The content knowledge mathematics teachers need*

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Abstract. We describe the mathematical content knowledge a teacher needs in order to achieve a basic level of competence in mathematics teaching. We also explain why content knowledge is essential for this purpose, how Textbook School Mathematics (TSM) stands in the way of providing teachers with this knowledge, and the relationship of this concept of content knowledge with pedagogical content knowledge (PCK).

Keywords. Pedagogical content knowledge, fundamental principles of mathematics, Textbook School Mathematics, definitions, and reasoning.

Contents

1 Introduction

This is the first in a projected series of papers that examine the content knowledge that mathematics teachers need in order to achieve a basic level of competence in mathematics teaching.[1] We share the belief with Ball, Thames, and Phelps (2008) that "Teachers must know the subject they teach. Indeed, there may be nothing more foundational to teacher competency." (*Ibid.*, page 404.) In subsequent articles, we will discuss specifically how to teach various topics from this perspective, such as long division, percent, ratio, rate, proportional reasoning, congruence and similarity, and slope.

What mathematics teachers need to know for teaching is a contentious issue in mathematics education. It is indeed a tall order to prescribe the content knowledge—beyond what is in the standard *school mathematics*[2] curriculum—that would enable a teacher to teach "effectively" in a school classroom. It becomes all the more forbidding when the desired level of effectiveness is not specified.

Since 1998, I have been engaged in providing a detailed answer to a far simpler question: "What is the mathematical knowledge that teachers need in order to achieve *teaching competence on the most basic level*"? I will give a more precise description of "basic teaching competence" in the next section, but it is much easier to begin by

[1] There should be no misunderstanding about what is being asserted: having this content knowledge is *necessary* for competent teaching.

[2] By *school mathematics*, we mean the mathematics of K–12.

describing several examples of teaching that I consider to be *below* this basic level.[3] One example is to teach a concept through several grades without ever giving that concept a precise definition, e.g., fraction, decimal, variable, slope, etc. This used to be the universal practice before the advent of the Common Core State Standards for Mathematics (**CCSSM** for short, see Common Core, 2010), but given the poor state of school textbooks, it is possible that this is still happening in many classrooms. Another is the failure to draw a sharp distinction between what is being *defined* and what is being *proved*, e.g., the assertion that a fraction is a division (of the numerator by the denominator), or the statement that $b^0 = 1$ (for a positive number b), or the statement that the graph of a quadratic function is a parabola (i.e., is this the definition of a "parabola" or is this a theorem that *proves* that the graph is a well-defined curve called a "parabola"?). Yet another is the careless blurring of the fine line between what is true and what is merely plausible. One of many such examples is the not uncommon attempt to show that, *without a definition of the division of fractions,* one can nevertheless arrive at the invert-and-multiply rule. Thus,

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{2}{3} \times \frac{5}{4}$$

because:

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3} \times (3 \times 5)}{\frac{4}{5} \times (3 \times 5)} = \frac{2 \times 5}{4 \times 3} = \frac{2}{3} \times \frac{5}{4} \qquad (1)$$

[3] These examples illustrate the almost universal bad practice forced on teachers by school textbooks from roughly 1970 to 2010; see Subsection 2.3 below.

This chain of pseudo-reasoning is most seductive, but it suffers from a multitude of errors, the most glaring being the justification for the first equality: it is supposed to be based on equivalent fractions. Unfortunately, equivalent fractions only guarantees that if m and n are *whole numbers*, then

$$\frac{m}{n} = \frac{m \times (3 \times 5)}{n \times (3 \times 5)} \tag{2}$$

What is needed to justify the first equality in (1), however, is for m and n in (2) to be equal to $\frac{2}{3}$ and $\frac{4}{5}$, respectively, and $\frac{2}{3}$ and $\frac{4}{5}$ are emphatically *not* whole numbers. As a final example of the kind of teaching that is below the most basic level, perhaps the failure to provide the reasoning for truly basic facts such as $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ or $(-a)(-b) = ab$ requires no further comment.

It is unfortunately a fact that, because of our collective dereliction of duty, most of our teachers have been forced to teach at a level below the basic level of competence (see, e.g., Wu, 2011b).

My effort to find out what content knowledge teachers need in order to achieve basic teaching competence took a practical turn when I began to provide professional development to preservice and inservice mathematics teachers of all grade levels in 2000; it lasted until 2013. Current practices in mathematics professional development have been to concentrate on instructional strategies (U.S. Department of Education, 2009, p. 89; Wu, 1999). Moreover, the teachers I taught have consistently told me that, whatever content-based professional development they got, it would be given in short workshops (half-day or one day, rarely two days) on specific topics. There have also been extended workshops lasting several weeks for teachers on "immersion in mathematics" devoted to problem-solving or doing mathematical research on topics sufficiently close to school mathematics (e.g., PCMI, 2016, or PROMYS, 2016). I made the decision from the beginning that I could better serve teachers by breaking with tradition. I would teach them, *systematically*, the mathematics they have to teach, but in a way that is both mathematically correct and adaptable to their classrooms. Such an endeavor requires long-term effort, e.g., three-week institutes strictly devoted to the *mathematics* of one or two major topics, with follow-up sessions throughout the year, or course-sequences in the mathematics departments of universities (see Wu, 1998 and Wu, 2011a for elementary teachers; Wu, 2010a and 2010b for middle school teachers; and the Appendix of Wu, 2011c for high school teachers).

It did not take me long to realize that these efforts will ultimately go nowhere unless we have on record at least one default model of a logical, coherent presentation of school mathematics *that is adaptable to the K–12 classroom.* Without such a presentation it is difficult to make the case that school mathematics, despite the need to be cognitively sensitive to the learning trajectory of school students, is nevertheless a discipline that respects mathematical integrity. In other words, the concepts and skills of school mathematics *can* be developed logically from one level to the next, and the transparency that one expects of mathematics proper is also attainable there. Without such a detailed presentation, our insistence that reasoning—and therewith problem-solving—must be everywhere in the school curriculum would also sound a bit hollow. Incidentally, an explanation of the need for such a presentation from the perspective of professional development will also be given on page 85.

For these reasons, I have embarked on a project of writing a series of textbooks for teachers that will cover all of school mathematics. Three have already appeared (Wu, 2011a, 2016a and 2016b), and three more to round out the series will probably be in print by 2018 (Wu, to appear).[4] This article will attempt to explain from the vantage point of what may be called **principle-based mathematics** (to be explained in detail on page 11) the content knowledge that teachers need in order to carry out their basic duty of teaching mathematics. In the process, we will also make contact with Shulman's concept of pedagogical content knowledge (Shulman, 1986) and its refinement in Ball, Thames, and Phelps, 2008.

If there is one thing I have learned through my many years of involvement with teachers, it is the melancholic realization that—as of 2016—relatively few educators and mathematicians seem to be aware of the *urgency* of the need to provide this content knowledge to mathematics teachers (compare the last paragraph of Section

[4] Although these professional development materials were written well before the CCSSM, they are compatible with the CCSSM because the first two served as a reference for the writing of the CCSSM. The CCSSM came to the same conclusion on numerous topics as these materials (fractions, rational numbers, use of symbols, middle school and high school geometry, etc.).

5). Our failure to do this has indirectly forced school students to memorize things that are unreasonable and incoherent, and therefore ultimately *unlearnable*. Yet we expect students to be proficient in "sense-making", "problem solving", and attaining "conceptual understanding", and when such irrational expectations are not met, we evaluate these same students and pass judgment on their inability to learn. It is time to stop inflicting such cruel and unusual punishment on the young. There is another victim of this strange education philosophy too: the teachers. In my experience, many of them are unhappy with the limitations in their content knowledge and are *eager to expand their mathematical horizon*, only to be frustrated by the overwhelming scarcity of resources to help them. We have let our teachers down for far too long.

Let us take a modest first step to making amends by providing a better mathematical education for teachers.

This article is organized as follows. Section 2 describes, on the one hand, the mathematical knowledge base of most teachers at present (which we call TSM; see pp. 23) ff.) and, on the other, the minimum mathematical knowledge that teachers need in order to achieve basic teaching competence. We also provide some threadbare data that is available to show why this knowledge would be beneficial to student learning. Section 3 attempts to give a more detailed description of the chasm that separates the two kinds of content knowledge. Section 4 explains what we mean by "knowing" a concept or a skill, and Section 5 makes some comments on the state of professional development at present and the hard work that lies ahead if our goal of providing teachers with this minimum knowledge is to be achieved. The last section, Section 6, makes contact with pedagogical content knowledge.

2 The two basic requirements

What is the mathematical knowledge that teachers need in order to teach at a basic competence level, and how to assess whether teachers *know* it? We will postpone the answer to the latter question to Section 4 but will try to answer the former in this section. Broadly speaking, this knowledge should enable teachers to teach procedural knowledge as well as the reasoning that supports it. It therefore asks for a knowledge of the most basic facts (e.g., the standard algorithms, operations on fractions, standard algebraic identities, and foundational theorems such as the Pythagorean Theorem or the angle sum of a triangle being $180°$) as well as correct, grade-level appropriate *mathematical* explanations for them, and the ability to "distinguish right from wrong", e.g., spot errors in "routine" situations related to these facts and be able to correct them. In particular, we will explicitly leave out from our considerations the more refined aspects of teaching (insofar as they are related to content knowledge) such as the ability to find more than one explanation for an assertion, give fruitful guidance to students' extemporaneous mathematical discussions, make up good examples or mathematical questions to pique students' interest, or make up good assessment items that probe students' understanding.

In the preceding section, it has already been mentioned in passing that the content knowledge that meets such a modest demand of basic teaching competence must satisfy, at least, both of the following requirements:

(1) It closely parallels what is taught in the school classroom.

(2) It respects the integrity of mathematics.

The first point should be self-evident: teachers should not be required to create new mathematics for their lessons, any more than violinists should compose the music they perform.[5] They should have a ready reference for what they teach. An additional reason for making this point explicit is that the mathematics community generally holds the conviction that teaching teachers the kind of mathematics *it* deems important will lead to educational improvement. The idea that, once teachers know the *good* stuff, they will somehow know the *elementary* stuff (school mathematics) better and therefore teach better[6] has led to the disastrous consequence that preservice teachers are typically not taught the mathematics of the K–12 curriculum in college. Another consequence is that many mathematicians, in their attempt to improve K–12 education, adopt the default position of teaching (preservice and inservice) teachers college topics that are elementary at the college level but are nevertheless too advanced for

[5] I am paraphrasing something said by Harold Stevenson at a TIMSS conference; see Math Forum@Drexel, 1998.

[6] This is the Intellectual Trickle Down Theory as described on page 41 of Wu, 2015.

the K–12 curriculum, such as finite geometry, discrete mathematics, number theory, etc. There is as yet no widespread recognition that *the mathematics of the K–12 curriculum is not a proper subset of the mathematics taught in college* (see p. 404 of Ball, Thames, and Phelps (2008); Wu, 2006; and pp. 42–47 of Wu, 2015), and therefore preservice mathematics teachers need explicit instruction on school mathematics.

The second point about teachers' need for content knowledge that respects the integrity of mathematics is even more of a no-brainer. If the goal of mathematics education is to teach students *mathematics*, then it is incumbent on us not to teach them anything less than *correct* mathematics. Therefore teachers' content knowledge cannot afford to be polluted by any kind of mathematics that has no mathematical integrity. Those not familiar with school mathematics or the state of school mathematics textbooks may be shocked that one would consider something this obvious to be worthy of discussion. Unfortunately, the reality is that our teachers' content knowledge—due to reasons to be explained in Subsection 2.3—has been a very flawed version of mathematics for a long time. There is some reason to believe that this kind of flawed mathematical knowledge is also shared by many education researchers so that these flaws cease to be noticeable in the education literature after a while. We bring up the issue of mathematical integrity precisely because we wish to provide a proper context for a fresh analysis of this body of flawed mathematical knowledge. This analysis will also reveal why it is so difficult for teachers to acquire the content knowledge they need.

Because we are mainly concerned with the nature of the content knowledge mathematics teachers need for basic teaching competence, we will leave out any discussion about the scope of the content knowledge that *a teacher of a particular grade* needs for this purpose. Without getting into details, we can nevertheless agree with the recommendation of the National Mathematics Advisory Panel that "teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching and the connections of that content to other important mathematics, both prior to and beyond the level they are assigned to teach" (National Mathematics Advisory Panel, 2008, page 38).

In the first subsection below, we will propose a workable definition of *mathematical integrity*. Because the idea of emphasizing *definitions* is so new in K–12, we make a few additional comments in Subsection 2.2 on this advocacy to preclude any misunderstanding. Then using this definition of mathematical integrity, we briefly describe in Subsection 2.3 the current state of most teachers' content knowledge. In the last subsection (pp. 28 ff.), we give some indication of why teaching correct mathematics is beneficial to mathematics learning.

2.1 Five fundamental principles

Any detailed discussion of teachers' content knowledge requires first of all a definition of "mathematical integrity". Like the concept of "beauty" in art and music, it is not likely that there will ever be a comprehensive definition of "mathematical integrity" that is agreeable to everyone. Nevertheless, we can propose a usable and reasonably short definition that *most* working mathematicians would consider unobjectionable. With this in mind, here are five **fundamental principles** that we believe characterize mathematical integrity (see Wu, 2011b):

(A) Every concept is precisely defined.

(B) Every assertion is supported by reasoning.

(C) Every assertion is precise.

(D) The presentation of mathematical topics is coherent.

(E) The presentation of mathematical topics is purposeful.

Before we amplify on these principles, let it be mentioned that, strictly speaking, these are fundamental principles that undergird what is called *pure mathematics.* For so-called *applied mathematics*, each of these principles will acquire a slightly different flavor. Nevertheless, for reasons to be discussed in Appendix 1 (page 88), it suffices to limit ourselves to (A)–(E) if our goal is to safeguard the mathematical integrity of *school mathematics.*

The first three principles, (A)–(C), are closely interrelated and therefore have to be discussed together. In mathematics, the starting point for any reasoning is a collection of *precise definitions* of concepts[7] and a collection of explicit assumptions

[7] Or *undefined terms* at the beginning of an axiomatic development.

or facts already known to be true. It is the unambiguous nature of the definitions, assumptions, or facts that enables them to serve as the foundation for correct logical deductions. The process of making logical deductions from precise definitions, assumptions, and facts in order to arrive at a desired conclusion is what we call *reasoning*, and reasoning is the vehicle that drives problem-solving.[8] It is therefore in the nature of mathematics that, without precise definitions, reasoning cannot get off the ground and therefore there will be no problem-solving. Those who lament students' inability to solve problems should look no further than the defective curricula around us that offer no precise (and correct) definitions for the most basic bread-and-butter concepts such as fractions, decimals, negative numbers, constant speed, slope, etc. (See Subsections 3.1 and 3.2 on pp. 36 and 46, respectively.)

It is easy to explain in everyday language why any mathematical discussion must rest on precise definitions. In a rational discourse, *we must know exactly what we are talking about,* and precise definitions serve the purpose of reminding us what we are talking about. Precision becomes even more critical when the discussion turns to *abstract* concepts and skills, which is what happens in the mathematics of middle school and high school. We need precision to minimize misunderstanding in the teaching and learning of mathematics because the precision helps to delimit, *exactly*, what each concept or assertion does or does not say. While human communication,

[8] In mathematics, there is no difference between *proving* and *problem-solving*.

being *human*, cannot maintain such precision at all times in a school classroom, there will come a time in any discussion of mathematics when such precision becomes absolutely indispensable. This is a persuasive argument that teachers should learn to judiciously nurture precision in the school classroom.

Beyond definitions, precision manifests itself in school mathematics in almost every conceivable way, and there is no end of such examples. Thus the domain of definition of the function $\log x$ is not $\{x \geq 0\}$ but $\{x > 0\}$; indeed the difference between the two is only one number, namely, 0, but that is the difference between nonsense and being correct. Another example: it seems plausible that if we have an inequality between numbers, let us say $a < b$, and if c is another number, then we have $ca < cb$. As is well-known, this is not correct because if c is negative, then the opposite is true, i.e., $ca > cb$, and if $c = 0$, then $ca = cb$. Therefore this assertion must be *precisely* announced as follows:

> Suppose two numbers a and b satisfy $a < b$. If $c > 0$, then $ca < cb$, but if $c < 0$, then $ca > cb$.

As a final example, if the three sides of a triangle are (of length) 20, 67.1, and 70, then it is not a right triangle. If you draw such a triangle using any unit of length (e.g., 20 cm, 67.1 cm, 70 cm) and measure the angles, you are most likely going to conclude, within the margin of error in measurements, that this is a right triangle. Yet, because $20^2 + 67.1^2 \neq 70^2$ (the left side is 4902.41 while the right side is 4900), we know by the Pythagorean Theorem that this cannot be a right triangle.

As for the critical role of reasoning in mathematics education, suffice it to note that rote-learning—the one quality in education that is universally decried—is nothing but the attempt to memorize in the absence of reasoning. When every assertion is seen to be supported by reasoning, students realize that mathematics is learnable after all because it is not faith-based and submission to another person's whimsical dictates is not required. For example, every elementary student has probably wondered *why* we cannot add fractions in the same simple way that we multiply fractions, i.e., why $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ but $\frac{a}{b} + \frac{c}{d} \neq \frac{a+c}{b+d}$. If people in education had ever given serious thought to this question, they would have realized the urgency of defining precisely the meaning of adding and multiplying fractions and then *proving the addition and multiplication formulas for fractions* (see Wu, 1998). Such a realization might have changed the landscape of teaching fractions several decades earlier. Needless to say, the same goes for all the arithmetic operations for fractions and for *rational numbers*[9] and indeed, for every assertion in school mathematics.

Next, let us turn to the concept of *coherence* in (D). The term "coherence" is often invoked in recent education discussions, but perhaps without realizing that it is quite subtle and can only be explained in terms of technical details. Roughly, it means that the body of knowledge we call mathematics, far from being a random collection

[9] *Rational numbers* is the correct terminology for *fractions and negative fractions;* it should not be conflated with *fractions*. Fractions are nonnegative rational numbers.

of facts, is a tapestry in which all the concepts and skills are logically interwoven to form a single piece. For example, the concept of division, when presented correctly, *is* essentially the same for whole numbers, fractions, rational numbers, and real numbers (this fact is emphasized in Wu, 2011a). And even for complex numbers. Right there, we see why coherence is vital for the teaching and learning of mathematics because it means that, if the concept of division is taught correctly the first time for whole numbers, it will spare learners the need to learn division anew on subsequent occasions. This message bears repeating because the division of fractions is still a much feared concept at the moment.[10] For another example, although the standard algorithms for whole numbers may seem to be four unrelated and unfathomable skills, they are all unified by a single idea: how to reduce all multi-digit computations to single-digit computations (cf. Wu, 2011a, Chapter 3). From this perspective, the success and the beauty of the standard algorithms are nothing short of stunning. They teach students the important lesson of reducing the complex to the simple, which is after all a main driving force behind all scientific investigations. Had this kind of coherence about the standard algorithms been widely understood among teachers and routinely taught in textbooks, it is doubtful that the Math Wars of the nineties would have erupted at all. We can push this line of reasoning one step further: the four arithmetic operations on fractions may seem to be unrelated skills

[10] "Ours is not to reason why, just invert and multiply."

until one realizes that they are conceptually the same as those on the whole numbers (this fact is especially emphasized in Part 2 of Wu, 2011a). Insofar as the whole of mathematics is coherent, there is no end of such examples, and some of them will naturally emerge in the discussions of the next section (Section 3). However, it should be obvious from this brief discussion that teachers must be aware of the coherence of mathematics if they want to be effective in the classroom.

Finally, the concept of *purposefulness* may also be a characteristic of mathematics that is hidden from a casual observer's view, but it is one of the main forces that shape mathematics from the most elementary part to the most advanced. Mathematics is goal-oriented, and every concept or skill is there for a *mathematical* purpose. This is especially true of school mathematics because the intense competition among the various topics to stay in the school curriculum naturally weeds out all but those that serve a compelling purpose. One of the most striking examples is the concept of *basic rigid motions* in the plane—translation, rotation, and reflection—that are standard topics in middle school. In TSM, these rigid motions are regarded as fun activities that shed light on the beauty of tessellations and Escher's prints (cf. page 33 of Conference Board of the Mathematical Sciences, 2001), and they lead to so-called *transformational geometry*, a novelty whose charm quickly gets lost in the technicalities of high school mathematics. But when these basic rigid motions are properly realized as the cornerstone for the concept of congruence in the plane, the *mathemat-*

ics of these rigid motions comes to the forefront and they become the thread that unifies middle and high school geometry (cf. Wu, 2010a, Chapter 4; CCSSM, 8.G and High School-Geometry; Wu, 2016a, Chapter 4). Another example is rational numbers (fractions and negative fractions). They are not just "another collection of numbers" that students must put up with, but are rather the agents that render the computations that one normally performs in solving linear equations (for example) entirely routine. (This may be likened to what the standard algorithms do for computations with whole numbers.) A final example is the concept of place value. In the way it is commonly taught in schools, this is a concept that primary students must accept, *by rote*, at all costs. Would it not be more productive to explain to them, no matter how informally, the fact that we *need place value* in order to count (and write) to any number by using only 10 symbols $\{0, 1, 2, \ldots 9\}$ *and* to also make number computations manageable at the same time? (See Sections 1.1 and 1.2 in Wu, 2011a, and pp. 13–31 in Wu, 2013a.)

2.2 Two caveats

Before we proceed further, we should clear up a common misunderstanding concerning the use of definitions in school mathematics. At present, there is great resistance to the idea of making the formulation of precise definitions a main focus of K–12 mathematics. Some textbook writers go so far as to refuse to let any reasoning be based on precise definitions because—as the saying goes—the definition of a concept emerges only *after* many explorations. Therefore some amplification on this idea is necessary.

Our insistence on the use of precise definitions as the basis for reasoning is not meant to be, literally, applicable to all of K–12 but only to roughly *grade* 5 *and up*.[11] These are the grades where reasoning begins to assume a critical role and the non-learning of mathematics starts to become most pronounced. We hasten to add that we do not by any means imply that definitions and reasoning do not matter in grades K–4; emphatically they do. After all, the foundation of learning how to reason from precise definitions must be laid in those grades. However, at least in K–3, the pedagogical and psychological components of teaching may be even more important than the content component. Therefore, a discussion of definitions and reasoning in the early grades will have to be more nuanced than is possible in the limited space we have here.

A second point we should make is that the use of definitions and the presentation of proofs in grades 5–12 must respect the reality of the school classroom. It is time to recall requirements (1) and (2) at the beginning of Section 2: we want mathematics that is both correct *and* usable in the school classroom. We therefore expect definitions to be introduced with motivation and background information, in ways that are

[11] In making this assertion, I am trying to be as conservative as possible. Larry Francis pointed out to me, for example, that the definition of a fraction as a certain point on the number line is essentially given in the third grade of the CCSSM: 3.NF.2.

grade-level appropriate.

We can illustrate with the teaching of fractions. By no later than the fifth grade, we expect a fraction to be defined as a point on the number line constructed in a prescribed way (see Wu, 1998, and Wu, 2011a, Part 2, and CCSSM, 5.NF).[12] But does this mean a fifth grade teacher should ram this definition down students' throats on day one of a fifth grade class? Not at all. We would expect something more persuasive to precede it. For example, when a textbook *for teachers* introduces this definition, it devotes six and a half pages to explaining the genesis and the need for such a definition (see Wu, 2011a, pp. 177–183). In fact, by the time this book gets to fractions, it has already spent a chapter explaining the virtues of the number line as a tool for codifying the mathematics of whole numbers (Wu, 2011a, Chapter 8). For another example, when the same book for teachers defines what fraction division means, it spends four pages reviewing the relevant definitions of subtraction and division for whole numbers and giving an intuitive meaning to the division of "simple" fractions (Wu, 2011a, pp. 283–286).

A final example is about the definition of a genuinely abstract concept, that of the **probability of an event.** This is without a doubt a difficult concept for middle school students. Therefore in a book for teachers (Wu, 2016a, pp. 121–141), no *general* definition of probability is given in the first twelve pages of the exposition

[12] Also see the preceding footnote.

on this topic. Instead, these twelve pages are devoted entirely to examples of coin tossing and dice throwing, and the probability *of each example* is defined specifically for that example; these definitions are relatively easy to accept because experiments can be performed to test the plausibility of each of these definitions. When the general definition is finally given at the end of these twelve pages of examples, the abstract pattern of the earlier definitions of the probability for each individual example is already in clear evidence, and the general definition becomes nothing but a summary of the earlier ones.

It remains to point out that the motivation for definitions in *student* textbooks will have to be even more expansive and more considerate. While one would not expect such elaborate preparation for the introduction of each and every definition, these three examples do serve the purpose of clarifying the recommendation that precise definitions be given in grades 5–12.

2.3 Textbook School Mathematics (TSM)

School textbooks are a powerful force in teachers' lives because teachers' lessons usually follow the textbooks. It is unfortunately the case that the mathematics encoded in the school textbooks of roughly the four decades from 1970 to 2010 is a very defective version of mathematics. Let us call it **Textbook School Mathematics (TSM)** (Wu, 2014a, Introduction; Wu, 2015). Because colleges and universities—as pointed out on page 11—make scant effort to help preservice teachers revisit and revamp their knowledge of TSM, what teachers know about school mathematics generally consists of nothing more than TSM. Consequently, teachers have no choice but to teach their students what they themselves were taught as school students so that they too imprint TSM on their own students. It therefore comes to pass that this body of defective mathematics knowledge gets recycled in schools from generation to generation.

In order for teachers to acquire a content knowledge base that respects mathematical integrity, i.e., satisfies condition (2), we must begin by helping them to recognize and replace their knowledge of TSM.

It is a legitimate question whether the concept of TSM has any validity. Does it exist? This question becomes all the more pressing when one realizes that the mathematics education reform of the 1990s (National Council of Teachers of Mathematics, 1989 and 2000) took place within the last four decades and the reform was a revolt against the school mathematics of the 1970s and 1980s. How can TSM possibly span both eras, pre-reform and post-reform? We will leave a more detailed answer to these questions to Appendix 2 (page 91) so as not to interrupt the present discussion of teachers' content knowledge. However, a little reflection will immediately reveal that the following features are equally common in pre-reform or post-reform texts: lack of *precise definitions* (e.g., fractions, negative numbers, the meaning of division of fractions, decimals, constant rate, percent, slope, etc.), the absence of *precise reasoning* for major skills (e.g., how to add or multiply fractions, how to multiply or divide decimals, why negative times negative is positive, how to write down the equation of a line passing two given points, how to locate the maximum or minimum of a quadratic function, etc.), and the failure to explain the *purpose* of studying major topics such as the standard algorithms, rounding off whole numbers or decimals, functions, exponential notation of numbers (why write $\sqrt{b}$ as $b^{1/2}$?), trigonometric functions (are right triangles *that* important?), etc. (Also see Wu, 2014a.)

The most egregious errors of TSM lie in rational numbers (especially in fractions), linear equations of two variables and linear functions of one variable, and middle school and high school geometry. Since these topics will be discussed at some length in the next section, what we are going to do here is describe how TSM, in its treatment of the laws of exponents in high school algebra, manages to violate all five fundamental principles of mathematics.

The *laws of exponents* in question state that for all $a, b > 0$ and for all *real numbers* s and t, we have:

$$\boldsymbol{(E1)} \quad \boldsymbol{a^s \cdot a^t \;=\; a^{s+t}}$$

$$\boldsymbol{(E2)} \quad \boldsymbol{(a^s)^t \;=\; a^{st}}$$

$$\boldsymbol{(E3)} \quad \boldsymbol{(a \cdot b)^s \;=\; a^s \cdot b^s}$$

The starting point is of course the easily verified simpler versions of (E1)–(E3) for all $a, b > 0$ and for all *positive integers* m and n,

$$(\boldsymbol{E1'}) \quad \boldsymbol{a^m \cdot a^n \; = \; a^{m+n}}$$

$$(\boldsymbol{E2'}) \quad \boldsymbol{(a^m)^n \; = \; a^{mn}}$$

$$(\boldsymbol{E3'}) \quad \boldsymbol{(a \cdot b)^n \; = \; a^n \cdot b^n}$$

The first order of business in generalizing $(E1')$–$(E3')$ to $(E1)$–$(E3)$ is to define a^0 and a^{-n} for any positive integer n. The way TSM tries to motivate the definition $a^0 = 1$ is by either asking students to believe that the validity of patterns $(\ldots a^3 = a^4/a$, $a^2 = a^3/a$, $a = a^2/a)$ also validates $a^0 = a^1/a = 1$, or by claiming that since $(E1')$ holds, we must have $a^2a^0 = a^{2+0} = a^2$ so that by dividing both sides of $a^2a^0 = a^2$ by a^2, we get $a^0 = 1$. This kind of *speculative reasoning* is of course an integral part of doing mathematics provided it is clearly understood to be *speculative*. However, precision not being a main concern of TSM, this motivation for the definition of a^0 is presented—informally to be sure—as "reasoning", and the result is that this motivation for a definition is commonly misconstrued as a proof of the theorem that *for any* $a > 0$, $a^0 = 1$. The same comment applies to the definitions of $a^{-n} = 1/a^n$ and $a^{1/n} = \sqrt[n]{a}$. Such imprecision contributes to teachers' confusion between what a *definition* is and what a *theorem* is.[13]

Once the concept of a^r has been defined for all rational numbers, the next step is to explain, to the extent possible, why $(E1)$–$(E3)$ are valid for all rational numbers s and t. Unfortunately, TSM simply dumps these laws of exponents for rational exponents

[13] I have personally witnessed this confusion not just in the U.S. but also in Australia and China.

on students with nary a word of explanation. Let us be clear: we do not want these laws for rational exponents to be completely proved in a high school classroom either, because these proofs are long and tedious (see, e.g., Wu, 2010b, pp. 183–191). Yet some special cases are so important that they deserve to be proved in full, e.g., the following special case of $(E3)$:

$$\sqrt[n]{a}\,\sqrt[n]{b} \;=\; \sqrt[n]{ab} \;\text{ for all positive integers } n \tag{3}$$

This equality, especially the case $n = 2$, is almost ubiquitous in the middle and high school mathematics curriculum, but it seems to be the case that either TSM assumes (3) without comment or, if a proof is attempted, it is not correct.[14]

Now the laws of exponents are taken up in textbooks long after the concept of a function has been taught. Therefore, there is no excuse for not pointing out, emphatically, that these laws are in fact remarkable properties of the exponential functions. Yet TSM introduces these laws almost always as "number facts", and even when it gets around to discussing exponential functions, no special effort is made to finally establish the relation of these so-called number facts with the exponential functions. Thus the real *purpose* of studying these laws of exponents (i.e., they are characteristic properties of exponential functions) goes by the wayside and students are likely to lose sight of the fact that it is automatic in mathematics to isolate the properties common

[14]Part of the difficulty of obtaining a correct proof of (3) is that the *uniqueness* of the positive n-th root of a is part of the definition of $\sqrt[n]{a}$, but TSM seems unwilling to confront the concept of uniqueness.

to a given class of functions. In this light, the laws of exponents are to exponential functions as the addition theorems (of sine and cosine) are to the trigonometric functions.[15] This is mathematical *coherence* in action. But, instead, TSM makes students believe that the exponential notation is a just a game we play in order to rewrite $\sqrt[n]{a}$ in the fancy notation $a^{1/n}$. Without any exposure to the reasoning behind the laws of exponents, students end up seeing these laws as undecipherable statements about a quaint notation that they must commit to memory.

There is an additional flaw in TSM in its failure to at least comment on the meaning of a^s when s is an irrational number such as π or $\sqrt{3}$. See the discussion in Chapter 9 of Wu, 2016b, that presents a more reasonable way to address the laws of exponents overall.

2.4 The data

Since the two requirements (1) and (2) on page 11 for the content knowledge that teachers need pull in opposite directions, it is by no means obvious how to provide teachers with this knowledge. Following Poon, 2014, let us call content knowledge that satisfies both requirements *principle-based mathematics*. TSM certainly satisfies requirement (1) of principle-based mathematics, but it fails requirement (2) in spectacular ways as we have just seen. Conversely, one can easily cobble together a coherent exposition of all the standard topics in school mathematics by making a judi-

[15] Or, more generally, as the addition theorems are to complex exponential functions.

cious selection of various pieces from the required courses of a university math major, but the result will not come close to resembling school mathematics, i.e., it cannot satisfy requirement (1). For example, to college math majors, a rational number—in particular a fraction—is just an equivalence class of ordered pairs of integers, but that is not something we would try to teach to fourth or fifth graders. Similarly, to these majors, the maximum of a quadratic function can be simply obtained by differentiating the function and setting the derivative equal to zero to obtain the point at which the function achieves the maximum. However, tenth or eleventh graders have to learn how to locate this maximum point without the benefit of calculus. And so on. Incidentally, these examples also give an indication of why school mathematics cannot be a proper subset of college mathematics (see page 12).

To the extent that the goal of school math education is to teach students *mathematics*, teachers cannot afford to teach them TSM, period. TSM is *incorrect* mathematics. *The need to replace teachers' knowledge of TSM by principle-based mathematics is therefore absolute.* Beyond such theoretical considerations, it would also be reassuring if we could get some indication from another source that principle-based mathematics is beneficial to mathematics learning. There *is* an indirect reassurance from the CCSSM. These standards have taken a major step in moving away from TSM to principle-based mathematics. One look at the standards on fractions (grade 3 to grade 6), rational numbers (grade 6 to grade 7), and geometry in grade 8 and high school will be enough to convince a reader of this fact. The belief in principle-based mathematics is therefore at least shared by some reasonable people. Beyond that, one would like to have some large scale data for this purpose. Thus far, there is little or no such data for the obvious reason that principle-based mathematics has not yet been available on a reasonable scale either in professional development for teachers or in the K-12 classroom. Perhaps more telling is the fact that, with rare exceptions (e.g., Hill, Rowan, and Ball, 2005; Ball, Hill, and Bass, 2005), the education research community has traditionally neglected content and its role in instruction (see the reference to the "missing paradigm" on page 6 of Shulman, 1986). What data we have is so meager that it borders on the anecdotal.

In her Berkeley dissertation (Poon, 2014), Rebecca Poon explored the impact of content knowledge training on student learning. She did a case study of four teachers (three in 4th grade and one in 6th grade) who received (to varying degrees) training in principle-based mathematics. Three were on the West coast (but not in California) and one on the East Coast. Through personal interviews and teachers' notes, she studied how these teachers taught one topic: the division interpretation of a fraction. This allowed her to sample the teachers' ability to faithfully implement the basic message of principle-based mathematics, especially definitions, precision, and reasoning. Then she looked at their students' state test scores and compared them to the scores of other *comparably matched*[16] students who were taught by teachers without any training in principle-based mathematics. Her conclusion is that "the average effect of PBM (principle-based mathematics) training on student achievement was significant and substantial" (*ibid.*, page 63), but there are uncertainties about whether the positive effect on student achievement can be attributed exclusively to the training in principle-based mathematics.

The article Alm and Jones, 2015, would seem to be the only relevant published article we can cite. The authors reported a success story about students in remedial courses in a small liberal arts college when principle-based mathematics (based on Wu, 2011a) was taught. They attribute the success to the emphasis on the use of precise definitions (particularly in fractions) and coherence (of fractions and algebra). The authors added:

> The a priori case that students are better off learning better mathematics is clear enough. The a posteriori case that student learning in the classroom is actually improved is more complicated (but anecdotal evidence and our observations certainly support it). In particular, small sample sizes are a major issue. We are currently working on constructing a multiyear study over several cohorts to measure the practical effectiveness of the approach described here. (*Ibid.*, footnote on page 1364.)

[16] This is a long story. Please see Sections 4.3, 6.1–6.3 of Poon, 2014.

My own summer institutes from 2000 to 2013 were devoted to principle-based mathematics. Over the years, teachers from those institutes have let me know how the institutes had impacted their students' learning, but none—with two exceptions—provided me with usable data. I will now briefly mention the results from those two exceptions. I will also mention the data from another teacher at the end.

Kyle Kirkman (`kirkmanks1@gmail.com`) was a first-year K–6 Math Interventionist in 2015–2016 at the Pan-American Charter School of Phoenix, AZ. The school uses the *Galileo K-12* Online Assessment System from Assessment Technology Incorporated (ATI). His charge was to work with *RTI* (Response to Intervention) students to bring them up to grade level. Students' progress is monitored by the "growth" of their test scores, measured in the following way. For each quarter (of the school year), students take a Galileo K-12 test at the beginning and another one at the end, and the score of the latter minus the score of the former is by definition their *growth* in the quarter. (The Galileo K-12 test at the end of the first quarter doubles as the test at the beginning of the second quarter, the test at the end of the second quarter doubles as the test at the beginning of the third quarter, and so on.) The following tables (numbers are rounded to the nearest one) summarize the comparison of the **average growth** of Kirkman's RTI students with that of the non-RTI students. Some comments will also be found after the tables.

Fall-QT 1, 2015:

	Gr K	Gr 1	Gr 2	Gr 3	Gr 4	Gr 5	Gr 6	**K–6 Av**
Non-RTI students	126	−31	15	49	30	37	8	33
RTI students	236	50	55	74	59	139	68	97
RTI student growth minus non-RTI student growth	**110**	**80**	**40**	**25**	**29**	**102**	**59**	**64**

Fall-QT 2, 2015:

	Gr K	Gr 1	Gr 2	Gr 3	Gr 4	Gr 5	Gr 6	**K–6 Av**
Non-RTI students	−13	63	48	41	98	15	63	45
RTI students	19	119	30	36	95	−10	64	50
RTI student growth minus non-RTI student growth	**32**	**56**	**−19**	**−5**	**−13**	**−25**	**2**	**6**

Spring, 2016:

	Gr K	Gr 1	Gr 2	Gr 3	Gr 4	Gr 5	Gr 6	**K–6 Av**
Non-RTI students	108	97	63	51	85	58	1	66
RTI students	188	177	78	96	142	100	52	119
RTI student growth minus non-RTI student growth	**80**	**80**	**15**	**45**	**57**	**42**	**51**	**53**

The average growth of the RTI students obviously far exceeds that of the non-RTI students except in the second table. Kirkman explained that in the second quarter, he stopped working with his students of the first quarter and got a new group of students. Moreover, in an effort to work with more students, he moved students in and out of his class in shorter intervals than a quarter. The strategy backfired, as the table shows. In the Spring, he worked with the same group of students all through the semester, and the Galileo K-12 test at the end of the third quarter was not administered.

He described how his knowledge of principle-based mathematics helped him:

> I have learned that precise mathematical definitions are critical. If precision is lacking, students will fill in any missing or vague elements of the definition with whatever happens to be present in their paradigm that seems to fit the idea. Not all of mathematics is intuitive in nature, so this can definitely lead to erroneous conclusions.

Larry Francis (`larrydotfrancis@gmail.com`) taught Title 1 math intervention groups, in 2014–2015, in grades 1 to 5 at Helman Elementary School of Ashland, OR. Grouped by grades, students came to his classroom for 30 minutes four times each week. Below is a comparison of the average grade-by-grade gains in percentile scores on the 2014–15 fall-spring easyCBM™ CCSS benchmark tests of his Title 1 students compared with those of their classmates in their home classrooms (classroom A and classroom B). In 9 out of 10 classrooms, these previously under-performing Title 1 students out-performed their classmates, sometimes dramatically. Title 1 students' scores have been removed from their respective classrooms' scores for this comparison. Furthermore, the fall-spring numbers are the nationally normed percentile scores according to easyCBM™.

	grade 1	grade 2	grade 3	grade 4	grade 5
title 1 math	19	10	11	2	16
classroom A	2	-12	7	-8	1
classroom B	-3	-8	3	4	12

According to Francis, "Precise definitions were crucial. Helping first and second graders with counting doesn't mean you need to tell them a bunch of definitions, but you need to make it clear that fundamentally a number is a thing you count with." What he learned from the summer institutes is "to reorganize my knowledge of arithmetic into a much more mathematical form. I continued to ‘know’ almost all the old things I used to know, but your [institutes] got me to reorganize that knowledge. . . . I am sure that reorganizing my knowledge contributed to my students’ successes."

Finally, I have some data from a teacher who was not at any of my summer institutes. Katie Bunsey (`kate.bunsey@lakewoodcityschools.org`) teaches fourth and fifth grades at Hayes Elementary School of Lakewood, OH. I happen to have been mentoring her, long distance, for the past three years on whole numbers and fraction using Wu, 2011a. She has just reported to me her fifth grade students’ 2016 math scores on the Ohio State Assessment (administered by AIR):

- 77% of her students scored proficient or above, whereas only 62% of Ohio’s fifth graders scored proficient or above, and only 63% of her school district’s fifth graders were proficient or above.

- Among those students who had her for two years (in their fourth and fifth grades), 84% were proficient or above, but among those who had her for only one year (in fifth grade), only 70% were proficient or above.

- Her students comprised only 16% of the district’s 5th grade population (65 out of 397), but 27% (respectively, 21%) of the district’s students who scored

Proficient (resp., Accelerated) were her students.

For the record, let it be mentioned that, together with an evaluation specialist, I did apply for grants (to NSF-EHR in 2010, and to IES in 2013) to evaluate the effectiveness of principle-based mathematics in the classroom. They were not funded.

3 TSM confronts mathematical integrity

We will discuss in this section, in considerable detail, the chasm that separates TSM from principle-based mathematics. It reveals the vast distance we will have to travel if we want to provide mathematics teachers with the content knowledge they need in order to competently discharge their basic obligation as teachers. What should stand out in the following discussion is the damage TSM has inflicted on mathematics learning. TSM is not learnable except by rote, as all irrational ideas are not learnable except by rote. If nothing else, this recognition should be incentive enough for us to join forces to undo this damage by eradicating TSM.

3.1 The importance of definitions: the case of fractions

Consider the teaching of fractions *in grade 5 and up.* In TSM, a fraction is not given a precise definition *that can be used as the starting point for logical reasoning.* The resulting absence of reasoning in the teaching of fractions therefore opens the floodgates to mathematics-with-no-reasoning, a.k.a. *rote-learning,* regardless of all the hands-on activities, analogies, and metaphors that rush in to fill this vacuum (cf. Wu, 2010c).

Although such a statement about the teaching of fractions is generally accepted by most as a given, it may nevertheless strike others as being too harsh. Let us therefore back it up by giving a more detailed analysis.

In TSM, fractions are usually introduced with pictures galore and fascinating stories about the different ways fractions are used in everyday life, together with the statement that a fraction can be interpreted as at least one of three things: parts-of-a-whole, quotient (division), and ratio. Here is one example:

> A fraction has three distinct meanings.
>
> Part-whole. The part-whole interpretation of a fraction such as $\frac{2}{3}$ indicates that a whole has been partitioned into three equal parts and two of those parts are being considered.
>
> Quotient. The fraction $\frac{2}{3}$ may also be considered as a quotient, $2 \div 3$. This interpretation also arises from a partitioning situation. Suppose you have some big cookies to give to three people. ...[If] you only have two cookies, one way to solve the problem is to divide each cookie into three equal parts and give each person $\frac{1}{3}$ of each cookie so that at the end, each person gets $\frac{1}{3} + \frac{1}{3}$ or $\frac{2}{3}$ cookies. So $2 \div 3 = \frac{2}{3}$.
>
> Ratio. The fraction $\frac{2}{3}$ may also represent a ratio situation, such as there are two boys for every three girls. (Reys, Lindquist, Lambdin, and Smith,

2009, page 266.)

The same viewpoint persists in the research literature. The usual introduction of the concept of a fraction is by way of the same multiple-representation approach:

> Rational numbers[17] can be interpreted in at least these six ways (referred to as subconstructs): a part-to-whole comparison, a decimal, a ratio, an indicated division (quotient), an operator, and a measure of continuous or discrete quantities. Kieren (1976) contends that a complete understanding of rational numbers requires not only an understanding of each of these separate subconstructs but also of how they interrelate. (Behr, Lesh, Post, and Silver, 1983, p. 92.)

The mathematical flaws of these "multiple interpretations" of fractions are analyzed in Wu, 2016a, pp. 5–6, and Wu, 2011a, page 178, respectively, but we are here to focus on the impact of such teaching on student learning. The overriding fact is that none of this information answers students' burning question about *what a fraction is.* To ask students to accept a fraction as part-whole, quotient, and ratio all at once is pedagogically untenable. First of all, the part-whole interpretation involves two whole numbers: the number of equal parts the whole has been divided and also the number of parts that are taken, so are we trying to tell them that a fraction is two numbers? The same is true for the ratio interpretation: the fraction $\frac{2}{3}$ is the

[17] This term is being used erroneously for *fractions.*

number 2 (the number of boys) *and* the number 3 (the number of girls). Two numbers again.[18] The pedagogical issue with the "quotient" interpretation is far more subtle and therefore far more insidious in the long run. Students' knowledge of "quotient" (division) is based entirely on their experience with whole numbers, where it is always about $24 \div 6$, $72 \div 4$, or $45 \div 15$. In other words, the dividend is known ahead of time to be a multiple of the divisor so that the "equal group" interpretation of division can make sense. Now teaching is generally about building on students' prior knowledge, and this time the prior knowledge is about "quotient". Keeping this in mind, can we as competent teachers ask students to divide 2 cookies into 3 equal groups? 6 cookies or 9 cookies, that is for sure. But *2* cookies? This is pedagogically unsound to say the least, because students' prior knowledge would not allow them to absorb this information. But since TSM insists on ramming this unnatural demand down their throats, right here TSM is pulling the rug from under their feet. Indeed, if they had any illusion at all about mathematics being *learnable* in the sense of a careful scaffolding of its steps with reasons given for the progression from one step to the next, it has been wiped out in one fell swoop. The formidable task they face is to try to understand a new gadget called a *fraction* by first submitting themselves to the uncomfortable proposition that whatever they have strived to learn about "quotient" is simply not good enough. Now they must ask themselves: what else must they

[18] Teachers that I have worked with told me consistently that students have difficulty conceptualizing a fraction as a single number.

unlearn before they can climb the mathematical ladder? Such thoughts cannot be an auspicious beginning for the arduous journey ahead.

It may be thought that the preceding analysis of "quotient" is not accurate because students do know about dividing an arbitrary whole number by a nonzero whole number before coming to fractions. For example, $5 \div 3$ is so-called "1 $R2$",[19] i.e., quotient 1 and remainder 2. In this light, $2 \div 3$ would be the two numbers 0 and 2, as in 0 $R2$. This then leads back to an earlier impasse about the meaning of the fraction $\frac{2}{3}$: this meaning of the fraction is *two numbers* 0 and 2. It is an insurmountable task to relate "0 and 2" to the concept of part-whole (i.e., partition the whole into 3 equal parts and consider 2 of them) and, failing to do that, we are guaranteeing non-learning again.

If we may use an analogy, asking students to believe at the outset that a fraction is three dissimilar things all at once is akin to asking them to look at a picture of a house obtained by superimposing three different views of the same house on each other. *Students get no clarity.* Such multiple representations of a fraction also beg the question: In a given situation, which representation should students use? Or should they use all three to make sure? Teaching based on TSM cannot provide answers to these natural questions.

Leaving students in this state of puzzlement, TSM nevertheless asks them to freely

[19]The multiple errors inherent in this notation 1 $R2$ should be better known. See, e.g., Wu, 2014b, p. 6

compute with fractions and use them to solve word problems. Can competent teaching afford to make students do things by rote for six or seven years (from grade 5 to grade 12) without informing them what they are doing? To make matters worse—or perhaps because of the lack of definition of a fraction—definitions for all concepts related to fractions seem to be completely missing as well. For example, students never get a precise definition for the intuitive and basic concept of "one fraction being bigger than another". Instead, they are taught that *if* they change both fractions to fractions with the same denominator, then they can see which is bigger. Now the reason this is worth pointing out is that it exemplifies a recurrent theme in TSM: Never mind whether you know what you are doing or not, because we are going to tell you *what to do*, and then you will get the right answer. As for the arithmetic operations on fractions, the plaintive refrain of "Ours is not to reason why, just invert and multiply" says it all: in TSM, one does not teach the definition for the division of fractions. The case of fraction addition, however, deserves a closer look (Wu, 1998, p. 24; Wu, 2011a, p. 228), and we will do just that.

In place of a precise definition of the addition of two fractions, TSM usually provides profuse verbal descriptions and pictorial illustrations of putting parts-of-a-whole together. Competent teaching on the most basic level however demands that, at this juncture of students' mathematics learning trajectory, they be exposed to a clear and logical argument that *leads from the definition of the sum of two fractions to* *an explicit formula for the sum.* Unhappily, without a precise definition of a fraction and a precise definition of the addition of fractions in TSM, such a demand cannot be met. What students get in place of reasoning is a formula for the sum involving LCD (*least common denominator*) that has to be memorized by rote. This is where fraction phobia seems to begin. Being cognizant of this fact, some have gone so far as to advocate de-emphasizing the addition of fractions, perhaps with a view towards reducing students' anxiety (e.g., National Council of Teachers of Mathematics, 1989, p. 96). In a climate of no-definitions and therefore no reasoning, any attempt at *teaching fractions for understanding*—no matter how well-intentioned—becomes an oxymoron.

It remains to explain why we believe students in grade 5 and up deserve to learn about the reasoning that leads from the *definition* of the sum of two fractions to the explicit formula for the sum. In a nutshell, this is the basic survival skill in navigating the mathematical waters of roughly grades 6 to 12. It therefore behooves students to begin acquiring this skill through the study of fractions. We have to recognize that the mathematics in grade 5 and beyond will be increasingly abstract and will be increasingly dependent on having precise definitions and logical deductions therefrom to make sense of the abstractions. The concept of fractions is the first genuine abstraction students face in school mathematics because fractions do not show up *naturally* in the real world (think of $\frac{7}{13}$ or $\frac{21}{11}$); if we want students to learn what a fraction is, it is incumbent on us to tell them, *precisely, what we want them to know about fractions.* This is what precise definitions can accomplish. If our goal is to nurture students' mastery of abstractions, then we can do no better than employ precise definitions in the teaching of fractions. Indeed, once students enter the world of fractions around the 4th or 5th grade, the march towards abstraction in the school curriculum becomes inexorable. Fractions are followed by negative numbers (particularly the multiplication and division of negative numbers), the use of "variables"[20] and the concept of *generality*, transformations of the plane and basic isometries, congruence and especially similarity, functions and their graphs, principle of mathematical induction, complex numbers, etc. The learning of each and every one of these concepts will require extra effort on the part of students—in the same way that the learning of fractions requires extra effort—because of the elevation in the level of abstraction. Competent teaching must therefore take note of students' battles ahead and prepare them accordingly.

Let it be known in no uncertain terms that we do not argue against appropriate use of stories, hand-on activities, and multiple representations to round off the intuitive picture of a concept *if a precise definition is part of the presentation and the primacy of the definition is understood* (see the protracted discussion of the definition of a fraction in Wu, 2011a, pp. 173–182, or Wu, 2016a, pp. 2–10). However, TSM

[20]Please see the discussion of "variables" on page 51.

promotes the idea that students can learn what an abstract concept such as fraction is, *without a definition,* solely by being exposed to a multitude of stories and activities to illustrate these multiple "meanings". This idea is predicated on the assumption that mathematics can be learned by what we call **inductive guessing**. This is the process of letting students work informally with a given concept to guess the properties this concept *might possess* and allowing their guesses to coalesce, over time, to form a complete picture of the concept. *But no precise definitions.* The fact that mathematics learning largely fails to materialize when fractions are taught exclusively by inductive guessing is by now beyond dispute. For example, fraction phobia has become almost a national pastime (there are numerous strips in the Peanuts and FoxTrot comic strips on fraction phobia). This failure has dramatically crystallized in a TIMSS fraction item for eighth grade, as pointed out in Askey, 2013. To my knowledge, there is no data to establish a causal relationship between inductive guessing and students' non-learning, but the ongoing school mathematics education crisis (cf. National Academy of Sciences, National Academy of Engineering, and Institute of Medicine, 2010; National Mathematics Advisory Panel, 2008) would seem to *strongly suggest* that such a causal relationship does exist.

The case against the sole reliance on *inductive guessing* in mathematics learning is rooted in the fact that correct reasoning requires a precise hypothesis as the starting point and a precise conclusion as the endpoint. If an abstract concept is nothing but the amalgamation of impressions accrued from inductive guessing, then it would be, by its very nature, imprecise because impressions vary from person to person. Consequently it cannot be *reliably* used in either the hypothesis or the conclusion of any reasoning and, without reasoning, there would be no mathematics. The virtue of a precise definition for an abstract concept is therefore that it "tames" the abstractness by providing precise information about *what the concept is*, no more and no less. Moreover, it is in the nature of mathematics that, once a definition is given, it will not change with time. If a fraction is defined in grade 4 to be a point on the number line constructed in a specific manner, then students can count on its being the same in every grade thereafter. This property of permanence makes the concept learnable because it allows students to stop wasting time trying to guess what a fraction might be in another situation but concentrate instead on getting to know fractions by using them in logical reasoning. In this way, students will get to derive *all* the known properties of fractions, including, in particular, what it means to add fractions and why the formula for adding fractions without using LCD is correct (cf. Wu, 2011a, Part 2, especially Section 14.1). No guesses, and no *deus ex machina*. Such an experience will give students the confidence that mathematics is learnable, and this confidence will in turn empower them to conquer the many more abstractions to come.

It remains to make a comment about the definition of a fraction as a point on the number line constructed in a prescribed manner (cf. Jensen, 2003; Wu, 1998; Wu, 2011a). In the event that such a definition is adopted, it is imperative to use the *same* definition throughout the whole development of fractions, including multiplication, division, ratio, and percent. If we abandon this definition at any point and choose, for example, to represent a fraction as a rectangle to discuss multiplication (as some have done), then we would be sending the erroneous signal that a definition is something we use when it is convenient but, otherwise, it is not to be taken seriously. Worse, we will be showing clearly that mathematics has no coherence, because it does not always tell the same story about a concept (fraction) but changes its story line at will. This will wreak havoc with student learning.

3.2 Other garbled definitions in TSM

There are more garbled definitions in TSM than we can count that have a profound effect on teaching and learning, but we will limit ourselves to four of them: decimals, constant rate, variable, and slope.

First, decimals. A finite decimal such as 0.2037 is defined in TSM as "2 tenths, 3 thousandths and 7 ten-thousandths". In terms of student learning, this causes damage in at least two different ways. The first is that it leads to students' misconception of a decimal as a fragmented collection of little bits of 2 tenths, 3 thousandths, and 7 ten-thousandths when they should be learning that a decimal is a single number. This is because *thousandths,* and *ten-thousandths,* etc., are almost invisible quantities to students in elementary school, and they don't know how to integrate these new tidbits into a single number. Could such a fragmented conception of a decimal be a factor in students' difficulty in comparing decimals and computing with decimals? This would make for an interesting research project in cognition. Secondly, if the vague statement "2 tenths, 3 thousandths and 7 ten-thousandths" is phrased in precise language, then it will state clearly that 0.2037 is *the sum of the following fractions:*

$$0.2037 \;=\; \frac{2}{10}+\frac{0}{100}+\frac{3}{1000}+\frac{7}{10000} \tag{4}$$

Unfortunately, TSM teaches decimals and fractions separately, making believe that they are different kinds of numbers (this may be the reason TSM uses the imprecise language "2 tenths, 3 thousandths and 7 ten-thousandths" to hide the fact that a finite decimal is a fraction). Since there is no attempt in TSM to ensure that the arithmetic of decimals is taught only after fraction addition has been introduced, the teaching of finite decimals in TSM is *mathematically* unlearnable.

A correct definition of 0.2037—historically as well as mathematically—is that it is the fraction

$$\frac{2037}{10000}$$

which is of course equal to the sum of four fractions on the right side of (4). Likewise, all finite decimals are nothing but fractions whose denominators are powers of 10 (see Wu, 2011a, page 187; Wu, 2016a, p. 17; CCSSM, 4.NF.5 and 4.NF.6). We may summarize the need of a correct definition for finite decimals as follows. On the one hand, it restores the *coherence* of mathematics by showing that, instead of three kids of numbers—whole numbers, decimals, and fractions—there is only one kind of numbers, namely, fractions. On the other hand, the correct definition allows for simple (and correct) explanations of the addition and multiplication algorithms for finite decimals. For the multiplication algorithm, TSM has forced teachers to teach *by rote* the correct placement of the decimal point in the product, whereas it is a simple consequence of the product formula for fractions (see Wu, 2011a, p. 269 and Wu, 2016a, pp. 68–69).

We will next look at the absence of any definition for *constant speed* or, more generally, for *constant rate* in TSM. We will show that this absence has very serious consequences because it spawns the bogus concept of *proportional reasoning*. We can begin the discussion with a typical rate problem:

> (P1). David drove 936 miles in 13 hours. At the same rate, how long will it take him to drive 576 miles?

According to TSM, we teach students that the "rate" of 936 miles in 13 hours should immediately suggest that we look for the "unit rate", which is $936/13 = 72$ mph. Therefore, proportional reasoning tells us that the answer is $576/72 = 8$ hours. Very simple. But is it?

If the problem had asked instead:

> (P2). David drove 936 miles in 13 hours. At the same rate, how long will

it take him to drive 2808 miles?

then this would be a reasonable problem for the following reason. No matter how one interprets "at the same rate", one would agree that it carries the information that, in every 13 hours, David covers 936 miles. So in 26 hours, he would cover 1872 miles ($1872 = 2 \times 936$), and every 39 hours he would cover 2808 miles ($2808 = 3 \times 936$). The answer to (P2) is therefore 39 hours.

But to ask how long it would take David to drive 576 miles? This adds complexity to the problem that makes (P1) unsolvable. Indeed, suppose David cruised for the first 10 hours at 70 mph, so that at the end of 10 hours, he had driven 700 miles. Knowing that he should get to his destination in 13 hours, he sped up and managed to cover the remaining 236 miles in 3 hours.[21] That was how he drove the 936 miles in 13 hours. Now if you want to know, *at the same rate*, how long it would take him to drive 576 miles, he will have to ask you whether it is the rate in the first 576 miles of his trip or the last 576 miles, or somewhere in between. If the first 576 miles, for example, then at 70 mph, it will take him $576/70 = 8\frac{8}{35}$ hours. *Not* 8 hours as claimed. Can anyone dispute that $8\frac{8}{35}$ is as good an answer as 8 to (P1)? Moreover, if we consider his rate in the last 576 miles of his trip, then it will take him $7\frac{6}{7}$ hours to cover 576 miles. Obviously there are other possibilities. Therefore, as is, (P1) is a problem that admits many correct solutions and, as such, it is not an acceptable

[21] He drove the last stretch of 236 miles in New Mexico where the freeway speed limit is 75 mph most of the time.

mathematics problem.

What happens is that the given data that David drives 936 miles in 13 hours is not precise enough to yield a definitive answer to (P1). The *implicit assumption* in all such problems in TSM that would render a definitive answer possible is that David drives at the same *constant speed* throughout. By bringing out this implicit assumption, we reformulate (P1) to read:

> (P3). David drives at a constant speed and he drove 936 miles in 13 hours.
>
> At the same constant speed, how long will it take him to drive 576 miles?

In TSM, the assumption of constant speed in such problems is usually missing, and even when it is mentioned, the concept of "constant speed" is understood *intuitively* without a precise definition. The idea seems to be that if the *words* themselves sound familiar, then definitions will be superfluous. Since an expression such as “driving at 70 miles an hour" is part of everyday language and people already have a *vague* understanding of it, a precise definition would be considered unnecessary in TSM. The fact is that the definition of "constant speed" is quite subtle, but when it is defined precisely and *is put to use in the solution of* (P3),[22] the solution turns out to be very simple. In particular, the correct solution does not make use of "proportional reasoning" in any shape or form (Wu, 2016a, pp. 108–115, and especially Section 7.2 of Wu, 2016b) and, instead of the mysterious invocation of "unit rate", it shows how

[22] We cannot overemphasize the fact that we need definitions in mathematics because they furnish the foundation for logical reasoning.

the concept of "unit rate" follows naturally from the definition of constant speed. Most importantly, the correct solution of such rate problems restores *reasoning* to teaching and mathematics education.

The cavalier attitude that TSM takes toward definitions also materializes in another form. Freed of the responsibility to provide definitions, TSM is at liberty to create fictitious mathematical concepts, the most notorious among these being that of a **variable**, “a quantity that varies". The inability to master this concept, according to an informal survey of the teachers that I have come in contact with, has been a real stumbling block for teachers and students alike in the learning of algebra. Yet, they feel compelled to grapple with this concept because:

> Understanding the concept of *variable* is crucial to the study of algebra; a major problem in students' efforts to understand and do algebra results from their narrow interpretation of the term. (National Council of Teachers of Mathematics, 1989, p. 102.)

I believe it is time for mathematics education to face the reality that "variable" is not a mathematical concept but is a cultural vestige of the way mathematicians in the eighteenth and nineteenth centuries referred to elements in the domain of definition of a function. If the function $f(t)$ describes the location of a moving particle in 3-space at time t, then as t *changes its value*, so does $f(t)$. So it is suggestive to think of t as a "variable". However, it is wrong to believe that learning must always be built on students' prior existing knowledge. Sometimes learning requires a revision, or at least some form of modification, of this knowledge. For example, we routinely speak about *sunrise* and *sunset* in everyday life, which suggests unmistakably that the sun revolves around the earth. Few would object to these expressions. But it will not do—purely for the sake of building on this prior misperception—to tell students in a science class that indeed the sun rises and sets because it revolves around the earth. At that point of students' education, it is time for them to recognize the limitations of the commonly used suggestive language and embrace the correct scientific information that it is the rotation of the earth that causes the illusion that the sun revolves around the earth. The truth is that the earth revolves around the sun.

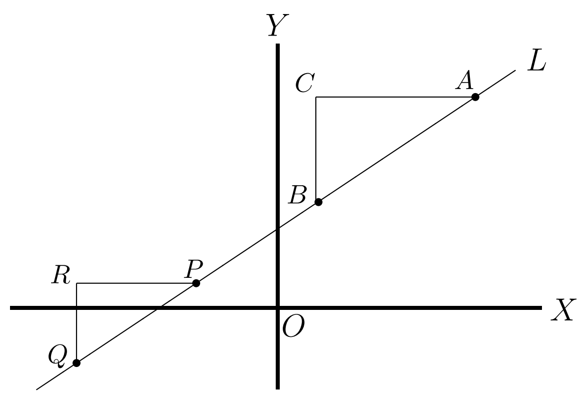
If we feel scandalized by a science class that does not clear up "sunrise" as a *human* misconception, then why do we complacently accept the teaching of "variable" in a mathematics class as a valid mathematical concept, or worse, that it is a “concept crucial to the study of algebra"? We do not advocate that we banish the word "variable" from mathematics because,

> . . . the word *variable* has been in use for more than three centuries and, sooner or later, you will run across it in the mathematics literature. The point is not to pretend that this word doesn't exist but, rather, to understand enough about the use of symbols to put so-called *"variables"* in the proper perspective. Think of the analogy with the concept of *alchemy*

> in chemistry; this word has been in use longer than *variable*. On the one hand, we do not want alchemy to be a basic building block of school chemistry, and, on the other hand, we want every school student to acquire enough knowledge about the structure of molecules to know why alchemy is an absurd idea. In a similar vein, while we do not make the concept of "variable" a basic building block of algebra, we want students to be so at ease with the use of symbols that they are not fazed by the abuse of the word "variable" because they know how to interpret it correctly. (Wu, 2016b, page 3.)

This discussion points to the need for school mathematics to move away from concepts without definitions—"variable" in this case—and engage students instead in the far more important issue of the correct use of symbols. When symbols are used correctly in school mathematics, "variable" as a *mathematical concept* will disappear from the school curriculum (cf. Wu, 2010b, Section 1; Wu, 2016b, Chapter 1).

Our final example of the mishandling of definitions in TSM is the concept of the *slope* of a line in the coordinate plane. Students' difficulty with slope is well-documented (cf. Postelnicu, 2011), but it does not seem that the education research that looks into this difficulty has taken note of a serious mathematical flaw in the usual definition given in TSM (the two papers of Newton and Poon, 2015a and 2015b, are among the exceptions). The TSM definition states that the *slope* of a (nonvertical) line L in the coordinate plane is the following "rise-over-run": let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be two distinct points on L, then the "rise-over-run" is $\frac{RQ}{RP}$ (the "rise" RQ and the "run" RP), where R is the point of intersection of the vertical line through Q and the horizontal line through P, and it would be minus this quantity if the line slants the other way. In a more compact form, the slope is the following ratio: $\dfrac{p_2 - q_2}{p_1 - q_1}$.



What is obviously missing this definition is the assurance that this ratio is the same regardless of which two points on L are chosen. In other words, suppose we take two other points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ on L instead of P and Q, then the ratio computed with A and B is the same as the one above computed with P and Q. More precisely, we should have:

$$\frac{p_2 - q_2}{p_1 - q_1} = \frac{a_2 - b_2}{a_1 - b_1} \tag{5}$$

This equality is important because if the slope of L is really a property of *the line L itself,* then it has to be the same number regardless of which two points on

L are chosen. Fortunately, equation (5) is indeed correct (see Section 4.3 of Wu, 2016b), but its proof requires some knowledge of similar triangles. The latter fact is not mentioned in TSM.

The reason a correct definition of slope, in the sense of making explicit equation (5), is important for mathematics learning is twofold. The first is that the general confusion about slope appears to include the misconception that it is a pair of numbers, "rise" and "run", but not *a single number* attached to the line itself.[23] In this light, one virtue of providing a proof of equation (5) is to reinforce the message that these are *numbers* that we are trying to prove to be equal. Such a proof may help to dislodge those students from this misconception.[24] A second reason is that it is difficult to solve problems related to *slope* without the explicit knowledge that slope can be computed by *choosing any two points on the line* that suit one's purpose. (Compare Wu, 2016b, pp. 72–76 on the proof of the graph of $ax + by = c$ being a line.) The lack of this knowledge is the cause of students' well-known difficulty with learning all aspects of the graphs of linear equations. For example, they are forced to memorize by brute force—often without success—the four forms of the equation of a line (two-point, point-slope, slope-intercept, and standard) because they are not taught any reasoning in connection with any part of the concept of slope. Accord-

[23] This echoes the phenomenon mentioned in Section 3.1 about students' confusion over a fraction also being a pair of numbers.

[24] It may be mentioned that the particular definition of slope in Section 4.3 of Wu, 2016b, brings out the fact from the beginning that the slope is a single number.

ing to a recent survey (Postelnicu and Greenes, 2012) of students' understanding of (straight) lines in introductory algebra, the most difficult problems for students are those requiring the *identification of the slope of a line from its graph.* That these research findings could actually be correct is almost unfathomable. Think about this for a moment: to compute the slope of a line, all you have to do is grab *any* two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ on the line and form the ratio $\frac{p_2 - q_2}{p_1 - q_1}$. This is trivial, but only if you happen to know, emphatically, that you can take *any two points* on the line for this purpose.

The correct use of definitions in school mathematics does matter after all.

3.3 Geometry in middle school and high school

The non-learning that has been taking place in the high school geometry course of TSM is perhaps too well-known to require comments (see, e.g., Schoenfeld, 1988). Incidentally, there may never be a better argument for the importance of teachers' content knowledge than Schoenfeld's account of what passes for "geometry teaching" in a TSM classroom. This kind of non-learning actually has its roots in the middle school curriculum and beyond. In this subsection, we will briefly summarize the three main issues and leave the more extended discussion to Section 4.1 of Wu, 2016a.

> (A) In TSM, the high school geometry course sticks out like a sore thumb among other courses in school mathematics. In the latter, reasoning is

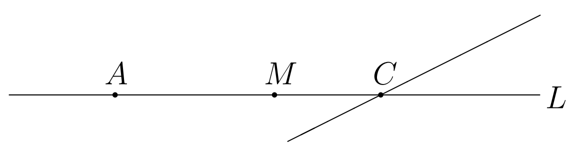
lacking and the opportunity to write a proof is nearly nonexistent, but in the former, literally *everything* demands a proof. This incongruity breeds non-learning.

(B) The discord between what is taught in middle school geometry regarding congruence, similarity, and scale drawing and what is taught about the same topics in the high school geometry course is too great for an average student to overcome.

(C) The high school geometry course is taught in isolation, as if it were unrelated to the rest of the school curriculum. In reality, certain geometric tools are critically needed for the teaching of slope of a line and the graphs of quadratic functions. The failure of the typical course to meet this need is an unfortunate missed opportunity to broaden its appeal and make it relevant to school mathematics.

We will add a few comments to round off the picture. Regarding (A), it has been a recurrent theme of this article to emphasize the overall lack of reasoning in TSM. Therefore students' transition into the high school geometry course may be likened to a non-swimmer being thrown into a lake in icy January and told to sink or swim. Trauma and bad results are pre-ordained. To make matters worse, the TSM high school geometry course also insists on starting with axioms and proving a series of boring and geometrically obvious theorems at the beginning.[25] For illustration, we will make use of the well-known text of Moise and Downs, 1964. We hasten to add that the text of Moise and Downs is on a higher plane than TSM, but it does following the tradition of "trying to prove everything". It was written in response to the call of the New Math of the 1960s (see Raimi, 2005 and Wikipedia, New Math). It purports to use a modified version of Hilbert's axioms of 1899 (cf. Hilbert, 1950) to prove *every* theorem in plane geometry. With this in mind, we find on page 177 the following theorem.

Theorem 6-5. *If M is a point between points A and C on a line L, then M and A are on the same side of any other line that contains C.*



One can imagine that not much of the discussion in the first 176 pages can be stimulating to the average beginner.

To give a little context to this discussion, let me relate a personal experiment. I taught the mathematics of the secondary curriculum (see Wu, 2011c, pp. 44–54) to pre-service high school teachers many times in 2006–2010. In these courses, proofs are provided for all the theorems, including all the major geometry theorems to be found

[25]For the lack of space, we will not take up the opposite kind of TSM geometry course which is all hands-on activities without a single proof. See, for example, Serra, 1997.

in a high school course and beyond (e.g., the nine-point circle). One day I suddenly popped the following question to a class of about twenty pre-service teachers: "You know that proofs in the high school geometry course are considered very difficult. Now that you have proved many geometric theorems much harder than those in your high school course, can you tell me whether you still find the proofs of these geometric theorems to be too hard?" It took them a few seconds to even understand my question, because (they later told me) having been with me for almost a year up to that point and having been conditioned to proving everything, they had ceased to differentiate between a geometric theorem from a non-geometric one. That was the reason they didn't understand what I was referring to. Naturally, their answer was *no*. The geometric proofs were not harder.

If principle-based mathematics is taught in K–12, the overall situation regarding (A) will improve considerably because students would be already accustomed to reasoning and proofs *before* they take the high school geometry course. The course itself can be improved too. One proposal of a new foundation for the course is to use the **basic isometries** (rotations, reflections, and translations) to define congruence, and use congruence and dilation to define similarity. Congruence and similarity now become tactile concepts rather than abstract inscrutable ones, and the classical criteria for triangle congruence (SAS, ASA, SSS) can now be proved as theorems. In addition, by assuming sufficiently many facts to get the geometric development started, we also avoid having to prove many uninteresting and possibly subtle theorems at the beginning, such as Theorem 6-5 in Moise and Downs, 1964. (For more details, see Wu, 2016a, Chapters 4 and 5; Wu, 2013b.) It is easy to believe that such a new foundation will provide an easier access to geometry for students, but obtaining data to verify this fact may be less easy. It will have to be large scale, long-term, and therefore expensive. However, the fact that the CCSSM also came to the same conclusion regarding such a new foundation gives us hope that there will be ample data on this issue in the years ahead.

Congruence and similarity provide a natural segue to (B) above on the discontinuity between middle school geometry and high school geometry in TSM.

There are two major disruptions in the transition from middle school geometry to high school geometry. First, TSM defines *congruence* as same size and same shape, and *similarity* as same shape but not necessarily the same size. These statements are intuitive and attractive, but they are comically inadequate as *mathematical definitions* because they lack *precision.* For example, if we draw the acute triangle with three sides of lengths 20, 67.1, and 70 with respect to any unit of measurement (see page 16), then it will appear to have the same size and same shape as the right triangle of sides 20, $30\sqrt{5}$, and 70.[26] But these two triangles are not congruent. Of course, such a "definition" of congruence or similarity has the virtue that it is applicable

[26]Note that $30\sqrt{5} = 67.082\ldots$

to any shape, curved or otherwise. But in high school congruence and similarity are suddenly defined precisely for polygons in terms of corresponding angles and corresponding sides, *and for nothing else.* Does this mean that one can only reason about polygons when it comes to the concepts of congruence and similarity but that there is no way to express whether two parabolas, for example, are congruent or similar? This jarring discrepancy does no service to the *coherence* of mathematics or to mathematics learning.

It goes without saying that, with such inadequate definitions, the middle school geometry of TSM cannot sustain any *reasoning* about congruence or similarity. And there is none.

A second major disruption in the teaching of congruence and similarity lies in the way TSM treats the basic isometries in middle school and high school. In middle school, basic isometries are taught only for the purpose of fun activities and art appreciation, e.g., the sometimes subtle symmetries exhibited in Escher's prints and how the beauty of tessellations in church windows and Islamic mosaic art is enhanced by the different kinds of symmetries. Nothing about the *purposefulness* of the basic isometries in school mathematics. Consequently, teachers who are immersed in TSM get the mistaken idea that the basic isometries are valuable only for so-called *transformational geometry*, which is roughly about doing the fun activities of moving geometric figures around the plane—using a coordinate system if necessary—and identifying symmetries in art works. To these teachers, the basic isometries are not about mathematics at all because the isometries appear to have nothing to do with the proofs of theorems in the high school course. While there are references to basic isometries near the end of some high school textbooks, they are mostly ornamental. In TSM, the basic isometries are long forgotten by the time of the high school geometry course. In this climate, it is therefore not surprising that, when in 2012 the Department of Education of a state on the East Coast produced a document on CCSSM geometry for its high school teachers, all 80 pages of it were devoted to transformational geometry but not a word about the serious business of using the basic isometries to understand congruence and proofs in high school geometry.

It should be quite clear that teachers' knowledge of TSM geometry will not enable them to teach the geometry of middle school or high school in any sensible way. We must help them to revamp their knowledge base. This is another reminder that teachers' content knowledge does matter.

Finally we briefly discuss the critical role of geometry in making sense of the algebra of linear and quadratic functions. The need of similar triangles for an understanding of the slope of a line has already been brought out in Section 3.2. The CCSSM have already asked for a reshuffling of our middle school curriculum so that 8th graders are at ease with the AA criterion for triangle similarity when they take up the graph of linear equations in two variables (CCSSM, 8.G.5). As for quadratic functions, the long and short of it is that, the graph of $f(x) = ax^2 + bx + c$ is a translation (in the sense of basic isometries) of $F_a(x) = ax^2$, and furthermore, the graphs of $F_a(x) = ax^2$ (where $a > 0$) are similar to each other under a dilation with center at the origin O (Sections 10.2 and 10.3 in Wu, 2016b). These two facts together clarify the structure of quadratic functions: at least conceptually, every quadratic function is qualitatively the same as the function $F_1(x) = x^2$.

At the moment, the above approach to quadratic functions is inaccessible to students because translations are not precisely defined in the usual high school geometry course, and similarity between graphs of quadratic function does not make mathematical sense because similarity applies only to polygons. The chasm between TSM and principle-based mathematics is real indeed.

3.4 How coherence and purposefulness impact learning

It is easy to explain, in theory, the reason that mathematics developed *coherently* and *purposefully* will improve student learning. Obviously, when some events are told as a coherent story and the narrative is propelled forward with a purpose, they will be more memorable to readers than if the same events are presented as a laundry list. This is why even a rushed reading of *Don Quixote*—all one thousand pages of it—will leave readers with vivid memories of the Don's amazing exploits, whereas reading pages of a phone book, no matter how conscientiously done, will leave the readers with no memorable highlights whatsoever. We will present two examples that are consistent with such a narrative. The first one shows how *in*coherent mathematics can impede mathematics learning, and the other suggests that, by infusing the teaching of a seemingly boring topic with *purposefulness*, one can make it more learnable.

The first example is the way TSM teaches equivalent fractions, the basic tool students need to put any two fractions on a common footing (Wu, 2011a, Section 13.4). To see, for example why $\frac{7}{3} = \frac{14}{6}$, TSM provides the following explanation:

$$\frac{7}{3} = 1 \times \frac{7}{3} = \boxed{\frac{2}{2} \times \frac{7}{3} = \frac{2 \times 7}{2 \times 3}} = \frac{14}{6} \tag{6}$$

The problem with such an "explanation" lies in the step enclosed in the box: It assumes, inexplicably, that before students know what it means to *add* two fractions with unequal denominators, they already know how to *multiply* them ($\frac{2}{2}$ and $\frac{7}{3}$) by the so-called *product formula*, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. In greater detail, a coherent mathematical progression through fractions could reach the product formula by at least one of two ways: either

definition of fraction $\longrightarrow$ equivalent fractions $\longrightarrow$ definition of fraction multiplication using "fraction of a fraction" $\longrightarrow$ the product formula

(Jensen, 2003, Section 7.1; Wu, 2016a, pp. 60 ff.; CCSSM, 5.NF.4), or

definition of fraction $\longrightarrow$ definition of fraction multiplication using *area* of a rectangle $\longrightarrow$ the product formula

(Wu, 1998, page 25; Wu, 2011a, pp. 263 ff.). In either case, the proof of the product formula is a difficult one for fifth graders and should by no means be used to explain something as basic as equivalent fractions. What TSM has done in (6) is to shred the basic structure of mathematics for the expediency of making equivalent fractions *look* easy. However, there is a price to pay: Students get the idea that, although they don't know what " $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ " means, they are supposed to believe it because it *looks right*. This naturally suggests to them that, in mathematics, *if it looks right, it must be true*. So why not $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$? It is a popular sport to lament that even some freshmen in prestigious universities make such a ghastly mistake, but this is not students' fall from grace. This is our collective handiwork!

Before giving the second example, we first make a few observations about purposefulness in the context of mathematics learning. The first is that mathematical research is overwhelmingly about investigations with a purpose. The purpose of an investigation is always front-and-center because it provides a focal point for the researcher's thinking. Serious mathematical work is rarely the result of a random walk through the mathematical jungle to pick up low-hanging fruits. The school curriculum, being the distillation of serious mathematical work through the ages, should reflect as much as possible the purposefulness of such investigations. There is an additional connection between *research* and *learning*: they are fundamentally two sides of the same coin. They are both driven by curiosity, and researchers and learners alike try to peer into the (to them) unknown.[27] For this reason, learners will benefit from knowing the purpose of learning a new concept or a new skill because the purpose helps them to focus their own thinking too. Teachers should be aware of this aspect of mathematics learning, and for this reason, should get to know the purpose behind every concept and every theorem.[28]

Now consider the teaching of the skill of *rounding* (to the nearest hundred, nearest thousand, etc.) in TSM. Personally I have never come across a teacher who is not bored by this skill as presented in TSM; it seems to be totally pointless and mechanical, but it doesn't have to be that way. Imagine a teacher engaging students in a discussion of what they hear from TV or radio about the temperature of the day. Ask students why they often hear things like "today is a mild day in the 70s". Why not say "today's temperature will be 74"? Make them realize that such precision is both *unattainable* and *unnecessary*. Ask them if they would change the way they dress if the temperature were 72 instead of 74, but also point out that they would likely change if the temperature were "in the 60s" rather than "in the 70s". Next, ask them in case the temperature is 68 degrees whether they would describe it as "about 60 degrees" or "about 70 degrees"? Pursuing this line of give-and-take, a teacher can lead students to naturally *round to the nearest ten* without using any jargon or imposing

[27]Of course there is also a big difference: whereas a helping hand is usually available if the learner fails to learn, there is no such helping hand when a researcher gets stuck!

[28]In this day and age when inquiry-based learning is encouraged, we hope that such learning can be conducted by emphasizing the *purpose* of any inquiry.

any rigid rules. Then tell them after the fact that what they did is what is known as "rounding to the nearest ten", and that they did so because they were in fact trying to strike a happy medium between being informative and being sensible. In any case, by the time the teacher gets around to summarizing their findings of "rounding to the nearest ten" into some simple rules ("34 will be rounded down to 30, but 78 will be rounded up to 80"), the rules will sound neither boring nor pointless. They have a *purpose.*

The teacher can likewise talk about the population of a city like Berkeley. In Wikipedia, the estimated 2014 population is 118,853. Ask the students how much faith they have in this estimate. Consider the daily births and deaths, the expected influx and outflow of people, and other issues such as the homeless population and undocumented immigrants. Ask them whether they think it is appropriate to list the estimate as 118,*853.* Do they think the last three digits, 853, mean anything? If not, how do they want to list it? 118,000 or 119,000? In fact, for the purpose of general information, wouldn't an estimate of 120,000 make more sense? Now let them know they are *learning to make a decision* about whether to round to the nearest thousand or nearest ten thousand. In this case, the skill of rounding serves the *purpose* of making sense of the world around us. It doesn't have to be a fossilized skill from TSM at all.

For more details on rounding and estimation from this perspective, see Wu, 2011a, Chapter 10.

4 What does it mean to *know* a fact in mathematics

We started off this article by asking what mathematics teachers need to know in order to achieve basic teaching competence. Having described in some detail the nature of this content knowledge, we now bring closure by addressing what it means to *know* a fact mathematically (this discussion should be compared with Ball, Hill, and Bass, 2005). In mathematics education, *knowing* a fact commonly means *knowing it by heart* (having memorized it). In mathematics, however, the same word means much more. To say you **know a fact** in mathematics means you know:

(a) what it says *precisely*,

(b) what it says *intuitively*,

(c) why it is true,

(d) why it is worth knowing,

(e) in what ways it can be put to use,

(f) how to put it in the proper perspective.

See Wu, 2013a, page 11. As in the case of "mathematical integrity", there is no pretense that such a characterization of "know" will be accepted by all mathematicians, but undoubtedly most would find it acceptable. The idea is that *knowing a fact* means being able to tell the *whole story about this fact* rather than just a few sound bites. It should also be said that there may not always be a good answer to each and every one of (d)–(f) in every situation. Moreover, since we are asking for content knowledge to ensure teaching competence *at the most basic level,* we will ignore (f) in subsequent discussions as its answer tends to be more sophisticated (see, e.g., pp. 25 and 38 of Wu, 2013a). That said, I believe a teacher should certainly make an effort to raise these questions all the time and try to get as many of them answered as possible. (Incidentally, the ability to answer most of these questions most of the time is intimately related to the coherence of mathematics.)

Again, it should come as no surprise that these are questions all mathematics researchers ask themselves again and again throughout the course of their work. Recalling once more the kinship between research and learning, we recognize that many students will be pondering the same questions (regardless of whether or not they can explicitly articulate them) when they are confronted by a new concept or a new theorem. A teacher must come prepared for these questions.

We give an example: what should a teacher know about the theorem on equivalent fractions? We will answer the preceding questions (a)–(e) in the same order:

(a) Given a fraction $\frac{m}{n}$, then for any nonzero whole number c, $\frac{m}{n} = \frac{cm}{cn}$. (In a fifth grade classroom, one will have to begin by using concrete numbers rather than symbols.)

(b) Don't get hung-up on the fraction symbol, e.g., $\frac{2}{3}$; it is the corresponding *point* on the number line that counts. A fraction is *a certain point on the number line,* and the symbol is nothing more than a representation of the point. Also get used to recognizing $\frac{2}{3}$ as $\frac{24}{36}$ or $\frac{18}{27}$. The moral is: nether the numerator nor the denominator in a fraction symbol means all that much by itself; it is the relative size of the numerator *and* the denominator that matters. For example, if we ask for *half* of $\frac{2}{3}$ of an apple pie, it is obvious: $\frac{1}{3}$ of the pie. Now if we ask for *a fifth* of the same amount of apple pie, is it any harder? Not much, because $\frac{2}{3} = \frac{10}{15}$, so a fifth of $\frac{10}{15}$ of the apple pie is $\frac{2}{15}$ of the pie.

(c) To prove $\frac{2}{3} = \frac{5\times 2}{5\times 3}$, for example, we ask whether the 2nd point in the sequence of thirds on the number line is the *same* point as the 10th point in the sequence of fifteenths. If we divide each segment between consecutive points in the sequence of thirds into 5 segments of equal length, the unit $[0, 1]$ is immediately divided into 15 equal parts and we get the sequence of fifteenths. Now count carefully, and the truth of the assertion is obvious.

The general proof is no different.

(d) If you work with fractions at all, you will be seeing equivalent fractions all day long. This theorem figures prominently in every discussion of fractions, including the hows and whys of the arithmetic operations on fractions: $+$, $-$, $\times$, $\div$.

(e) Each time you get stuck on a problem involving fractions, your conditioned reflex ought to be: can I use equivalent fractions to get me out of this jam? More often than you can imagine, this strategy will work. For example, see (b) above.

5 Professional development

It may be self-evident at this point, but we will nevertheless demonstrate presently, that any professional development (PD) that manages to pry open the grip of TSM on teachers and introduce them to principle-based mathematics will not be easy to come by. Let us consider two examples.

First, suppose some high school teachers want you to help them learn about quadratic functions. TSM being what it is, you know they are likely to have been misinformed about the need to understand the graphs of quadratic functions for the purpose of understanding the functions themselves. Equally likely, they may not realize that the study of quadratic *equations* is a very small part of the study of quadratic functions. They may also have been misinformed about the technique of completing the square, the fact that it is just as important for the study of *functions* as for the derivation of the quadratic formula. There will be a lot to talk about, but you feel comfortable telling the teachers that two days of PD should be enough.

Next, suppose some elementary teachers come to you and ask for PD that explains why definitions are important. You probably do a double-take before replying because that is a big job! In your thinking, you may likely begin with the definition of fractions. Considering how much misinformation about fractions has been handed out in TSM, you figure that one day may not be enough to explain why the various TSM "definitions" of a fraction are not mathematically acceptable, and why the definition in terms of the number line will promote better learning. Because the teachers are likely not to have come across any definitions for the addition, subtraction, multiplication, and division of fractions either, you want to take this opportunity to explain how this absence has led to “Ours is not to reason why, just invert and multiply", among other things. You want to convince them that having definitions for these operations is as important as getting the computational formulas because the definitions will make it possible to *explain* these formulas. This will take more time, because you cannot just *tell* them what the definitions of the operations are and move on; you must also explain the associated reasoning in detail because they have never seen it before. There is another reason you cannot rush them: they have been living with mathematics-without-definitions all through K–16 as well as all through their professional lives. You cannot change someone’s habits of twenty-some years overnight. You will need even more time.

But it is not just fractions that need definitions; whole numbers do too. In TSM, even concepts in the whole numbers do not have definitions. Few teachers will remember the definition of adding or multiplying two whole numbers (see, e.g., Case 12 of Schifter, Bastable, and Russell, 1999), much less why these definitions are relevant. After all, can the algorithms not be taught simply by rote? Therefore few will be able to explain the virtues of the standard algorithms for addition and multiplication, among other things. In fact, even fewer will be able to give a precise definition of the long division *algorithm*. Recall: to define an algorithm one must state the *precise procedure* as well as the *desired* outcome in a general context. To the extent that neither appears to have been done in the education literature, you begin to realize that explaining the significance of definitions is much more than changing the teachers' perception about "definitions" per se. You are in fact called upon to revamp their mathematical knowledge base—which is steeped in TSM—into principle-based mathematics. You have to change their belief system *and* rebuild their content knowledge from the ground up. Clearly *two weeks* will not be enough.

These examples hint at the difference between the run-of-the-mill kind of PD and the kind that aims at providing teachers with principle-based mathematics. The fundamental difficulty with the latter is the stranglehold that TSM has had on teachers for such a long time; if we want to enable teachers to teach correct mathematics, we will have to retrofit their knowledge base. This is hard, unpleasant work.[29] Let us

[29]For an example of why there can be no short-cuts in this kind of professional development, see the analysis of Garet et al., 2011, in Wu, 2011c, pp. 20–31.

start with preservice teachers. They have had thirteen years of TSM by the time they get to college. Even if their undergraduate program offers courses on K–12 mathematics, these courses will have the burden of convincing them, point-counter-point, that the TSM they are familiar with is not correct and therefore not learnable by students, so that they had better replace it with something that is logical and coherent. This is a hard sell because, in all the years preservice teachers were in school, they saw with their own eyes that "mathematics" (i.e., TSM) was nothing more than a bag of tricks to memorize in order to score well on standardized tests and move on to the next class. They had no conception of the *logical and coherent progression of ideas* in principle-based mathematics. For example, they have all been taught that it is legitimate to "prove" equivalent fractions, i.e., $\frac{m}{n} = \frac{mc}{nc}$, by the following string of equalities (see equation (6) on page 64),

$$\frac{m}{n} = \frac{m}{n} \times 1 = \frac{m}{n} \times \frac{c}{c} = \frac{mc}{nc}.$$

Now imagine the hard work that is necessary to retroactively explain to them the fatal mathematical *incoherence* in this one line.

If we try to teach fractions without *directly* confronting pre-service teachers with such fatal errors but only tell them what the correct reasoning is, will they realize on their own that *what they think they know* is wrong? If not, how then can we expect them to turn around and be advocates for principle-based mathematics? Changing teachers' minds about the precision, reasoning, and coherence of mathematics is clearly more than making a few tweaks here and there in the TSM they know. We will have to retrace essentially all the mathematics they have ever learned in school and *revamp it systematically* before any new ideas of principle-based mathematics can hope to sink in. At this point, perhaps what was said in Section 1 about the need for *long-term* PD will begin to make sense.

The last I heard, the pervasive dominance of TSM in school mathematics is largely unknown and unmentioned in the education literature and in Schools of Education, and the need for content-based professional development is widely ignored. Certainly the urgent need of professional development to explicitly undo the ills of TSM is unheard of. In addition, there is as yet no awareness in most mathematics departments that the standard math majors do not necessarily make good high school teachers. If there are still any doubts about this fact, the recent study of Newton and Poon, 2015a, should lay them to rest. Beyond this awareness, there is the obstacle of finding the right personnel to do this kind of PD. We have a long way to go.

The issues facing the PD of *inservice* teachers are even more dire. Districts do not invest (or do not have the funds to invest) in long-term professional development, and the teachers in the trenches do not have the time and energy to make the intensive effort to relearn the content during the regular school year. Unless something extraordinary happens soon, TSM will continue to be the default content in teaching and learning in schools for the foreseeable future.

Finally, we should address the naive question of why not just expunge TSM by the most direct method possible, namely, by rewriting school textbooks? To properly answer this question will take a separate article, but the short answer is that textbook publishers worry about their bottom line but not necessarily about good education. For slightly more details, see Keeghan, 2012, and pp. 84ff. of Wu, 2015.

In summary, we have isolated the singularly destructive presence of TSM in school mathematics—especially its wanton disregard of definitions and reasoning—as a target for the mathematical reconstruction of the average teacher's knowledge base. Some may question whether this critique of TSM and the advocacy of its obliteration are necessary or appropriate. Our answer is affirmative, very much so. The school curriculum is a vast terrain, and teachers' misconceptions from TSM in this terrain are not confined to a few spots or a few chosen pathways; they are minefields that lay waste to the entire territory. Any attempt at professional development without *confronting and removing TSM*, such as the above "proof" of equivalent fractions or the pseudo-definition of the slope of a line mentioned in Section 3.2, runs the danger of "floating down a smooth-flowing river, so broad that you can seldom see either bank; but, when from time to time a promontory comes into view, you are surprised that it is a new one, as you have been unconscious of movement."[30] It would be irresponsible of us to usher complacent teachers through a tour of the K–12 landscape that they

[30]Bertrand Russell's critique of George Santayana's literary style; see Russell, 1956, p. 96.

think they recognize through the lens of their TSM-infused misconceptions *without* explicitly making them realize that they must now leave these misconceptions behind. We want teachers and teacher-educators to become aware of the pressing need to eradicate TSM.

Having said that, I am compelled to point out in the spirit of full disclosure that the emphasis on the need to replace TSM—especially the malpractice of pretending to do mathematics without definitions and reasoning—and the urgency of the need to implement (content-based) PD to help teachers dislodge TSM are strictly my personal conviction thus far. These issues are not to be found in other recent discussions of teachers' content knowledge, e.g., Common Core, 2012, Conference Board of Mathematical Sciences, 2001 and 2010, National Council of Teachers of Mathematics, 2014, and Zimba, 2016. *Caveat emptor.*

6 Pedagogical content knowledge (PCK)

So far, we have focused our attention on the reality of teaching and learning in the classroom. However, the question about what mathematical content knowledge teachers need has theoretical implications as well. In his well-known address (Shulman 1986), Shulman initiated an inquiry into the kind of content knowledge that all teachers need for teaching. He introduced the concept of **pedagogical content knowledge (PCK)**, which is roughly the bridge that leads from content expertise to the process of teaching. The starting point is thus subject matter content knowledge. According to Shulman:

> We assume that most teachers begin with some expertise in the content they teach. ... Our central question concerns the transition from expert student to novice teacher. ... How does the novice teacher (or even the seasoned veteran) draw on expertise in the subject matter in the process of teaching? (*ibid.*, p. 8).

The precise nature of this "expertise in the content" is therefore foundational to his work on teacher education. This naturally leads us to ask what "some expertise in the content they teach" might mean and what it entails.

To the extent that Shulman was looking into all content disciplines all at once, a precise definition of this content expertise in general is out of the question because such a definition would have to be specific to each discipline. However, since we are now only considering the teaching of *mathematics*, it is incumbent on us to be as precise as possible about what constitutes *mathematical content expertise.* At this point, the picture can get murky. Since the content knowledge that an overwhelming majority of teachers possess is TSM, can mathematics teacher education be built on a foundation of TSM? Obviously not. So how then should the discussion of PCK *in mathematics* proceed? Can we assume that this requisite content knowledge is *principle-based mathematics?* An affirmative answer will bring clarification to the concept of PCK in mathematics and clear the way for us to get to work on providing the minimum content knowledge for PCK. Unfortunately, this remains very much an open question at the moment.

We can perhaps more deeply appreciate the preceding concerns if we take up the refinement of PCK in mathematics teaching proposed in Ball, Thames, and Phelps, 2008. These authors isolated what they called **subject matter knowledge for teaching** (*ibid.*, page 402) as the *content* foundation of PCK. In their work, this knowledge is further subdivided into three categories. In our effort to understand what this subject matter knowledge for teaching consists of *in mathematical terms*, however, we find it more revealing to turn to a series of questions posed on page 402 of their article.

> Where, for example, do teachers develop explicit and fluent use of mathematical notation? Where do they learn to inspect definitions and to establish the equivalence of alternative definitions for a given concept? Where do they learn definitions for fractions and compare their utility? Where do they learn what constitutes a good mathematical explanation? Do they learn why 1 is not considered prime or how and why the long division algorithm works? (Ball, Thames, and Phelps, 2008, page 402)

In the view of Ball et al., the *subject matter knowledge for teaching* that they have in mind is the home for answers to questions such as these. Let us therefore try to answer them one by one.

• "Where, for example, do teachers develop explicit and fluent use of mathematical notation?"

The authors have put their fingers on a key issue in the school curriculum: how to properly use mathematical symbols. Since TSM is cavalier with the symbolic notation—lack of *precision*—it ends up with the bogus concept of a “variable" (see page 51).[31] Clearly TSM is very far from being the requisite *subject matter knowledge for teaching*, at least in this instance. The need to address the use of mathematical notation naturally comes with the requirement of *precision* in principle-based mathematics. In fact, precision suggests that symbols be used, albeit gently, in the elementary classroom in the statements of the commutative laws and associative laws for whole numbers and fractions (cf. Wu, 2011a, page 42; also see Section 1.3). When in the middle grades the use of symbols becomes both necessary and intensive, teachers must come to terms with a fundamental fact regarding the use of symbols:

> *Each time one uses a symbol, one must specify precisely what the symbol stands for.*

This is given the name *the basic protocol in the use of symbols* in Wu, 2016b, page 4 (also see Wu, 2010b, Section 1). When such precision is duly observed, the usual

[31] Some go even further and define a "variable" as a symbol without qualification, and sentences involving symbols-without-qualification are then called **open sentences** (e.g., UCSMP, 1990, page 4). But the concept of "open sentence" is not needed for doing mathematics.

symbolic computations in school mathematics are demystified as nothing more than computations with numbers. The whole of Chapters 1 and 3 and Section 2.1 of Wu, 2016b are devoted to an explanation of this fact from different angles. To the extent that this aspect of principle-based mathematics seems to be neglected in the mathematics literature—not to mention the education literature—the concerns of Ball et al. are entirely justified. We must teach teachers more than TSM.

• "Where do they learn to inspect definitions and to establish the equivalence of alternative definitions for a given concept?"

This question does not even make sense in TSM because TSM has shown no appetite for definitions. So once again, teachers who know only TSM will not possess the *subject matter knowledge for teaching.*

In professional development materials, establishing the equivalence of definitions is a very rare occurrence even in principle-based mathematics because such an occasion is not commonly called for. For example, because there is as yet no usable definition of a fraction in school mathematics other than that using the number line (see the discussion of the following question), we are not in a position to compare the pedagogical pros and cons of different definitions or prove their equivalence, no matter how desirable such a discussion may be. A slight exception is the equivalence of the two definitions of fraction multiplication that is implicit in the discussion on page 64 of the product formula, i.e., the definition using "fraction of a fraction" (Wu, 2016a, p. 58; CCSSM, 5.NF.4), and the definition using the area of a rectangle (Wu, 2011a, p. 263). This equivalence is mentioned on page 262 of Wu, 2011a, but no proof was offered. The equivalence is implicitly proved by combining Section 17.3 of Wu, 2011a, and Theorem 1.6 on page 65 of Wu, 2016a. Indeed, the former proves that the area definition implies the "fraction of a fraction" definition, while the latter proves the converse. In any case, this kind of knowledge is beyond principle-based mathematics even if it is compatible with it.

- "Where do they learn definitions for fractions and compare their utility?"

Again, not in TSM, because there is no definition for a fraction in TSM. See the discussion in Subsection 3.1 The first part of this question implicitly assumes that there are usable definitions of a fraction in school mathematics. As of 2016, the assumption is correct, but unfortunately there is only one such definition at the moment, which was the one was put forth in Wu, 1998, and subsequently put to use in Jensen 2003, Wu, 2011a, and Wu 2016a, and put to partial use in Siegler et al., 2010. It would also appear to be the one in CCSSM, 3.NF.2. So as far as mathematics is concerned, a comparison of different definitions of a fraction is not yet a reality in 2016.

The second part of this question suggests that, perhaps, the authors meant to ask whether any of the existing TSM "interpretations" of a fraction (see Subsection 3.1) can be used as a *definition* of a fraction and, if so, how do they compare? Let us first consider this question in the context of advanced mathematics. Then one of them—the quotient interpretation—can indeed serve as a definition, but perhaps not others. It is known (in advanced mathematics) that a fraction $\frac{m}{n}$ can be defined as a division, $m \div n$, but this has to be done with great care. For example, $m \div n$ cannot be recklessly tossed around as in TSM (see page 39), but has to be defined abstractly as the solution of $nx = m$. Then this solution can be proved to be equal to the fraction $\frac{m}{n}$, which is understood to be the equivalence class of the ordered pair (m, n). However, even this brief description is enough to reveal that such a discussion is way beyond the level of school mathematics and is therefore inappropriate for the consumption by teachers. In summary then, the answer as of 2016 is that there is only one usable definition of a fraction in school mathematics.

- "Where do they learn what constitutes a good mathematical explanation?"

A "mathematical explanation" is of course just a "proof". Given the paucity of reasoning in TSM, one does not look for proofs in TSM. So emphatically TSM does not provide the *subject matter knowledge for teaching* that Ball et al. are looking for. If I understand the question, Ball and her co-authors are asking how teachers can learn to decide whether a proof is *correct* or not and, if correct, how to present it in an accessible way to students. Let us start with the former.

The ability to reason is not an instinctive one, and has to be carefully nurtured. My own observation is that among teachers, especially elementary teachers, their prolonged immersion in TSM has often rendered them incapable of routinely asking *why*, much less looking for the answer. In the mathematical education of teachers, I believe we have to help teachers regain their reasoning faculty in at least two ways. First, they have to get used to the *mechanics* of proofs by a process of total immersion: learn the proof of every assertion in the mathematics they teach. Second, they have to get a feel for the overall architecture of mathematics by working through a systematic logical development of school mathematics.

To illustrate the first point, consider the assertion that, even without a definition of fraction division, one can derive the invert-and-multiply rule (see page 5):

$$\boxed{\frac{\frac{2}{3}}{\frac{4}{5}} \;=\; \frac{\frac{2}{3}\times(3\times 5)}{\frac{4}{5}\times(3\times 5)}} \;=\; \frac{2\times 5}{4\times 3} \;=\; \frac{2}{3}\times\frac{5}{4}$$

If teachers' sensibilities in reasoning have been heightened by a prolonged exposure to proofs, their conditioned reflex would be immediately alarmed by the fact that, in the first equality above, the left side of the equality, $\frac{2}{3}/\frac{4}{5}$ is as yet undefined. They know that to say $A = B$ is to say they already know what each of A and B is before asserting that they are equal. Therefore they would see right away that there is no way the equality can make sense. Next, let us see why they need to have an overview of the hierarchical structure of school mathematics. Look at the TSM proof of a special case of equivalent fractions, $\frac{7}{3} = \frac{14}{6}$, quoted on page 64:

$$\frac{7}{3} \;=\; 1\times\frac{7}{3} \;=\; \boxed{\frac{2}{2}\times\frac{7}{3} \;=\; \frac{2\times 7}{2\times 3}} \;=\; \frac{14}{6}$$

Now equivalent fractions comes near the beginning of every discussion of fractions, but this proof makes use the product formula for the multiplication of fractions. That should be enough to raise a red flag to teachers reading this proof, because they should have an overall understanding of the logical structure of fractions: no matter how fractions are developed, multiplication is never easy and the product formula require hard work. They should suspect right away that it is probably wrong to make use of a result that only appears down the road to prove something that is foundational.

In 2016, most teachers only know TSM but not principle-based mathematics. If we expect them to know the *subject matter knowledge for teaching*, we must begin by helping them go through an immersion in proofs and a point-by-point systematic development of the mathematics they teach. It was exactly the lack of any systematic exposition of principle-based mathematics that provided the initial impetus for the writing of the six-volume work: Wu, 2011a; Wu, 2016a and 2016b; Wu, (to appear).

• "Do they learn why 1 is not considered prime or how and why the long division algorithm works?"

As usual, this question has no answer in TSM. The fact that 1 is not defined to be a prime has to do with the uniqueness of prime factorization (see Section 3.1 of Wu, 2016a), but TSM skirts any explicit mention of either *existence* or *uniqueness* (e.g., what $\sqrt{2}$ means is never seriously discussed in TSM). Next, if we reformulate the second part of this question about the long division algorithm in mathematical terms, then what it asks for is a formal statement of the algorithm as a *theorem* (i.e., with a hypothesis and a conclusion) as well as a *proof* of this theorem. This by itself is a most remarkable question because it seems not to have been previously raised in the education literature (and therefore never answered either). At its best, TSM provides heuristic arguments for the "division house" by using analogies or metaphors, but nothing remotely resembling a *proof*, i.e., a sequence of precise steps that progresses logically from hypothesis to conclusion. One of the difficulties is that, in TSM, the hypothesis of such a theorem has never been clearly stated. Moreover, the conclusion in TSM of the division-with-remainder of 125 by 4 is "31 $R1$", but there is no known mathematical reasoning that has the nonsensical statement "31 $R1$" as a conclusion (see p. 6 of Wu (2014b)). Now, it is possible in principle-based mathematics to explicitly describe the algorithm and—following the description—to systematically present a sequence of *simpler* divisions-with-remainders that ends with the equality $125 = (31 \times 4) + 1$. This may be the proof that Ball et al. are looking for. See Section 7.3 of Wu, 2011a for a formulation of such a theorem, and *ibid.*, Section 7.5 for its proof. The recognition of the long division algorithm as a theorem and a knowledge of its proof should without a doubt be part of every elementary teacher's minimal content knowledge.

In summary: If we are interested in the kind of content knowledge that can provide answers to the preceding five questions from Ball, Thames, and Phelps, 2008, then we must abandon TSM and look for something even more comprehensive than principle-based mathematics. Our conclusion is therefore that the subject matter knowledge for teaching that Ball et al. assumed to be foundational for PCK is a bit beyond principle-based mathematics.

In Section 5, we expressed the pessimism that we may not have a system in place, nor the requisite personnel, to provide mathematics teachers with the content knowledge for achieving basic teaching competence. If the preceding analysis of Ball, Thames, and Phelps, 2008, is correct, then fundamental to both Shulman's theory of PCK in mathematics and its refinement in Ball, Thames, and Phelps, 2008, is a content expertise for teaching that exceeds principle-based mathematics. We must therefore pool our resources together to *try to* provide this basic content knowledge for teachers before we can seriously contemplate tackling PCK. Let us begin by teaching them principle-based mathematics.

In a 2005 article (Shulman, 2005), Shulman said tongue-in-cheek that "Teacher education does not exist" because educators had failed to converge on a set of "signature pedagogies" that characterize all of teacher education. In the same vein, we can say that *teacher education in mathematics does not exist* because we haven't found (yet) a way to give teachers the content knowledge they need to achieve a basic level of competence in mathematics teaching.

Appendix 1. Applied mathematics

The five principles (A)–(E) of Section 2.1 may be said to be foundational to the integrity of *pure mathematics*, which is the discipline that is driven principally by its internal logic and its internal imperatives (the sense of beauty, the sense of structure, etc.). However, its allied discipline of *applied mathematics*, which mediates between pure mathematics on the one hand and science and technology on the other, is never far from school mathematics. Consider, for example, the following problem:

> Two shuttle trains traveling at constant speed go between cities A and B which are 15 miles apart. It takes the first train 10 hours to make the trip, but it takes the second train 12 hours. Suppose now the first train is at city A and the second train is at city B and they take off at the same time on parallel tracks. How long will it be before they meet?

Notice that, as stated, this problem cannot be solved because we don't know precisely what "distance between cities" means, and we are also not given the lengths of the trains. Does the "distance between cities" mean the distance between city centers or the shortest distance between the outskirts of the cities, or is it the distance between the train stations? Let us assume that it is the latter. Now suppose the first train is 528 feet long ($= 1/10$ miles). Then the train doesn't travel 15 miles in going from city A to city B; it only travels $(15 - \frac{1}{10})$ miles and the given data actually means this train travels $(15 - \frac{1}{10})$ miles in 10 hours. Similarly, suppose the second train has length 264 feet ($= 1/20$ miles). Then this train travels $(15 - \frac{1}{20})$ miles in 12 hours. Now a little reflection will reveal that, if "meeting" of the trains means the meeting of the *fronts* of the trains, then they will meet after they have traveled a combined distance of $(15 - \frac{1}{10} - \frac{1}{20})$ miles. Without proceeding further with this analysis, it is quite clear that a 7th grade *school mathematics problem* cannot afford to be this unwieldy.

In order to make the problem manageable to a middle school classroom, the standard simplification is to imagine that both trains are *points* without length. By further assuming the precise distance between the train stations of the two cities to be 15 miles, we are now given that these train will travel 15 miles in 10 and 12 hours, respectively. With these simplifications understood, then this problem can be solved in the usual way as a typical *mathematics problem*.

This process of translating a word problem into a "doable" *mathematics problem* by making "reasonable simplification" is what is formally called "modeling". Applied mathematics may be said to be the study of mathematical problems whose solutions require modeling. The train problem is a rather trivial example of problems in applied mathematics. Most such problems arise from science or technology, and the modeling that is required for their solutions usually require a heavy dose of *scientific* knowledge. To solve these problems, we will have to deal with concepts whose primary definitions lie "in the real world", so to speak, outside mathematics. For example, in dealing with the electric field in Newtonian physics, mathematicians may believe that the electric field is precisely defined by the gradient of the solution of Poisson's equation. But in physics, what truly matters is the force exerted on a test charge by the field. Therefore if the same problem is taken up in 19th century electrodynamics, the modeling of the field changes. The mathematical definition of the field will now involve Maxwell's equation and the combined electric plus magnetic force on a test charge. There will be other variations if the context changes to non-quantum special relativity or special relativistic quantum field theory. All the while, the electric field "out there" remains the same. Moreover, the reasoning used will involve a substantial amount of science in addition to mathematics, and the purpose behind the problem would likely lie more in science rather than in pure mathematics. The fundamental principles that characterize the integrity of applied mathematics will therefore be a slightly modified version of each of (A)–(E), as we have just indicated.

However, given the lack of coordination between the teaching of school science and school mathematics as of 2016, the chances of being able to do substantive applied *school* mathematics are essentially nonexistent, because such problems inevitably involve serious science. Problems like the train problem above typically constitute the only kind of applied mathematics that can be taught in K–12, and the modeling that is required for their solution is no more than certain *formal* conventions that—like the modeling of a train by a point—once set up, can be learned quickly.[32] As the preceding solution of the train problem shows, once these conventions are understood, the usual applied problems in school mathematics quickly become part of pure mathematics again.

It is for this reason that we believe that the fundamental principles (A)–(E) are sufficient to characterize the integrity of the mathematics of K–12 in year 2016.

Appendix 2. The existence of TSM

To people not directly involved with the professional development for mathematics teachers or the evaluation of school mathematics textbooks, TSM is an unbelievable concept: could a *nation's* textbooks be so bad for so long? Could it be that someone is taking poetic license to create this concept for purposes that are not entirely intellectual? This appendix addresses these doubts and suggest projects for research to confirm or refute the validity of this concept.

The most reliable way to identify TSM is to read, in succession, several textbooks for the same grade from major publishers. Using this article as a guide, the reader will not fail to notice the many similarities—and the anti-mathematical qualities—among

[32]This may explain why it is almost impossible to find sensible assessment items on modeling.

these books. In order to generate data for research, however, we will have to suggest a far cruder methodology. We are going to write down a small list of observable characteristics to be used for detecting the presence of TSM. Note that it is easy to expand this list. For middle school mathematics, simply look up "TSM" in the indices[33] of Wu, 2016a and 2016b. For high school mathematics, the volumes of Wu, (to appear), will serve the same purpose when they are finally published. Because the volume Wu (2011a) was published before the term TSM was coined, it is slightly more difficult to come up with a list for elementary school mathematics. Nevertheless, it is not difficult to single out the many implicit references to TSM in Wu, 2011a (e.g., pp. 106, 177–178, 206, 228, 332, 335, etc.).

What we suggest is to use the items on the following list to check the school mathematics textbooks from the major publishers. If over 75% of these books (in a fixed grade band) contain the error described by each item on this list (that is relevant to the grade band), then the validity of TSM would be beyond doubt. Moreover, one can get further confirmation by a survey of teachers using these items. Again if over 75% of the teachers confirm that these errors were exactly what they were taught when they were students, that would be a double confirmation of the validity of the concept of TSM. For this kind of research, the participation of a very competent

[33] These indices are not in those volumes but are obtainable from `http://www.ams.org/publications/authors/books/postpub/mbk-98` and `http://www.ams.org/publications/authors/books/postpub/mbk-99`.

mathematician will be crucial.

It should also be pointed out that many of the errors in the following list are recorded in the lessons of the teachers in the casebooks of Barnett, Goldstein, and Jackson, 1994; Merseth, 2003; Schifter, Bastable, and Russell, 1999; and Stein, Smith, Henningsen, and Silver, 2000.

Here is the list:

(I) Missing or garbled basic definitions. (By "definition", we mean as in Subsection 2.1 a precise and mathematically correct statement about a concept that is put to use in the textbook for reasoning.)

> Number; division-with-remainder; fraction; decimal; one fraction *bigger than* another; addition, subtraction, multiplication, and division of fractions; ratio; percent; constant speed; negative fraction; addition, subtraction, multiplication, and division of rational numbers; variable; expression; equation; polynomial; length of curve, area of region in a plane, and volume of solid in 3-space; scale drawing; slope of a line; half-plane of a line in the plane; the graph of an inequality, equation, or function.

(II) Wrong instructions.

(a) Writing a division-with-remainder, e.g., 17 by 5, as $17 \div 5 = 3\ R2$.

(b) Add two fractions by the use of the least common denominator of the fractions.

(c) Introduce mixed numbers before fraction addition.

(d) Expanding the product of two linear polynomials by the mnemonic device of FOIL.

(e) Teach *order of operations* as a major skill by the mnemonic device of PEMDAS.

(f) Define slope of a line as *rise-over-run* without emphasizing that it is *a single number* attached to the line.

(g) *Define* in a high school algebra text that two lines in the plane are perpendicular if and only if the product of their slopes is -1.

(III) Lack of reasoning (proof) for any of the following basic facts:

(a) The long division algorithm for whole numbers.

(b) The product formula of fractions: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

(c) The invert-and-multiply rule for the division of fractions.

(d) The multiplication algorithm for the product of two finite decimals.

(e) The theorem $(-x)(-y) = xy$ for rational numbers x and y.

(f) The theorem $\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$ for all rational numbers a and b $(b \neq 0)$.

(g) The theorem that the graph of $ax + by = c$ is a line.

(h) The theorem that the graph of a linear inequality is a half-plane.

(i) The theorem that the solution of a system of two linear equations in two variables is the point of intersection of the two lines defined by the linear system.

(j) The theorem that a linear function attains its maximum or minimum at a vertex of the feasibility region in linear programming.

(k) The formula for the vertex of the graph of a quadratic function.

(ℓ) For any positive a and b and any positive integer n, $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.

(m) The Factor Theorem for polynomials of one variable.

(n) The addition formulas for sine and cosine for *all* angles (i.e., not just acute angles).

(IV) Lack of purpose for basic skills or concepts.

(a) Why round off whole numbers or decimals?

(b) Why do we need negative numbers?

(c) Why do we need absolute values?

(d) Why teach rotations, translations, and reflections in middle school if they seem to be useful only for art appreciation?

(e) Why do we need to know the *slope* of a line?

(f) Why change the notation of $\sqrt[n]{a}$ to $a^{1/n}$ and $\frac{1}{a}$ to a^{-1}?

(V) Incoherence in the teaching of geometry.

Congruence is defined to be same size and same shape in middle school, but in the high school geometry course, it is *redefined* as equal sides and equal angles for polygons but nothing else. There is no explanation as to why once students are in high school, they will no longer be concerned about the congruence of curved figures. Similarly, *similarity* is defined to be same shape but not necessarily the same size in middle school, but in the high school geometry course, it is *redefined* as proportional sides and equal angles for polygons but nothing else. (See Subsection 3.3 for a more nuanced discussion.)

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