}.$$

Together, we see that $67\frac{1}{2}\%$ differs from $\frac{2}{3}$ by at most 1%. We have just seen, in a very superficial way perhaps, how (c) and (d) are put to work.

Complex fractions appear routinely in all kinds of situations, either in mathematics or in everyday life, and one needs to be able to compute with them with total ease. For example, again anticipating the discussion of percent in the next section and decimals in Chapter 4, we see immediately that a common statement such as “The Federal Reserve has announced that it would raise the prime interest rate from 7.2% to 7.7%” makes use of complex fractions, because 7.2% and 7.7% are nothing but shorthand notations for the complex fractions

$$\frac{7\frac{2}{10}}{100} \quad \text{and} \quad \frac{7\frac{7}{10}}{100}.$$

Furthermore, complex fractions are important from a a purely mathematical point of view because they are the bridge between ordinary fractions and general quotients such as $\frac{\pi}{\sqrt{2}}$. It will be seen that the assertions (a)–(f) above are crucial for an understanding of the algebraic operations with real numbers; see §11 following.

Exercise 9.1 Give algebraic proofs of (b), (c), (e) and (f).

Exercise 9.2 If $A = \frac{11}{5}$, $B = \frac{2}{7}$, $C = \frac{22}{21}$, $D = 2\frac{4}{5}$, $E = \frac{11}{7}$, and $F = \frac{5}{2}$, directly verify (a)–(e) above.

Exercise 9.3 If A, B, $\ldots E$ are fractions, then $\frac{A}{BC} + \frac{E}{BD} = \frac{AD+CE}{BCD}$. (This is the analogue of equation (11) in §6.)

10 “Of” and Percent

We are now in a position to explain two everyday expressions in terms of the precise language of fractions. What does it mean, for example, when someone says “two-fifths of the people” or “65% of the students” ?

Let us take up the first sentence first. There is universal agreement that, when someone refers to “two-fifths of the people in a room”, she means : “the total number of people in 2 of the parts when the people in the room are divided into 5 equal parts”. There is no reason to lose sleep over *why* this is so, any more than to do the same over why *red* was chosen as the color of stop lights. After all, language is nothing more than a collection of commonly adopted conventions. Our goal is not to probe the linguistic subtleties but to translate this phrase into precise mathematics instead. Let us say that there there are n people in the room. According to interpretation (26) (or alternatively the combination of (18) and (19)), the meaning of the product $\frac{k}{\ell} \times n$ is the total number of people in k of the parts when the n people are divided into ℓ equal parts. Therefore we see that

$$\text{“two-fifths of } n \text{ people”} \quad \text{means} \quad \text{“}\left(\frac{2}{5} \times n\right) \text{ people”}$$

Similarly, “two-thirds of k cars” means “$(\frac{2}{3} \times k)$ cars”, etc.

> The discerning reader would have noticed that, in most cases, such a statement as “two-fifths of the people” does not make strict sense because the total number of people is unlikely to be divisible by 5. For example, “two-fifths of 72 people” would mean 28.8 people. The purpose of this remark is therefore to serve notice that common expressions have to be interpreted liberally, even as we try to make mathematical sense of them.

It should be noted that this attitude towards everyday expressions is not shared by all educators. Whereas we have tried to achieve a coherent presentation of fractions and then, using precise arguments, interpret the mathematical facts we have proved in terms of ordinary language, it is often thought that for school mathematics a better way to teach it is to reverse the roles of mathematics and everyday language. In this view, everyday language is the primary source, and mathematics — especially fractions — is no more than a symbolic reflection of the language. So doing mathematics becomes a kind of guessing game: how well we can do mathematics depends on how well we understand the hidden meanings of everyday language. A good illustration of this point of view is given by the article on “Of-ing fractions” by J. Moynahan in *What is Happening in Math Class?*, (Deborah Schifter, editor, Teachers College Press, 1996), in which the author describes a class discussion in a *sixth-grade* classroom about what students did to “solve the following problems”:

1. The Davis family attended a picnic. Their family made up 1/3 of the 15 people at the picnic. How many Davises were at

> the picnic?
> 2. John ate 1/8 of the 16 hot dogs . How many hot dogs did John eat?
> 3. One-fourth of the hot dogs were served without relish. How many were served without relish?

Apparently, the students solved the problems in groups and then were challenged by the teacher to explain what mathematical operation should take the place of the preposition “of”: is it $+$ or $\times$? According to the teacher, knowing that “of” means *multiply* was important, because “If the algorithm for multiplying fractions was to make sense, they need to understand that ‘of’ means multiply” and “understanding had to come first”.

This is a case of an attempt to improve the teaching of fractions, specifically the multiplication of fractions, not by trying to attach precise mathematical meaning to the relevant concepts — which is a fundamental requirement of mathematics — but by appealing to students’ *a priori* understanding of everyday language (in this case, the meaning of “of”). It should be a matter of concern in mathematics education when a *sixth-grade* class does not show interest in a clear explanation of what it means to “multiply” two fractions and why this way of multiplying fractions should be interpreted as in (26). Instead, the formula $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ was taken as preordained, and all that remained was for children to fathom its mysteries. Would the teacher or those sixth graders feel the same way if fraction multiplication were defined as $\frac{a}{b} \times \frac{c}{d} = \frac{a^2c^2}{bd}$ instead? There is no indication at all that a *mathematical* understanding of this formula was considered a necessity for the students, only a linguistic one. This is hardly a good way to promote the learning of mathematics.

By contrast, we took great pains to carefully define the meaning of the multiplication of fractions, and to argue on mathematical grounds why the interpretation (26) of the multiplication is correct. We did not probe the psychology behind the linguistic usage of the preposition *of*, because linguistic usage may not be *exactly* the same for everyone. Insofar as we believe that mathematics is a discipline of precision, it is best not to let the correctness of a mathematical statement rest on the vagaries of everyday language. We also believe that it is essential for students to learn that the correctness of mathematical statements can be ascertained by purely mathematical reasoning alone without extra-mathematical considerations.

We next turn to percent. Suppose you go to a supermarket to buy beef and you want as little fat as possible. One package says “$\frac{3}{28}$ of this package is fat”[17] and the other says “Fat content: $\frac{2}{15}$”. Of course in real life, such oddball labelings do not occur, but the point of this story is to explain why they do not occur. So let us go on. Which of the two has less fat per unit weight? You know your fractions by now, so mentally you compute by using the cross-multiplication algorithm: $3 \times 15 = 45 < 56 = 2 \times 28$, therefore $\frac{3}{28} < \frac{2}{15}$ and you pick the first package.

This kind of labeling is unacceptable, of course, because important information should be clear at a glance without requiring mental computations. One way to resolve this difficulty would be for the first meat packing company to also measure fat by dividing each unit weight into 15 instead of 28 equal parts. Had it done that, it would have found that $1\frac{17}{28}$ parts out of 15 consist of fat and would have therefore labelled the package as “$1\frac{17}{28}/15$ of this package is fat”. (Notice how complex fractions appear naturally.) Then it would indeed be clear at a glance that the second package contains more fat inasmuch as $2 > 1\frac{17}{28}$. This then give rise to the idea that if all meat packing companies agree to use the same denominator for their fat-content declarations, all shoppers would be able to make comparisons at a glance. In fact, the same goes for all kinds of quantitative comparisons: why not use the same denominator across the board?

Activity: Verify that the preceding statement about $\dfrac{3}{28} = \dfrac{1\frac{17}{28}}{15}$ is correct.

Quite miraculously, such a general agreement was reached at some point in the past, and people seemed happy to use 100 as the standard denominator. They even devised a new notation and created a new word in the process: instead of $\frac{15}{100}$, they agreed to write 15% and call it *fifteen percent* (Latin: *per centum*, meaning “by the hundred”), and instead of $\dfrac{5\frac{1}{2}}{100}$, they agreed to write $5\frac{1}{2}$% and call it *five-and-a-half percent*, etc. Thus instead of $\frac{3}{28}$, we want to express it as $C\%$, where C is some fraction, i.e., we want $\frac{3}{28} = C\% = C \times \frac{1}{100}$, where the last equality uses (a) of §9. By (25) in §8, we are guaranteed that

[17] Such measurements are understood to be by weight.

there is such a C, and in fact,

$$C = 100 \times \frac{3}{28} = 10\frac{5}{7}.$$

The first package of beef would then be labeled: “$10\frac{5}{7}\%$ of this package is fat”. Similarly, the second package would be labeled: “Fat content: $13\frac{1}{3}\%$”, because if $\frac{2}{15} = \frac{y}{100}$, we get as before that

$$y = \frac{100 \times 2}{15} = 13\frac{1}{3}.$$

Now every shopper can tell without a moment of hesitation that the second package has more fat per unit weight because $13\frac{1}{3} > 13 > 11 > 10\frac{5}{14}$,

It remains to point out that there is an element of unreality in this story: ordinary labeling of percentages in a commercial context usually rounds off to the nearest digit. Instead of $13\frac{1}{3}$ and $10\frac{5}{14}$, it would be respectively, 13% and 10%. The rounding-off notwithstanding, the method of computation of percentages illustrated above is essential knowledge. It makes plain the importance of the assertions in §9.

We have so far intepreted the notation of “%” as the denominator of a complex fraction or fraction, e.g., 13% means $\frac{13}{100}$. In ordinary language, however, “percent” is often used as an adjective, as in “I am getting 5 percent interest from my bank”. There are educators and mathematicians alike who would argue that “5%” is not a number but an “action” or an “operator”. Either way, once you understand the underlying mathematics, you are not likely to get confused no matter how “percent” is used.

Example 1. Express $\frac{5}{16}$ as a percent.

Solution. If $\frac{5}{16} = C\%$, then $\frac{5}{16} = C \times \frac{1}{100}$. By the discussion following (25) of §8,

$$C = 100 \times \frac{5}{16} = 31\frac{1}{4}.$$

So $\frac{5}{16} = 31\frac{1}{4}\%$.

Example 2. The price of the stocks of a diapers company went down by 12%, and then went back up by 12% the next day. Did it get back to its original price?

Solution. Say the price of the stocks is D dollars. If it went down by 12%, then the low price is $D - 12\% \times D = 88\% \times D$ dollars. If the low price went up by 12%, then the new price is

$$\begin{aligned}(88\% \times D) + 12\% \times (88\% \times D) &= 1 \times (88\% \times D) + 12\% \times (88\% \times D)\\ &= [1 + 12\%](88\% \times D)\\ &= 112\% \times 88\% \times D\\ &= \frac{9856}{10000} \times D\end{aligned}$$

Because $\frac{9856}{10000} < 1$, the new price is less than D dollars and therefore not as high as the original price.

It may be of interest to find out by how many precent this stock had to go up the next day in order to climb back to the original price. Suppose it went up by $x\%$ the next day, then the new price would be computed exactly as above, except that 12 would be replaced by x everywhere:

$$\begin{aligned}(88\% \times D) + x\% \times (88\% \times D) &= 1 \times (88\% \times D) + x\% \times (88\% \times D)\\ &= [1 + x\%](88\% \times D)\\ &= (100 + x)\% \times 88\% \times D\\ &= \frac{(100 + x) \times 88}{10000} \times D\end{aligned}$$

Thus the new price will be D dollars exactly when $(100+x) \times 88 = 10000$, or when $8800 + 88x = 10000$, which means $88x = 1200$. Therefore $x = \frac{1200}{88} = \frac{150}{11} = 13\frac{7}{11}$. In other words, the price of the stock would have to go up by $13\frac{7}{11}\%$ in order to equal its original price.

Before leaving the subject of percents, let us pursue an idea by way of a concrete example. Suppose we try to convert $\frac{3}{8}$ to percent. As usual, if $\frac{3}{8} = C\%$, then

$$C = 100 \times \frac{3}{8} = 37\frac{1}{2}.$$

Thus $\frac{3}{8} = 37\frac{1}{2}\%$. Now the complex fraction is somewhat unsightly, and we can use Example 1 of §9 to rewrite $37\frac{1}{2}\%$ as

$$\frac{3}{8} = 37\% + \frac{1}{2}\% = \frac{37}{100} + \frac{\frac{50}{100}}{100} = \frac{37}{100} + \frac{50}{100^2}.$$

This then allows us to avoid complex fractions at the expense of adding an extra term $\frac{50}{100^2}$, which is also a kind of "super-percent". Or we can use powers of 10 as denominator and rewrite:

$$\frac{3}{8} = \frac{30}{100} + \frac{7}{100} + \frac{5}{1000} = \frac{3}{10} + \frac{7}{10^2} + \frac{5}{10^3}.$$

You recognize the last expression as a "decimal". Thus from the expression of a fraction as a percent, we easily get to decimal expansion of the fraction. This is a topic we will discuss at length in the Chapter 4, but the connection of decimals with percents deserves to be singled out right from the start.

Exercise 10.1 What percent is 18 of 84? 72 of 120? What is 15 percent of 75? And 16 percent of what number is 24?

Exercise 10.2 Express the following as percents: (a) $\frac{1}{4}$, $\frac{7}{5}$, $\frac{3}{16}$, $\frac{17}{32}$, $\frac{34}{25}$, $\frac{24}{125}$, $\frac{18}{125}$. (b) $\frac{5}{12}$, $\frac{24}{7}$, $\frac{8}{15}$, $\frac{7}{3}$, $\frac{5}{6}$, $\frac{7}{48}$. Do you notice a difference between the answers to the two groups in terms of the considerations immediately preceding these exercises? Can you guess an explanation?

Exercise 10.3 A shop plans to have a sale. One suggestion is to give all customers a 15% discount after sales tax has been computed. Another suggestion is to give a 20% discount before sales tax. If the sales tax is 5%, which suggestion would give the customer a greater saving?

Exercise 10.4 A bike is priced at $469.80 including an 8 percent sales tax. How much is the price of the bike before sales tax and how much is the sales tax?

11 Ratio, Rates, and the Fundamental Assumption of School Mathematics (FASM)

The concept of a "ratio" is almost never defined in textbooks or professional development materials.[18] Let us therefore begin with this provisional definition: the ratio of two numbers A and B is just the division of A by B. We have been working extensively on the concept of division in this chapter

[18] Some textbooks define a ratio as a "quotient", which would seem to be in complete accord with this monograph. Unfortunately, it is also true that in these books the concept of a "quotient" is left undefined.

(cf. §§4, 8, and 10), so that this definition would seem to be rock solid. Suppose however we consider the ratio of the circumference of a circle of radius 1 and the diagonal of the unit square: as is well-known, this ratio is $\frac{2\pi}{\sqrt{2}}$, which is unfortunately not the quotient of a fraction by a fraction because neither π nor $\sqrt{2}$ is a fraction.[19] This then calls for more work before we can give a legitimate definition of a ratio.

We will begin with some general comments about fractions and complex fractions. We do so **informally**, so that we would on occasion invoke concepts and results not yet developed in this monograph. On the whole, the discussion is kept on an intuitive level, so that even if you encounter a phrase or two that seem unfamiliar, you should just ignore them and forge ahead.

By Chapter 5, we will have treated both positive and negative fractions in some detail, and will in particular extend the validity of all the identities in §9 to allow A, B, $\ldots F$ to be negative fractions as well. Now, the positive and negative fractions together are called *rational numbers*. In real life, these are the only numbers one encounters. For example, although we know the diagonal of the unit square is $\sqrt{2}$, which as mentioned above is *irrational* (i.e., not a rational number), this diagonal in an everyday context would usually be taken to be 1.414 or similar approximations to the real value of $\sqrt{2}$ ($= 1.41421356237309504 88\ldots$). Therefore, rational numbers occupy a position of singular importance among numbers. Nevertheless, we must acknowledge that there are many *real numbers* — they are by tradition the name given to all the points on the number line — which are irrational. Those points which correspond to rational numbers as prescribed in §1 of this chapter do *not* comprise the whole number line. Put another way, there are many lengths which cannot be represented by a segment with fractions as endpoints. In addition to $\sqrt{2}$, the square root of any whole number which is not a perfect square is irrational, as is the length of a semicircular arc of a circle with radius 1, which is π of course. (We will prove in §5 of Chapter 3 that $\sqrt{2}$ is irrational.) There are other irrationals that can be manufactured at will (again, see §5 of Chapter 3). We should also note that other roots of whole numbers such as $\sqrt[5]{11}$, or $\sqrt{\sqrt[3]{5} + \sqrt{8}}$ are all irrational. The irrational numbers and rational numbers together form the *real number system*, the understanding of which took human beings more than two thousand years to achieve.

[19] Neither fact is obvious, especially the one about π.

It is a fact that school mathematics is essentially the mathematics of rational numbers and that the real number system is almost never discussed in the K–12 curriculum. One cannot avoid irrational numbers entirely, however. Numbers such as $\sqrt{2}$ or $\sqrt[3]{5}$ and π naturally come up, for example, in the solutions of quadratic or cubic equations and discussions of area and circumference of a circle, so that computations with these numbers are unavoidable. Such being the case, how does the school mathematics curriculum cope with this situation? It does so by **implicitly** invoking what we propose to call *The Fundamental Assumption in School Mathematics* (FASM):

> All the information about the arithmetic operations on fractions can be extrapolated to all real numbers.

This is a profound assumption. For example, school students are supposed to know what it means to multiply $\sqrt{2}$ with π or divide one by the other. This is clearly a very big assumption considering the effort we had to spend merely to understand the multiplication and division of fractions. Moreover, school students are also expected to manipulate these irrational numbers as if they were integers. Thus a typical student would write down the following without a moment's thought:

$$\frac{\pi}{7}+\frac{\sqrt{2}}{3\sqrt{5}}=\frac{3\pi\sqrt{5}+7\sqrt{2}}{21\sqrt{5}},$$

or,

$$\sqrt{2}(\sqrt{3}+\pi)=\sqrt{2}\,\sqrt{3}+\sqrt{2}\,\pi,$$

or,

$$37\times\pi=\pi\times 37.$$

If pressed as to how these can be justified, they would likely answer that for the first equality, they assume that (d) of §9 would hold for all real numbers, for the second, that the distributive law would hold for all real numbers, and for the last equality, the commutative law of multiplication is valid for arbitrary numbers as well. In other words, FASM is implicitly at work.

We comment in passing that the second equality reveals why it is important to have a general formula for the addition of two fractions as in (8) of §6, and *why the common way to define the addition of fractions by seeking the lcm of the denominators distorts what fraction addition means.*

Now FASM is in fact correct, but the explanation is a bit subtle. Here, we will attempt only the barest outline of such an explanation. Take the case of the product $\sqrt{2}\,\pi$: we know this is a point on the number line, but *which* point? Instead of giving a one-sentence answer, we describe a step-by-step procedure to get as close to this point as we please, or as we will say from now on: *to approximate* this point. First, a general comment. One of the basic facts about the real numbers is that **every real number can be approximated by a sequence of rational numbers**. In the case of $\sqrt{2}$, we can explicitly write down such a sequence. Using the infinite decimal expansion of $\sqrt{2}$, which is $1.4142135623730950488\ldots$, we define the sequence $\{a_n\}$ to be:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1.4 \\ a_3 &= 1.41 \\ a_4 &= 1.414 \\ a_5 &= 1.4142 \\ &\vdots \\ a_{15} &= 1.41421356237309 \\ a_{16} &= 1.414213562373095, \quad \text{etc.} \end{aligned}$$

Note that each a_n is in fact a fraction, because for example, $a_5 = \frac{14142}{10000}$. For $\pi = 3.14159265358979323846\ldots$, we produce a similar sequence of fractions $\{b_n\}$, so that

$$\begin{aligned} b_1 &= 3 \\ b_2 &= 3.1 \\ b_3 &= 3.14 \\ b_4 &= 3.141 \\ b_5 &= 3.1415 \\ &\vdots \\ b_{15} &= 3.14159265358979 \\ b_{16} &= 3.141592653589793, \quad \text{etc.} \end{aligned}$$

Then it is plausible, and in fact provable, that the sequence of points $\{a_1b_1, a_2b_2, a_3b_3, \ldots\}$ would approximate *some* point on the number line, and this point is what is usually denoted by $\sqrt{2}\,\pi$. In greater detail:

$$\begin{aligned}
a_1 b_1 &= 1 \times 3 = 3 \\
a_2 b_2 &= 1.4 \times 3.1 = 4.34 \\
a_3 b_3 &= 1.41 \times 3.14 = 4.4274 \\
a_4 b_4 &= 1.414 \times 3.141 = 4.441374 \\
a_5 b_5 &= 1.4142 \times 3.1415 = 4.4427093 \\
a_6 b_6 &= 1.41421 \times 3.14159 = 4.4428679939 \\
&\vdots \\
a_{15} b_{15} &= 1.41421356237309 \times 3.14159265358979 \\
&= 4.4428829381583458058597947511 \\
a_{16} b_{16} &= 1.414213562373095 \times 3.141592653589793 \\
&= 4.442882938158365756463749819335 \\
a_{17} b_{17} &= 1.4142135623730950 \times 3.1415926535897932 \\
&= 4.44288293815836603930646229395400
\end{aligned}$$

Even without considering $a_n b_n$ for $n > 17$, it is reasonably clear that this sequence of products will be $4.44288293815836 \pm 10^{-14}$. In terms of everyday usage, we can say with confidence that we *know* what this point $\sqrt{2}\,\pi$ is because the approximation 4.443 usually suffices. In addition, we also know that we can get as close to this point as we want, provided we are willing to multiply out $a_n b_n$ for n large.

Let us now briefly indicate why the following equation is valid:

$$\frac{\pi}{7} + \frac{\sqrt{2}}{3\sqrt{5}} = \frac{3\pi\sqrt{5} + 7\sqrt{2}}{21\sqrt{5}}.$$

Let $\sqrt{5}$ be approximated by a sequence of fractions $\{c_n\}$, and let $\{a_n\}$ and $\{b_n\}$ be the preceding sequences of fractions which approximate $\sqrt{2}$ and π, respectively. Then from (d) of §9, we know that for each n:

$$\frac{b_n}{7} + \frac{a_n}{3c_n} = \frac{3b_n c_n + 7a_n}{21c_n}$$

Here we see clearly why we need validity of the identities in §9 not just for fractions A, B, $\ldots F$ but for complex fractions as well. We now let n get arbitrarily large. As it does so, the left side (by arguments similar to those already encountered above) approximates

$$\frac{\pi}{7} + \frac{\sqrt{2}}{3\sqrt{5}}$$

while the right side approximates

$$\frac{3\pi\sqrt{5} + 7\sqrt{2}}{21\sqrt{5}}.$$

So these two points must be the same point on the number line.

We hardly need to add that the underlying principle of the preceding argument applies to any irrational number and not just π, $\sqrt{2}$, or $\sqrt{5}$. Moreover, the analogs of identities (a)–(f) in §9 can be proved in this manner for arbitrary *real numbers* A, B, $\ldots F$.

The complexity of the preceding discussion, incomplete as it is, gives a good reason for the mathematics curriculum in K–12 not to discuss the real numbers in any details. Having said this, we must also profess total puzzlement as to how the school mathematics curriculum could pretend that the problem with the transition from rational numbers to real numbers does not exist. This is an inexcusable neglect of the logitudinal mathematical coherence in school mathematics. Until better textbooks are written, you should be aware of overall presence of FASM as an *unspoken assumption*.

The most important consequence of FASM for our present purpose is that we can now extend the scope of the division of fractions (as in §9) to include the division of any two real numbers.

Henceforth, given two real numbers A and B (rational or irrational), we can talk about the division of A by B, namely, $\frac{A}{B}$ (assuming $B \neq 0$). Recall that this is quite a complicated concept, namely, it is approximated by $\frac{A_n}{B_n}$, where $\{A_n\}$, $\{B_n\}$ are sequences of rational numbers which approximate A and B, respectively, and the meaning of $\frac{A_n}{B_n}$ is given by (27) of §8. What is important for school mathematics is however the fact that, *on a formal level*, FASM together with the identities of §9 allow us to treat the division of real numbers *operationally* as the division of two whole numbers. Therefore, the division of real numbers can hardly be simpler from a computational point of view. With this understood, we are now in a position to give precise meanings to four concepts that cause a great deal of anxiety among students and teachers. *The word "number" in the following will mean real numbers in general.*

Percent. Given a division of two numbers $\frac{k}{\ell}$, which typically expresses parts of a whole, we can write it as $\frac{k}{\ell} = A \times \frac{1}{100}$ for some number A, by the real-number counterpart of (27). Then "k-ℓth of something" is sometimes also expressed as "A-percent of something" (cf. §10). For example, because $\frac{2}{5} = 40 \times \frac{1}{100}$, two-fifths is often expressed as 40 percent.

Ratio. The ratio of two quantities A and B is the division of A by B in the sense we have just described, $\frac{A}{B}$. Intuitively, the ratio of A and B is the *multiplicative way* of comparing the two numbers A and B (in contrast with the *additive* way of comparing, which would ask for $A - B$ instead). By tradition, this ratio is also written as $A : B$, and this strange notation may have been responsible for most of the misunderstanding connected with this concept (but see the historical discussion below).

Rate. The ratio of two quantities of "different types", in one sense or another, is usually singled out and given a separate name called a rate. For example, the ratio of the total number of miles traveled by the total number of hours of the trip is called the *average speed,* and because miles and hours are considered "different", average speed is usually cited as the prototypical example of a rate.

Proportion. The equality of two or more ratios.

It is possible to make this discussion appear more profound by more verbiage — and the public agonizing over these terms by teachers and educators alike almost invites this kind of verbal extravagance. But we have spent the effort to explain the meaning of division, and once that is done, there is nothing left to agonize over. Perhaps we can compare this situation with our treatment of fractions. Our main effort was spent in constructing a clear definition of a fraction as a point on the number line obtained in a prescribed manner. Once done, such a definition obviates any further need for a prolix discussion of the supposed profundity of the fraction concept.

The following two problems would further clarify the situation regarding ratios and rates.

Problem. A school district has a teacher-student ratio of $1 : 24$. If the number of students stay constant, how many more teachers does the district need to hire in order improve the ratio to $1 : 18$?

Problem. Paul rode his motorbike to Lanterntown 40 miles away from home. He maintained a steady speed of 15 miles per hour. On the way back, he decided to increase his speed to 18 miles an hour. What is the *average speed* of his roundtrip in the sense of total distance divided by total travel time?

These two problems will be discussed and solved in the next section. The reason for bringing them up now is that the first problem is regarded as a typical *ratio problem* and the second a *rate problem*. Because "teachers" and "students" are considered to be of the same type, comparing them is then a ratio. On the other hand, average speed is a comparison of miles and hours which are of "different types", and so the second problem is one about rate. In the education literature as well as school texts, the terms "ratio" and "rate" are flaunted as key concepts but almost never explained, and students are asked to know the difference. This may be the reason why these terms produce anxiety, and this is the problem we try to address here.

As the definitions given above of ratio and rate make it abundantly obvious, a rate or a ratio is just a *division* of two numbers, and because we have made the extra effort to explain what division means, these definitions at least have the merit of being correct. Armed with a correct definition for each, we can hope to be able to probe more deeply into their interpretations by the use of mathematical reasoning. Now a division is a division (just as a rose is a rose), and it is not necessary to indulge in hairsplitting discussions as to why some divisions are different from others unless such discussions are called for. But so far, we see no mathematical merit in differentiating *rate* from *ratio*. For example, we discussed speed extensively in §8 as a partitive division (between fractions), and the need for the definition of "rates" certainly never arosed in that discussion. You will have further evidence of this point of view when the two preceding problems (on teacher-student ratio and traveling to Lanterntown) are analyzed and solved in the next section in terms of precise definitions, mathematical reasoning, and common sense, but without once mentioning the difference between rate and ratio.

It should be stressed once more that the concept of a ratio is unambiguously defined in this monograph as a division of two numbers, whereas elsewhere it is usually shrouded in mysticism. Here are a random selection of the definitions that can be found in the literature:

> A ratio is a comparative index; it always makes a statement about one measurement in relation to another.
>
> A ratio is a comparison of any two quantities. A ratio may be used

> to convey an idea that cannot be expressed as a single number.
>
> A ratio is a comparison of two quantities that tells the scale between them. Ratios may be expressed as quotients, fractions, decimals, percent, or given in the form of $a : b$.
>
> A ratio is a way to describe a relationship between numbers. If there are 13 boys and 15 girls in a classroom, then the ratio of boys to girls is 13 to 15.

From a historical perspective, one can see why the concept of a ratio has this mystical character. The word *ratio* appears in Euclid's *Elements* (*The Thirteen Books of the Elements*, Volumes 1-3, Dover Publications, 1956). At the beginning of Book V, we find the following:

> A **ratio** is a sort of relation in respect of size between two magnitudes of the same kind.
>
> Magnitudes are said to be **in the same ratio**, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and the third, and any equimultiples whatever of the second and the fourth, the former equimultiples alike exceed , are alike equal to, or alike fall short of the latter equimultiples respectively taken in corresponding order.
>
> Let magnitudes which have the same ratio be called **proportional**.

If we are willing to accept what Euclid says about ratio and the equality of ratios (i.e., "be in the same ratio"), then his definition of "proportional" is perfectly understandable because it is exactly the modern definition. However, his definition of a ratio as "a sort of relation in respect of size" sounds exactly like any of the contemporary versions quoted above in terms of its lack of information. This is no accident, because what happened in the more than two thousand years after Euclid (c. 300 B.C.E.) was that everybody *copied Euclid*, including our own textbooks. We see, for instance, that the insistence that ratios be comparisons of magnitudes of the same type is nothing but an echo of Euclid's definition. Now, the reason Euclid had to express himself in such strange fashion (e.g., "…exceed , are alike equal to, or alike fall short of the latter equimultiples respectively taken in corresponding order") is because the concept of a "real number" did not exist in Euclid's times. Not only did he not have the algebraic notation to express his mathematical thoughts, but he also had no inkling that all numbers, rational or irrational, could be added, subtracted, multiplied and divided like fractions. Examine, for example, the way Euclid made use of ratios in arguments with similar triangles, clearly if he had felt comfortable with the concept of division between numbers, he would have defined ratio as a division. In other words, Euclid's cryptic definition of a ratio was the inevitable consequence of the mathematical limitations of his time. Nevertheless, we continue to copy him.

The conductor Arturo Toscanini (1867-1957) suffered from extreme myopia all his life and he had to hold music scores an inch or two from his face before he could read them. As a result, he had to conduct without a score (but which his phenomenal memory allowed him to do with ease). Because of his fame, however, soon every conductor was forced to imitate him and conduct without a score. Toscanini was once caught making the biting remark: "They copy my weakness."

We can say likewise that, twenty-three centuries after Euclid, our school mathematics textbooks continue to copy Euclid's weakness. But mathematics in the past hundred years has progressed far beyond the mathematics of Euclid's time. We now have a robust understanding of real numbers, we have excellent symbolic notations, and mathematics is done by starting with clearly defined concepts and each step is logically explained. It is no longer necessary to copy Euclid's weakness anymore.

> *Let us lay to rest the obscurantism surrounding the concept of a ratio once and for all and steer our children away from the unknowable and the incomprehensible.* **A ratio is a division***, no less and no more.*

12 Word Problems

In this section, we discuss several word problems to illustrate the applications of fractions. Notice that in Problems 3 and 6, FASM is implicitly invoked.

PROBLEM 1.[20] SUPPOSE A HIGH SCHOOL MATH CLASS CAN ONLY MAKE USE OF 60% OF ITS CLASS TIME FOR TEACHING MATHEMATICS, BUT AN 8TH GRADE MATH CLASS IS EVEN LESS PRODUCTIVE PEDAGOGICALLY AND CAN USE ONLY 40%. ASSUMING THAT BOTH KINDS OF MATH CLASSES COVER THE SAME NUMBER OF PAGES OF A 360-PAGE ALGEBRA I TEXTBOOK FOR EACH HOUR OF ACTUAL MATHEMATICS INSTRUCTION, AND THAT HIGH SCHOOL MATH CLASSES COVER THE WHOLE BOOK IN A YEAR, HOW MANY PAGES OF THE SAME BOOK WILL BE COVERED BY 8TH GRADE MATH CLASSES IN A YEAR?

Suppose there are a total of H hours in each math class per year, then a high school math class has $(\frac{60}{100} \times H)$ hours devoted to the actual teaching of mathematics per year. (see §10 for the interpretation of percent). Thus in $\frac{60H}{100}$ hours, it covers the whole 360-page book. Therefore, the number of pages covered per hour of mathematics instruction is $[300/(\frac{60H}{100})]$ pages.[21] By hypothesis, the number of pages which a 8th grade math class can cover per hour of actual mathematics instruction is also $[300/(\frac{60H}{100})]$. However, the latter class has only $(\frac{40}{100} \times H)$ hours which can be used for the teaching of mathematics per year, so the total number of pages covered in one year in the 8th grade math class is:

$$\begin{aligned} (\frac{40H}{100}) \times \frac{360}{(\frac{60H}{100})} &= \frac{40H}{100} \times \frac{360 \times 100}{60H} \\ &= \frac{(40 \times 360) \times (100H)}{60 \times (100H)} \\ &= 240 \quad \text{pages.} \end{aligned}$$

Discussion. The usual approach taught in schools is to use so-called *proportional reasoning*: if the total number of pages covered in the 8th grade math class is x, then

$$\frac{\frac{60}{100}}{\frac{40}{100}} = \frac{360}{x} \tag{29}$$

so that

$$x = \frac{\frac{40}{100}}{\frac{60}{100}} \times 360 = 240\,,$$

[20] This problem is due to Jerome Dancis.

[21] Observe the natural appearance of complex fractions and the partitive interpretation of the division of fractions.

which is the same as before. However, the setting up of (29) is a mystery to beginners unless the underlying reasoning is carefully explained, as follows. We wish to compute the number of pages covered per hour of actual mathematics instruction in both the high school and 8th grade classes. Let us say that there are a total of H hours of class time per year. The high school class devotes $(\frac{60}{100} \times H)$ hours to cover 360 pages, and so the number of pages covered per instruction hour is $[360/\frac{60H}{100}]$. The eighth grade class devotes $(\frac{40}{100} \times H)$ hours to cover x pages, and so the number of pages covered per per mathematics instruction hour is $[x/\frac{40H}{100}]$. By assumption, the number of pages covered per mathematics instruction hour is the same for both classes. Therefore,

$$\frac{360}{\frac{60H}{100}} = \frac{x}{\frac{40H}{100}}.$$

By the cross-multiplication algorithm (5) of §5, this is the same as

$$\frac{360}{x} = \frac{\frac{60}{100} \times H}{\frac{40}{100} \times H} = \frac{\frac{60}{100}}{\frac{40}{100}},$$

which is the same as (29).

Problem 2. Paul rode his motorbike to Lanterntown 40 miles away from home. He maintained a steady speed of 15 miles per hour. On the way back, he decided to increase his speed to 18 miles an hour. What is the *average speed* of his roundtrip in the sense of total distance divided by total travel time?

According to the definition of average speed, we need to compute the total distance traveled in the roundtrip and the amount of travel time. Since each way is 40 miles, the distance of the round trip is $40 + 40 = 80$ miles. We have to find the total travel time. On the way out to Lanterntown, Paul covered 15 miles each hour. The number of hours needed to cover 40 miles is therefore (see the discussion of motion in §8) is $\frac{40}{15}$ hours. On the way back, Paul's speed is 18 miles per hour, and so the travel times by similar reasoning is $\frac{40}{18}$ hours. The roundtrip therefore took $\frac{40}{15} + \frac{40}{18}$ hours altogether. Hence

$$\begin{aligned} \text{average speed} &= \frac{80}{\frac{40}{15} + \frac{40}{18}} = \frac{80}{\frac{12\times 40 + 10 \times 40}{180}} = \frac{80}{\frac{880}{180}} \\ &= \frac{80 \times 180}{880} = \frac{180}{11} = 16\frac{4}{11} \end{aligned}$$

miles per hour.

Discussion. A common mistake of students is to say that the average speed is the "average of the two speeds" and therefore equal to $\frac{15+18}{2} = \frac{33}{2} = 16\frac{1}{2}$ miles per hour. This may be the right place to remind students of the importance of *precise definitions* in mathematics. The fact that "average speed" tends to suggest "the average of two speeds" is of course confusing, but such confusion in no way lessens students' basic obligation to get to know the precise meaning of each term. A related example is that, although "complex fraction" vaguely suggests "a fraction of complex numbers", its precise meaning (§9) is different. One cannot overemphasize the importance of learning precise definitions.

PROBLEM 3. A TRAIN GOES BETWEEN TWO TOWNS IN CONSTANT SPEED. BY INCREASING THE SPEED BY A THIRD, THE TRAVEL TIME IS SHORTENED BY HOW MANY PERCENT?

Again, we just have to systematically worked through the definitions (compare the discussion of motion near the end of §8). There are two sets of travel times: the initial travel time ($\mathcal{I}$) and the subsequent faster travel time ($\mathcal{F}$). The amount of time shortened as a result of the increase in speed is $\mathcal{I} - \mathcal{F}$. We want *in percent* the quotient

$$\frac{\mathcal{I} - \mathcal{F}}{\mathcal{I}}.$$

Let us proceed to compute each item involved. We do not know the distance between the two towns, so for the sake of discussion, let us call it D. For the same reason, let call the initial speed s. Then

$$\mathcal{I} = \frac{D}{s}.$$

In order to get some intuitive feeling for the problem, let us try something simple like $D = 120$ miles and $s = 60$ mph. The original travel time is then $\frac{120}{60} = 2$ hours. If the speed is increased by a third, then the new speed is $60 + (\frac{1}{3} \times 60) = 80$ mph, so that the new travel time would be $\frac{120}{80} = 1\frac{1}{2}$ hours. The amount of time saved is $2 - 1\frac{1}{2} = \frac{1}{2}$ hour which (by inspection) is 25% of the original travel time of 2 hours. Next we try $D = 200$ miles and $s = 30$ mph. The the two travel times are $6\frac{2}{3}$ hours and $\frac{200}{40} = 5$ hours. The time saved is then $6\frac{2}{3} - 5 = 1\frac{2}{3}$ hours. Because

$$\frac{1\frac{2}{3}}{6\frac{2}{3}} = \frac{\frac{5}{3}}{\frac{20}{3}} = \frac{1}{4},$$

again the time saved is 25% of the original travel time. One should do many such special cases until one detects a certain pattern. In this case, it is 25% over and over again, so that it becomes a matter of verifying this answer in general.

We now do the general case. The increased speed is a third greater than s, hence it is $s + \frac{1}{3}s = \frac{4}{3}s$. The shorter travel time is then

$$\mathcal{F} = \frac{D}{\frac{4}{3}s} = \frac{3}{4}\frac{D}{s}.$$

Therefore, we have

$$\frac{\mathcal{I} - \mathcal{F}}{\mathcal{I}} = \frac{\frac{D}{s} - \frac{3}{4}\frac{D}{s}}{\frac{D}{s}} = \frac{\frac{1}{4}\frac{D}{s}}{\frac{D}{s}} = \frac{1}{4},$$

by (b) of §9. In terms of percent, it is easily seen to be 25%. So the traveled time is shortened by 25 percent.

Problem 4. A school district has a teacher-student ratio of $1:24$. If the number of students stay constant, how many more teachers does the district need to hire in order improve the ratio to $1:18$?

As before, we have to work through the definitions patiently. We are not given how many teachers or students there are, so we just give them names, say, there are T teachers and S students. We *are* given that

$$\frac{T}{S} = \frac{1}{24}.$$

Note that we are using the definition of ratio as a division (and of course the interpretation of a fraction as division in §4). We want to increase the number of teachers so that the new teacher-student ratio would be $\frac{1}{18}$. Let us try to do this problem using some concrete numbers, say $S = 360$. Then from $\frac{T}{360} = \frac{1}{24}$, we conclude $T = 15$, i.e., there are 15 teachers. How many more than 15 would increase the teacher-student ratio to $\frac{1}{18}$? We can find out by trial and error:

$$\frac{16}{360} = \frac{2}{45}, \quad \frac{17}{360}, \quad \frac{18}{360} = \frac{1}{20}, \quad \frac{19}{360}, \quad \frac{20}{360} = \frac{1}{18}.$$

So 20 is the answer, which is to say, 5 new teachers should be hired. Now we strive for understanding: why 5? Suppose N new teachers should be added in order to bring the techer-student ratio up to $\frac{1}{18}$. Then

$$\frac{15+N}{360} = \frac{1}{18}.$$

We can use the cross-multiplication algorithm to get $18 \times (15 + N) = 360$. By the distributive law, $18 \times 15 + 18N = 360$, or $270 + 18N = 360$. So $18N = 360 - 270 = 90$, and $N = 5$. So that was why 5 was the correct answer.

> We have just given a convincing argument of why it is important that the distributive law ((10) of §2 in Chapter 1) must be valid for *any* three numbers. Otherwise the above expansion of $18 \times (15 + N) = 18 \times 15 + 18N$ could not have been performed for the simple reason that we did not know (until we solved for N) what N was going to be. But because this law is true for any three numbers regardless of what they are, the above expansion was valid. Thus the generality of this and other laws — the fact that they are valid for all numbers — arises from a real mathematical need and is not an empty gesture.

We can next try $S = 500$, $S = 600$, etc. Now you will not get T to be a whole number each time, and neither would N come out as a whole number. See the indented fine-print comment at the beginning of §10. However, the idea is to get comfortable with the problem itself through the use of concrete numbers and os we should not be distracted by these peripheral issues.

Next we do the general case. Suppose N new teachers are hired, then the number of teachers is now $T + N$. We want N to satisfy

$$\frac{T+N}{S} = \frac{1}{18}. \tag{30}$$

It looks hopeless to solve for N until we remember that $\frac{T+N}{S} = \frac{T}{S} + \frac{N}{S}$, by (8) of §6. Thus we can rewrite (30) as

$$\frac{N}{S} = \frac{1}{18} - \frac{T}{S} = \frac{1}{18} - \frac{1}{24} = \frac{1}{72},$$

Note that in computing the subtraction, we used 72 as the common denominator instead of 18×24 because 72 happens to be a multiple of 18 and 24. If 18×24 is used, the answer would be $\frac{6}{432}$, and the answer would still be the same. In any case, we have an answer in the form of $N = S \times \frac{N}{S} = \frac{1}{72} S$, which means that the number of new teacher that must be hired is $\frac{1}{72}$ of the whole student body (see (17) in §7.1). Can we get an answer more explicit than this? A little reflection would reveal that with so little information given, this is all one can expect.

The next two problems are from Russia.

Problem 5. Fresh cucumbers contain 99% water by weight. 300 lbs. of cucumbers are placed in storage, but by the time they are brought to market, it is found that they contain only 98% of water by weight. How much do these cucumbers weigh?

Since 99% of 300 lbs. is just water, there are $\frac{99}{100} \times 300 = 297$ lbs. of water (§10) and hence only $300 - 297 = 3$ lbs. of solid. By the time the cucumbers are brought to market, some water has evaporated but the 3 lbs. of solid remain unchanged, of course. Since 98% is water, the solid is now 2% of the total weight. Hence if the total weight at market time is w lbs., we see that $3 = \frac{2}{100} \times w$. Using (27), we see that

$$w = \frac{100 \times 3}{2} = 150 \text{ lbs.}$$

Discussion. A mindless application of proportional reasoning would have produced the following: Let w be the weight of the cucumbers when they are brought to market. Then,

$$\frac{99/100}{300} = \frac{98/100}{w} .$$

Of course this gives $w = \frac{98}{99} \times 300 = 296.97\ldots.$ This is one reason why proportional reasoning should be taught only after its underlying reasoning has been clearly explained.

Problem 6. There is a bottle of wine and a kettle of tea. A spoon of tea is taken from the kettle and poured into the bottle of wine. The mixture is thoroughly stirred and a spoonful

OF THE MIXTURE IS TAKEN FROM THE BOTTLE AND POURED INTO THE KETTLE. IS THERE MORE TEA IN THE BOTTLE OR MORE WINE IN THE KETTLE? DO THE SAME PROBLEM AGAIN, BUT *without* ASSUMING THAT THE MIXTURE HAS BEEN STIRRED.

Let us do the stirred version first. Let the amount of wine in the bottle, the amount of tea in the kettle, and the capacity of the spoon be b cc, k cc, and s cc, respectively ("cc" means "cubic centimeter). Using b, k and s, we can compute the amount of tea in the bottle and the amount of wine in the kettle. If you are uncomfortable with the operation with symbols, we can start off with some concrete numbers to get a feel for the problem. So let us say, $b = 1000$ cc, $k = 2500$ cc, and $s = 5$ cc. After a spoonful of tea has been added to the bottle of wine, the amount of liquid in the bottle is $1000 + 5 = 1005$ cc. The fraction of tea in the mixture is therefore $\frac{5}{1005}$, and the fraction of wine in the mixture is $\frac{1000}{1005}$. A spoonful of of the thoroughly stirred mixture would therefore contain $5 \times \frac{5}{1005} = \frac{25}{1005}$ cc of tea and $5 \times \frac{1000}{1005}$ cc of wine (see §10). When this spoonful is poured into the kettle of tea, there would be $5 \times \frac{1000}{1005} = \frac{5000}{1005}$ cc of wine in the bottle. On the other hand, the mixture in the bottle originally had 5 cc of tea, but since $(5 \times \frac{5}{1005}) = \frac{25}{1005}$ cc have been taken away, the amount of tea left in the bottle is

$$5 - \frac{25}{1005} = \frac{5000}{1005} \quad \text{cc},$$

which is the same as the amount of wine in the kettle. If necessary, do this all over again with different choices of values for b, k, and s.

By the way, the number 2500 never appeared in the above solution.

Now we can begin the general argument. After a spoonful of tea has been added to the bottle of wine, the amount of liquid in the bottle is $(b + s)$ cc. The fraction of tea in the mixture is $\frac{s}{b+s}$, and the fraction of wine in the mixture is $\frac{b}{b+s}$. A spoonful of of the thoroughly stirred mixture would therefore contain $[(\frac{s}{b+s})s]$ cc of tea and $[(\frac{b}{b+s})s]$ cc of wine (see §10). When this spoonful is poured into the kettle of tea, there would be

$$(\frac{b}{b+s})s = \frac{bs}{b+s} \quad \text{cc of wine in the bottle.}$$

On the other hand, the mixture in the bottle originally had s cc (1 spoonful) of tea, but since $[(\frac{s}{b+s})s]$ cc has been taken away, the amount of tea left in the bottle is

$$s - (\frac{s}{b+s})s = s - \frac{s^2}{b+s} = \frac{s(b+s)}{b+s} - \frac{s^2}{b+s} = \frac{sb+s^2-s^2}{b+s} = \frac{sb}{b+s} \text{ cc},$$

which is the same as the amount of wine in the kettle.

We also notice that k did not figure in the solution of the problem.

Now the "unstirred" case. Suppose the spoonful of mixture contains α cc of tea and β cc of wine. The $\alpha + \beta = s$, where as before s denotes the capacity of the spoon. Therefore when the spoonful of mixture is poured into the kettle, the amount of wine in the bottle is β cc. On the other hand, the bottle of mixture originally had s cc of tea. But with α cc of the tea taken away by the spoon, only $(s - \alpha)$ cc of tea is left in the bottle. Since $(s - \alpha) = \beta$, the amount of tea in the bottle is equal to the amount of wine in the kettle, as before.

The surprising aspect of the second solution is that, since it does not depend on any assumption about whether or not the mixture has been stirred, it supersedes the first solution. Thus the precise calculations of the first solution were completely unnecessary! Nevertheless, the first solution is a valuable exercise in thinking about fractions and should not be thought of as a waste of time.

Exercise 12.1 Colin and Brynn saw a CD set that they wanted to buy, but neither had enough money. Brynn could pay for 70% of the cost, and if Colin would contribute $\frac{2}{3}$ of what he had, they could take home the set and Colin would have \$9 left. How much is the CD set, and how much money did Colin and Brynn have individually?

Exercise 12.2 A law firm has a men to women ratio of $5:1$. The firm wants to reduce it to $4:1$. What percentage increase in women would make this increase possible?

Exercise 12.3 Because of drought, each faucet is fitted with a water-saving device to reduce the rate of water flow by 35%. How long does it take to fill a tank if it used to take 15 minutes (assuming the faucet is fully open in either case)?

Exercise 12.4 For alcoholic beverage, "200% proof" means "100%" by volume, so that "120%" proof means "60% by volume". Suppose 150 bottles of 120%-proof vodka was left in the vault without the lid on, and by the time the mistake was discovered, 90% of the alcohol had evaporated. Assuming for simplicity that there was no evaporation of the remaining fluid, what is the proof of the vodka now, and how many bottles' worth is there?

Exercise 12.5 A water tank contains 271 gallons of water when it is $\frac{19}{23}$ of its full capacity. What is its full capacity?

Exercise 12.6 Mr. Dennis took his students to a concert and he was disconcerted by the fact that only $\frac{2}{5}$ of the students showed up. If 52 students showed up, how many students did Mr. Dennis have?

Exercise 12.7 Given two bottles of liquor (of different sizes), one is 50 proof and the other 140 proof. Suppose the two bottles contain the same amount of non-alcoholic fluid, say 2 cups, what is the amount of alcohol in each bottle in terms of cups?

Exercise 12.8 There are two recipes for making banana bread. One calls for $\frac{5}{8}$ cups of sugar for 4 cups of flour, and another calls for $\frac{9}{10}$ cups of sugar for 6 cups of flour. All other things being equal, which banana bread would taste sweeter?

> If you ever write a recipe like this, you should not get into the culinary business!! Learn to use easier fractions, such as $\frac{3}{4}$ or $\frac{2}{3}$, but certainly not $\frac{9}{10}$. On the other hand, if you decide that you do not want to do this problem because the numbers are not "real-world", then (1) you do not know much about fractions, and (2) unless you are willing to learn more about fractions, you should think twice before pursuing a career as a teacher. By the way, there are actually two ways to think about which banana bread is sweeter: which has more sugar in each cup of the sugar-flour mixture, or which sugar-flour ratio is bigger. See the Theorem in §5.

13 APPENDIX: Some Remarks on the Teaching of Fractions in Elementary School

> *The following is an abbreviated version of an article written in October of 1999.*

It is widely recognized that there are at least two major bottlenecks in the mathematics education of grades K–8: the teaching of fractions and the introduction of algebra. Both are in need of an overhaul. I hope to make a contribution to the former problem by devising a new approach to elevate teachers' understanding of fractions. The need for a better knowledge of fractions among teachers has no better illustration than the the following story related by Herbert Clemens (1995):

> Last August, I began a week of fractions classes at a workshop for elementary teachers with a graph paper explanation of why $\frac{2}{7} \div \frac{1}{9} = 2\frac{4}{7}$. The reaction of my audience astounded me. Several of the teachers present were simply terrified. None of my protestations about this being a preview, none of my "Don't worry" statements had any effect.

This situation cries out for improvement.

Through the years, there has been no want of attempts from the mathematics education community to improve on the teaching of fractions (Lamon 1999, Bezuk-Cramer 1989, Lappan Bouck 1989, among others), but much work remains to be done. In analyzing these attempts and the existing school texts on fractions, one detects certain persistent problematic areas in both the theory and practice, and they can be briefly described as follows:

> (1) The concept of a fraction is never clearly defined and its affinity with the whole numbers is not sufficiently stressed, if at all.
>
> (2) The conceptual complexities associated with the common usage of fractions are emphasized from the beginning at the expense of the underlying mathematical simplicity of the concept.
>
> (3) The rules of the four arithmetic operations seem to be made up on an *ad hoc* basis, unrelated to the usual four operations on positive integers with which students are familiar.
>
> (4) In general, *mathematical* explanations of essentially all aspects of fractions are lacking.

These four problems are interrelated and are all fundamentally mathematical in nature. For example, if one never gives a clearcut definition of a fraction, one is forced to "talk around" every possible interpretation of the many guises of fractions in daily life in an effort to overcompensate. A good example is the over-stretching of a common expression such as "a third of a group of fifteen people" into a main theme in the teaching of fractions (Moynahan 1996). Or, instead of offering *mathematical* explanations to children of why the usual algorithms are logically valid—a simple task *if* one starts from a precise definition of a fraction,—algorithms are justified through "connections among real-world experiences, concrete models and diagrams, oral language, and symbols (p. 181 of Huinker 1998; see also Lappan & Bouck 1998 and Sharp 1998). Why not do the obvious thing by offering a bona fide explanation? It is almost as if one makes the concession from the start: "We will offer everything but the real thing".

Let us look more closely at the way fractions are introduced in the classroom. Children are told that a fraction $\frac{c}{d}$, with positive integers c and d, is simultaneously at least five different objects (cf. Lamon 1999 and Reys et al. 1998):

> (a) parts of a whole: when an object is equally divided into d parts, then $\frac{c}{d}$ denotes c of those d parts.
>
> (b) the size of a portion when an object of size c is divided into d equal portions.
>
> (c) the quotient of the integer c divided by d.
>
> (d) the ratio of c to d.
>
> (e) an operator: an instruction that carries out a process, such as "$\frac{2}{3}$ of".

It is quite mystifying to me how this glaring "crisis of confidence" in fractions among children could have been been consistently overlooked. Clearly, even those children endowed with an overabundance of faith would find it hard to believe that a concept could be so versatile as to fit all these descriptions. More importantly, such an introduction to a new topic in mathematics is contrary to every mode of mathematical exposition that is deemed acceptable by modern standards. Yet, even Hans Freudenthal, a good mathematician before he switched over to mathematics education, made no mention of this central credibility problem in his Olympian ruminations on fractions (Freudenthal 1983). Of the existence of such crisis of confidence there is no doubt. In 1996, a newsletter for teachers from the mathematics department of the University of Rhode Island devoted five pages of its January issue to "Ratios and Rational Numbers" ([3]). The editor writes:

> This is a collection of reactions and responses to the following note from a newly appointed teacher who wishes to remain anonymous:
>
> "On the first day of my teaching career, I defined a rational number to my eighth grade class as a number that can be expressed as a ratio of integers. A student asked me: What exactly are ratios? How do ratios differ from fractions? I gave some answers that I was not satisfied with. So I consulted some other teachers and texts. The result was confusion ..."

This is followed by three pages worth of input from teachers as well as the editor on this topic, each detailing his or her inconclusive findings after consulting existing texts and dictionaries (!). In a similar vein, Lamon (1999) writes: "As one moves from whole number into fraction, the variety and complexity of the siutation that give meaning to the symbols increases dramatically. Understanding of rational numbers involves the coordination of many different but interconnected ideas and interpretations. There are many different meanings that end up looking alike when they are written in fraction symbol" (pp. 30–31). All the while, students are told that no one single idea or interpretation is sufficiently clear to explain the "meaning" of a fraction. This is akin to telling someone how to get to a small town by car by offering fifty suggestions on what to watch for each time a fork in the road comes up and how to interpret the road signs along the way, **when a single clearly drawn road map would have gotten the job done**. Given these facts, is it any wonder that Lappan-Bouck (1998) and Lamon (1999) would lament that students "do" fractions without any idea of what they are doing? If we do not give our students correct information, it is a foregone conclusion that they will not learn. For example, it is certainly difficult for children to learn how to add two "operators" in the sense of (e) when all they know up to that point is how to add two numbers.

Sometimes one could "get by" a mathematical concept without a precise definition if its rules of operation are clearly explained. Conjecturally, that was how Europeans in the 14th and 15th centuries dealt with negative numbers. In the case of fractions, however, this is not true even when interpretation (b) of fractions is used. The worst case is the rule of adding two fractions. In book after book (with very few exceptions, such as Lang (1988)), $\frac{a}{b} + \frac{c}{d}$ is defined as $(pa + cq)/m$, where $m = \mathrm{lcm}\{\mathrm{b}, \mathrm{d}\}$ and $m = bp = cq$. Now at least two things are wrong with this definition. First, it turns off many students because they cannot differentiate between lcm and gcd. This definition therefore sets up an entirely unnecesary roadblock in students' path of learning. Second, from a mathematical point of view, this definition is seriously flawed because it tacitly implies that without the concept of the lcm of two integers, fractions cannot be added. If we push this reasoning another step, we would arrive at the absurd conclusion that unless an integral domain has the unique factorization property, its quotient field cannot be defined.

Informal surveys among teachers consistently reveal that many of their students simply give up learning fractions at the point of the introduction of addition. It is probably not just a matter of being confused by gcd and lcm, but more likely a feeling of bewilderment and disgust at being forced to learn a new way of doing addition that seems to bear no relation to what they already know about addition, namely, the addition of whole numbers. This then brings us to the problem area (3) at the beginning of this article. We see, for example, that Bezuk and Cramer (1989) willingly concede that "Children must adopt new rules for fractions that often conflict with well-established ideas about whole number" (p.156). In mathematics, one of the ultimate goals is to achieve simplicity. In the context of learning, it is highly desirable, perhaps even mandatory, that we convey this message of simplicity to students. However, when we tell students that a concept as simple as the addition of whole numbers must be different for fractions, we are certainly misleading them in the worst way. Even when students are willing to suspend disbelief and go along on such a weird journey, they pay a dear price. Indeed, there are recurrent reports of students at the University of California at Berkeley and at Stanford University who claim in their homework and exam papers that $\frac{a}{b} + \frac{a}{c} = \frac{a}{b+c}$ and $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$.

All in all, a mathematician approaching the subject of fractions in school mathematics cannot help but be struck by the total absence of the characteristic features of mathematics: precise definitions as starting point, logical progression from topic to topic, and most importantly, explanations that accompany each step. This is not to say that the teaching of fractions in elementary school should be rigidly formal from the beginning. Fractions should be informally introduced as early as the second grade (because even second graders need to worry about drinking "half a glass" of orange juice!), and there is no harm done in allowing children to get acquainted with fractions in an intuitive manner up to, say, the fourth grade. An analogy may be helpful here. The initial exploration of fractions may be taken as the "data-collecting phase" of a working scientist: just take it all in and worry about the meaning later. In time, however, the point will be reached when the said scientist must sit down to organize and theorize about his or her data. So it is that when students reach the fifth grade ([2]) or the sixth grade ([1]), their mathematical development cannot go forward unless "miracles" such as having one object $\frac{c}{d}$ enjoying the five different properties of (a)–(e) above are fully explained, and rules such as $\frac{a}{b}/\frac{c}{d} = \frac{ad}{bc}$ justified. And it at this critical juncture of students' mathematical education that I hope to make a contribution.

The work done on the teaching of fractions thus far has come mainly from the education community. Perhaps because of the recent emphasis on situated learning, fractions tend to be discussed *at the source*, in the sense that attention is invariably focussed on the interpretation of fractions in a "real world" setting. Since fractions are used in many contexts in many ways, students are led through myriad interpretations of a fraction from the beginning in order to get some idea of what a fraction is. At the end, a fraction is never defined and so the complexities tend to confuse rather than clarify (cf. (2) at the beginning of the article). More to the point, such an approach deprives students the opportunity to learn about an essential aspect of doing mathematics, namely, **when confronted with complications, try to abstract in order to achieve understanding**. Students' first serious encounter with the computation of fractions — generally in the fifth and sixth grades — would be the right moment in the school curriculum to begin emphasizing the abstract component of mathematics and make the abstraction a key point of classroom instruction. By so doing, one would also be giving students a head start in their quest for learning algebra. The ability to abstract, so essential in algebra, should be taught as early as possible in the school curriculum, which would mean during the teaching of fractions. By giving abstraction its due in teaching fractions, we would be easing students' passage to algebra as well.

It takes no insight to conclude that two things have to happen if mathematics education in K-8 is to improve: there must be textbooks that treats fractions logically, and teachers must have the requisite mathematical knowledge to guide their students through this rather sophisticated subject. I propose to take up the latter problem by writing a monograph to improve teachers' understanding of fractions.

The first and main objective of this monograph is to give a treatment of fractions and decimals for teachers of grades 5–8 which is mathematically correct in the sense that everything is explained and the explanations are sufficiently elementary to be understood by elementary school teachers. In view of what has already been said above, an analogy may further explain what this monograph hopes to accomplish. Imagine that we are mounting an exhibit of Rembrandt's paintings, and a vigorous discussion is taking place about the proper lighting to use and the kind of frames that would show off the paintings to best advantage. Good ideas are also being offered on the printing of a handsome catalogue for the exhibit and the proper way to publicize the exhibit in order to attract a wider audience. Then someone takes a closer look at the paintings and realizes that all these good ideas might go to waste because some of the paintings are fakes. So finally people see the need to focus on the most basic part of the exhibit—the paintings—before allowing the exhibit to go public. In like manner, what this monograph would try to do is to call attention to the need of putting the *mathematics* of fractions in proper order before lavishing the pedagogical strategies and classroom activities on the actual teaching.

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