Some remarks on the preparation of mathematics teachers in China*

Hung-Hsi Wu

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Abstract. The preceding chapters contain important information about the education of mathematics teachers in China. From an American perspective, the emphasis on teachers' content knowledge and the concept of professional growth and lifelong learning are particularly striking. We believe that such a general framework for the professional development of teachers is one that the U.S. can use to its benefit, *provided* that one can ensure that the content knowledge is what teachers really need in their classrooms and that it meets the minimum requirements of mathematics. We will briefly explain this concern in some American contexts and will also make some speculations on the Chinese situation.

Keywords. Content knowledge, pedagogical content knowledge, fundamental principles of mathematics, Textbook School Mathematics (TSM), professional development, lesson planning, and lesson observation.

A nation's mathematics education is only as good as its mathematics teachers. The professional development of teachers is therefore serious business. This is an inherently complex subject, because, into the complex human dynamics between students and their teachers, professional development must try to inject an optimal strategy for the transfer of knowledge from teachers to students—no matter how this "transfer" is defined. (The last word on the pedagogical aspect of professional development will never be written.) What further complicates matters in this context is that, mathematics being the highly technical discipline that it is, mathematical content knowledge is bound to play a dominant role in mathematics instruction. Therefore content knowledge has to play a key role in any successful mathematics professional development. Unfortunately, there has been some serious misunderstanding in the U.S. over what this "content knowledge for teaching" ought to be, apparently all through the last century and up to 2017 (cf. Wu, 2011b, also Shulman, 1986, and Ball, Thames, and Phelps, 2008). The first section of Chapter 4 of this volume points to the additional difficulty that this body of knowledge also seems to vary across nations:

> These different studies point to the difficulty to reach international agreement on a definition of mathematical knowledge for teaching and how to acquire it. (Chapter 4, page 9)

The complexity of mathematics professional development cannot be denied.

For all these reasons, it is not likely that any one nation will ever have a monopoly on excellence in the professional development of mathematics teachers. A continuous exchange of ideas between nations will always be beneficial to the health of the enterprise that we call school mathematics education. From this perspective, the present volume promises to provoke a fruitful international dialog. It seems to this author that there are some striking features in the Chinese system that the U.S. should diligently study, and perhaps emulate, for its own benefit. At the same time, other elements in the Chinese system may be ripe for a reappraisal. The following sections will amplify on these claims.

1 Some noteworthy features of the Chinese system

The pre-service preparation of mathematics teachers in China puts great emphasis on the acquisition of content knowledge but gives less attention to the acquisition of experience in student teaching or *pedagogical content knowledge* (**PCK**, see Shulman, 1986). The rationale for the emphasis on content knowledge is straightforward and unimpeachable:

> [The consensus is] that it is impossible to develop pedagogical content knowledge without appropriate content knowledge. (Chapter 6, page 21)

For elementary *mathematics* teachers, a survey of 16 major teacher-training programs shows that, in general, the required mathematics courses include:

> Mathematical Analysis, Spatial Analytic Geometry, Projective Geometry, Non-Euclidean Geometry, Theory of Probability, The Structure of Algebra, a Brief History of Mathematics, Mathematical Modeling, Advanced Algebra, and Elementary Number Theory. (Chapter 5, page 8)

Also included are modified versions of Calculus and Linear Algebra.[1] As for secondary

[1] Later in this chapter, there will be a few comments on the appropriateness of some of these mathematical requirements.

teachers, one learns that in the teaching program of a "superior" normal university, the "credit point of the mathematics curriculum is three times that of the teacher education curriculum". (Chapter 6, page 6) The required courses include:

> Ordinary Differential Equations, Classical Geometry, Complex Analysis, Probability and Statistics, Abstract Algebra I and II, Differential Geometry, Number Theory, Real Analysis, and Combinatorics and Graph Theory. (Chapter 6, Table 1 on page 7)

The 2011 **Teacher Education Curriculum Standards** try to address the imbalance of content over PCK in pre-service education. It is believed that "mathematics teacher preparation in China has undergone a significant transition from solely focusing on content knowledge to balancing content and practice-based knowledge" (Chapter 6, page 22). However, how to develop high-quality courses by integrating mathematics content and practice-based mathematic pedagogical knowledge remains an ongoing challenge. That said, it is well to observe that this imbalance is offset by the system of in-service professional development in China, which has a long tradition of supporting *professional growth and lifelong education* for the acquisition of classroom management skills and PCK. There is a well-established multi-tiered system of mentorship by experienced teachers, collaborative lesson planning with peers, lesson observation and post-lesson reflection and improvement (Chapter 9, page 21), and developing and observing public lessons (Chapter 11). Experienced teachers can further hone their craft by engaging in a Master Teacher Workstation program (Chapter 9). Lifelong learning for Chinese teachers is in fact a requirement:

> Once a teacher starts teaching career, s/he has to actively and constantly engage in a variety of professional development (**PD**) activities which are mandatory, including pairing up with an experienced teacher of the same grade level for one-on-one mentoring, participating in teaching research groups in which teachers study textbooks and develop lessons together, and giving public lessons for colleagues and administrators to observe. The PD activities are continuous, practice-oriented, relevance-focused, and integrated as a whole to be an important part of a teacher's entire teaching professional life. A teacher never stops engaging in PD activities unless s/he leaves the teaching profession. (Chapter 11, page 1)

Lifelong learning is therefore an integral part of the Chinese teaching culture so that all teachers come to feel that they are not alone but are surrounded by a strong support group. They gain much of their knowledge for teaching through school-based continuous professional development activities:

> It is very common in China that teachers observe each other's class and then discuss teaching ideas and strategies on the purpose of learning knowledge for teaching from each other and growing professionally to-

> gether. It could happen between a new teacher and an experienced teacher. (Chapter 11, page 2)

> In [the research group] activities, Chinese teachers usually discuss how to improve classroom instruction. The emergence of knowledge is quite similar to PCK . . . That is, when a teacher copes with a special topic, he/she will organize, adjust and present subject matter knowledge to do task design by considering a special group of students' interest and capacity. (Chapter 10, page 2)

To Chinese mathematics teachers, doing lesson preparation, observation, and post-lesson reflection together is therefore a way of life. Their American counterparts may likely read about such a teaching culture with longing and envy.

2 Some concerns from an American perspective

Here are the two features in the Chinese preparation of mathematics teachers that should be of special interest to the U.S.:

(A) A strong emphasis on content knowledge in pre-service PD.

(B) Making professional growth and lifelong education integral parts of the teaching culture.

Although many Chinese educators consider the advocacy of (A) at the expense of PCK to be a weakness of their system, the opposite has occurred in the U.S. where there has been a longstanding over-emphasis on pedagogy over content knowledge. At the present time (2017) when the urgent need for mathematically (and scientifically) knowledgeable teachers in the U.S. is beyond dispute, the current strategy in the empowerment of mathematics teachers still tends to be of the "unleashing students' potential through innovative teaching" variety. *Not* about learning correct mathematics for their teaching. In this climate, (A) should come as a welcome correction.

Although (B) has not been central to current discussions about mathematics education in the U.S., it should be. There is a teacher dropout crisis at the moment (cf. NPR Ed, 2014). In each of the years from 1988 to 2013, between 12.4 and 16.5 percent of public school teachers moved to a different school or left the teaching profession entirely (Table 1 of Goldring et al., 2014). More than 42% of new teachers leave teaching within 5 years of entry (Ingersoll, Merrill, and May, 2014, page 5). The reasons vary, but one of them is apparently teachers' strong desire for learning from each other, a desire which often goes unmet in the teaching profession. A telling statistic is that, of teachers who left teaching in 2012–2013 for a different occupation, about 41.7 percent of *public school* teachers reported that "opportunities for learning from colleagues" were better in their new (non-teaching) positions, compared with only 15.9 percent reporting that they were worse (Table 7 on page 13 of Goldring et al., 2014). Moreover, 45.7 percent reported that "opportunities for professional development" were better in their new positions.[2] In a 2014 survey of *all* teachers (not just mathematics teachers), it was found that 84 percent wanted to spend more time on lesson observation, 82 percent on coaching, and 74 percent on professional learning communities (Bill & Melinda Gates Foundation, 2014, page 5). It was almost as if teachers were asked whether they wanted (B) in their professional lives and they overwhelmingly voted *yes*! What these statistics point to is an undeniable need for the American teaching culture (at least in the public sector) to embrace some version of (B).

However, we cannot recommend (A) and (B) without reservation as of 2017 for at least two reasons. The first is that we have to clear up the meaning of "content knowledge for teaching" before (A) and (B) can be understood in the proper context. A second reason is that (B) would be meaningless without some large-scale restructuring of the teaching profession.

We will deal with the content issue first. Let us begin with the fact that, in the U.S., the mathematical knowledge that our teachers bring to their classrooms has typically not been *mathematics* but a corrupted version of mathematics that has resided in the standard school textbooks for (at least) the past four decades. This body of knowledge is as different from *mathematics* as margarine is from butter. We

[2] The same Table gives 21.2 percent of the teachers reporting that they were worse, but the standard error for this figure is between 30 and 50 percent.

propose to call it **Textbook School Mathematics (TSM)** (see Wu, 2011b and 2011c) to distinguish it from **school mathematics**, which is the mathematics of the K–12 curriculum. TSM is *unlearnable* because of its lack of what may be called *logical transparency*. In greater detail, the unlearnability of TSM is the result of its pervasive violation of the following five **Fundamental Principles of Mathematics (FPM)** (Wu, 2011b, pp. 379–380):

1. Every concept has a precise *definition*, and definitions furnish the basis for logical reasoning. (Definitions leave no doubt about what exactly students have to learn.)

2. Mathematical statements are *precise*. Precision makes possible the distinction between what is known and what is not known, and what is true and what is false. (Precision eliminates the need to *guess* in learning mathematics.)

3. Every assertion is supported by logical *reasoning*. There are no arbitrary or irrational decrees in mathematics. (Mathematics is learnable *because* it is reasonable.)

4. Mathematics is *coherent*. Mathematics is a living organism in which all the different parts are interconnected. (Mathematics is not a bag of isolated tricks for students to memorize.)

5. Mathematics is *purposeful*. Every concept and skill in the school mathematics

curriculum is there for a purpose. (Students get to know where they are headed.)

For those readers who are shocked to find that textbooks that responded to the mathematics education reform of 1989 are lumped together in this analysis with those from traditional major publishers, it suffices to point out that while the reform texts teach K–12 mathematics on a different pedagogical platform, their *mathematical content*—seen through the lens of FPM—is still just TSM.

Because TSM is an overriding theme of this chapter, we will digress a bit to illustrate how TSM violates the fundamental principles of mathematics. Some examples are given in Wu, 2011b and 2011c, but we will mention three of the most obvious to make our point here. The first illustrates (among other things) the lack of precision in TSM. TSM teaches that "22 divided by 5 has quotient 4 and remainder 2" should be written as $22 \div 5 = 4\ R2$. (The idea seems to be: since it is so convenient, why not just use the equal sign to denote "here is the result of my calculation"?) But by the same token, we will also have $42 \div 10 = 4\ R2$ and therefore $22 \div 5 = 42 \div 10$ because both are equal to $4\ R2$. In terms of fractions, we now have $42 \div 10 = \frac{42}{10} = \frac{21}{5}$. TSM has therefore led us to an apparent equality $22 \div 5 = 42 \div 10$ which implies this absurdity:

$$\frac{22}{5} = \frac{21}{5}$$

A second example illustrates the damage inflicted by TSM on student learning due to its failure to provide precise definitions of fundamental concepts. Thus a fraction in

TSM is "parts of a whole" or, more often, pieces of pizza. This then begs the question of how to explain the division of $\frac{3}{7}$ by $\frac{11}{5}$. There is no reasoning on earth that can possibly convince young learners how to divide "a *quantity* that consists of 3 of the parts when a pizza is divided into 7 equal parts", by another "*quantity* consisting of 11 of the parts when a pizza is divided into 5 equal parts". Such a *quantity* is neither beast nor man, nor in fact anything students know how to deal with. They are thus forced to conclude *ours is not to reason why, just invert and multiply.*

A final example on the teaching of high school geometry illustrates the frequent lack of coherence in TSM. Since students are almost never given a definition of a concept in TSM (e.g., "fraction"), they come into their high school geometry course having no experience with how a definition can be used for reasoning.[3] Since TSM does not give students any hint of the *logical* hierarchical structure of mathematics, they have no conception of what "axioms" are before they run into them in high school geometry. Since TSM hardly ever exposes students to reasoning before high school geometry, "proving a theorem" is also a completely foreign concept. Yet, in this geometry course, and *only* in this course in TSM, students are suddenly confronted with a litany of definitions, axioms, theorems and proofs. They are called upon to *prove every single assertion no matter how trivial*, and in fact, most of the early theorems are unbelievably trivial and therefore extremely difficult to "prove". Under

[3] And sometimes not even then.

these circumstances, geometry teaching and learning can easily degenerate into a farce (see, for example, the documentation in Schoenfeld, 1988). Being aware of this farce, some teachers and school districts have reacted by going to the opposite extreme of teaching high school geometry with *no proofs*, relying on the use of computer software to bring mathematical conviction to geometric theorems. Either way results in an incomplete and incoherent presentation of mathematics to students.

One can get an extensive documentation of how TSM abuses the mathematics of grades 6–8 by looking up "TSM" in the indices of Wu, 2016a and 2016b. A little extrapolation will give a fairly accurate picture of what TSM has done to the teaching and learning of mathematics in K–12 as a whole.

Important as FPM may be to school mathematics, we caution that teachers' content knowledge must do more than respect FPM. The mathematics we teach college math majors—groups, rings, fields, Dedekind cuts, Cauchy integral theorem, Gauss-Bonnet theorem—also respects FPM, but it would be bad education to make it the required content knowledge for all teachers because college mathematics is too advanced to be usable in the K–12 classroom. The content knowledge we need our mathematics teachers to possess therefore has to meet both of the following conditions:

(I) It closely parallels the K–12 mathematics curriculum.

(II) It is consistent with FPM.

For brevity, we will refer to the body of mathematical knowledge that satisfies both

(I) and (II) as **principle-based mathematics** (see Poon, 2014, and Wu, 2017). TSM satisfies (I) but emphatically not (II), and the mathematics embodied in the usual requirements of a college math major satisfies (II) but definitely not (I). If we want the above outcomes (A) and (B) to materialize in the preparation of American mathematics teachers, then *"content knowledge" will have to be understood to be principle-based mathematics.* The relevance of principle-based mathematics to (A)—a strong emphasis on content knowledge in pre-service PD— is obvious, but its relevance to (B), regarding professional growth and lifelong education among in-service teachers, should be no less clear. If teachers only know TSM and not principle-based mathematics, then their collaborative lesson planning will produce nothing more than pedagogical embellishments of TSM. Their critiques of each other's teaching in their lesson observations will all be based on TSM, but TSM is too inherently defective to serve as an arbiter of mathematical truth. For example, one can imagine one teacher suggesting to another, "Don't you think you could have used proportional reasoning to give a simple solution of the problem?".[4] Also keep in mind that, since most professional developers only know TSM, in-service mathematics PD cannot help but reinforce teachers' knowledge of TSM. Therefore, if teachers only know TSM, the upshot of any effort to foster lifelong learning and continuous education is likely to result in the creation of a robust environment for TSM to thrive and metastasize

[4]Proportional reasoning is a mainstay of TSM, but it is not correct mathematics. See Section 7.2 (pp. 144–154) of Wu, 2016b, for an explanation of why it is not correct.

throughout the body of school mathematics education. This is not a consummation devoutly to be wished. For the good of school mathematics education, we cannot import (A) and (B) into the American system without first getting rid of TSM.

Before exploring the *content* issue any further, we must point out that it is impractical to make (B) a reality in the teaching culture without first introducing major changes in the present education system. If teachers are going to engage in mutual classroom visits, collaborative lesson planning, and attending professional development institutes as a way of life, they will need *time* in their job to engage in such time-consuming activities, valuable as these may be. It would appear that, as of 2017, the Chinese education system allows its teachers to do that but the American system does not, because:

> Chinese teachers have much larger classes: typically around twice the size of U.S. classes.... (National Research Council, 2010, page 5) Chinese teachers have fewer classes than do U.S. teachers, typically just two or three per day, whereas U.S. teachers are in their classrooms for most, if not all, of their day. (*ibid.,* page 6)

The long and short of the matter is that (B) will not materialize in the American system for free and, to make it happen, we will have to make some hard choices. First, we will have to choose between smaller classes and less opportunities for professional growth, or larger classes and more time for professional growth. The debate about the pros and cons of class size has been going on for a long time (Mishel and Rothstein, 2002), and at some point, I hope teachers themselves will be able to make the right choice for themselves.

A second consideration is that if we opt for larger classes, the demand on each teacher's mathematical and pedagogical competence will likely increase. This will require an upgrade of the teaching corps. Such an upgrade will not happen unless we can make teaching a more attractive profession. Needless to say, that is a major social issue that is mostly beyond the concerns of this chapter. For example, even one of the easiest parts of this problem, raising teachers' salaries, already stirs up a political hornets' nest. However, there is at least one thing that is very much pertinent to the present discussion. One of the many reasons teachers are dissatisfied with their profession is that they are not given the respect due them as professionals. For example, teachers have little input into school decision-making and "have only limited authority over key workplace decisions" such as "which courses they are assigned—or misassigned—to teach" (Ingersoll, 1999, page 34). Any changes along this line will require an overhaul of school management and the education administrative bureaucracy. Do we have the political will to face up to this challenge?

Let us return to the content issue and make a few comments on the feasibility of bringing about the necessary changes to make (A) and (B) a reality. To help pre-service teachers change their knowledge base from TSM to principle-based mathematics, institutions of higher learning across the land will have to be aware of the deleterious effects of TSM on school mathematics education and make the commitment to change. This will require cooperation between schools of education and departments of mathematics in these institutions as well as the willingness to make financial investments by the institutions' administrations. As of 2017, there seems to be no sign of such awareness and, in any case, the cooperation between the mathematics and education communities may not be easy to come by.

An additional difficulty in getting rid of TSM is the need for a default version of *principle-based mathematics* to demonstrate that there can be a systematic exposition of mathematics that toes the line of the school curriculum *and* is consonant with FPM. This is because, although principle-based mathematics is a branch of mathematics, it is different from standard college mathematics, in the same way that although the basic theory of electrical engineering is part of physics, electrical engineering is *not* part of physics (see Wu, 2011b, Section 3). The two demands on principle-based mathematics, (I) and (II) above, pull in opposite directions, so that something as simple as the concept of a fraction in *abstract algebra* easily satisfies (II) but is too sophisticated to be used in the elementary classroom (i.e., does not satisfy (I)). It is at times not entirely straightforward to find a way to develop mathematics that is consistent with FPM and also suitable for use in the school classroom. The very existence of principle-based mathematics therefore cannot be taken for granted.

At the moment, an overwhelming majority of teachers and university mathematics educators were themselves educated in TSM, and many have come to believe that TSM *is* mathematics. These teachers accordingly teach their own students TSM and these educators carry out their research in terms of TSM (e.g., a great deal of research effort was invested in the vain attempt to make fractions-as-pizzas teachable and learnable). Unfortunately, both actions end up imprinting TSM on the next generation and perpetuating a vicious cycle (see Wu, 2011c, page 9 for a more detailed discussion of the vicious cycle). Having a default version of *principle-based mathematics* will make it possible to teach it to pre-service teachers and break this vicious cycle.

There is another benefit of having available a default version of principle-based mathematics that seems not to be fully appreciated, namely, it will facilitate the development of teachers' PCK as described, for example, in Chapters 4–8 of this volume. This is because PCK is the *bridge* between teachers' *content knowledge* and their *pedagogical practices* in the classroom. At the moment, the education literature seems to be uncertain about exactly what this bridge is, or might be. The general idea (Chapter 6, page 112) that PCK is needed to transform mathematics in "its academic form featuring rigorous deduction and logical reasoning"—which is "is precise but cold"—into something "interesting, beautiful, and easy to access" in the school classroom speaks to the misconception about what constitutes the "content" in *pedagogical content knowledge* (PCK). If we correctly understand this "content" to be *principle-based mathematics*, then the fact that this content is already conceived and developed in accordance with students' learning trajectories (i.e., condition (I) of principle-based mathematics) means that it will be, by design, already much easier to access than the supposedly "precise but cold" mathematics. Moreover, since the reasoning in principle-based mathematics is introduced at the level of the school curriculum for each grade, teachers and educators should have little doubt about whether it is appropriate for students to learn this "logical reasoning". Instead, they can concentrate their efforts on the pedagogical issues of how to facilitate their students' learning about this reasoning. Therefore, making principle-based mathematics the *content* of PCK adds clarity to what PCK is by closing the gap between content knowledge and pedagogical practices in the classroom.

Finally, we note that there has been an ongoing effort to write an exposition of principle-based mathematics for K–12; see Wu, 2011b (elementary school), 2016a and 2016b (middle school), and *to appear* (high school). It is hoped that the series will be completed by 2018.

3 Speculation on content knowledge preparation

We will look into the content-knowledge preparation of mathematics teachers in China and propose a few related research directions.

In Chapter 8, it is stated that school mathematics textbooks are very important to Chinese teachers:

> Mathematics textbooks are used in several important ways in the Chinese school education, including, as tools for teachers' professional development through studying textbooks; as self-paced learning materials for out-of-school children; and as the main resource for teaching and learning in the classroom. Both in urban and rural schools, mathematics textbooks and related teaching manuals are important resources for teachers' instructional and lesson planning (page 166).

Such an overt reliance on textbooks by teachers is natural, yet to someone from the U.S. who has experienced the devastation of TSM firsthand, this passage does raise some concerns: Could some incipient version of TSM be at work in Chinese school mathematics education? Such a question runs counter to the general tenor of the preceding chapters, and I raise it only because of my own sporadic and unorthodox encounters with Chinese school mathematics education.

I have visited China (essentially only Beijing) eight times and—except the one-week visit in 2010—each visit lasted between three and seven weeks. The first five visits, from 1976 to 1985, were entirely devoted to mathematics;[5] I went there in my

[5] My first visit was with a large mathematics delegation and one subgroup was entrusted with the task of reporting on Chinese mathematics education. If memory serves, I was not with that subgroup, but I am not absolutely certain.

capacity as a geometer. The last three, from 2006 to 2011, were wholly or partly related to school mathematics education. In seven of the eight visits, I got to meet with officials from the Ministry of Education. Now, during my visits up to 1985, I never once suggested that I knew anything about school education, yet each time, those officials would voluntarily bring up the question of what to do about what they called the "duck-stuffing" phenomenon (*tianya*, i.e., rote-learning) in school math classrooms. The fact that such an issue about *school education* would be discussed at all during my first visit in May of 1976, four months before the end of the Cultural Revolution, must be regarded, in retrospect, as nothing short of stunning. Given the wholesale disruption of all phases of education during the ten years of the Cultural Revolution, the raising of this question about *education* in that setting could not help but make an indelible impression on me. The officials' concern was obviously genuine, so the phenomenon itself had to have been deep-rooted. The fact that the same question—in the same terminology of *tianya*—would continue to be raised by different officials in each of my subsequent visits with them could only mean that the Ministry's search for a solution to the problem was still ongoing.

Before 1992, I was a professional mathematician full-time, and my knowledge of school mathematics education was nearly nonexistent. My answers to the Ministry officials' question about "duck-stuffing" in those days were at best *pro forma*, but the concept of "duck-stuffing" became permanently lodged in my mind. It is because of this *idée fixe* that I am now led to wonder about the possible presence of some form of TSM in Chinese school mathematics education. Obviously I do not have any evidence that Chinese school mathematics education is suffering from TSM, yet there are a few indicators—from my firsthand experiences and from what I read in the preceding chapters—that suggest that this idea may not be so outlandish after all but may actually be worthy of serious investigation.

Let me recount my limited firsthand experiences. In my 2011 visit to China, I once had a roundtable discussion with Ministry officials and local teachers (in Beijing). I brought up the subject of *reasoning* in teaching. Around that time, I happened to be wrestling with the teaching of the "laws of exponents" to American secondary teachers. In TSM, these laws are factoids to be memorized, and the exponential notation is just one more new notation like a host of other notations. In the TSM tradition, a main emphasis is to drill students on becoming fluent in re-writing radicals of numbers in exponential notation, and that is an end in itself. TSM treats these laws as *number facts* that have nothing to do with the real reason for learning about these laws—the amazing properties of the exponential function. I was faced with the problem of finding a way to make teachers aware of the real mathematical issues behind these laws, and to make my arguments sufficiently persuasive so that they would rethink their TSM-infused knowledge.[6] For this reason, I inquired specifically

[6]The eventual outcome of this effort is recorded in Wu, 2016b, Chapter 9.

about how these laws were taught in Chinese classrooms. My memory of the teachers' responses is not too clear after all these years, but I think they said that their main emphasis in general was on teaching students how to use the laws to solve problems but not so much on why the laws are true or even what they mean. I was a bit surprised, so I asked whether, in their lessons, they would at least consider *defining* clearly the meaning of fractional exponents and *proving* certain special cases such as the ubiquitous identity: $x^{1/n}\, y^{1/n} = (xy)^{1/n}$. After a short silence, one teacher finally spoke up and explained that if she were to do this, students would ask whether this material would be on *gaokao*,[7] and if not, then they would simply tune her out. The other teachers nodded and murmured in agreement.

Some days after that meeting, I was given the opportunity to visit several classes in a high school and, afterwards, I had a short session with some teachers for an exchange of views. Very likely, something I heard in one of the classes made me bring up the need to clearly differentiate between a *definition* and a *theorem* in teaching. I said that, in America, our teachers were not always able to draw that distinction correctly and that, in Australia, many teachers also seemed to have the same problem. For example, I asked whether "$3^0 = 1$" was a definition or a theorem.[8] A teacher said confidently that it was clearly a theorem. This is an answer that the American

[7] This is the examination that is a prerequisite for entering almost all higher education institutions in China at the undergraduate level. It is usually taken by students in their last year of high school.

[8] I raised this question because this is a definition that is usually presented as a theorem in TSM.

TSM textbooks would emphatically agree with.

Because I had no firsthand knowledge of Chinese school mathematics textbooks, I reflected on these two anecdotes and wondered whether they were related, and also how students' preoccupation *gaokao* might impact textbooks. Would these books make any effort to discuss things not directly related to the procedures of solving problems on *gaokao*?

According to what I read in the preceding chapters, the answer to this question is more complex than I would have hoped. Unlike the American situation, textbooks *per se* seem not to be the main issue in Chinese mathematics education. Textbooks in China don't seem to include a lot of information about PCK and "a large amount of materials used in classroom instruction are literally added by teachers based on their experience." (Chapter 11). This is consistent with the information about the state of elementary mathematics education in Chapter 5, for example. In that chapter, one finds some reactions of pre-service elementary teachers to the rigorous *mathematics* course requirements for pre-service elementary teachers. One teacher said, "Some of the courses are so boring, also difficult." Another teacher said, "I don't know why we need to learn these. Useless." (page 99). These comments point to an obvious gap between the mathematics taught in the required mathematics courses and the mathematical knowledge teachers actually *need for teaching in a school classroom.* As a matter of fact, one could have easily arrived at the same conclusion by inspecting the course requirements for elementary teachers on p. 93 in Chapter 5 and those for secondary teachers on page 116 of Chapter 6 (See pp. 372–373 of Wu, 2011b, for an analogous discussion.) Some deans of normal universities acknowledged that this gap is substantial and real, and have tried to forge a closer connection between these mathematics courses and teachers' pedagogical needs (Chapter 5, page 99).

In this light, we now better understand the quote from Chapter 8 at the beginning of this section: the disconnect between the rigorous mathematics in university pre-service courses and actual on-the-job pedagogical needs has driven teachers to rely on school textbooks and collaborative lesson planning (Chapter 9) to develop their own content knowledge and PCK for teaching. The burning question is whether teachers do so by isolating themselves and turning their collective backs on the mathematics community, and if such is the case, whether the mathematical knowledge so produced is consistent with FPM. In the case of the laws of exponents, for example, could experienced teachers' perception of the futility of teaching these laws *as principle-based mathematics* persuade new teachers to concentrate on teaching these laws as *procedures* because this is "what works" in a *gaokao*-fixated classroom? If so, might this tradition set up a vicious cycle so that such an approach to teaching the laws of exponents becomes a permanent fixture in Chinese mathematics education? Could similar classroom "realities" of this nature also come to inform the teaching of other mathematical topics in teachers' reservoir of PCK? Might not this kind of "reality-based" instruction be the source of the "duck-stuffing" phenomenon?

We need evidence-based answers to these questions. What we know from the American experience is that such isolation from the mathematics community was in fact a main cause that triggered the development of TSM—*Textbook* School Mathematics. In the U.S., the abandonment of FPM for pedagogical expediency was apparently what led teachers and educators to certain practices of TSM (e.g., fractions as pieces of pizza) which, when codified in textbooks, eventually became the orthodoxy in American school mathematics education. In this context, it may be useful to recall the two basic requirements of principle-based mathematics:

(I) Principle-based mathematics closely parallels the school mathematics curriculum.

(II) Principle-based mathematics is consistent with FPM.

Our present discussion of the isolation of teachers from the mathematics community puts in perspective why both requirements (I) and (II) are absolutely essential to a sound *content knowledge for teaching*, and why it is important for teachers to learn principle-based mathematics.

Now, back to the main thread of our discussion: could the knowledge (content knowledge as well as PCK) created by mathematics teachers in such isolation have something to do with "duck-stuffing"?

If the concern of the Ministry of Education is to be taken seriously, real research effort should be devoted to settling this question one way or the other. The most direct approach would be to evaluate current textbooks to see whether they are consistent with FPM. This will require the cooperation of the mathematics and education communities. But, as we mentioned above, looking at textbooks is not enough, because we would need to collect a representative sample of actual classroom lessons to see the kind of mathematics instruction school students are actually getting. In year 2017, it should not be difficult to obtain video records of such lessons for a detailed study. Given the vast scope of such a study, perhaps one should break it down into three parts: elementary school, junior high school (middle school), and senior high school. Again, the video analysis will require a team of both mathematicians and educators.

Another study worth undertaking would be the effect of *gaokao* on mathematics education. Teaching-to-the-test is hardly a new phenomenon; it has galvanized the attention of American educators for a long time. Clearly, the *gaokao* is not going away, so there should be serious research on how to minimize its impact on general mathematics education. One school of thought believes that, if a standardized test is to serve a *useful* purpose for school mathematics education, it has to be low-stakes (see Wu, 2012, pp. 15–17). Since the *gaokao* is by definition a high-stakes test, it may be time to consider the possibility of having two tiers of mathematics education in the last two years of senior high school: students who intend to go to college take the *gaokao* but those who don't follow a different curriculum.

It is important to recognize that those who want to excel on *gaokao* will have the *obligation* to learn more than what the *gaokao* dictates. These advanced students must learn that excellence in the sciences and mathematics cannot be achieved solely by a superior ability to do *problems that are handed to them.* They will have to learn mathematics consistent with FPM, especially its reasoning and coherence, to nurture their creativity. We hope the (yet-to-be-written) curriculum for this group of students will leave no doubt about such an expectation. Classrooms in which principle-based mathematics is taught will certainly steer mathematics education away from "duck-stuffing". For the other group not going to college, being freed from the tyranny of the *gaokao* will allow them to settle down and learn about—not all the technical skills to negotiate the tricky problems in the *gaokao*—but the two components of mathematics education that validate the presence of thirteen years of mathematics in their school curriculum: the reasoning and critical thinking that are needed for decision making. These two qualities are indispensable to life in the high-tech age but they will not survive a duck-stuffing education. Because many of the "duck-stuffing" skills are increasingly being taken over by computers, students who do not go to institutions of higher learning have no choice but to learn to reason and think critically—things that computers have not yet mastered. This is not the place to enter into a detailed discussion of a curriculum that can promote reasoning and critical thinking, but Part

II of Wu, 2011d, can serve to give *some* idea of its potential.

Such a sea-change in the basic structure of school mathematics education cannot be carried out without detailed research into all the associated social questions it raises. For example, will such a two-tiered education be socially acceptable and, in fact, politically feasible? Fortunately, some nations (Japan, Germany, etc.) already practice variations of such a two-tiered system, so there is no need to reinvent the wheel. There should also be research into the creation of a complete series of school textbooks that are less *technically-oriented* but which nevertheless meet the high standards of FPM. Altogether, this would be a vast and Herculean undertaking, and until these preliminary hurdles have been overcome, it would be inadvisable to launch such a reform.

Finally, I will strongly suggest that all teachers be taught principle-based mathematics in their teacher preparation programs. Back in 1972, when Begle first looked into the disconnect between teachers' "mathematical" knowledge and the effectiveness of their teaching, he concluded:

> ... teachers should be provided with a solid understanding of the courses they are expected to teach. (Begle, 1972, page 8)

Subsequent works on PD for teachers have basically borne out Begle's dictum (Wu, 2011b). Now if principle-based mathematics is to be made part of the mathematical preparation of teachers, then adjustments will have to be made to the rest of the required *mathematics* courses listed in section 1 (of this chapter). There is a need to lower the *technical* level and deepen the conceptual level of most of these courses by replacing them with *general surveys* for the benefit of teaching school mathematics. Recall one teacher's comment on these courses: "Some of courses are so boring, also difficult". For example, it is hard to imagine that many elementary teachers would be enthralled by the need to prove theorems in a course on Non-Euclidean Geometry or Projective Geometry when they will probably never come across the hyperbolic axiom or Desargues' Theorem in their professional life. Would not a single survey course on geometry that gives the history of the Parallel Postulate with some hands-on activities on hyperbolic geometry, spherical geometry, and projective geometry better serve elementary teachers? A similar comment can be made about the requirement for secondary teachers to take Complex Analysis, Abstract Algebra II, Differential Geometry, and Combinatorics and Graph Theory. However, to make a comprehensive reform possible, research will be needed to create a cohesive program that attends to the goal of broadening teachers' mathematical knowledge without sacrificing the relevance of this knowledge to their professional practice. This research will also have to include the creation of a series of textbooks for the new teacher preparation program. There is a tremendous amount of work ahead, but for the good of the next generation, it needs to be done.

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Department of Mathematics, University of California

Berkeley, CA 94720-3840

wu@berkeley.edu