The impact of Common Core Standards on the mathematics education of teachers

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In June of 2010, the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO) jointly released a set of **Common Core Standards** for mathematics and reading, and asked the states for voluntary adoption.

Four states—Alaska, Minnesota, Texas, and Virginia—rejected these standards, but over 40 other states have signed on in the meantime. Implementation will take place in 2014.

This lecture is about the Common Core Mathematics Standards (**CCMS**).

As of 2011, almost every state has its own set of math standards, and *they are all different from each other in some ways*. Given the mobility of families, the differences are damaging to students' math education; they also obstruct any meaningful attempt to improve school math textbooks.

The need of a good set of common standards is real.

But is CCMS good enough?

CCMS has its flaws— that is inevitable—but its **mathematical quality** is, overall, far superior to the existing state standards.

• It does not engage in the usual educational one-upmanship of teaching each topic earlier than other standards.

• It emphasizes what counts the most in mathematics education:

(a) Restore mathematical clarity and precision to school mathematics.

(b) Maintain logical continuity from grade to grade, and infuse reasoning in the presentation of each topic.

What stands in the way of a successful implementation of CCMS?

Textbooks

Assessment

Teacher Quality

As of 2011:

Quality of math textbooks: Very poor in general[1].

Quality of assessment: Poor in general[2].

Teachers' content knowledge (of mathematics): Fragile, due to reasons to be discussed.

1. See, for example, Appendix B in Chapter 3, *Foundations for Success*, The National Mathematics Advisory Panel Reports of the Task Groups and Subcommittees, 2008. http://www2.ed.gov/about/bdscomm/list/mathpanel/reports.html

2. See Chapter 8, loc. cit.

How to get adequate math textbooks to support CCMS is a major, major issue.

Textbook publishers are driven by one thing only: the bottom line. This translates into, *not* exposition of higher mathematical quality, but more adoptable textbooks, i.e., books that most teachers feel are easy to use, which is distinct from books that make more mathematical sense.

If you know how to deal with the bottom-line mentality in education, please call me 24/7.

The problem with state assessments deserves to be discussed at length, and all by itself.

The way the Common Core Assessment is shaping up, there is ample room for concern: Are there *knowledgeable* mathematicians involved to do quality control? Will students be over-tested?

Every state should be on full alert.

How to get mathematically knowledgeable teachers to implement CCMS is the main subject of this lecture.

The problem is every bit as intractable as the other problems, but it is something over which we academics have *some* control.

Our goal is to help produce teachers who are proficient in **school mathematics (SM)**.

By *SM*, we mean the mathematics of the standard school math curriculum, which is roughly,

$$\text{whole numbers} \longrightarrow \text{fractions} \longrightarrow \text{rational numbers}$$

$$\longrightarrow \left\{ \begin{array}{l} \text{algebra} \\ \text{geometry} \end{array} \right\} \longrightarrow \text{trigonometry and pre-calculus}$$

Of course one may add to this a small amount of statistics.

A university mathematician's typical reaction to school mathematics is the following:

> it is elementary,
>
> therefore trivial,
>
> therefore if we teach future teachers "real" mathematics "the right way", they will understand this elementary stuff.

This mistake has been made for a long time, most infamously in the New Math era.

Fundamental Fact:

Much of SM is not part of university mathematics.

We can see this by considering a typical problem in SM:

Convert $\frac{5}{27}$ to a decimal with 6 decimal digits.

$$
\begin{aligned}
\frac{5}{27} &= \frac{5 \times 10^6}{27 \times 10^6} \qquad \text{(equivalent fractions)} \\
&= \frac{5 \times 10^6}{27} \times \frac{1}{10^6} \qquad \left(\text{product formula: } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}\right)
\end{aligned}
$$

Caution: The fraction $\dfrac{5 \times 10^6}{27}$ itself is **not** the long division of 5,000,000 by 27. It is, rather, the size of one part when 5×10^6 is divided into 27 equal parts.

If we want to bring in long division, we have to do it the right way.

$$
\begin{array}{cccccccccc}
 & & & 1 & 8 & 5 & 1 & 8 & 5 \\
\cline{2-9}
27 &) & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 2 & 7 & & & & & \\
\cline{3-5}
 & & 2 & 3 & 0 & & & & \\
 & & 2 & 1 & 6 & & & & \\
\cline{3-6}
 & & & 1 & 4 & 0 & & & \\
 & & & 1 & 3 & 5 & & & \\
\cline{4-7}
 & & & & & 5 & 0 & & \\
 & & & & & 2 & 7 & & \\
\cline{5-8}
 & & & & & 2 & 3 & 0 &
\end{array}
$$

We obtain: The long division of 5,000,000 by 27 has quotient 185185 and remainder 5. (The repetition of the row 5 0 means that the whole process of the long division will repeat itself.)

School textbooks tell you to express the long division as

$$5,000,000 \div 27 \ = \ 185185 \text{ R } 5$$

This doesn't make sense. The left side is supposed to be a single number while the right side is at least two numbers: 185185 and 5. (There are other things wrong with this "equality".)

The correct symbolic expression for the long division is:

$$5,000,000 \ = \ (185185 \times 27) + 5$$

We can now return to the conversion of $\frac{5}{27}$ to a decimal.

$$\begin{aligned}
\frac{5}{27} &= \frac{5 \times 10^6}{27 \times 10^6} \qquad \text{(equivalent fractions)} \\
&= \frac{5 \times 10^6}{27} \times \frac{1}{10^6} \qquad \left(\text{product formula: } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}\right) \\
&= \frac{(185185 \times 27) + 5}{27} \times \frac{1}{10^6}
\end{aligned}$$

$$
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\frac{5}{27} &= \frac{5 \times 10^6}{27 \times 10^6} && \text{(equivalent fractions)} \\
&= \frac{5 \times 10^6}{27} \times \frac{1}{10^6} && \left(\text{product formula: } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}\right) \\
&= \frac{(185185 \times 27) + 5}{27} \times \frac{1}{10^6} \\
&= \left(185185 + \frac{5}{27}\right) \times \frac{1}{10^6}
\end{aligned}
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&= \frac{(185185 \times 27) + 5}{27} \times \frac{1}{10^6} \\
&= \left(185185 + \frac{5}{27}\right) \times \frac{1}{10^6} \\
&= \frac{185185}{10^6} + \left(\frac{5}{27} \times \frac{1}{10^6}\right)
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&= \frac{(185185 \times 27) + 5}{27} \times \frac{1}{10^6} \\
&= \left(185185 + \frac{5}{27}\right) \times \frac{1}{10^6} \\
&= \frac{185185}{10^6} + \left(\frac{5}{27} \times \frac{1}{10^6}\right) \\
&= 0.185185 + \left(\text{a positive number} < \frac{1}{10^6}\right)
\end{aligned}
$$

Some observations:

(A) SM is different from university mathematics:

> This conversion is a basic topic in SM, but it does not sit comfortably in any standard university-level math course.
>
> In SM, the product formula is not a definition of fraction multiplication, but a nontrivial **theorem**.

(B) The reasoning behind the conversion:

requires an understanding of what long division means,

shows that this mysterious sounding conversion is nothing more than expressing a given fraction as another fraction with denominator equal to 10^6, and

shows the conversion to be a consequence of the product formula in *fraction* multiplication.

How is this conversion taught in school textbooks, and therefore in school classrooms?

As a rote skill that makes the long division algorithm even more mystifying: Throw in as many zeros as you want and add a decimal point to the quotient.

$$
\begin{array}{rcccccccc}
 & & & .1 & 8 & 5 & 1 & 8 & 5 \\
\cline{2-9}
27 &) & 5 & .0 & 0 & 0 & 0 & 0 & 0 \\
 & & 2 & 7 & & & & & \\
\cline{3-5}
 & & 2 & 3 & 0 & & & & \\
 & & 2 & 1 & 6 & & & & \\
\cline{3-6}
 & & & 1 & 4 & 0 & & & \\
 & & & 1 & 3 & 5 & & & \\
\cline{4-7}
 & & & & & 5 & 0 & & \\
 & & & & & 2 & 7 & & \\
\cline{6-8}
 & & & & & 2 & 3 & 0 & \\
\end{array}
$$

To us, the rote skill is easily recognizable as a neat summary of the multistep reasoning. However, when the reasoning is suppressed—as in school textbooks and standard educational materials,—the rote skill becomes totally incomprehensible because we have

a fraction $\frac{5}{27}$ (which is a piece of pizza),

a decimal $0.185185\ldots$ (a senseless sequence of digits created by long division).

In what sense are they equal, and why?

All of us were taught such rote procedures, and some survived to learn more mathematics. Many others were not so fortunate.

The mathematics in school math textbooks is as incomprehensible as this rote procedure much too often for comfort.[3] This is a perversion of SM, and we shall call it

Textbook School Mathematics (TSM).

3. It is difficult to quantify the frequency of incomprehensibility, but one can assert with confidence that the estimate in footnote 1 above, to the effect that there is on average one error every two pages is incredibly conservative.

To summarize:

SM: It is elementary, perfectly understandable, and not trivial.

TSM: It is too often irrational, and therefore incomprehensible.

It has been said with ample justification that

TSM is our de facto national curriculum.

Everybody in mathematics and math education must share the guilt of allowing TSM to be the only kind of mathematics taught in K–12.

With rare exceptions, TSM is also the only kind of mathematics known and discussed in math education.

If our schools can teach SM instead of TSM, then probably there wouldn't be any *Mathematics Education Crisis*.

We must get out of **this** curriculum as fast as we can.

Yet almost all schools continue to teach TSM.

Why?

One reason is that those who know the difference between SM and TSM—the knowledgeable mathematicians—have not taken the time to look at the scandalous state of school math textbooks.

Another reason is the irresponsible way universities prepare our teachers.[4]

Let us look at the *life cycle of school teachers* from the time they were students.

4. See, for example, H. Wu, The Mis-Education of Mathematics Teachers, *Notices Amer. Math.* Soc. 58 (2011), 372-384. http://math.berkeley.edu/∼wu/

They learned TSM as students in K-12.

$\longrightarrow$ They learn *university mathematics* in college, but not SM.

$\longrightarrow$ They must fall back on the TSM they learned in K-12 when they become teachers.

$\longrightarrow$ Their students learn TSM from them.

$\longrightarrow$ The next generation of teachers only know TSM.

Mathematics educators are themselves victims of TSM.

The above *life cycle of school teachers* is equally applicable to educators. This is how TSM gets recycled in mathematics education as well.

The only hope of breaking this vicious cycle is to **teach future teachers and educators SM in universities**.

This sounds simple, but there is campus politics. There is also the lack of a default version of SM:

> There is, as yet, no systematic exposition of K-12 mathematics that meets
>
> the needs of the school classroom and
>
> the minimum requirements of mathematics
>
> in terms of clarity, precision, reasoning, and cohesiveness.[5]

5. There is some attempt at doing this for K-6 mathematics. See, e.g., H. Wu, *Understanding Numbers in Elementary School Mathematics*, Amer. Math. Soc., 2011.

But the most serious missing component is the contribution of truly competent mathematicians who want to improve school math education and possess the requisite knowledge of schools.

This knowledge is best illustrated by the recognition that since fractions in SM are taught to ten-year olds, its mathematical development

> must be sensitive to their knowledge base and mathematical sophistication, and
>
> must differ significantly from that in university algebra courses.

It would be fair to say that we mathematicians have had a dismal record in educating teachers.

Our efforts have produced books that range from **overly formal and inappropriate** for use by teachers, to **mathematically oversimplistic** in trying too hard to be *pedagogically correct*.

As an example of the latter, there is a volume written under the auspices of CBMS, *The Mathematical Education of Teachers* **(MET)**. This is *the* standard reference in math education for the professional development of mathematics teachers.

The authors are a mix of mathematicians and educators, which could in principle produce something that is balanced and beneficial to teachers. However, except for the broad recommendations on the need of more mathematics courses, its detailed guidance—on the whole—falls far short of the ideal.

MET is being revised for a second edition. For the good of math education, the failings of MET should be thoroughly discussed. I will illustrate with one example.

Rigid motions, **symmetry**, and **congruence**

are a staple of the middle school curriculum nationwide. These concepts are casually brought up and nonchalantly discarded in the middle school classroom, e.g., congruence is "same size and same shape", and symmetry is for appreciating beauty in art.

Mathematics cannot be done on this basis.

What is missing is mathematical guidance on

how to delineate their logical interrelationship, and
how to bring out their relevance in mathematics.

Page 33 of MET: "The study of rigid motions can lead to an understanding of congruence . . . Geometry should also be studied as it occurs outside of mathematics, such as in nature and in art. There are many examples that could be studied, such as in the artwork of various cultures (*examples omitted*). Geometric transformations can be found in many designs, and recognizing these transformations adds, for prospective teachers, a legitimacy to the study of transformation by middle grades students."

Page 111: "A careful study of the meaning of congruence and of how congruence can be established should be included."

Two salient points made by MET:

> The justification for the study of rigid motion and transformation is *not a mathematical one*, but must be sought in art and nature.
>
> The "meaning" or "understanding" of congruence that MET has in mind is left to the reader.

Mathematical questions left unanswered:

What is the *mathematical* connection between “same size same shape” and SAS, ASA, SSS in high school?

What is meant by two *curved* geometric figures being *congruent*?

Precisely, how are rigid motions related to congruence?

Congruence and rigid motions *are* important mathematical topics in SM, but their importance is completely hidden in TSM.

MET seems to be unaware of the disconnect between SM and TSM, and therefore sees no need to offer help where help is urgently needed.

(Writers of state standards also seem to be completely unaware of this disconnect, as are many commentators on CCMS.)

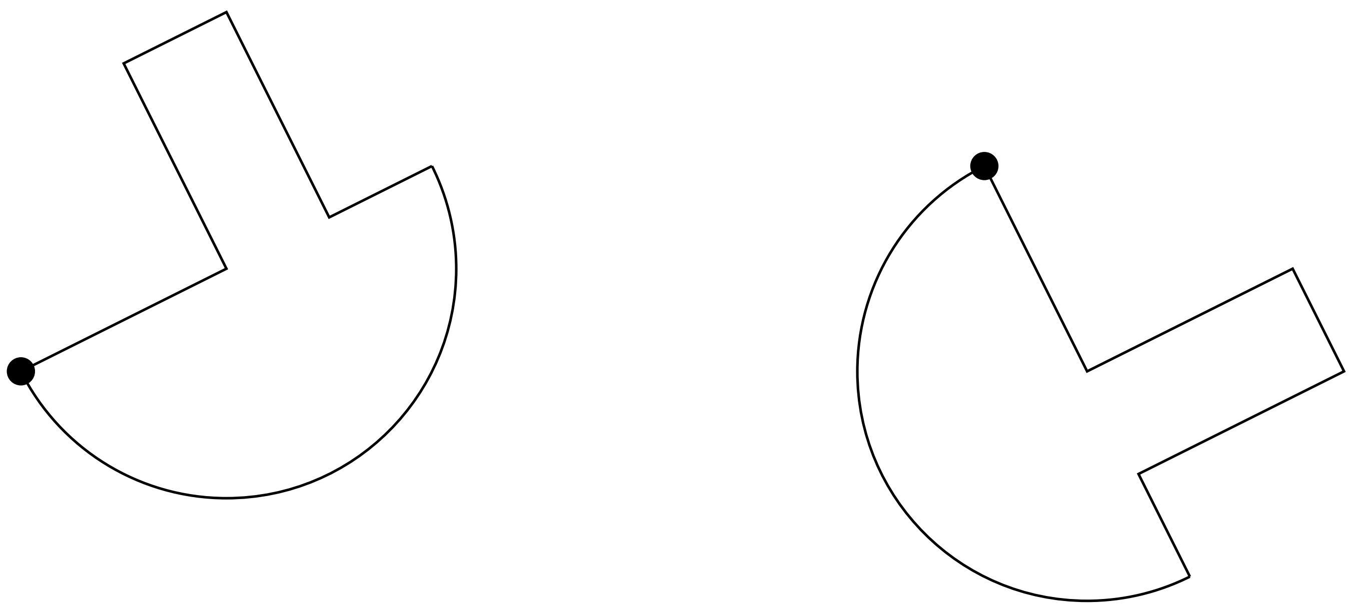
Here is how CCMS tries to align the teaching of geometry in grade 8 and high school with SM:

Grade 8

(1) Introduce the three basic rigid motions—**translation, reflection, rotation**—by hands-on activities, allowing students to gain an *intuitive* understanding of these concepts.

(2) Introduce the concept of the **composition** of basic rigid motions, again by hands-on activities.

(3) Define **congruence** as the composition of a finite number basic rigid motions, emphasizing that congruent figures are intuitively "the same size and same shape". For example, one can explain why the following figures are congruent :



A translation that brings the left black dot to the right black dot, followed by a 90 degree clockwise rotation, would bring the left figure to the right figure. The figures are therefore congruent.

(4) Prove ASA and SAS for triangles by hands-on activities.

High School Geometry is built on the eighth grade foundation:

(1) Define **transformation** of the plane.

(2) Define **translation, reflection, rotation** (basic rigid motions) as specific transformations.

(3) Define the **composition** of transformations.

(4) Define **congruence** as the composition of a finite number of basic rigid motions.

(5) Make explicit the *assumption* that basic rigid motions map lines to lines and segments to segments, and are length-preserving and degree-preserving transformations.

(6) Prove ASA, SAS, and SSS as *theorems* about triangles.

The development of plane geometry can essentially proceed as usual at this point.

CCMS gives students the opportunity to see that, beyond art and nature, congruence serves a serious mathematical purpose, and the ASA, SAS, SSS criteria for triangle congruence are more than rote skills. They are an integral part of the fabric that we call *geometry*.

More importantly, CCMS lets students see a different view of the Euclidean plane, one that gets at the geometric essence of the plane:

The plane is what it is, precisely because it possesses these three kinds of basic rigid motions.

CCMS thus tries to make sense of school geometry for students.

It tries to change TSM to SM.

It also exhibits so-called **sense making** in the context of serious mathematics and not as a slogan:

> Sense making has to begin at the most basic level of the curriculum, and should not be a separate headline in mathematics education.

The last comment about sense making is very germane to our task at hand:

> How can we produce a corps of teachers that understand the core message of CCMS,
>
> make sense of TSM, and
>
> can transform TSM into SM?

CCMS can say all it wants, but without such a corps of teachers to implement its vision, it will remain another document to collect dust on the bookshelf.

Universities—schools of education and mathematics departments in particular—have to begin teaching teachers SM.

They cannot do that if they continue to be ignorant of the chasm between SM and TSM.

Helping teachers to replace the mis-information they gathered from thirteen years of schooling in TSM with correct mathematical information about SM has to be the primary obligation of every kind of professional development as of 2011.

Universities have to begin taking this obligation seriously.

All professional developers must also take this obligation seriously.

From this vantage point, we can see why the kind of one day or two day workshops provided by school districts for math teachers do not constitute effective professional development. These tend to be pedagogical embellishments of teachers' defective knowledge from TSM; they do not increase teachers' instructional capacity.

One cannot improve on rotten meat by spraying it with perfume. It is still rotten meat.

In general, the more serious kind of professional development has different emphases, including children's mathematical thinking, classroom strategies, skillful use of manipulatives, skillful use of technology, teacher-student communication, refined teaching practices, etc. These are important aspects of a teacher's pedagogical equipment.

However, such emphases in the absence of a solid knowledge of SM is no different from discussing strategies in competitive running with children who don't yet know how to walk.

The dilemma we face is this:

- To provide teachers with the needed mathematical knowledge, we need the expertise of research mathematicians.

- Yet research mathematicians have no incentive to acquire the requisite knowledge of schools to get this job done and research universities cannot afford to put the education of math teachers as a top priority.

Like all good things in life—freedom, for example—effective mathematics professional development is something we must fight for, everyday.

Good math teachers will materialize only when we are determined to negotiate a balance between highly charged conflicting demands.

Do we have the will to do it?