What is different about the Common Core Mathematics Standards?

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*I am grateful to Larry Francis for many useful suggestions.

Many sets of state and national math standards have come and gone in the past twenty years. I imagine you are all veterans in this game.

By 2014, the Common Core Mathematics Standards (**CCMS**) will be phased in. How will it impact teaching in the classroom?

At least among the better standards, change usually means reshuffling or wordsmithing the same collection of statements. If some standards are moved up to an earlier grade, then the new set of standards is considered to be *better*.

The underlying assumption is that **the mathematics of the school curriculum is in good shape and it is only a matter of putting all the pieces together "the right way"**.

In this model, a set of *rigorous standards* is one in which each topic is taught as early as possible.

CCMS does not follow this model.

For example, CCMS does not even mandate the teaching of Algebra I in grade 8. This has become a source of consternation in some quarters.

CCMS' main contribution lies in righting many of the wrongs in the mathematics of the existing curriculum.

Getting the math right is a serious issue. If we don't get it right, our students cannot learn. **Garbage in, garbage out.**

You have heard of the problem with *proofs in geometry*. You have heard of *algebra being the unattained civil right.* You have heard of *fraction-phobia*. All that because of **Garbage in, garbage out.**

We as a nation have been suffering from this educational malaise for decades.

Beyond the frequent absence of reasoning, the disconnectedness in the presentation of mathematical topics has turned a coherent subject into nothing more than a bag of tricks.

CCMS succeeds in most instances to restore the mathematical continuity from grade to grade, e.g., the development of fractions in grades 3-6, or the seamless transition from the geometry of grade 8 to high school geometry.

The mathematics in CCMS finally begins to look like

mathematics.

Goals of this presentation:

(1) Give some examples to illustrate the kind of change that CCMS initiates.

(2) Discuss how this change impacts professional development (**PD**) and district-wide policy.

Three examples of change:

(A) Adding fractions.

(B) Solving equations.

(C) Slope of a line.

I will compare CCMS with existing state standards on these topics. For definiteness, I will use the California Standards, but the same comments can be made with "California" replaced by any state.

(A) Adding fractions. Here are the California standards on the addition of fractions:

Gr5 NS 2.0 Students perform calculations and solve problems involving addition, subtraction, and simple multiplication and division of fractions and decimals:

> 2.3 Solve simple problems, including ones arising in concrete situations, involving the addition and subtraction of fractions and mixed numbers (like and unlike denominators of 20 or less), and express answers in the simplest form.

Gr6 NS 2.0 Students calculate and solve problems involving addition, subtraction, multiplication, and division:

> 2.1 Solve problems involving addition, subtraction, multiplication, and division of positive fractions and explain why a particular operation was used for a given situation.

That is all. No details are necessary because we all know what to do. *But do we?*

This is what students across the land have been getting:

To add $\frac{\mathbf{7}}{\mathbf{8}} + \frac{\mathbf{5}}{\mathbf{6}}$, take the **LCD** of 8 and 6, which is 24. Note that $24 = 3 \times 8$ and $24 = 4 \times 6$. Therefore

$$\frac{7}{8} + \frac{5}{6} = \frac{(3 \times 7) + (4 \times 5)}{24} = \frac{41}{24}$$

Does this make any sense to you? We know it makes no sense to students because most of math-phobia reputedly starts with the addition of fractions.

Adding is supposed to "combine things". The concept of "combining" is so basic that it is always taught at the beginning of arithmetic.

But did you see any "combining" in the preceding description of $\frac{7}{8} + \frac{5}{6}$?

Children who have made the effort to master the addition of whole numbers naturally expect that the **addition** of fractions will be more of the same, i.e., "combining things". So how are they supposed to learn this inscrutable procedure?

Before we discuss how CCMS handles the same situation, I should emphasize that, by themselves, there is nothing wrong with the above California standards. They are mathematically correct.

However, given the present educational environment, one cannot hope to use such standards to achieve any educational improvement. Indeed, the school mathematics curriculum nation-wide has remained as unlearnable as this way of adding $\frac{7}{8} + \frac{5}{6}$ in all the years of having mathematics standards.

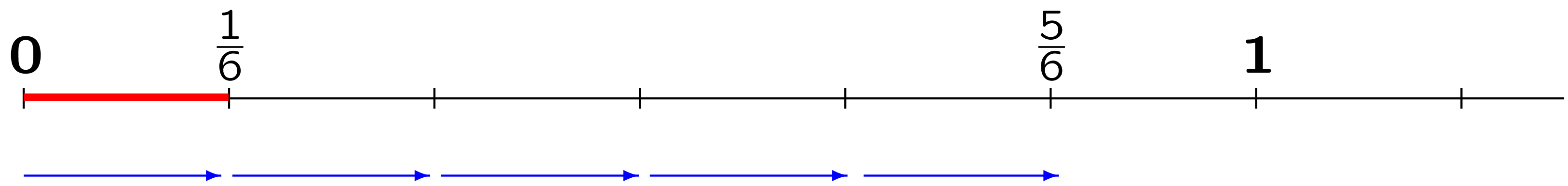
Later, we will put this remark in perspective.

CCMS approaches the addition of fractions as follows.

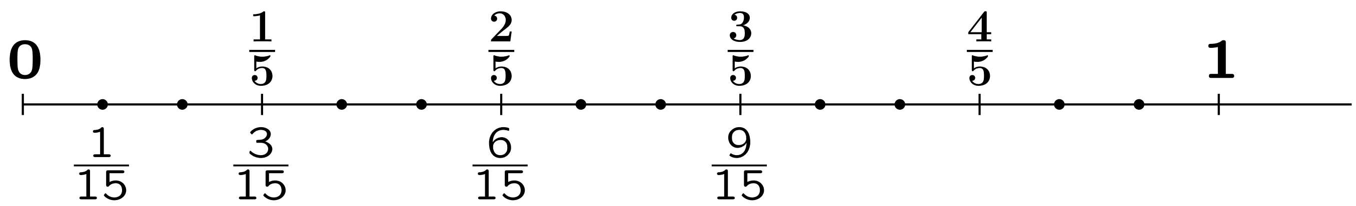
Grade 3 (*paraphrase*) Understand a fraction as a number on the number line. Explain *equivalence of fractions* in special cases.

> Represent $\frac{1}{n}$ as the point next to 0 when [0,1] is divided into n equal parts, then $\frac{m}{n}$ is the m-th division point to the right of 0. Therefore, identifying $\frac{1}{n}$ with $[0, \frac{1}{n}]$ and $\frac{m}{n}$ with $[0, \frac{m}{n}]$, we think of $\frac{m}{n}$ as "m copies of $\frac{1}{n}$", i.e., $\frac{m}{n}$ is joining m copies of $\frac{1}{n}$ together.

Why $\frac{5}{6}$ is 5 copies of $\frac{1}{6}$:



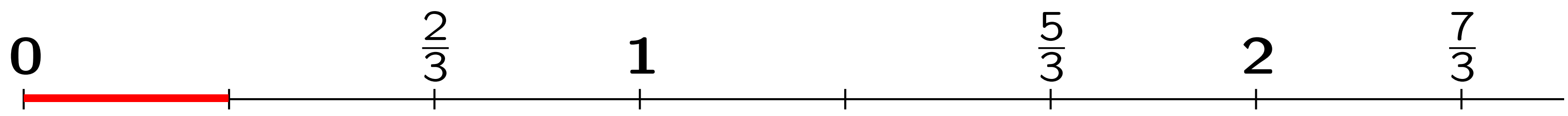
Why $\frac{2}{5}$ is equal to $\frac{3\times 2}{3\times 5}$:



Grade 4 Explain why a fraction $\frac{a}{b}$ is **equivalent** to a fraction $\frac{n \times a}{n \times b}$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size, i.e., same point on the number line.

Define addition of fractions as joining parts that refer to the same whole. Then for two fractions with *like denominators*,

$$\frac{m}{n} + \frac{k}{n} = \frac{m+k}{n}.$$



Why $\frac{2}{3} + \frac{5}{3} = \frac{7}{3}$:

$\frac{2}{3}$ is the length of 2 pieces of .

$\frac{5}{3}$ is the length of 5 pieces of .

Combining them gets us the length of $2 + 5 = 7$ pieces of , which is exactly $\frac{7}{3}$.

Grade 5 Add and subtract fractions with unlike denominators by replacing given fractions with equivalent fractions, so that we have fractions with like denominators. For example,

$$\frac{2}{3} + \frac{5}{4} \;=\; \frac{8}{12} + \frac{15}{12} \;=\; \frac{23}{12}.$$

In general, given fractions $\frac{a}{b}$ and $\frac{c}{d}$, we have:

$$\frac{a}{b} + \frac{c}{d} \;=\; \frac{ad + bc}{bd}.$$

Altogether, these standards guide students through three grades to get them to know the meaning of adding fractions: Addition *is* putting things together, even for fractions, and the logical development ends with the formula $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. **No mention of LCD.**

A teacher teaching from CCMS has to be aware how a child learns about "combining things" and, more importantly, has to know the *mathematics* so that she can teach in a way that respect the child's intuition about "combining things".

Notice the unbroken line of cognitive development from grade 3 to grade 4, and finally to grade 5.

This is an example of the continuity from grade to grade in CCMS. Such continuity can be observed in many other topics.

(B) Solving equations. Solving equations is the most basic part of algebra. The California Standards have this to say:

Gr6 AF 1.0 Students write verbal expressions and sentences as algebraic expressions and equations; they evaluate algebraic expressions, solve simple linear equations, and graph and interpret their results.

Gr8 4.0 Students simplify expressions before solving linear equations and inequalities in one variable.

Again, these standards are mathematically above reproach.

However, embedded in them is the assumption that we all know how to solve equations. So, just say it, and equation-solving will be taught correctly in school classrooms.

In reality, how are equations solved in school classrooms?

Consider $3x-1=5x+2$. This is an *open sentence*, because x is an unspecified number. To transpose -1 from left to right, we add 1 to both sides. But $3x-1$ is not a number, and neither is $5x+2$, how to explain $(3x-1)+1=(5x+2)+1$? You use a balance to "weigh" both sides of the equation on the weighing platforms. **You muddle through.** Then you get:

$$3x = 5x+3$$

Reasoning like this some more, you get $(-2x)=3$ and $x=-\frac{3}{2}$.

There are many other questionable intermediate steps, but we will gloss over them for now.

What does it mean to solve an equation? Why should we believe that

$$\text{if} \quad 3x - 1 = 5x + 2, \quad \text{then} \quad (3x - 1) + 1 = (5x + 2) + 1\,?$$

What is x? Why should the associative, commutative and distributive laws be applicable to "open sentences"?

These are uncomfortable questions.

There are in fact no answers, because the above steps for solving the equation are mathematically illegitimate.

There is a correct way to solve equations, but before worrying about that, we first ask **what an equation is, and what it means to solve an equation**.

An equation such as "$3x-1=5x+2$" is always an abbreviation of a QUESTION:

What are all the numbers x so that $3x-1=5x+2$?

Each such x is a *fixed number!*

CCSS High Schol Algebra explains how to answer the preceding question:

A-REI 1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Thus it is about *numbers*, not open sentences.

Let us decode the preceding standard. We are asking: What are all the numbers x so that $3x - 1 = 5x + 2$? (Such an x is called a **solution**.)

We don't know if there is any solution. **So suppose there is one, a certain number x, so that $3x - 1$ is equal to $5x + 2$.** Remember this x is a **fixed** number. Therefore $3x - 1 = 5x + 2$ is an equality about ordinary numbers, and we can apply to it everything we know about numbers, including the associative, commutative, and distributive laws, and the fact that

if a, b, c are numbers and $a = b$,
then $a + c = b + c$ and $ca = cb$.

For this x, we have $3x - 1 = 5x + 2$. So:

$$\begin{aligned}
(3x + (-1)) + 1 &= (5x + 2) + 1 \\
3x + ((-1) + 1) &= 5x + (2 + 1) \qquad \text{(assoc. law)} \\
3x &= 5x + 3 \\
(-5x) + 3x &= (-5x) + (5x + 3) \\
-2x &= 3 \qquad \text{(dist. law and assoc. law)} \\
x &= -\frac{3}{2} \qquad \left(\tfrac{3}{-2} = -\tfrac{3}{2}\right)
\end{aligned}$$

We have proved:

If x is a solution of $3x - 1 = 5x + 2$, then $x = -\frac{3}{2}$.

Do we **know** that x is a solution of $3x - 1 = 5x + 2$? Not yet.
But it is now easy to check this:

$$3\left(-\frac{3}{2}\right) - 1 = 5\left(-\frac{3}{2}\right) + 2$$

because both sides are equal to $-\frac{11}{2}$.

Thus $-\frac{3}{2}$ is a solution of $3x - 1 = 5x + 2$.

One should note that there were people in California who knew about the problem with the traditional way of solving equations. In the California Mathematics Framework (2005, pp. 175-176), http://www.cde.ca.gov/ci/ma/cf/documents/math-ch3-8-12.pdf, there is a precise explanation of the correct way to do this.

There is no evidence, however, that publishers, curriculum developers, and teachers ever read the Framework. This observation will be put in its proper context below.

(C) Slope of a line. California is one of the states whose Mathematics Standards mandate the teaching of Algebra I in eighth grade. To do that, the concept of the **slope** of a line has to be introduced by grade 8. Here is what the CA Standards have to say about slope in the grade 7:

Gr7 AF Graph linear functions, noting that the vertical change (change in y-value) per unit of horizontal change (change in x-value) is always the same and know that the ratio (rise over run) is called the slope of a graph.

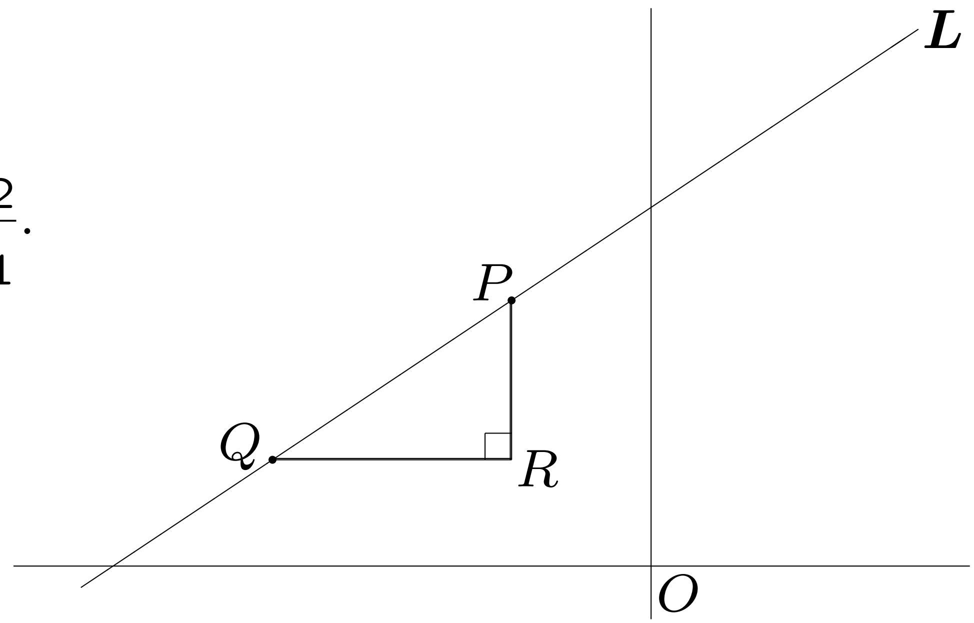
Once again, note the unfailing confidence California has placed in book publishers, curriculum developers, and teachers. It assumes that they can all make sense of this standard.

In particular, students will see why the vertical change per unit of horizontal change is **always** the same.

What if I tell you that this is confidence misplaced, not just in California but nation-wide?

Let a line $\boldsymbol{L}$ be given. Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be distinct points on $\boldsymbol{L}$. Then the usual definition is:

Slope of $\boldsymbol{L}$ is $\dfrac{p_2 - q_2}{p_1 - q_1}$.



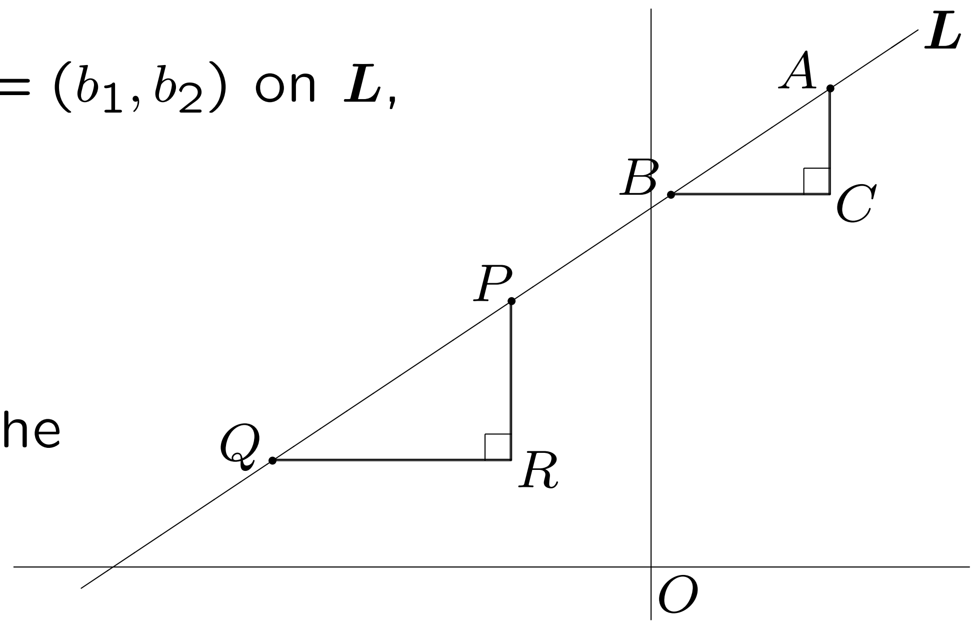
Almost all textbooks stop right here and consider this to be an adequate definition of "slope".

Lines have lots of points. What if two different points A and B are chosen instead?

If $A = (a_1, a_2)$ and $B = (b_1, b_2)$ on $\boldsymbol{L}$, unless we can **prove**

$$\frac{p_2 - q_2}{p_1 - q_1} = \frac{a_2 - b_2}{a_1 - b_1},$$

we don't know what the what the slope of a line is.



This proof requires the concept of *similar triangles*: $\triangle ABC \sim \triangle PQR$.

Textbooks want students to conflate the slope of a line $\boldsymbol{L}$ with the slope of *two chosen points* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *on* $\boldsymbol{L}$.

If students get used to blurring the distinction between two concepts as different as these, they may never be able to learn any mathematics of value again. Without precision, there is no mathematics.

In the short term, students will not understand why the graph of $ax+by=c$ is a line, and consequently will have trouble struggling to memorize the four forms of the equation of a line.

It remains to point out that California undercuts its own good intentions by not formally introducing the concept of similar triangles in the standards for K-8.

This labor-saving device is what makes it possible, in California, to teach Algebra I in grade 8. The unfortunate consequence is that students suffer because their learning of algebra is not well supported by the standards.

(The California Mathematics Framework (2005, p. 181) http://www.cde.ca.gov/ci/ma/cf/documents/math-ch3-8-12.pdf, does point out this gap in the standards.)

What does CCMS do in this situation?

It introduces eighth graders to an *intuitive* approach to congruence and similarity. Get them comfortable with the angle-angle criterion for similar triangles. Then:

8EE 6. Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

Because CCMS spends so much time laying the foundation of similarity, it does not get to quadratic equations until grade 9.

I should mention in passing that CCMS makes the effort to teach similarity in grade 8 not just for making sense of the concept of slope. It also serves the larger purpose of laying the groundwork for high school geometry.

This is part of CCMS's overall effort to maintain grade-to-grade continuity.

In summary, these three examples illustrate a perennial problem in school mathematics education:

> The mathematics defined by school textbooks is too often inscrutable and beyond the reach of human reason.

I call this the **Textbook School Mathematics (TSM)**. This has been the de facto national school curriculum for a long time.

The discussion of these three examples also brings a ray of hope, however. They give an indication that:

> Every topic in school mathematics *can* be made transparent and reasonable.

CCMS is taking a first step to break down TSM and make school mathematics transparent and reasonable again.

We give some references.

For a general discussion of fractions and Example 1, See Part 2 of H. Wu, *Understanding Numbers in Elementary School Mathematics*, American Mathematical Society, 2011.
http://www.ams.org/bookstore-getitem/item=mbk-79

For Examples 2 and 3, see Section 4 and Section 3, respectively, of Introduction to school algebra,
http://math.berkeley.edu/~wu/Algebrasummary.pdf

Thus far, TSM gets recycled from generation to generation because, in college, pre-service teachers are not taught what they need to know in order to break away from TSM.

We in the universities force them to go back to school and teach TSM, as we have done in the past for decades.

One example of how TSM has corrupted many teachers' understanding of mathematics: I received an inquiry in 2010 about why CCMS put similar triangles in grade 8 before approaching the algebra of linear equations:

"After 13 years of teaching high school algebra, I wonder why you see similarity as critically important to algebra I mastery — that certainly never occurred to me as a teacher of algebra. . . . What makes you say that a student needs to understand similar triangles in order to write the equation of a straight line between two points?"

It is not the teacher's fault.

The whole education establishment has conspired to put teachers into the school classroom without teaching them what they need to know.

TSM is also perpetuated by major textbook publishers. It is a long story, but ultimately, they have their own reasons to maintain the status quo.

This is why the responses to state standards, in terms of teaching and textbooks, have always been more of the same TSM. This is also why documents like the California Mathematics Framework that try to promote progress cannot produce any results, because if it a document is merely advisory but not high-stakes, it doesn't count.

CCMS is our last hope of breaking the vicious cycle of TSM for a long time to come.

What next?

Pre-service PD must improve, even if we have *nothing* to show for it as yet. But we are trying.

A more pertinent question on this occasion is what we can do for you who are already in the field.

We have to improve in-service PD because TSM is incompatible with CCMS. To this end, in-service PD can no longer be business-as-usual.

At the moment, in-service PD means different things to different people.

To some, it means games, fun activities, new manipulatives, pedagogical strategies, and projects that teachers can directly bring back to their classrooms.

To others, it is designed to make teachers feel good about themselves, make them feel that that already know mathematics, and make them believe that mathematics can be learned without hard work.

To most, it is something you do to get it over with.

The better kind of in-service PD deals with the essential issues of children's mathematical thinking, skillful use of technology, teacher-student communication, and refined teaching practices.

But the kind of PD that is most urgently needed in 2011 to help teachers implement CCMS is one that *leads them away from TSM:*

> It shows how to teach the basic mathematical topics of the school curriculum with precision, reasoning, and coherence, and it does so *from the vantage point of the school classroom.*

We emphasize: It is not enough to have PD that teaches mathematics with precision, reasoning, and coherence. University math courses for math majors do that!

Effective PD must also address topics in school mathematics from the vantage point of the school classroom. *Listening to a professor lecture on fractions as the quotient field of the ring of integers does not constitute effective PD.*

We need PD that combines the best of both worlds. At the same time, we also need teachers to be on board, because learning this kind of mathematics requires *sustained effort.* There is no other way.

This state of affairs is not easy to achieve as of 2011, and this where district policy comes in.

Let me digress. You may have heard of the playwright Tony Kushner, who wrote the Pulitzer Prize winning play Angels in America (1993). Recently he gave an interview and had the following to say about acting:

> Good acting training should be sadistic. You have to unlearn a lot of what you think you know about acting,
> . . .
> (http://magazine.columbia.edu/print/793)

I think this is more or less true of good PD as of 2011:

> Good professional development should make you unlearn a lot of what you think you know about school mathematics.

In an ideal world, we do things right from the start. In that scenario, teachers never learn TSM.

But in the less-than-ideal world of math education in 2011, I am afraid CCMS will be making many of our teachers unlearn a lot of things picked up from TSM.

I hope the district administrators in this audience are willing to help this effort by doing a serious re-thinking of in-service PD. The rethinking includes the realization that:

1. Math teachers need content-based PD.

2. Such PD must be sustained over a long period of time.

3. The decision on the choice of professional developers takes into account the input from competent mathematicians who are knowledgeable about schools.

Leadership is critical.

Many professional developers have been advertising “content-based PD” for years, but “content” is easier said than done. The professional developer must know mathematics (rare), and must not be afraid to be demanding.

Half-day workshops, if content-based, are better than nothing. Better: half-day workshops each week for several weeks. Better still: one-day workshops each week for several weeks.

Ideally, give one-week or two-week workshops in the summer, with daily homework assignments and with full pay for teachers.

Effective PD will have to be the collaborative efforts of all parties concerned, including mathematicians, who don't normally come to the table.

Informed decisions that separate the wheat from the chaff will depend on getting good advice on such a highly technical subject as mathematics.

Good results will also depend on teachers' willingness to take PD seriously as a learning experience, not just a social one.

The fate of CCMS is hanging in the balance:

> Can we get teachers who can make sense of the mathematics they teach, who can teach mathematics in a way that is clear, precise, and supported by reasoning every step of the way?

Our children are waiting for an affirmative answer.