Teaching Fractions According to the Common Core Standards

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*I am very grateful to David Collins and Larry Francis for very extensive corrections.

Preface

Le juge: Accusé, vous tâcherez d'être bref.
L'accusé: Je tâcherai d'être clair.
—G. Courteline[1]

This document gives an expanded view of how **the Common Core Standards on fractions in grades 3-7** may be taught. As of 2014, it may be unique in that *it is written for the classroom teachers* by someone who has been teaching fractions to elementary and middle school teachers since year 2000 in a way that is in almost complete agreement with the Common Core Standards. The specific standards that are addressed in this article are listed at the beginning of each grade in san serif front.

Students' learning of fractions may be divided roughly into two stages. In the initial stage — that would be grade 3 and part of grade 4 in the Common Core Standards — students are exposed to the various ways fractions are used and how simple computations can be made on the basis of simple analogies and intuitive reasoning. They learn to represent fractions with fraction strips (made of paper or just drawings), fraction bars, rectangles, number lines, and other manipulatives. Even in the exploratory and experiential stage, however, one can help students form good habits, such as always paying attention to a fixed *unit* (the whole) throughout a discussion and always being as precise as practicable. Regarding the latter issue of precision, we want students to know at the outset that the *shape* of the rectangle is not the "whole"; its area is. (This is one reason that a pizza is not a good model for the whole, because there is very little flexibility in dividing the area of a circle into equal parts except by using circular sectors. The other reason is that, unlike rectangles, it cannot be used to model fraction multiplication.)

The second stage is where the formal mathematical development of fractions begins, somewhere in grade 4 according to these Common Core Standards. In grade 4,

[1] Quoted in the classic, *Commutative Algebra*, of Zariski-Samuel. Literal translation: The judge: "The defendant, you will try to be brief." The defendant replies, "I will try to be clear."

the fact that a fraction is a *number* begins to assume overriding importance on account of the extensive computations students must make with fractions at that point of the school curriculum. They have to learn to add, subtract, multiply, and divide fractions and use these operations to solve problems. Students need a clear-cut model of a fraction (or as one says in mathematics, a *definition* of a fraction) in order to come to grips with all the arithmetic operations. The shift of emphasis from multiple models of a fraction in the initial stage to an almost exclusive model of a fraction as a point on the number line can be done gradually and gracefully beginning somewhere in grade 4. This shift is implicit in the Common Core Standards. Once a fraction is firmly established as a number, then more sophisticated interpretations of a fraction (which, in a mathematical context, simply mean "theorems") begin to emerge. Foremost among them is the division interpretation: we must explain, *logically*, to students in grade 5 and grade 6 that $\frac{m}{n}$, in addition to being the totality of m parts when the whole is partitioned into n equal parts, is also the number obtained when "m is divided by n", where the last phrase must be carefully explained with the help of the number line. If we can make students realize that this is a subtle *theorem* that requires delicate reasoning, they will be relieved to know that they need not feel bad about not having such a "conceptual understanding" of a fraction as they are usually led to believe. Maybe they will then begin to feel that the subject of fractions is one they *can* learn after all. That, by itself, would already be a minor triumph in school math education.

The most sophisticated part of the study of fractions occurs naturally in grades 6 and 7, where the concept of fraction division is fully explained. Division is the foundation on which the concepts of percent, ratio, and rate are built. Needless to say, it is the latter concepts that play a dominant role in applications. The discussion given here of these concepts is, I hope, at once simple and comprehensive. I would like to call attention to the fact that all three concepts—percent, ratio, and rate—are defined simply as numbers and that, when they are so defined, problems involving them suddenly become transparent. See page 67 to page 74. In particular, one should be aware that the only kind of rate that can be meaningfully discussed in K-12 is *constant rate*. A great deal of effort therefore goes into the explanation of the meaning of constant rate because the misunderstanding surrounding this concept is monumental as of 2011. In Grade 7, the difficult topic of converting a fraction to a decimal is taken up. This is really a topic in college mathematics, but its elementary aspect *can* be explained. Due to the absence of such an explanation in the literature, the discussion here is more detailed and more complete than is found elsewhere.

In spite of the apparent length of this article, I would like to be explicit about the fact that *this document is not a textbook.* I have tried to give enough of an indication of the most basic facts about fractions and, in the process, had to give up on mentioning the fine points of instruction that must accompany any teacher's actual lessons in the classroom. For example, nowhere in this document did I mention the overriding importance of the *unit* (on the number line). Another example is the discussion of the addition of mixed numbers on page 33, which only mentions the method of converting mixed numbers to improper fractions. Needless to say, students should also know how to add mixed numbers by adding the whole numbers and the proper fractions separately. The same for the subtraction of mixed numbers. For such details, I will have to refer the reader to the following volume by the author:

> H. Wu, *Understanding Numbers in Elementary School Mathematics*,
> American Mathematical Society, 2011.
> http://www.ams.org/bookstore-getitem/item=mbk-79

Specifically, see Sections 8.2 and 12.4 for a discussion of the importance of the unit, and Sections 14.3 and 16.1 for the addition and subtraction of mixed numbers. In general, this reference—though written before the Common Core Standards—provides a development of fractions that is essentially in total agreement with the Common Core Standards.

THIRD GRADE

Number and Operation — Fractions 3.NF

Develop understanding of fractions as numbers.

1. Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.

a. Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.

b. Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

3. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

b. Recognize and generate simple equivalent fractions, e.g., $1/2 = 2/4, 4/6 = 2/3$. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

c. Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. Examples: Express 3 in the form $3 = 3/1$; recognize that $6/1 = 6$; locate $4/4$ and 1 at the same point of a number line diagram.

d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.

In grade 3, students are introduced for the first time to the (part-whole) concept of a fraction and the language associated with its use. Among the many ways to model a fraction, those involving a rectangle and the number line are singled out for discussion; these are not models of "shape" — a rectangle and a line segment — but models of *area* and *length*. In other words, even in an intuitive discussion of fractions for beginning students, we should instill the right way to think about them: these are *numbers*. One advantage of the number line model is that it allows an unambiguous formulation of the basic concepts of "equal", "smaller" and "bigger" among fractions. Simple experimentations on the number line will expose students to the phenomenon of equivalent fractions.

The meaning of fractions

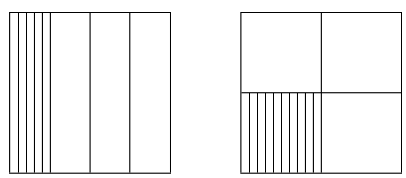
Students can be introduced to fractions informally by the use of *discrete objects*, such as pencils, pies, chairs, etc. If the whole is a collection of 4 pencils, then one pencil is $\frac{1}{4}$ of the whole. If the whole is 5 chairs, then one chair is $\frac{1}{5}$ of the whole, etc. This is an appropriate way to introduce students to the so-called **unit fractions**, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ...

The pros and cons of using discrete objects to model fractions are clear. It has the virtue of simplicity, but it limits students to thinking only about "how many" but not "how much". Thus if the whole is 4 pencils, we can introduce the fractions $\frac{2}{4}$ and $\frac{3}{4}$ by counting the number of pencils, but it would be unnatural to introduce, using this whole, the fractions $\frac{1}{3}$ or $\frac{1}{5}$, much less $\frac{1}{11}$. At some point, students will have to learn about *continuous models* involving length, area, or volume. This is a territory fraught with pitfalls, so teachers have to get to know the terrain and to tread carefully.

Let us start with the easiest example (pedagogically) in this regard: let the whole be *the area* of a square with each side having length 1, to be called the **unit square**, it being understood that the area of the unit square is (by definition) the number 1. One can introduce the concept of area to third graders through the idea of **congruence**, i.e., "same size and same shape." It is easy to convince them (and it is even correct to boot) that

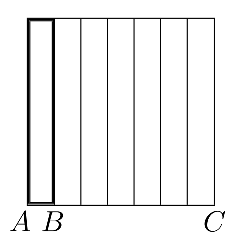
congruent figures have the same area.

Remember that we are doing things informally at this stage, so it is perfectly fine to simply use many hands-on activities to illustrate what "same size and same shape" means: one figure can be put exactly on top of another. Now suppose in the following drawings of the square, it is always understood that each division of a side is an **equi-division**, in the sense that the segments are all of the same length. Then it is equally easy to convince third graders that either of the following shaded regions represents the fraction $\frac{1}{4}$:



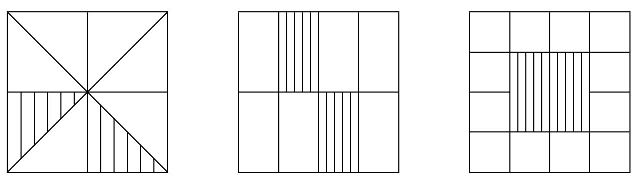
Take the left picture, for example. There are 4 thin congruent rectangles and the unit square is divided into four congruent parts. Since the rectangles are congruent, they each have the same area. So the whole (i.e., the area of the unit square, which is 1) has been divided into 4 equal parts (i.e., into 4 parts of equal area). By definition, the area of the shaded region is $\frac{1}{4}$, because it is one part when the whole is divided into 4 equal parts. It is in this sense that each rectangle represents $\frac{1}{4}$. The same discussion can be given to the right picture.

We can now easily draw an accurate model of $\frac{1}{7}$ in a third grade classroom. *Starting with a short segment* AB, we reproduce the segment 6 more times to get a long segment AC, as shown. Declare the length AC to be 1, and using AC as a side, draw a square as shown. Then the thickened rectangle represents $\frac{1}{7}$ in the above sense.



Of course, we can do the same with $\frac{1}{11}$, or $\frac{1}{17}$, or in fact any unit fraction. In general, many such drawings or activities with other manipulatives will strengthen students' grasp of the concept of a unit fraction: if we divide the whole into k equal parts for a whole number $k \neq 0$, then one part is the fraction $\frac{1}{k}$.[2]

It is very tempting to follow the common practice to let a unit square itself, rather than *the area of the unit square*, to be the whole. It seems to be so much simpler! There is a mathematical reason that one should not do that: a fraction, like a whole number, is a number that one does calculations with, but a square is a geometric figure and cannot be a number. There is also a pedagogical reason not to do this, and we explain why not with a picture. By misleading students into thinking that a fraction is a geometric figure and is therefore a *shape*, we lure them into believing that "equal parts" must mean "same size and same shape". Now consider the following pictures. Each large square is assumed to be the unit square. It is not difficult to verify, by reasoning with area the way we have done thus far, that the area of each of the following shaded regions in the respective large squares is $\frac{1}{4}$:

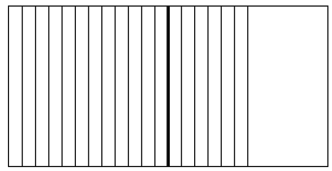


If students begin to buy into the idea that "division into equal parts" must mean "division into congruent parts", then it would be difficult to convince them that any of the shaded regions above represents $\frac{1}{4}$.

Another pitfall one should avoid at the initial stage of teaching fractions is the failure to emphasize that in a discussion of fractions, every fraction must be understood to be a fraction with respect to an unambiguous whole. One should never give students the impression that one can deal with fractions referring to different wholes in a given discussion without clearly specifying what these wholes are. In this light, it would not do to give third graders a problem such as the following:

`What fraction is represented by the following shaded area?`

[2]This document occasionally uses symbolic notation in order to correctly convey a mathematical idea. It does not imply that the symbolic notation should be used in all third grade classrooms.



In this problem, it is assumed that students can guess that the area of the left square is the whole (in which case, the shaded area represents the fraction $\frac{3}{2}$. This assumption is unjustified because students could equally well assume that the area of the big rectangle (consisting of two squares) is the whole, in which case the shaded area would represent the fraction $\frac{3}{4}$.

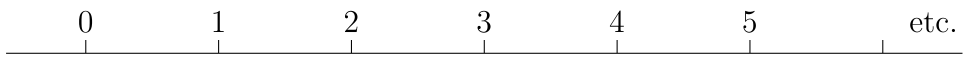
Now that we have unit fractions, we can introduce the general concept of a fraction such as $\frac{3}{4}$. With a fixed whole understood, we have the unit fraction $\frac{1}{4}$. The **numerator** 3 of $\frac{3}{4}$ tells us that $\frac{3}{4}$ is what you get by combining 3 of the $\frac{1}{4}$'s together. In other words, $\frac{3}{4}$ is what you get by putting 3 parts together when the whole is divided into 4 equal parts. In general, an arbitrary fraction such as $\frac{5}{3}$ is what one gets by combining 5 parts together when the whole is divided into 3 equal parts. Likewise, $\frac{7}{2}$ is what one gets by combining 7 parts together when the whole is divided into 2 equal parts.

There is no need to introduce the concepts of "proper fraction" and "improper fraction" at the beginning. Doing so may confuse students.

The number line

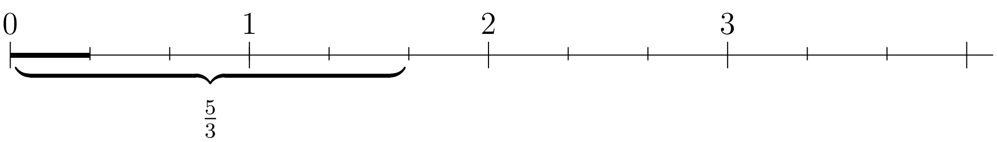
Unit fractions are the basic building blocks of fractions, in the same sense that the number 1 is the basic building block of the whole numbers, i.e., to the extent that every whole number is obtained by combining a sufficient number of 1's, we now explain how we can obtain any fraction by combining a sufficient number of unit fractions. To this end, it will be most advantageous to use the number line model for fractions. Its advantages over the area models (such as pizza and rectangles) are that it is much easier to divide the whole into equal parts because only length is involved, and that addition and subtraction of fractions are much more easily modeled on the number line. It may be that it will take children in third grade longer to get used to the number line than the rectangular area model. One should therefore make allowance for the extra instruction time.

On the number line, the number 1 is the **unit** and the segment from 0 to 1 is the **unit segment** $[0,1]$. The other whole numbers then march to the right so that the segments between consecutive whole numbers, from 0 to 1, 1 to 2, 2 to 3, etc., are of the same length, as shown.

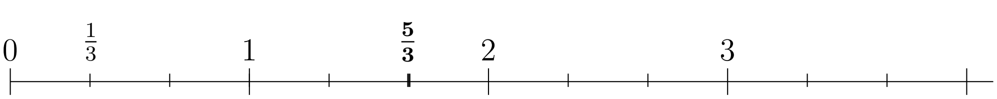


The *whole* is the *length* of the unit segment $[0,1]$ (and *not* the unit segment itself). One can teach students to think of the number line as an infinite ruler.

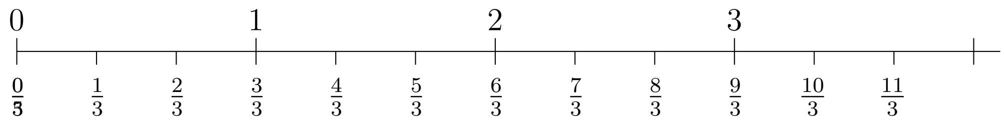
Now, consider a typical fraction $\frac{5}{3}$. Relative to the whole which is the length of the unit segment $[0,1]$, then $\frac{5}{3}$ is the symbol that denotes the combined length of *5* segments where each segment is a part when $[0,1]$ is divided into *3* parts of equal length. In other words, if we think of the thickened segment as "the unit segment", then $\frac{5}{3}$ would just be "the number 5" in the usual way we count up to 5 in the context of whole numbers.



Once we agree to the CONVENTION that each time we divide the unit segment into 3 **equal parts** (i.e., segments of equal lengths) we use the segment that has 0 as an endpoint as the *reference segment*, then we may as well forget the thickened segment itself and use its right endpoint as the reference. We will further agree to call this endpoint $\frac{1}{3}$. Then for obvious reasons such as those given in the preceding paragraph, the endpoint of the preceding bracketed segment will be denoted by $\frac{5}{3}$. The "5" signifies that is the length of 5 of the reference segments.

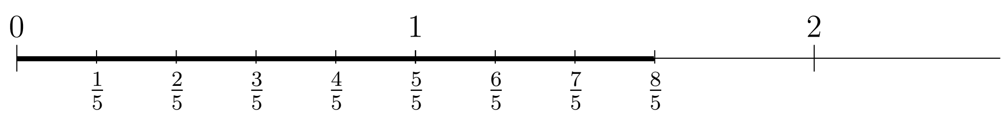


The "3" in $\frac{5}{3}$ is called the **denominator** of $\frac{5}{3}$. In this way, every fraction with denominator equal to 3 is now regarded as a certain point on the number line, as shown. (The labeling of 0 as $\frac{0}{3}$ is a CONVENTION.)



Notice that, from the point of view of the number line, fractions are not different from whole numbers: If we start with 1 as the reference point, going to the right 5 times the length of $[0, 1$ gets us to the number 5, and if we start with $\frac{1}{3}$ as the reference point, going to the right 5 times the length of $[0, \frac{1}{3}]$ gets us to the number $\frac{5}{3}$.

If a fraction has a denominator different from 3, we can identify it with another point on the number line in exactly the same way. For example, the point which is the fraction $\frac{8}{5}$ can be located as follows: the denominator indicates that the unit segment $[0, 1]$ is divided into 5 segments of equal length, and the **numerator** 8 indicates that $\frac{8}{5}$ is the length of the segment when 8 of the above-mentioned segments are put together end-to-end, as shown:



Recalling that the number 1 is actually the length of the unit segment (i.e., the whole), we see that the meaning of the unit 1 determines how each fraction should be interpreted. For example, if 1 is the volume in a glass (let us say in pints), then $\frac{1}{3}$ is a third of a glass of water in pints. If on the other hand, 1 is the width of the classroom (let us say in feet), then $\frac{1}{3}$ is a third of the width of the classroom in feet. And so on. The flexibility in the interpretation of fractions as points on the number line, according to the meaning of the unit, is a main attraction of the number line for the discussion of fractions. It would be a good idea to give plenty of practice problems on different interpretations of fractions (as points on the number line) according to meaning of the unit.

We make two immediate observations about the number line representation of fractions that are germane to our purpose. One is that each whole numbers (as a point on the line) is now labeled by other symbols. For example,

> the point on the number line designated by 2 is now also designated by $\frac{2}{1}$, $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, and in general, by $\frac{2n}{n}$, for any whole number $n \neq 0$.

We now introduce a concept and a notation that make it possible to express the preceding statement more simply. We say two fractions (as points on the number line) are **equal** if they are the same point on the line. The symbol to use in this case is the usual one, =. Thus we can say,

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \cdots = \frac{2n}{n} = \cdots, \text{ for any whole number } n \neq 0.$$

Similarly,

$$3 = \frac{3}{1} = \frac{6}{2} = \frac{9}{3} = \frac{12}{4} = \cdots = \frac{3n}{n} = \cdots, \text{ for any whole number } n \neq 0,$$

$$4 = \frac{4}{1} = \frac{8}{2} = \frac{12}{3} = \frac{16}{4} = \cdots = \frac{4n}{n} = \cdots, \text{ for any whole number } n \neq 0,$$

and so on. In this notation, it should be pointed out explicitly to students that

$$1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \cdots$$

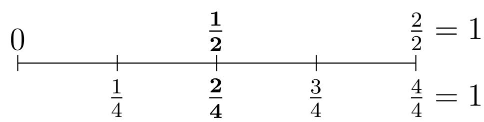
For reasons having to do with tradition, equal fractions are usually called **equivalent fractions.** We hasten to add that *this precise concept of equality is far from a routine matter.* It is only with the fractions as points on the number line that we can say, precisely, what it means for two fractions to be "equal". The usual statement in school textbooks that "two fractions are equal if they name the same amount" is not one that can be used as a basis for mathematical reasoning, because it begs the question: what does it mean to "name" the "same amount"? In view of recent concerns in the education research literature about students' abuse of the equal sign, we want to focus on the root cause of this so-called problem: we should teach students the precise meaning of the equality of any two concepts, carefully, at each stage. Obviously, the subject of fractions is a good starting point.

The equalities $2 = \frac{2}{1} = \frac{4}{2} =$ etc., is the simplest manifestation of the concept of *equivalent fractions*, to be taken up later in this grade.

A second observation is the need to reinforce the idea that every fraction is the "piecing together of unit fractions". To explain this more precisely, take, for example, $\frac{3}{5}$. It is the length of the segment when 3 segments, each being a copy of the one from 0 to $\frac{1}{5}$, are put together end-to-end. To simplify matters, we express this more simply by saying $\frac{3}{5}$ is **3 copies of the unit fraction** $\frac{1}{5}$. Similarly, $\frac{8}{5}$ is 8 copies of the unit fraction $\frac{1}{5}$.

Equivalent fractions

Students can experiment on the number line and discover that, even among "simple" fractions, many are equal. For example, the fraction $\frac{1}{2}$ is equal to $\frac{2}{4}$.



They would also discover, as we did in the preceding section, that

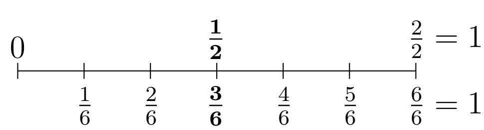
$$2 \;=\; \frac{2n}{n} \quad \text{for any whole number } n \neq 0,$$

$$3 \;=\; \frac{3n}{n} \quad \text{for any whole number } n \neq 0.$$

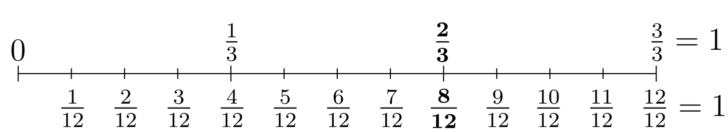
In fact, if they take any whole number such as 17, then the same thing prevails:

$$17 \;=\; \frac{17n}{n} \quad \text{for any whole number } n \neq 0.$$

Suppose they choose to subdivide the two segments from 0 to $\frac{1}{2}$, and from $\frac{1}{2}$ to 1 into 3 parts of equal length, then the unit segment will be divided into 6 parts of equal length. They will get $\frac{1}{2} = \frac{3}{6}$.



In general, given any fraction such as $\frac{2}{3}$, if we subdivide each of the segments of length $\frac{1}{3}$ into (let us say) 4 parts of equal length, then we will get $\frac{2}{3} = \frac{4\times 2}{4\times 3}$.



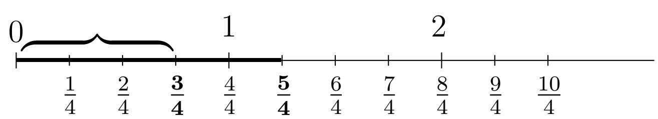
If students get plenty of practice of this type, viz., showing $\frac{3}{4} = \frac{6}{8}$, or $\frac{2}{5} = \frac{6}{15}$, they will be ready for the general discussion of equivalent fractions in the next grade.

Comparing fractions

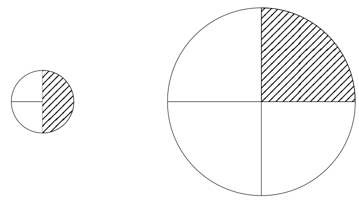
Given two fractions — thus two points on the number line — the one to the left of the other is said to be **smaller** than the other and of course the one on the right is said to **bigger** than the other. The symbols to use are $<$ and $>$, respectively. Thus,

$$\frac{1}{4} < \frac{3}{4}, \quad \text{and} \quad 2 > \frac{3}{2}.$$

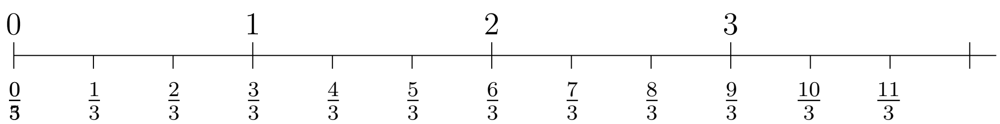
These concepts of "smaller than" and "bigger than" should be carefully explained to students because in the usual way fractions are taught, it is never made clear what these concepts mean. This meaning of "smaller than" (or "bigger than") makes perfect sense, intuitively, because if one fraction, say $\frac{3}{4}$, is to the left of $\frac{5}{4}$, then the segment from 0 to $\frac{3}{4}$ is of course shorter than the segment from 0 to $\frac{5}{4}$. Therefore the above meaning of $\frac{3}{4} < \frac{5}{4}$ respects this intuition.



We should point out that one cannot **compare** two fractions, i.e., determine which is bigger, unless they both refer to the *same* unit (i.e., same whole). Of course by the very meaning of comparing fractions that we have just given, two fractions are already on the same number line before we can say which is to the left of the other, and being on the same number line implies that the fractions refer to the same unit. Nevertheless this is something students can easily forget. We can illustrate the danger of comparing fractions that refer to different wholes in a given discussion with the following classic example. A student can claim that $\frac{1}{4} > \frac{1}{2}$, because a fourth of the pizza on the right is bigger than a half of the pizza on the left, as shown:



There are some simple comparisons that are accessible to third graders. One is that if two fractions have the same denominator, then the one with larger numerator is the larger fraction (in other words, bigger). This comes from the way fractions with the same denominator are placed on the number line, e.g.,



A second accessible fact about comparison is that, for unit fractions, the greater the denominator, the smaller the fraction. This is usually explained by saying "in order for more (identical) pieces to make the same whole, the pieces must be smaller." This has great intuitive appeal after students have inspected, for example, how half a pizza is larger than a third of a pizza and a few similar examples involving small numbers, but the extrapolation of this fact to large numbers is not backed by logical reasoning. It is however possible to introduce third graders to some standard reasoning in order to make the conclusion a little bit more convincing. For example, let us show why $\frac{1}{6} < \frac{1}{5}$. We reason as follows. We combine 5 copies of each fraction to itself. As we have seen,

$$5 \text{ copies of } \tfrac{1}{5} \;=\; \frac{5}{5} \;=\; 1$$

while

$$5 \text{ copies of } \tfrac{1}{6} \;=\; \frac{5}{6} \;<\; 1$$

It is then reasonable to conclude that $\frac{1}{6} < \frac{1}{5}$.

Next, students can compare any two fractions with the same numerator and conclude that the one with the smaller denominator is bigger. For example, why is $\frac{11}{5}$ bigger than $\frac{11}{6}$? This is because we already know $\frac{1}{6} < \frac{1}{5}$. Therefore if we combine 11 copies of each fraction to itself, we get:

$$\frac{11}{6} \;=\; 11 \text{ copies of } \tfrac{1}{6} \;<\; 11 \text{ copies of } \tfrac{1}{5} \;=\; \frac{11}{5}.$$

In some cases, students may be able to compare fractions by reasoning about how the fractions are related to benchmarks such as $\frac{1}{2}$ and 1. For example, students could reason that $\frac{7}{8} < \frac{13}{12}$ because $\frac{7}{8}$ is less than 1 (and is therefore to the left of 1) but $\frac{13}{12}$ is greater than 1 (and is therefore to the right of 1). Conclusion: $\frac{7}{8}$ is to the left of $\frac{13}{12}$ and therefore $\frac{7}{8} < \frac{13}{12}$.

FOURTH GRADE

Number and Operations — Fractions 4.NF[3]

Extend understanding of fraction equivalence and ordering.

1. Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

2. Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as $1/2$. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.

Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.

3. Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

b. Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. Examples: $3/8 = 1/8 + 1/8 + 1/8$; $3/8 = 1/8 + 2/8$; $2\frac{1}{8} = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8$.

c. Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.

d. Solve word problems involving addition and subtraction of fractions referring to

[3]Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.

the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

4. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

a. Understand a fraction a/b as a multiple of $1/b$. For example, use a visual fraction model to represent $5/4$ as the product $5 \times (1/4)$, recording the conclusion by the equation $5/4 = 5 \times (1/4)$.

b. Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number. For example, use a visual fraction model to express $3 \times (2/5)$ as $6 \times (1/5)$, recognizing this product as $6/5$. (In general, $n \times (a/b) = (n \times a)/b$.)

c. Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. For example, if each person at a party will eat $3/8$ of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?

Understand decimal notation for fractions, and compare decimal fractions.

5. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[4] For example, express $3/10$ as $30/100$, and add $3/10 + 4/100 = 34/100$.

6. Use decimal notation for fractions with denominators 10 or 100. For example, rewrite 0.62 as $62/100$; describe a length as 0.62 meters; locate 0.62 on a number line diagram.

7. Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>, =,$ or $<$, and justify the conclusions, e.g., by

[4] Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with un-like denominators in general is not a requirement at this grade.

using a visual model.

The main topics of grade 4 are **the fundamental fact about equivalent fractions** (a fraction is not changed when its numerator and denominator are multiplied by the same nonzero whole number), its application to the comparison, addition, and subtraction of fractions, the meaning of multiplying a fraction by a whole number, and the introduction of (finite) decimals.

The above-mentioned fundamental fact about equivalent fractions is the one theme that ties the various strands within fractions together. This fact will be made abundantly clear in the following discussion, but it should also be brought out more explicitly in classroom instructions than has been done so far.

Equivalent fractions

The *fundamental fact* states that a fraction is not changed (i.e., its position on the number line is not changed) when its numerator and its denominator are both multiplied by the same nonzero whole number. Thus if the fraction is $\frac{9}{5}$, then

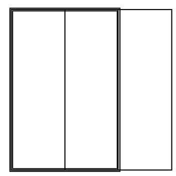
$$\frac{9}{5} = \frac{18}{10} = \frac{27}{15} = \frac{36}{20} = \frac{90}{50} = \frac{108}{60}, \quad \text{etc.}$$

In some 4th grade classrooms, it may be appropriate to state it in the usual symbolic form:

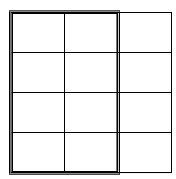
> For any fraction $\frac{a}{b}$ where a and b are whole numbers and $b \neq 0$, and for any nonzero whole number n, $\frac{a}{b} = \frac{na}{nb}$.

This will of course necessitate the explanation about na means $n \times a$, etc. For the benefit of students' mathematics learning, especially the learning of algebra in middle school, some symbolic notation has to be gently introduced starting with grades 3–5. Students cannot afford to wait until they come to algebra before they are exposed to abstract symbols.

Students can use fraction strips or the area model to try to understand this fact. For example, let us see why $\frac{2}{3} = \frac{4\times 2}{4\times 3}$ using the area model. So the whole is the area of the unit square, and if the square is vertically divided into three rectangles of equal area, then $\frac{2}{3}$ is represented by the thickened rectangle:



Now divide the unit square into 4 *horizontal* rectangles of equal area, then the unit square is now divided into 4×3 small rectangles of equal area, and the thickened rectangle is paved by 4×2 of these small rectangles, as shown:



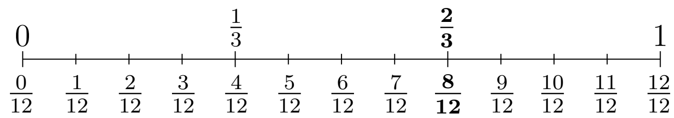
Therefore the thickened rectangle, which is $\frac{2}{3}$, is now also

$$\frac{4 \times 2}{4 \times 3}$$

because it is also the totality of 4×2 parts when the whole is divided into 4×3 parts of equal area.

When the fraction $\frac{2}{3}$ is replaced by a more complicated one such as $\frac{17}{4}$, and the number 4 is replaced by a large number such as 12, it may be a bit difficult to visualize how this argument will be carried out. A more uniform approach to this fundamental fact that makes such reasoning almost routine may be one that uses the number line. We now repeat the preceding reasoning using the number line. Then we first have the fraction $\frac{2}{3}$,

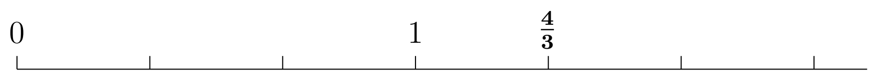
To show $\frac{2}{3} = \frac{4\times 2}{4\times 3}$, the number 4 suggests that we subdivide each segment of length $\frac{1}{3}$ into 4 parts of equal length. Then $\frac{2}{3}$ is seen to be exactly $\frac{8}{12}$, which is of course just $\frac{4\times 2}{4\times 3}$.



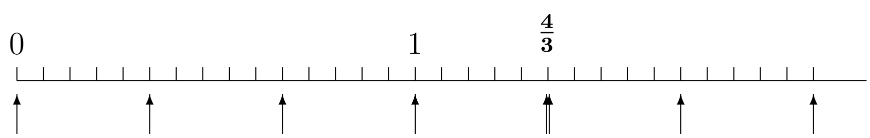
The number line representation makes it very clear how to demonstrate this fundamental fact for other fractions. Let us show, for example, why

$$\frac{4}{3} = \frac{5 \times 4}{5 \times 3}.$$

So $\frac{4}{3}$ is 4 copies of $\frac{1}{3}$, and we want to know why it is also 20 copies of $\frac{1}{15}$. On the number line, we have that $\frac{4}{3}$ is the 4th multiple of $\frac{1}{3}$.



Now divide each of these segments of length $\frac{1}{3}$ into 5 parts of equal length (the "5" is because we are looking at $\frac{5\times 4}{5\times 3}$); call these parts *small segments*. All small segments have the same length. Here is a picture of these *small segments* with the vertical arrows indicating the thirds:



The small segments now give a division of the unit segment into 5×3 parts of equal length, and the fraction $\frac{4}{3}$ is 5×4 copies of these small segments. Therefore $\frac{4}{3}$ is $\frac{5\times 4}{5\times 3} = \frac{20}{15}$.

For fourth grade, working out many such examples would be sufficient to give students the underlying reasoning behind the fundamental fact. A general proof using symbols would not be necessary.

The fundamental fact is usually presented in terms of division rather than multiplication, i.e., it is stated that

$$\frac{20}{15} = \frac{20 \div 5}{15 \div 5},$$

or

$$\frac{28}{36} = \frac{28 \div 4}{36 \div 4}$$

To see why the latter is true, for example, we reason as follows. We know $28 \div 4 = 7$ and $36 \div 4 = 9$. Therefore we have to show

$$\frac{28}{36} = \frac{7}{9}$$

Recognizing that $28 = 4 \times 7$ and $36 = 4 \times 9$, we see that this is the statement that

$$\frac{4 \times 7}{4 \times 9} = \frac{7}{9}.$$

This immediately follows from the fundamental fact. Again, working out many such examples would be sufficient in the fourth grade.

The ability to see that fractions such as $\frac{28}{36}$ can be simplified to $\frac{7}{9}$ is a basic skill associated with the fundamental fact, but the importance of this skill is usually exaggerated out of proportion in textbooks and perhaps also by some teachers. In this connection, there is a tradition in elementary education that *every* fraction must be written in **reduced form**, i.e., the numerator and the denominator are both divisible only by 1 and by no other whole number. This is not a defensible tradition as far as mathematics is concerned because there is no theorem in mathematics that outlaws non-reduced fractions. A more restricted form of this tradition, to the effect that if the numerator and denominator of a fraction are single-digit numbers then the fraction should be written in reduced form, may be justifiable on non-mathematical grounds as a classroom convention.

Once students have learned fraction multiplication (in Grade 5), they will be able to verify to their own satisfaction that the fundamental fact can be more easily remembered as just multiplying the fraction by 1, in the sense that *if multiplication of fractions is already understood*, then, for example,

$$\frac{28}{36} = \frac{7}{9}$$

can also be obtained by

$$\frac{7}{9} = \frac{7}{9} \times 1 = \frac{7}{9} \times \frac{4}{4} = \frac{28}{36}$$

However, students should be made aware that, while this is an excellent mnemonic device to remember what the fundamental fact says, it is not a valid *explanation* for it because they do not as yet know how to add two fractions, much less how to multiply them. A useful analogy to understand this situation is to ask yourself: if an ESL student comes to you and asks what “huge” means, would you answer with “gargantuan”?

An application

From the point of view of the conceptual understanding of fractions, the most important message of the fundamental fact about equivalent fractions may be that, *any two fractions may be considered as fractions with the same denominator.* For example, if the given fractions are $\frac{2}{9}$ and $\frac{3}{8}$, then by the fundamental fact, they may be rewritten as two fractions with the same denominator 72:

$$\frac{16}{72} \left(= \frac{8 \times 2}{8 \times 9}\right) \quad \text{and} \quad \frac{27}{72} \left(= \frac{9 \times 3}{9 \times 8}\right)$$

This understanding simplifies many considerations in the study of fractions.

Let us apply this understanding right away. We will compare two fractions such as $\frac{5}{8}$ and $\frac{7}{12}$, i.e., we ask which of the following is smaller:

5 copies of $\frac{1}{8}$, or 7 copies of $\frac{1}{12}$.

If we were comparing $\frac{5}{12}$ and $\frac{7}{12}$, i.e.,

5 copies of $\frac{1}{12}$ and 7 copies of $\frac{1}{12}$,

we would be able to do it on the spot. The problem is therefore that the denominators 8 and 12 of the given fractions are *different*. Recalling our understanding above, we realize that this is no impediment at all because we can rewrite both fractions as

$$\frac{60}{96} \left(= \frac{12 \times 5}{12 \times 8}\right) \quad \text{and} \quad \frac{56}{96} \left(\frac{8 \times 7}{8 \times 12}\right)$$

Then clearly $\frac{56}{96}$ is smaller, i.e., $\frac{7}{12}$ is smaller. Thus

$$\frac{7}{12} < \frac{5}{8}$$

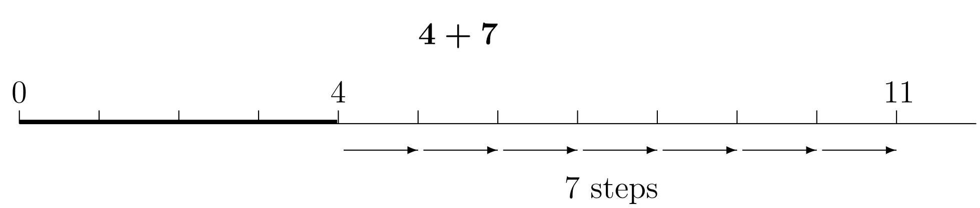
Observe that this comparison was achieved by comparing the two "cross products" 7×8 and 12×5.

The above method is clearly applicable to the comparison of any two fractions. In some fourth grade classes, it may be possible to use symbolic notation to put this method in the form of an algorithm, namely, to compare two fractions $\frac{a}{b}$ and $\frac{c}{d}$, we only look at the "cross products" ad and bc. (*Reminder:* This is not a trick, but is rather the logical consequence of rewriting the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ as two fractions with the same denominator and then examining their numerators.) If $ad < bc$, then $\frac{a}{b} < \frac{c}{d}$. Conversely, if $\frac{a}{b} < \frac{c}{d}$, then $ad < bc$. This is the **cross-multiplication algorithm**.

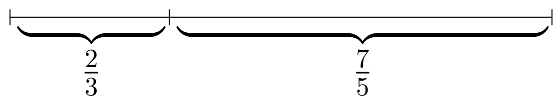
Adding and subtracting fractions

We now give a second application of the fundamental fact about equivalent fractions to the addition and subtraction of fractions.

We want to emphasize at the outset that adding fractions is not essentially different from adding whole numbers. Observe that the addition of two whole numbers, such as 4 and 7, can be modeled by the length of the total segment obtained by joining together two segments of lengths 4 and 7, for the following reason. The usual meaning of the sum $4 + 7$ is to start with 4 and count 7 more steps until we reach 11, and that is the sum. The corresponding geometric model is then to put a segment of length 4 and a segment of length 7 together, end-to-end, on the number line, and observe that the total length is 11.



Of course putting two segments together end-to-end on the number line expresses the intuitive feeling of the addition of two whole numbers, which is to "combine things". We now build on this concept, that "addition is combining things" by introducing the concept of fraction addition, e.g., $\frac{2}{3} + \frac{7}{5}$, as the total length of the segment obtained by putting together two segments end-to-end on a horizontal line, one of length $\frac{2}{3}$ and another $\frac{7}{5}$.



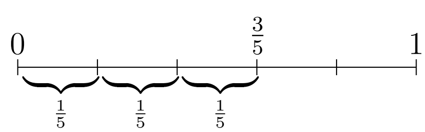
There is no need to worry at this point about "how much is $\frac{2}{3} + \frac{7}{5}$ exactly?". Just knowing what the sum means is enough. This is analogous to introducing students to the concept of whole number multiplication, e.g., 51×78, is adding 78 to itself 51 times; the exact determination of 51×78 can be dealt with when it is time to introduce the multiplication algorithm.

What is worth pointing out is the fact that students can now see adding fractions as a direct continuation of the concept of adding whole numbers. *This kind of conceptual continuity in mathematics makes learning easier* because it shows students that the new concept dovetails with what they already know.

With the availability of the concept of addition, we can clarify the role of unit fractions in the study of fractions. We observe that among whole numbers, every whole number is just the repeated addition of 1 to itself, e.g., 23 is just 1 added to itself 23 times. In the same way, every fraction is the repeated addition of a unit fraction. For example,

$$\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$$

because $\frac{3}{5}$ is the total length of 3 copies of $\frac{1}{5}$:



In like manner,

$$\frac{4}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$$
$$\frac{5}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}, \quad \text{etc.}$$

We now see that, in terms of addition, unit fractions are the basic building blocks of fractions.

We can push this kind of reasoning to its logical conclusion: we now see that, for example,

$$\frac{7}{5} + \frac{4}{5} \;=\; \frac{7+4}{5}$$

The reason is that, by what we did above,

$$\begin{aligned}
\frac{7}{5} + \frac{4}{5} \;&=\; \overbrace{\frac{1}{5} + \cdots + \frac{1}{5}}^{7} \;+\; \overbrace{\frac{1}{5} + \cdots + \frac{1}{5}}^{4} \\
&=\; \frac{\overbrace{1 + 1 + \cdots + 1}^{7+4}}{5} \\
&=\; \frac{7+4}{5}
\end{aligned}$$

The same reasoning shows that,

> *if we add fractions with the same denominator, we get a fraction with the same denominator whose numerator is the sum of the numerators.*

There is another kind of addition we can perform at this point, namely, adding a whole number to a fraction, e.g., $5 + \frac{7}{6}$. For this, we have to recall from Grade 3 that $5 = \frac{6\times 5}{6} = \frac{30}{6}$. Of course the number 5 is also equal to $\frac{15}{3}$ or $\frac{20}{4}$, but the reason for choosing $\frac{30}{6}$ among all the possibilities is that we want to add 5 to $\frac{7}{6}$, and since the denominator of $\frac{7}{6}$ is 6 and since we only know how to add two fractions with the same denominator at this point, the choice of $\frac{30}{6}$ is the logical one. Such being the case,

$$5 + \frac{7}{6} \;=\; \frac{30}{6} + \frac{7}{6} \;=\; \frac{37}{6}$$

This then brings up one of the standard concepts, that of a **mixed number**: by definition, a mixed number is the sum of a whole number and a **proper fraction** (a fraction whose numerator is smaller than its denominator, or equivalently, a fraction < 1). There is a **notational convention** associated with mixed numbers: the + sign is omitted. Thus we write $5\frac{3}{4}$ in place of $5 + \frac{3}{4}$ and $7\frac{1}{5}$ in place of $7 + \frac{1}{5}$.

Two remarks are very pertinent to the teaching of mixed numbers. The first is that, in these Standards, *the concept of a mixed number is defined only after fraction addition has been defined.* Mixed numbers seem to be always introduced in school textbooks right after the introduction of fractions *before* the addition of fractions is even mentioned. A mixed number such as $7\frac{1}{5}$ would be introduced as "7 and $\frac{1}{5}$" without making precise that "and" means "add". Such disregard of students' logical learning trajectory has contributed to the nonlearning of fractions. A second remark is that, as a consequence of this disregard, the conversion between a mixed number and an **improper fraction** (a fraction whose numerator is at least as big as its denominator, or equivalently, a fraction at least as big as 1) is usually given as a rote skill. From our point of view, converting a mixed number to a fraction is just fraction addition, e.g.,

$$7\frac{1}{5} \;=\; 7+\frac{1}{5} \;=\; \frac{35}{5}+\frac{1}{5} \;=\; \frac{36}{5}$$

Conversely, given an improper fraction such as $\frac{47}{6}$, we convert it to a mixed number by using so-called **division-with-remainder**, i.e., if 47 is divided by 6, it has quotient 7 and remainder 5. The correct way to express this is $47 = (7 \times 6) + 5$. Therefore,

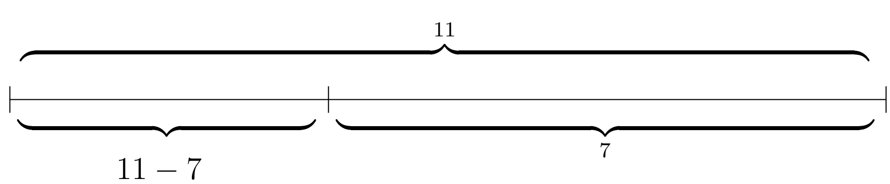
$$\frac{47}{6} \;=\; \frac{(7\times 6)+5}{6} \;=\; \frac{7\times 6}{6}+\frac{5}{6}$$

where the last step makes use of our knowledge of adding fractions with the same denominator (6, in this case). From equivalent fractions, we know $\frac{7\times 6}{6} = 7$. So

$$\frac{47}{6} \;=\; 7+\frac{5}{6} \;=\; 7\frac{5}{6},$$

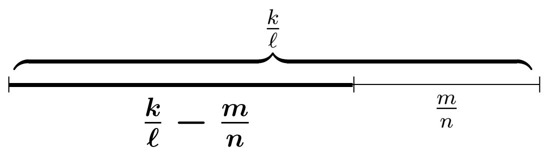
where the last step is by the definition of a mixed number. This method is applicable to all improper fractions.

Finally, we address the related concept of subtraction. Among whole numbers, subtraction is "taking away". Thus $11-7$ is "take 7 away from 11". In terms of the number line $11-7$ is the length of the segment when a segment of length 7 is taken away at one end from a segment of length 11:



Note that, because we are operating within whole numbers, we have to make sure that $11 \geq 7$ before we can consider $11 - 7$ (where the symbol "$\geq$" means "$>$ or $=$). In the same way, we define for fractions $\frac{k}{\ell} \geq \frac{m}{n}$,

$$\frac{k}{\ell} - \frac{m}{n} \;=\; \textit{the length of the remaining segment when a segment of length } \frac{m}{n} \textit{ is removed from one end of a segment of length } \frac{k}{\ell}$$



Again, we do not worry about computing the exact value of $\frac{k}{\ell} - \frac{m}{n}$ at this point; such a computation is saved for the 5th grade. In one special case though, we can do the subtraction precisely, which is the case of equal denominators. For example,

$$\frac{17}{6} - \frac{5}{6} \;=\; \frac{17-5}{6} \;=\; \frac{12}{6} \;=\; 2$$

This is because, taking 5 copies of $\frac{1}{6}$ away from 17 copies of $\frac{1}{6}$ leaves 12 copies of $\frac{1}{6}$, and this is the precise meaning of $\frac{17-5}{6}$.

Multiplication of a fraction by a whole number

Recall the meaning of multiplication among whole numbers: 3×7 means $7+7+7$ and the meaning of 5×4 is $4+4+4+4+4$. Now what meaning should we give to $3 \times \frac{2}{5}$? It would be reasonable to say:

$$3 \times \frac{2}{5} \;\stackrel{\text{def}}{=}\; \frac{2}{5} + \frac{2}{5} + \frac{2}{5}$$

and, similarly,

$$11 \times \frac{5}{4} \;\stackrel{\text{def}}{=}\; \underbrace{\frac{5}{4} + \frac{5}{4} + \cdots + \frac{5}{4}}_{11 \text{ times}}$$

where the symbol "$\stackrel{\text{def}}{=}$" means we are giving a definition. In general, we will agree to say that a whole number n multiplying a given fraction such as $\frac{5}{8}$ is n **copies of** $\mathbf{\frac{5}{8}}$, in the sense that it is equal to

$$\underbrace{\frac{5}{8}+\frac{5}{8}+\cdots+\frac{5}{8}+\frac{5}{8}}_{n \text{ times}}$$

Notice that in the above examples, because we know how to add fractions with the same denominator, we have:

$$3\times\frac{2}{5} \;=\; \frac{3\times 2}{5} \;=\; \frac{6}{5}$$

and similarly,

$$11\times\frac{5}{4} \;=\; \frac{11\times 5}{4} \;=\; \frac{55}{4}$$

We can now look at a fraction from a different perspective. For example,

$$\frac{7}{5} \;=\; 7\times\frac{1}{5}, \qquad \frac{11}{3} \;=\; 11\times\frac{1}{3}$$

We will express the above equalities by saying that $\frac{7}{5}$ is the **7th multiple of** $\mathbf{\frac{1}{5}}$ and $\frac{11}{3}$ is the **11th multiple of** $\mathbf{\frac{1}{3}}$. In general, a fraction $\frac{m}{n}$ is the $\boldsymbol{m}$**th multiple of** $\boldsymbol{\frac{1}{n}}$**.**

The concept of multiplication of a fraction by a whole number allows us to carry over certain ways of thinking about a problem from whole numbers to fractions. For example, the following problem is standard: if the capacity of a bucket is 2 gallons and if 43 buckets of water fill a tank, then the capacity of the tank is $43\times 2 = 86$ gallons of water. Now suppose the capacity of a bucket is $2\frac{3}{4}$ gallons and, again, 43 buckets of water fill a tank, what is the capacity of the tank? The answer is $43\times 2\frac{3}{4}$ gallons. This is because the capacity of the tank is

$$\underbrace{2\frac{3}{4}+2\frac{3}{4}+\cdots+2\frac{3}{4}}_{43} \quad \text{gallons}$$

Since $2\frac{3}{4}$ is a fraction, the above sum is $43\times 2\frac{3}{4}$. So the answer is $43\times 2\frac{3}{4}$ gallons, which is

$$43\times\left(2+\frac{3}{4}\right) \;=\; 43\times\frac{11}{4} \;=\; \frac{473}{4} \;=\; 118\frac{1}{4} \;\text{ gallons}$$

(Finite) decimals

Some decimals are special, for one reason or another. For example, those fractions with denominator equal to 10, 100, 1000, etc. This is not a surprise because we are in a *decimal* system so that all numbers related to 10, 100, 1000, etc., are automatically special. There are deeper reasons related to the fact that the arithmetic of these fractions is essentially that of the whole numbers; see the discussion in grade 5. In any case, these fractions are called **decimal fractions**. The terminology is confusing, so it should be used sparingly.

The fundamental fact about equivalent fractions immediately tells us that a fraction with 10 as denominator may be considered a fraction with 100 as denominator. For example,

$$\frac{7}{10} = \frac{10 \times 7}{10 \times 10} = \frac{70}{100}$$

This may be the occasion in a 4th grade classroom to discuss why this statement means that 7 dimes is the same as 70 cents. Another application is that we can now add any fractions whose denominators are 10 or 100, e.g.,

$$\frac{3}{10} + \frac{27}{100} = \frac{30}{100} + \frac{27}{100} = \frac{57}{100}$$

One can interpret this as saying that 3 dimes together with 27 cents make 57 cents.

Because fractions with denominators equal to 10, 100, 1000, etc., come up so often, someone[5] came up with the idea long ago that one could abbreviate the fractions

$$\frac{27}{10}, \quad \frac{27}{100}, \quad \frac{27}{1000}, \text{ etc.}$$

by using a so-called **decimal point** to

$$2.7, \quad 0.27, \quad 0.027,$$

respectively. The latter are called **decimals**, or more precisely, **finite decimals**. In other words, one could simply indicate the number of zeros in the denominator of decimal fractions by counting the number of digits to the right of the decimal point. Note that a decimal such as 2.70 is considered to have two **decimal digits**, namely,

[5] Specifically, the German Jesuit priest C. Clavius introduced this notation in 1593.

7 and 0. However, if we use the fundamental fact about equivalent fractions, then we see that $2.70 = 2.7$ because, *by definition,* $2.70 = \frac{270}{100}$ so that

$$2.70 \;=\; \frac{270}{100} \;=\; \frac{10 \times 27}{10 \times 10} \;=\; \frac{27}{10} \;=\; 2.7$$

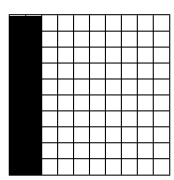
It is easy to compare decimals to see which is bigger provided one does not forget their meaning (i.e., they are decimal fractions) and how the fundamental fact about equivalent fractions allows us to consider any two fractions as fractions with the same denominator. For example, let us compare 0.2 and 0.09. By definition, we are comparing

$$\frac{2}{10} \quad \text{and} \quad \frac{9}{100}.$$

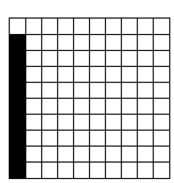
Since we already know that $\frac{2}{10} = \frac{20}{100}$, and since $20 > 9$. we know

$$\frac{20}{100} \;>\; \frac{9}{100} \quad \text{so that} \quad 0.2 > 0.09.$$

We hasten to add two comments. The first comment is that, in this special case, indeed one can resort to picture-drawing to get a more intuitive understanding of why $0.2 > 0.09$. For example, if we let the unit be the area of the unit square, and we divide it into 100 parts of equal area, then 0.2 — being the totality of two parts when the unit is divided into 10 parts of equal area — may be represented as the following darkened area:



On the other hand, 0.09 — being the totality of 9 parts when the unit is divided into 100 parts of equal area — is represented by the following darkened area:



So clearly, $0.2 > 0.09$. This is certainly useful knowledge. Our second comment is, however, that one cannot allow this picture-drawing activity to be all a 4th grader needs to know about the comparison of fractions. The previous argument using the meaning of a decimal (i.e., it is a decimal fraction) and the fundamental fact about equivalent fractions is by far the more important since it is universally applicable to all decimals whereas picture-drawing is not. So fourth graders' knowledge of decimals must include this argument.

5TH GRADE

Number and Operations — Fractions 5.NF

Use equivalent fractions as a strategy to add and subtract fractions.

1. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. For example, $2/3 + 5/4 = 8/12 + 15/12 = 23/12$. (In general, $a/b + c/d = (ad + bc)/bd$.)

2. Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. For example, recognize an incorrect result $2/5 + 1/2 = 3/7$, by observing that $3/7 < 1/2$.

Perform operations with multi-digit whole numbers and with decimals to hundredths.

7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

3. Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. For example, interpret 3/4 as the result of dividing 3 by 4, noting that $3/4$ multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each

person has a share of size $3/4$. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?

4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

a. Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$. For example, use a visual fraction model to show $(2/3) \times 4 = 8/3$, and create a story context for this equation. Do the same with $(2/3) \times (4/5) = 8/15$. (In general, $(a/b) \times (c/d) = ac/bd$.)

b. Find the area of a rectangle with fractional side lengths by tiling it with unit rectangles[6] of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

5. Interpret multiplication as scaling (resizing), by:

a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

6. Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[7]

a. Interpret division of a unit fraction by a non-zero whole number, and compute

[6] In the original, it is incorrectly stated as "squares".

[7] Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.

such quotients. For example, create a story context for $(1/3) \div 4$, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that $(1/3) \div 4 = 1/12$ because $(1/12) \times 4 = 1/3$.

b. Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for $4 \div (1/5)$, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that $4 \div (1/5) = 20$ because $20 \times (1/5) = 4$.

c. Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. For example, how much chocolate will each person get if 3 people share $1/2$ lb of chocolate equally? How many $1/3$-cup servings are in 2 cups of raisins?

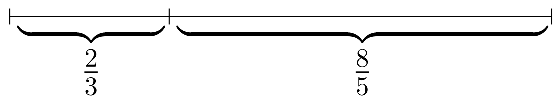
The study of fractions is the dominant theme of fifth grade mathematics and it requires a careful and detailed discussion. The main topics are:

the addition and subtraction of any two fractions,
the division interpretation of a fraction,
the multiplication of any two fractions and applications,
the arithmetic of decimals (to hundredths),
simple cases of the division of fractions.

We will take them up in succession.

Adding and subtracting fractions

The first thing to note about the addition and subtraction of fractions is that, conceptually, there is no difference between these same operations on whole numbers and fractions: addition means putting things together and subtraction means taking one thing from another. This fact should be emphasized in instructions. For addition, we take up the previous example of $\frac{2}{3} + \frac{8}{5}$ in grade 4 (page 24). The meaning (definition) of the sum is that it is the total length of the segment obtained by putting together these two segments end-to-end, one of length $\frac{2}{3}$ and the other of length $\frac{8}{5}$.



We observe at once that the addition of fractions obeys the commutative and associative laws.

At first glance, it is hard to tell what the total length is, but *if we know equivalent fractions* and *if we know how to add fractions with the same denominator*,[8] then we would immediately think of changing these two fraction $\frac{2}{3}$ and $\frac{8}{5}$ into two fractions with the same denominator by use of equivalent fractions. We recognize that $\frac{2}{3}$ and $\frac{8}{5}$ are equal, respectively, to $\frac{10}{15}$ and $\frac{24}{15}$, because

$$\frac{2}{3} = \frac{5 \times 2}{5 \times 3} = \frac{10}{15} \quad \text{and} \quad \frac{8}{5} = \frac{3 \times 8}{3 \times 5} = \frac{24}{15}.$$

Therefore, by definition, the sum $\frac{2}{3} + \frac{8}{5}$ is the total length of

10 copies of $\frac{1}{15}$, and

24 copies of $\frac{1}{15}$

So the total length is $10 + 24$ copies of $\frac{1}{15}$. In symbolic expressions, this becomes

$$\frac{2}{3} + \frac{8}{5} = \frac{10}{15} + \frac{24}{15} = \frac{10 + 24}{15} = \frac{34}{15}.$$

Once again, we call attention to the fact that the fraction addition $\frac{2}{3} + \frac{8}{5}$ becomes no different from the *whole number* addition of putting together 10 and 24 objects.

In general, if m, n, k, ℓ are whole numbers, then the same reasoning yields

$$\frac{k}{\ell} + \frac{m}{n} = \frac{kn}{\ell n} + \frac{\ell m}{\ell n} = \frac{kn + \ell m}{\ell n},$$

(the fact that $\ell \neq 0$ and $n \neq 0$ being always understood.)

Observe that the reason for changing $\frac{2}{3}$ and $\frac{8}{5}$ into $\frac{10}{15}$ and $\frac{24}{15}$ is to put both fractions "on equal footing" in the sense of having the same denominator. In general, taking the product of the denominators 3 and 5 to be the new denominator is both

[8] Knowing facts is as important in mathematics as in anything else.

efficient and simple. In special cases, however, there may be a more obvious choice of a common denominator. We give two examples. Consider $\frac{2}{3} + \frac{23}{18}$. Clearly we should use 18 as a common denominator in this case because 18 is already a multiple of 3, i.e.,

$$\frac{2}{3} + \frac{23}{18} \;=\; \frac{6 \times 2}{6 \times 3} + \frac{23}{18} \;=\; \frac{12 + 23}{18} \;=\; \frac{35}{18}$$

Had we taken 3×18 as the common denominator, we would have gotten

$$\frac{2}{3} + \frac{23}{18} \;=\; \frac{18 \times 2}{18 \times 3} + \frac{3 \times 23}{3 \times 18} \;=\; \frac{36 + 69}{54} \;=\; \frac{105}{54}.$$

But by the theorem on equivalent fractions, we have

$$\frac{105}{54} \;=\; \frac{3 \times 35}{3 \times 18} \;=\; \frac{35}{18},$$

so this is the same answer as before. Next, consider $\frac{23}{18} + \frac{5}{27}$. Now 27 is not a multiple of 18, but it is easily recognized that both are multiples of 9 so that, instead of taking 18×27 as the common denominator, it suffices to take $2 \times 3 \times 9$ $(= 54)$ because

$$54 \;=\; 3 \times 18 \quad \text{and} \quad 54 \;=\; 2 \times 27,$$

so that

$$\frac{23}{18} + \frac{5}{27} \;=\; \frac{3 \times 23}{3 \times 18} + \frac{2 \times 5}{2 \times 27} \;=\; \frac{69 + 10}{54} \;=\; \frac{79}{54}.$$

It can be checked that if 18×27 $(= 486)$ were used as the common denominator, then we would have gotten

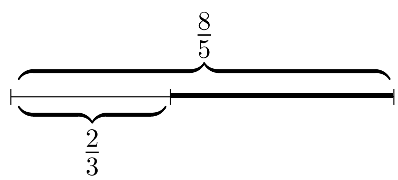
$$\frac{23}{18} + \frac{5}{27} \;=\; \frac{711}{486},$$

which is exactly the same answer because

$$\frac{711}{486} \;=\; \frac{9 \times 79}{9 \times 54} \;=\; \frac{79}{54}.$$

The numbers 18 and 54 are the so-called *lowest common denominators* of the fractions in the first and second examples, respectively. What we wish to point out is that the use of the lowest common denominator, while achieving some computational simplification, is by no means mandatory for getting the right answer. The skill of making use of the lowest common denominator for adding fractions should be taught as an enrichment topic if time is available, but it is not a required skill for the purpose of adding fractions by these Common Core Standards.

The concept of fraction subtraction is so similar to fraction addition that we will be brief. Consider $\frac{8}{5} - \frac{2}{3}$. As in the case of subtracting whole numbers, we must first check that $\frac{8}{5} \geq \frac{2}{3}$. Knowing that $\frac{8}{5} > 1$ while $\frac{2}{3} < 1$, we easily obtain $\frac{8}{5} > 1 > \frac{2}{3}$. (In general, one can use the cross-multiplication algorithm (page 24).) Then by the definition of subtraction on page 28, $\frac{8}{5} - \frac{2}{3}$ is the length of the thickened segment shown below after a segment of length $\frac{2}{3}$ has been taken from the segment of length $\frac{8}{5}$.



As in the case of the addition $\frac{2}{3} + \frac{8}{5}$, the same reasoning yields:

$$\frac{8}{5} - \frac{2}{3} \;=\; \frac{3 \times 8}{3 \times 5} - \frac{5 \times 2}{5 \times 3} \;=\; \frac{24 - 10}{15} \;=\; \frac{14}{15}$$

In particular, the fraction subtraction $\frac{2}{3} - \frac{8}{5}$ is now seen to be the same as the *whole number* subtraction of taking 10 from 24. The preceding comments about the use of the least common denominator in addition applies verbatim to subtraction.

Adding and subtracting (finite) decimals

Next, consider the addition of two decimals (up to hundredths). To be specific, consider $15.6 + 2.74$. By definition, we are looking at nothing other than the *addition of the fractions,*

$$\frac{156}{10} + \frac{274}{100}$$

This we can do:

$$
\begin{aligned}
15.6 + 2.74 &= \frac{156}{10} + \frac{274}{100} && \text{(by definition of a decimal)} \\
&= \frac{1560}{100} + \frac{274}{100} && \text{(equivalent fractions)} \qquad (1) \\
&= \frac{1560 + 274}{100} && \qquad (2) \\
&= \frac{1834}{100} \\
&= 18.34 && \qquad (3)
\end{aligned}
$$

We can now use this computation to explain the usual algorithm for computing 15.6+ 2.74, which is the following.

(a) Line up the two decimals by the decimal point:

$$
\begin{array}{ccccc}
 & 1 & 5 & .6 & \\
+ & & 2 & .7 & 4 \\
\hline
\end{array}
$$

(b) Add the numbers as if they were whole numbers by ignoring the decimal points:

$$
\begin{array}{ccccc}
 & 1 & 5 & 6 & 0 \\
+ & & 2 & 7 & 4 \\
\hline
 & 1 & 8 & 3 & 4
\end{array}
$$

(c) Put the decimal point in part (b) where it used to belong, and that is the answer:

$$
\begin{array}{cccccc}
 & 1 & 5 & . & 6 & \\
+ & & 2 & . & 7 & 4 \\
\hline
 & 1 & 8 & . & 3 & 4
\end{array}
$$

We now compare equalities (1)–(3) with steps (a)–(c). Of the two decimals, 15.6 has only one decimal digit and 2.74 has two, so the procedure in (a) of lining up the two decimals by their decimal points corresponds to increasing the number of decimal digits in 15.6 from one to two and writing it as 15.60, which is the same as changing $\frac{156}{10}$ to $\frac{1560}{100}$, and which is exactly the content of equality (1). Next, by the definition of a finite decimal, the procedure in (b) of adding the decimals as if they are whole numbers is just another way of saying: "look at the numerators of the decimal fractions $\frac{1560}{100}$ and $\frac{274}{100}$ and ignore the denominator 100, and just add the numerators". However, this is exactly how the addition of the two fractions with the same denominator is carried out, as in equality (2). Changing the resulting whole number back to a decimal in (c) is now seen to be nothing more than remembering that the denominator is 100 and therefore, by the definition of a decimal, we get back 18.34 in equality (3).

A little reflection will show that the reasoning above, ostensibly dealing only with decimals up to hundredths, is in fact perfectly general and is applicable to the addition of any two decimals.

There is of course an analogous algorithm for subtracting decimals corresponding to (a)–(c) above, and the reasoning behind this algorithm is so similar to the case of addition that we can safely skip it.

Division interpretation of a fraction

Next, we want to revisit the concept of a fraction.

We begin with a review of the concept of division between whole numbers. What does it mean to say $36 \div 9 = 4$? One way to model the division is to consider a water container with a capacity of 36 gallons. If we want to fill it with 9 bucketfuls of water, what should the capacity of the bucket be? Then 4 is the answer because if we divide 36 into 9 equal parts then the size of each part is 4 because

$$\begin{aligned} 36 &= \underbrace{4 + 4 + \cdots + 4}_{9} \\ &= 9 \times 4 \qquad\qquad \text{(by definition of multiplication)} \end{aligned}$$

This is the so-called **partitive interpretation of** (whole number) **division**.[9] Notice that students have only encountered whole number division up to this point, and the number 36, a multiple of 9, is designed to ensure that the **quotient** 4 is a whole number. *It would make no sense in this context to ask for* $37 \div 9$.

[9]We will deal with the *measurement interpretation* in the last section of grade 5 on division.

In a more formal setting, we formulate the concept of division between any two whole numbers m and n ($n \neq 0$) as follows.

> When m is a multiple of n, then $m \div n = q$ (where q is a whole number, called the **quotient** of the division) means that q is the size of one part when m is partitioned into n equal parts, i.e.,
>
> $$m \;=\; \underbrace{q + \cdots + q}_{n} \;=\; n \cdot q$$
>
> (The dot between n and q in $n \cdot q$ is for emphasis.)

With the availability of fractions, we are in a position to extend this meaning of $m \div n$ to *any* two whole numbers m and n ($n \neq 0$) without regard to whether m is a multiple of n or not. To do this, we have to give up the requirement that q be a whole number in $m \div n = q$ and allow it to be a fraction. Other than that, we can essentially repeat the whole definition:

> Let m and n be any two whole numbers ($n \neq 0$). Then $\boldsymbol{m \div n = q}$ (where q is a *fraction*, called the **quotient**) means that q is the size of one part when m is partitioned into n equal parts, i.e.,
>
> $$m \;=\; \underbrace{q + \cdots + q}_{n} \;=\; n \cdot q$$
>
> (The dot between n and q in $n \cdot q$ is for emphasis.)

We will now show that

> *the quotient of $m \div n$, in this extended sense, is equal to the fraction $\frac{m}{n}$.*

Thus we will need to explain why

> *the size of one part is $\frac{m}{n}$ when m is partitioned into n equal parts, i.e.,*
>
> $$m \;=\; \underbrace{\frac{m}{n} + \cdots + \frac{m}{n}}_{n} \;=\; n \cdot \frac{m}{n}$$

The fact that

$$\underbrace{\frac{m}{n} + \cdots + \frac{m}{n}}_{n} \;=\; \frac{nm}{n} \;=\; m$$

follows immediately from the addition of fractions (see page 26). The fact that

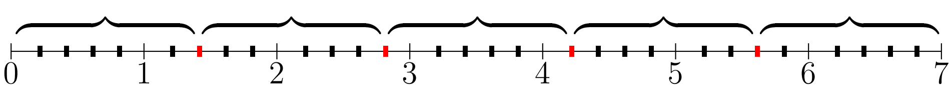
$$\underbrace{\frac{m}{n} + \cdots + \frac{m}{n}}_{n} \;=\; n \cdot \frac{m}{n}$$

is, however, a matter of definition (see page 28). Therefore we have shown that $m \div n = \frac{m}{n}$ is true in the sense described.

Let us show directly, without computations, that $m \div n = \frac{m}{n}$ in the partitive sense, i.e., we will show that

$$\frac{m}{n} \;=\; \textit{the size of one part when } m \textit{ is partitioned into } n \textit{ equal parts} \qquad (4)$$

without computations. The reasoning will go over better if we look at a specific case, e.g., $7 \div 5 = \frac{7}{5}$. Thus we will show that $\frac{7}{5}$ *is the length of one part when a segment of length 7 is divided into 5 equal parts.* To this end, we divide each segment between consecutive whole numbers on the number line into 5 equal parts (the number 5 being the denominator of the fraction $\frac{7}{5}$), then the segment from 0 to 7 is now divided into $7 \times 5 = 35$ equal parts and each part has length $\frac{1}{5}$. If we take every 7 of these parts at a time (indicated by the red markings below), then the red markings give a division of the segment from 0 to 7 into 5 equal parts. So each part of course has length $\frac{7}{5}$.



A similar reasoning yields the fact that, for instance, $\frac{2}{11}$ is the length of one part when (a segment of length) 2 is divided into 11 parts of the same length. The reasoning is perfectly general.

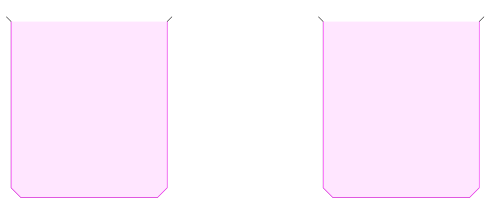
Equation (4) for any whole numbers m and $n \neq 0$ is the so-called **division interpretation of a fraction $\frac{m}{n}$**. Some comments about this interpretation are in order. Let us first compare this meaning of the fraction $\frac{m}{n}$ in equation (4) with its original meaning: $\frac{m}{n}$ is m copies of the unit fraction $\frac{1}{n}$. At the risk of belaboring the point, we observe that the original meaning of $\frac{m}{n}$ requires only the consideration a unit fraction $\frac{1}{n}$ and taking m copies; just focus on $\frac{1}{n}$ and there is no need to know how big m is. On the other hand, the meaning of $\frac{m}{n}$ given by equation (4) is that we *start* with a segment of length m and then divide it into n equal parts to find out how big one part is, and that turns out to be $\frac{m}{n}$. This is therefore a different view of a fraction. Moreover, it is common to express equation (4) as "$\frac{m}{n}$ is also equal to m divided by n", and no explanation is given. In addition to emphasizing that an explanation should be given, we wish to bring out the subtle point that the extended meaning of divisions such as "7 divided by 5" (where 7 is not a multiple of 5) should be carefully explained (as we did above) before any explanation of equation (4) is attempted.

The meaning of a fraction given in equation (4) helps us do problems of the following kind: *If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get?* Thus we are asked to divide the segment from 0 to 50 on the number line (whose unit is 1 pound) into 9 equal parts. This is where equation (4) comes in: it states unequivocally that one part is $\frac{50}{9}$ pounds, or $5\frac{5}{9}$ pounds. No guessing. Just simple reasoning on the basis of what is known to be true.

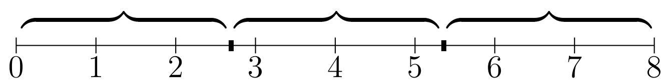
Multiplication of fractions

Having introduced the concept of multiplying a fraction by a whole number in grade 4 (page 28), we now tackle multiplication of fractions in general.

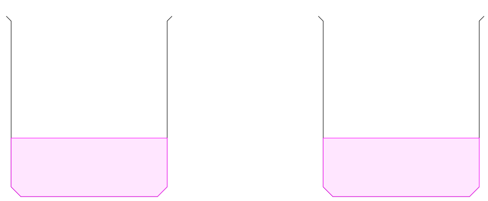
If you drink 2 cups of milk, how much milk in terms of fluid ounces (fl oz.) did you drink? One cup being 8 fl oz., you have drunk 2×8 fl oz., i.e., 16 fl oz.



With this example in mind, now suppose you drink two-thirds of a cup of milk, how much milk in terms of fl oz. did you drink? Notice that we are now asking for a precise (mathematical) answer about something expressed in colloquial English, and this calls for caution. Colloquial expressions are usually vague, and we must exercise care in rendering them into precise language. For students, the need to translate vague statements into precise language should gradually become instinctive, and the fifth grade would be a good place to start. "Two-thirds of a cup", when thought through carefully, really means: *divide the 8 fl oz. of liquid into 3 equal parts and take 2 of those parts".* Let this be our agreement once and for all. So how much is two-thirds of a cup in terms of fl oz.? By our agreement, it is the total amount in 2 parts when 8 is divided into 3 equal parts, or more precisely, when the segment of length 8 on the number line from 0 to 8 (where the unit is 1 fl oz.) is divided into 3 segments of equal length. By equation (4), each of these 3 segments has length $\frac{8}{3}$, so two-thirds of a cup is $\frac{2\times 8}{3}$ fl oz.



In terms of the cup of milk directly, this is the total amount in the following two portions of a third of a cup.



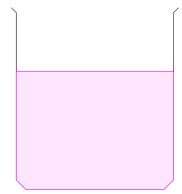
The parallel with the preceding situation should be obvious. Recall that we expressed two cups as 2×8 fl oz., so by analogy, we are tempted to say that

> the amount of milk in two-thirds of a cup of milk ought to be expressed as $\frac{2}{3} \times 8$ fl oz.

Because this thought is so reasonable, we accept it and move on. We see that multiplication by a fraction arises naturally. Summarizing our findings about $\frac{2}{3} \times 8$ fl oz., we have

$$\frac{2}{3} \times 8 \;=\; \frac{2 \times 8}{3}$$

Suppose now we do not start with a cup full of milk but only $5\frac{1}{3}$ fl oz.



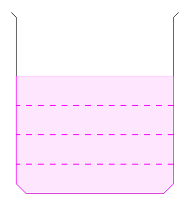
How much is three-quarters of this amount? By our agreement, this means we have to find out how much milk is in three parts when $5\frac{1}{3}$ fl oz. is divided into 4 equal parts. Because

$$5\frac{1}{3} \;=\; 5+\frac{1}{3} \;=\; \frac{16}{3} \;=\; \frac{4}{3}+\frac{4}{3}+\frac{4}{3}+\frac{4}{3},$$

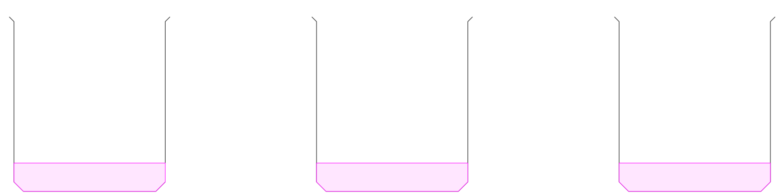
we see that $\frac{3}{4}$ of $5\frac{1}{3}$ is $\frac{4}{3}+\frac{4}{3}+\frac{4}{3} = \frac{3\times 4}{3} = 4$ fl oz. Recalling the agreement about fraction multiplication, we can express this fact symbolically as:

$$\frac{3}{4}\times\frac{16}{3} \;=\; 4 \;\text{ fl oz.}$$

Pictorially, we can represent this multiplication as follows. We first divide the above $5\frac{1}{3}$ fl oz. of milk into 4 parts of equal volume, thus:



Then the totality of 3 of these parts is the total amount in the following 3 parts when $5\frac{1}{3}$ fl oz. is divided into 4 equal parts:



Finally, we formalize the concept of **fraction multiplication** in general as follows. Given any two fractions $\frac{m}{n}$ and $\frac{k}{\ell}$, the meaning of their product is that

$$\frac{m}{n}\times\frac{k}{\ell} \;=\; \textit{the length of } m \textit{ parts when } \frac{k}{\ell} \textit{ is divided into } n \textit{ equal parts.}$$

We also agree to use the phrase $\boldsymbol{\frac{m}{n}}$ **of** $\boldsymbol{\frac{k}{\ell}}$ to abbreviate the statement on the right or, on occasions, we also express the right side as $\boldsymbol{\frac{m}{n}}$ **copies of** $\boldsymbol{\frac{k}{\ell}}$**.** Thus, by definition, $\frac{m}{n} \times \frac{k}{\ell}$ is $\frac{m}{n}$ copies of $\frac{k}{\ell}$.

It is important at this point for us to confirm that this definition of fraction multiplication does not conflict with the earlier definition of a whole number multiplying a fraction (page 28). Thus consider

$$m \times \frac{k}{\ell} \left(= \frac{m}{1} \times \frac{k}{\ell} \right).$$

According to the preceding definition, this is equal to the length of m parts when $\frac{k}{\ell}$ is divided into *1* equal part (which is of course just $\frac{k}{\ell}$ itself), i.e., this is equal to

$$\underbrace{\frac{k}{\ell} + \frac{k}{\ell} + \cdots + \frac{k}{\ell}}_{m}$$

But this is exactly the meaning of $m \times \frac{k}{\ell}$ as given on page 28. So we are on solid ground here.

We now clarify a special feature of fraction multiplication. The definition of multiplication implies, without any calculations, that

if $\frac{m}{n} > 1$, *then* $\frac{m}{n} \times \frac{k}{\ell} > \frac{k}{\ell}$,

if $\frac{m}{n} < 1$, *then* $\frac{m}{n} \times \frac{k}{\ell} < \frac{k}{\ell}$.

Indeed, consider the first assertion (the second one is entirely similar). If $\frac{m}{n} > 1$, then $m > n$, and therefore $m = n + c$ for a nonzero whole number c so that

$$\frac{m}{n} = \frac{n+c}{n} = 1 + \frac{c}{n}.$$

It follows that if we divided $\frac{k}{\ell}$ into n equal parts, the totality of m such parts is the combination of $\frac{k}{\ell}$ itself (which is the totality of n such parts) and an additional c such parts. By definition, $\frac{m}{n} \times \frac{k}{\ell}$ is thus the totality of c such parts more than $\frac{k}{\ell}$. More generally, this explanation also gives the intuitive picture that, e.g., $\frac{4}{3} \times A$ for any fraction A increases the size of A by a third of A, whereas $\frac{2}{5} \times A$ reduces the size of A by $\frac{3}{5}$ of A. This is the naive concept of "scaling", a full understanding of which will have to await the study of similarity in high school geometry.

We now prove the all-important **product formula**:

$$\frac{m}{n} \times \frac{k}{\ell} \;=\; \frac{mk}{n\ell} \tag{5}$$

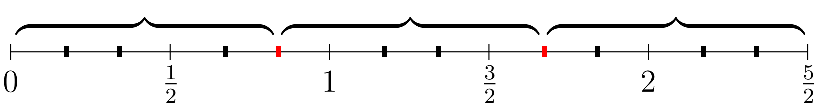
It will be best if we begin by explaining a special case:

$$\frac{4}{3} \times \frac{5}{2} \;=\; \frac{4 \times 5}{3 \times 2} \tag{6}$$

(“If the capacity of a bucket is $2\frac{1}{2}$ gallons and $1\frac{1}{3}$ buckets of water fill a container, what is the capacity of the container?”) We have to first find out what one part is when $\frac{5}{2}$ is divided into 3 equal parts. In the example above, we saw that when $\frac{16}{3}$ is divided into 4 equal parts, one part is simply $\frac{4}{3}$, because 16 being a multiple of 4, a fourth of “16 copies of $\frac{1}{3}$” is of course 4 copies of the same, and therefore $\frac{4}{3}$. Now we are confronted with computing a third of “5 copies of $\frac{1}{2}$” but 5 is unfortunately not a multiple of 3. At this point, the fundamental fact about equivalent fractions saves the day.[10] We know that

$$\frac{5}{2} \;=\; \frac{3 \times 5}{3 \times 2}$$

and the numerator 3×5 of $\frac{3\times 5}{3\times 2}$ is certainly a multiple of 3. This reasoning can be made more intuitive by drawing a picture that corresponds to how the equality $\frac{5}{2} = \frac{3\times 5}{3\times 2}$ is proved in the first place. Thus we subdivide each of the 5 segments of length $\frac{1}{2}$ in the picture below into 3 equal parts. This then results in the division of the segment from 0 to $\frac{5}{2}$ into 15 ($= 3 \times 5$) equal parts so that, by taking 5 of these parts at a time, we would get a division of the segment from 0 to $\frac{5}{2}$ into 3 equal parts:



It follows that if $\frac{5}{2}$ $(= \frac{3\times 5}{3\times 2})$ is divided into 3 equal parts, one part is $\frac{5}{3\times 2}$. Therefore 4 of these parts would be $\frac{4\times 5}{3\times 2}$, which is exactly equation (6).

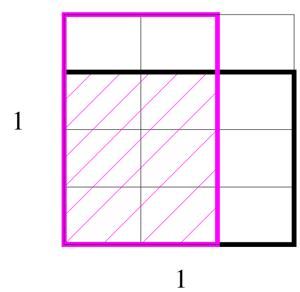
The reasoning behind the derivation of equation (6) turns out to be entirely adequate for explaining equation (5). However, in a fifth grade classroom, it will probably

[10]It is well to recall the assertion on page 19, to the effect that this fundamental fact ties the various strands within fractions together.

be more productive not to engage in symbolic arguments (that are needed for explaining equation (5)) but to continue with explanations of simple special cases of the type in equation (6). For example,

$$\frac{2}{5} \times \frac{4}{3} = \frac{2 \times 4}{5 \times 3}, \qquad \frac{5}{6} \times \frac{8}{7} = \frac{5 \times 8}{6 \times 7}, \quad \text{etc.}$$

It would be appropriate to say a few words about the pictorial representation of the product of two fractions such as $\frac{3}{4} \times \frac{2}{3}$ as the intersection of horizontal strips and vertical strips in the unit square:



This is usually taught with no explanation. Suffice it to say that the representation is indeed correct, but the explanation would be best given in the later section on Area of a rectangle.

Immediate applications of the product formula

Because the multiplication of whole numbers satisfies the commutative, associative, and distributive laws, the product formula (equation (5)) shows that the same is true of the multiplication of fractions. These laws are usually not held in high regard by students, but it so happens that the subsequent discussion will make strong use of these laws. To illustrate that these laws mean something, ask them:

Which is heavier, $\frac{7}{9}$ of $\frac{11}{4}$ kg of sand, or $\frac{11}{4}$ of $\frac{7}{9}$ kg of sand?

At first glance, it is not easy to decide. However, by the definition of fraction multiplication, $\frac{7}{9}$ of $\frac{11}{4}$ kg is

$$\left(\frac{7}{9} \times \frac{11}{4}\right) \quad \text{kg}$$

whereas $\frac{11}{4}$ of $\frac{7}{9}$ kg is

$$\left(\frac{11}{4} \times \frac{7}{9}\right) \quad \text{kg.}$$

Because multiplication is commutative, the two numbers are the same.

For computations with fractions, the product formula leads to the important skill of **cancellation**. This may be generically described as follows. Let c be a nonzero whole number and let $\frac{m}{n}$ and $\frac{k}{\ell}$ be two given fractions. Then:

$$\frac{c\,m}{n} \times \frac{k}{c\,\ell} \;=\; \frac{m}{n} \times \frac{k}{\ell} \tag{7}$$

Thus we "cancel the two c's from top and bottom". The validity of equation (7) is a direct consequence of the product formula and the fundamental fact about equivalent fractions:

$$\frac{c\,m}{n} \times \frac{k}{c\,\ell} \;=\; \frac{cmk}{cn\ell} \;=\; \frac{mk}{n\ell} \;=\; \frac{m}{n} \times \frac{k}{\ell}\,.$$

A particularly striking consequence of equation (7) is the fact that given any nonzero fraction $\frac{m}{n}$, we can always find a fraction which, when multiplied by $\frac{m}{n}$, yields the number 1. Indeed, the **reciprocal fraction** $\frac{n}{m}$ has the desired property, because applying equation (7) twice, we get:

$$\frac{n}{m} \times \frac{m}{n} \;=\; \frac{1}{1} \times \frac{1}{1} \;=\; 1$$

Next, we explain the multiplication algorithm for (finite) decimals. Consider, for example, 34.5×4.78. The usual multiplication algorithm says:

> (a) Multiply the two numbers as whole numbers by ignoring the decimal point (getting 164910).
>
> (b) Convert the answer in (a) to a decimal by the following rule: it should have $1 + 2(= 3)$ decimal digits because 35.4 has 1 decimal digit and 4.78 has 2 (getting 164.910).

The explanation of this algorithm is exceedingly simple provided we remember the definition (meaning) of a (finite) decimal. So:

$$
\begin{aligned}
34.5 \times 4.78 &= \frac{345}{10} \times \frac{478}{100} && \text{(definition of decimal)} \\
&= \frac{345 \times 478}{10 \times 100} && \text{(product formula)} && (8) \\
&= \frac{164910}{1000} \\
&= 164.910 && \text{(definition of decimal)} && (9)
\end{aligned}
$$

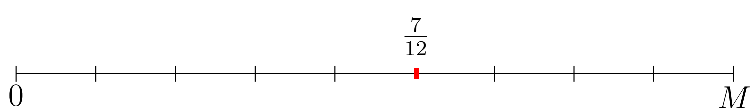
We see immediately that step (a) corresponds to equality (8) and step (b) corresponds to equality (9).

What is noteworthy about the preceding reasoning is that the correctness of the multiplication algorithm for finite decimals depends on the product formula for multiplying fractions.

There is no end of the applications of the product formula to word problems. Here is a typical one:

> *I was on a hiking trail, and after walking $\frac{7}{12}$ of a mile, I was $\frac{5}{9}$ of the way to the end. How long is the trail?*

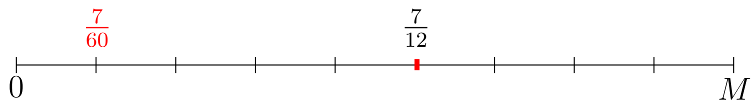
Let the trail be M miles long. There are at least two ways to find out what M is. The first is to draw a correct picture. On the number line where the unit stands for 1 mile, we divide the segment from 0 to M into 9 equal parts. It is given that after walking $\frac{7}{12}$ of a mile, I came to the 5th division point to the right of 0. Thus the fifth division point is the number $\frac{7}{12}$, as shown.



Now the first division point to the right of 0 is $\frac{1}{5}$ of the distance from 0 to $\frac{7}{12}$, so by the definition of fraction multiplication, the first division point is

$$\frac{1}{5} \times \frac{7}{12} = \frac{7}{60},$$

where we have used the product formula.



Since M is the 9th division point to the right of 0, we see that M is equal to:

$$\begin{aligned} 9 \times \frac{7}{60} &= \frac{9}{1} \times \frac{7}{60} \\ &= \frac{3}{1} \times \frac{7}{20} && \text{(equation (7))} \\ &= \frac{21}{20} = 1\frac{1}{20} && \text{(equivalent fraction)} \end{aligned}$$

Recalling that the unit is 1 mile, we see that the trail is $1\frac{1}{20}$ miles long.

A second solution makes use of the cancellation phenomenon (equation (7)) above. This solution is slightly more sophisticated, but it will be important for the consideration of division below. According to the definition of fraction multiplication, we can express symbolically the given information in the problem as $\frac{5}{9} \times M = \frac{7}{12}$ miles. Since the two numbers $\frac{5}{9} \times M$ and $\frac{7}{12}$ are the same number (i.e., same point on the number line), multiplying each by the same fraction would result in the same number again, according to the definition of fraction multiplication. Thus multiplying both sides of $\frac{5}{9} \times M = \frac{7}{12}$ by $\frac{9}{5}$, we get

$$\frac{9}{5} \times \left(\frac{5}{9} \times M\right) = \frac{9}{5} \times \frac{7}{12}\,.$$

By the associative law of multiplication, the left side is equal to

$$\left(\frac{9}{5} \times \frac{5}{9}\right) \times M = 1 \times M = M\,,$$

and therefore

$$M = \frac{63}{60} = 1\frac{1}{20}\,,$$

which is the same as before.

Area of a rectangle

The area formula of a rectangle with fractional side lengths may be the most substantive application of the product formula. The result is not in doubt: it is length times width, as everybody knows. It is the reasoning that is important. We briefly recall the usual assumptions of how areas are assigned to planar regions:

(a) The area of a planar region is always a number ≥ 0.

(b) The area of the unit square (the square whose sides have length 1) is by definition the number 1.

(c) If two regions are *congruent*, then their areas are equal.

(d) (*Additivity*) If two regions have at most (part of) their boundaries in common, then the area of the region obtained by combining the two is the sum of their individual areas.

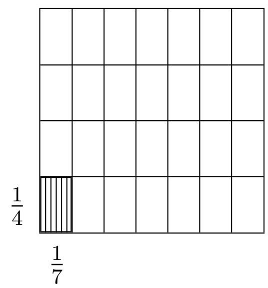
The computation will be guided at every turn by these four assumptions. Regarding (c), we shall not define "congruent regions" precisely in fifth grade except to use the intuitive meaning that congruent regions have the "same shape and same size", or that one can check congruence by moving one region without altering its shape to see if it can be made to coincide completely with the other. More precisely, we only need the fact that rectangles with the same side lengths are "congruent" and therefore have the same area.

We will compute the area of a rectangle with sides $\frac{3}{4}$ and $\frac{2}{7}$. It will be seen that the reasoning is equally applicable to the general case. We break up the computation into two steps.

(i) The area of a rectangle with sides $\frac{1}{4}$ and $\frac{1}{7}$.

(ii) The area of a rectangle with sides $\frac{3}{4}$ and $\frac{2}{7}$.

We begin with (i). To get a rectangle with sides $\frac{1}{4}$ and $\frac{1}{7}$, divide the vertical sides of a unit square into 4 equal parts and the horizontal sides into 7 equal parts. Joining the corresponding division points, both horizontally and vertically, leads to a partition of the unit square into 4×7 ($= 28$) congruent rectangles, and therefore 28 rectangles of equal areas, by (c). Observe that each small rectangle in this division has vertical side of length $\frac{1}{4}$ and horizontal side of length $\frac{1}{7}$, and this is exactly the rectangle we want.

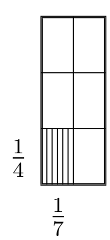


Now the total area of these 28 small rectangles is the area of the unit square, by (d), and the area of the unit square is 1 (by (b)). Look at this fact from another angle: we have divided the unit 1 (area of unit square) into 28 equal parts (28 equal areas), so by the definition of a fraction, each one of these 28 areas represent $\frac{1}{28}$, which is equal to $\frac{1}{4\times 7}$. Therefore the conclusion of (*i*) is:

$$\text{Area of rectangle with sides } \frac{1}{4} \text{ and } \frac{1}{7} \quad = \quad \frac{1}{4\times 7} \;=\; \frac{1}{4}\times\frac{1}{7} \tag{10}$$

where, in the last step, we made use of the product formula.

To perform the computation in Step (*ii*), *we change strategy completely.* Instead of partitioning the unit square, we use small rectangles of sides $\frac{1}{4}$ and $\frac{1}{7}$ to build a rectangle of sides $\frac{3}{4}$ and $\frac{2}{7}$. By the definition of $\frac{3}{4}$, it is the combination of 3 segments each of length $\frac{1}{4}$. Similarly, the side of length $\frac{2}{7}$ consists of 2 combined segments each of length $\frac{1}{7}$. Thus if we pile up 3 rows and 2 columns of small rectangles, each of which has sides of lengths $\frac{1}{4}$ and $\frac{1}{7}$, we obtain a rectangle of sides $\frac{3}{4}$ and $\frac{2}{7}$.



By equation (10), each of the small rectangles has area $\frac{1}{4\times 7}$. Since the big rectangle contains exactly 3×2 such congruent rectangles, its area is (by (d) above):

$$\underbrace{\frac{1}{4\times 7}+\frac{1}{4\times 7}+\cdots+\frac{1}{4\times 7}}_{3\times 2} \quad = \quad \frac{3\times 2}{4\times 7} \;=\; \frac{3}{4}\times\frac{2}{7}$$

where, in the last step, we again made use of the product formula. Therefore the conclusion of (*ii*) is:

$$\text{Area of rectangle with sides } \tfrac{3}{4} \text{ and } \tfrac{2}{7} \quad = \quad \frac{3}{4} \times \frac{2}{7}$$

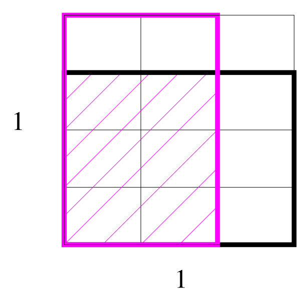
In other words, the area of the rectangle is the product of the (lengths of) the sides.

The general case follows this reasoning word for word. In most fifth grade classrooms, it would be beneficial to *state* the general formula: if m, n, k, ℓ are nonzero whole numbers, then

$$\text{Area of rectangle with sides } \tfrac{m}{n} \text{ and } \tfrac{k}{\ell} \quad = \quad \frac{m}{n} \times \frac{k}{\ell} \tag{11}$$

Instead of giving an explanation of equation (11) directly in terms of symbols, it would probably be more productive to compute the areas of several rectangles whose sides have lengths equal to reasonable fractions, e.g., $\frac{3}{2}$ and $\frac{1}{5}$, $\frac{7}{3}$ and $\frac{5}{6}$, etc.

Finally, we turn to the pictorial representation of fraction multiplication such as the following for $\frac{3}{4} \times \frac{2}{3}$, and explain why it is correct.



Precisely, let the unit 1 on the number line be the area of the unit square. Then what is done here is to divide the vertical sides of the unit square into 4 equal parts so that the black-thickened rectangle encompasses $\frac{3}{4}$ of the square in terms of area. Similarly, divide the horizontal sides into 3 equal parts so that the magenta-thickened rectangle encompasses $\frac{2}{3}$ of the unit square in terms of area. The intersection of these two rectangles—the magenta-shaded rectangle—is then a rectangle with side lengths $\frac{3}{4}$ and $\frac{2}{3}$, which then has area $\frac{3}{4} \times \frac{2}{3}$, as we have just shown.

To summarize: this pictorial representation of fraction multiplication is in terms of the unit being the area of the unit square, and it makes use of the fact that the area of a rectangle is the product of the side lengths.

Division of fractions (beginning)

We first bring closure to the discussion on page 41 of the extended meaning of the division of one whole number by another, $m \div n$ $(n \neq 0)$. We defined $36 \div 9 = 4$ to mean $36 = 9 \times 4$, and we modeled this equation by saying that if a container with a capacity of 36 gallons is filled by 9 bucketfuls of water, then the capacity of each bucket is 4 gallons. We can model the division $36 \div 9 = 4$ another way, however. If 36 gallons of water are poured into containers each with a capacity of 9 gallons, how many containers are needed? The answer is 4 again, because

$$\begin{aligned} 36 &= 9 \times 4 \\ &= 4 \times 9 && \text{(multiplication is commutative)} \\ &= 9 + 9 + 9 + 9 && \text{(definition of multiplication)} \end{aligned}$$

This is called the **measurement interpretation of** (whole number) **division**, because it measures how many 9's there are in 36 in the division $36 \div 9$. We can rephrase this as follows: the quotient q of $36 \div 9$ is the number so that there are q of 9 in 36. Symbolically, the quotient q is the number that satisfies $36 = q \times 9$. In view of the definition of fraction multiplication (page 46), the equation $36 = q \times 9$ continues to retain the same meaning that, even if *quotient* q is a fraction and not a whole number, 36 *is* still equal to q of 9. This then brings us the measurement interpretation of division for $m \div n$ where m and n are arbitrary whole numbers $(n \neq 0)$ and m is not necessarily a multiple of n.

> Let m and n be any two whole numbers $(n \neq 0)$. Then $m \div n = q$ (where q is a *fraction*, called the **quotient**) means that m is equal to q of n i.e.,
>
> $$m = q \cdot n$$

The reason is simple: by definition, $m \div n = q$ means $m = n \cdot q$. Since fraction multiplication is commutative, $m = q \cdot n$. By the definition of fraction multiplication (page 46), this says m is q of n.

We are now in a good position to introduce the division of fractions. Knowing ahead of time that the meaning of the arithmetic operations is basically the same for fractions as for whole numbers, we have the prior assurance that if in the preceding discussion, m, n, and q are fractions and not whole numbers, the meaning of $m \div n = q$ would stay the same. Thus, formally, let A, B, C be fractions ($C \neq 0$).

$$\textit{We say} \quad A \div C = B \quad \textit{if} \quad A = B \cdot C.$$

In grade 5 and grade 6, we will unravel this definition slowly. For grade 5, it suffices to look at some simple cases.

Consider $5 \div \frac{1}{3}$. To say C is a fraction so that $5 \div \frac{1}{3} = C$ is to say $5 = C \cdot \frac{1}{3}$. From the definition of fraction multiplication (page 46), we see that C copies of $\frac{1}{3}$ has to be equal to 5. We use common sense: 3 copies of $\frac{1}{3}$ is 1, and 5 copies of 1 is 5. So 3×5 copies of $\frac{1}{3}$ is equal to 5, and $C = 3 \times 5$. Anticipating a little bit of what is to come, we write:

$$5 \div \frac{1}{3} \;=\; 5 \times \frac{3}{1}$$

Next, try $2 \div \frac{1}{7}$. To say C is a fraction so that $2 \div \frac{1}{7} = C$ is to say $2 = C \cdot \frac{1}{7}$. From the definition of fraction multiplication (page 46), we see that C copies of $\frac{1}{7}$ has to be equal to 2. Again we appeal to common sense: 7 copies of $\frac{1}{7}$ is 1, and 2 copies of 1 is 2. So 2×7 copies of $\frac{1}{7}$ is equal to 2, and $C = 2 \times 7$. Once more anticipating what is to come, we write:

$$2 \div \frac{1}{7} \;=\; 2 \times \frac{7}{1}$$

Doing a few more examples like these, one can see that if m, n are any nonzero whole numbers, then

$$m \div \frac{1}{n} \;= m \times \frac{n}{1} \tag{12}$$

Recall that $m \div \frac{1}{n} \;= C$ means C copies of $\frac{1}{n}$ make up m. Thus if we ask roughly how many dinner guests we can serve with 8 pounds of beef steak if we anticipate that, on average, each guest consumes $\frac{1}{3}$ of a pound of steak, then we are asking if C copies of $\frac{1}{3}$ make up 8, what is C? The answer is:

$$8 \div \frac{1}{3} \;=\; 8 \times 3 \;=\; 24$$

guests, according to equation (12).

Next, we try something a little different. What is $\frac{1}{4} \div 5$? Now $\frac{1}{4} \div 5 = C$ means $\frac{1}{4} = C \times 5$. Knowing that multiplication is commutative, we have

$$\frac{1}{4} \;=\; 5 \times C.$$

(The reason for using the commutative law is because of our prior experience with the explanation of the measurement interpretation of division on page 55.) This means 5 copies of the fraction C has to be equal to $\frac{1}{4}$. What could C be? Now we appeal to common sense again: 5 copies of $\frac{1}{5}$ is 1, and clearly $\frac{1}{4}$ copy of 1 is $\frac{1}{4}$. Altogether,

$$5 \times \frac{1}{5} \;=\; 1$$

so that

$$\frac{1}{4} \times \left(5 \times \frac{1}{5}\right) \;=\; \frac{1}{4}.$$

Using the associative law and commutative law to rewrite the left side of this equation, we get:

$$5 \times \left(\frac{1}{5} \times \frac{1}{4}\right) \;=\; \frac{1}{4}$$

Thus it is a good guess that C should be $\frac{1}{5} \times \frac{1}{4}$. It is easy to check that this is correct. So

$$\frac{1}{4} \div 5 \;=\; \frac{1}{4} \times \frac{1}{5} \;=\; \frac{1}{5 \times 4},$$

where the last step uses the product formula. So the upshot of dividing a unit fraction such as $\frac{1}{4}$ by a whole number such as 5 is that we get the unit fraction whose denominator is the product 5×4.

Doing a few more examples of this type, we would also see the pattern that if m and n are any nonzero whole numbers, then

$$\frac{1}{m} \div n \;=\; \frac{1}{n} \cdot \frac{1}{m} \;=\; \frac{1}{nm} \tag{13}$$

We will explain equation (13) completely. Recall that if $\frac{1}{m} \div n = C$, then $\frac{1}{m} = C \cdot n = n \cdot C$. Therefore n copies of C is equal to $\frac{1}{m}$ and C is the size of one part when $\frac{1}{m}$ is partitioned into n equal parts. Therefore, on the one hand,

$\frac{1}{m} \div n$ *is the size of one part when* $\frac{1}{m}$ *is partitioned into* n *equal parts.*

On the other hand, by the definition of fraction multiplication (page 46), the size of one part when $\frac{1}{m}$ is partitioned into n equal parts is precisely $\frac{1}{n} \cdot \frac{1}{m}$. Putting these two facts together, we have:

$$\frac{1}{m} \div n \;=\; \frac{1}{n} \cdot \frac{1}{m} \;=\; \frac{1}{nm}\,.$$

In particular, this shows in general why equation (13) is correct.

6TH GRADE

Ratios and Proportional relationships 6.RP

Understand ratio concepts and use ratio reasoning to solve problems.

1. Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. For example, "The ratio of wings to beaks in the bird house at the zoo was $2 : 1$, because for every 2 wings there was 1 beak. "For every vote candidate A received, candidate C received nearly three votes.

2. Understand the concept of a unit rate a/b associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is $3/4$ cup of flour for each cup of sugar. "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."[11]

3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

b. Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?

c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

The Number System 6.NS

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

[11]Expectations for unit rates in this grade are limited to non-complex fractions.

1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share $1/2$ lb of chocolate equally? How many $3/4$-cup servings are in $2/3$ of a cup of yogurt? How wide is a rectangular strip of land with length $3/4$ mi and area $1/2$ square mi?

3. Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

The concept of fraction division lies at the foundation of the whole discussion of fractions in this grade. We will use a clear understanding of this concept to explain ratio, rate, and percent.

Division of fractions (conclusion)

We first recall the meaning of **fraction division** given on page 56:

Let A, B, C be fractions with $C \neq 0$. We say $A \div C = B$ if $A = B \cdot C$.

B is called the **quotient** of the division. Because $B \cdot C$ means B copies of C, the division $A \div C$ therefore measures how many copies of C there are in A. So this is literally the measurement interpretation of division.

Note that there is a subtle issue here. Given any two fractions A and C ($C \neq 0$), we do not know a priori if there is such a quotient B. We have to establish the fact that there is always such a B. The standard way to do this is in fact the usual procedure for solving an equation (provided it is done *correctly*), so the subsequent reasoning may be considered a good introduction to this kind of algebraic thinking. It is the following. *Suppose there is such a B.* Then on the basis of this hypothesis, we will get enough information about B to find out what B would be should it exist. Let us start with a simple example: $A = \frac{3}{4}$, $C = \frac{2}{3}$. Suppose B is a fraction so that

$$B \;=\; \frac{3}{4} \;\div\; \frac{2}{3}\,.$$

By definition of fraction division, this means B satisfies

$$\frac{3}{4} \;=\; B \times \frac{2}{3}\,, \tag{14}$$

so that there are B copies of $\frac{2}{3}$ in $\frac{3}{4}$. It is difficult to directly compare $\frac{2}{3}$ with $\frac{3}{4}$ until we remember the fundamental fact about equivalent fractions.[12] Thus we will compare instead,

$$\frac{8}{12}\,\left(=\frac{2}{3}\right) \quad \text{and} \quad \frac{9}{12}\,\left(=\frac{3}{4}\right),$$

and the task becomes one of finding a B that satisfies

$$\frac{9}{12} \;=\; B \times \frac{8}{12}\,. \tag{15}$$

So how many copies of $\frac{8}{12}$ would equal $\frac{9}{12}$? It should not be difficult to see that $\frac{9}{8}$ is the answer: $\frac{1}{8}$ of $\frac{8}{12}$ is $\frac{1}{12}$, and 9 copies of the latter is then $\frac{9}{12}$. Or, more simply,

$$\frac{9}{8} \times \frac{8}{12} \;=\; \frac{9}{12}\,,$$

where we have made use of the cancellation phenomenon, equation (7) on page 49. Thus

$$B \;=\; \frac{9}{8}\,.$$

Observe that we can rewrite B as

$$B \;=\; \frac{9}{8} \;=\; \frac{3 \times 3}{2 \times 4} \;=\; \frac{3}{2} \times \frac{3}{4} \tag{16}$$

To summarize: we have shown that *if* there is a fraction B which is the quotient of $\frac{3}{4} \div \frac{2}{3}$, then B is given by equation (16). This does not say $\frac{3}{2} \times \frac{3}{4}$ is the quotient, but only that if there is a quotient, this has to be *it*. It remains therefore to go through the motions of verifying that $\frac{3}{2} \times \frac{3}{4}$ indeed satisfies equation (14):

$$\left(\frac{3}{2} \times \frac{3}{4}\right) \times \frac{2}{3} \;=\; \left(\frac{3}{2} \times \frac{2}{3}\right) \times \frac{3}{4} \;=\; 1 \times \frac{3}{4} \;=\; \frac{3}{4}\,,$$

[12] Recall the assertion on page 19, to the effect that this fundamental fact ties the various strands within fractions together.

where we made use of the associative law and commutative law of multiplication in the first step.

It may be clearer to carry out the reasoning in a more formal way. So again starting with

$$B \;=\; \frac{3}{4} \div \frac{2}{3}\,,$$

we have, by definition, that B is the fraction that satisfies

$$\frac{3}{4} \;=\; B \times \frac{2}{3}\,,$$

which can be equivalently expressed as

$$\frac{3 \times 3}{3 \times 4} \;=\; B \times \frac{4 \times 2}{4 \times 3}\,,$$

so that

$$\frac{3 \times 3}{12} \;=\; B \times \frac{4 \times 2}{12}\,.$$

One can see by inspection and using the cancellation phenomenon of equation (7) on page 49 that if

$$B \;=\; \frac{3 \times 3}{4 \times 2}\,,$$

it would get the job done. Therefore,

$$\frac{3}{4} \div \frac{2}{3} \;=\; B \;=\; \frac{3 \times 3}{4 \times 2} \;=\; \frac{3}{4} \times \frac{3}{2}\,.$$

In a classroom, several such examples should be worked out so that, for example,

$$\frac{2}{5} \div \frac{7}{8} \;=\; \frac{2}{5} \times \frac{8}{7}$$

The preceding reasoning is enough to establish the general fact about fraction division: Given fractions $\frac{m}{n}$ and $\frac{k}{\ell}$, where $\frac{k}{\ell} \neq 0$. Then:

$$\frac{m}{n} \div \frac{k}{\ell} \;=\; \frac{m}{n} \times \frac{\ell}{k}\,.$$

This symbolic statement should be at least stated if not derived in such generality. As is well-known, this is the **invert and multiply rule**, a useful skill for computing fraction divisions.

The preceding reasoning is also important for another reason: embedded in it is the skill to solve all such standard word problems. For example:

A rod $43\frac{3}{8}$ meters long is cut into pieces which are $\frac{5}{3}$ meters long. How many such pieces can we get out of the rod?

Let B be a fraction so that the rod can be cut into A such pieces, each being $\frac{5}{3}$ meters long. Then the problem asks, if B copies of the $\frac{5}{3}$-meter piece make up $43\frac{3}{8}$ meters, what is B? We are therefore given the analog of equation (14):

$$43\frac{3}{8} \;=\; B \times \frac{5}{3} \tag{17}$$

By the definition of division, we see that B has to be the quotient of a division problem, namely,

$$\begin{aligned} B \;=\; 43\frac{3}{8} \div \frac{5}{3} \;&=\; \frac{347}{8} \div \frac{5}{3} \\ &=\; \frac{347}{8} \times \frac{3}{5} \qquad\qquad \text{(invert and multiply)} \\ &=\; 26\frac{1}{40} \end{aligned}$$

So the answer is 26 such pieces with "$\frac{1}{40}$" left over. What does this "$\frac{1}{40}$" mean? To answer this question, it is necessary to allow the mathematics to speak for itself, and sixth graders should begin to be exposed to such reasoning. By equation (17), we have

$$\begin{aligned} 43\frac{3}{8} \;=\; 26\frac{1}{40} \times \frac{5}{3} \;&=\; \left(26 + \frac{1}{40}\right) \times \frac{5}{3} \\ &=\; \left(26 \times \frac{5}{3}\right) + \left(\frac{1}{40} \times \frac{5}{3}\right) \qquad \text{(distributive law)} \end{aligned}$$

The last line says explicitly that $43\frac{3}{8}$ is equal to 26 copies of the $\frac{5}{3}$-meter piece together with $\frac{1}{40}$ of the $\frac{5}{3}$-meter piece. This is then the meaning of $\frac{1}{40}$ *as dictated by the mathematics.*

There are two observations that are worth making. First, we saw how effective invert-and-multiply can be as a tool in mathematical reasoning. A second observation is that any understanding of division ultimately rests on an understanding of multiplication. This point shows up not only in understanding *why* one has to use division to solve this kind of problem—it comes from the multiplicative statement in equation (17)—but also in uncovering the meaning of $\frac{1}{40}$. Understanding division requires a thorough grounding in multiplication.

We should mention a related kind of problem which relies on the "partitive interpretation of fraction division" (see page 41). Consider the following:

> *The capacity of a water tank is 61 gallons and it can be filled by* $19\frac{1}{2}$ *bucketfuls of water. What is the capacity of the bucket?*

Here is one way to solve this problem. Let the capacity of the bucket be V gallons. Then $19\frac{1}{2}\times V = 61$. We want the value of V. Knowing the precise definition of fraction division on page 60, we rewrite it (using the commutativity of multiplication) as

$$61 \;=\; V \times 19\frac{1}{2}$$

to exhibit V as the quotient of the division statement $61 \div 19\frac{1}{2} = V$. Then invert and multiply yields:

$$V \;=\; 61 \div \frac{39}{2} \;=\; 61 \times \frac{2}{39} \;=\; 3\frac{5}{39} \;\text{ gallons}$$

The usual way to motivate the introduction of division $(61 \div 19\frac{1}{2})$ for the solution is to think of a simpler problem: *The capacity of a water tank is 61 gallons and it can be filled by 19 bucketfuls of water. What is the capacity of the bucket?* In this case, it is easy to imagine that we are partitioning 61 into 19 equal parts and the size of one part is the capacity of the bucket. The solution is given by $61 \div 19$, so that by the reasoning on page 41, the capacity of the bucket is $\frac{61}{19} = 3\frac{4}{19}$ gallons. By analogy, the solution of the original problem with 19 replaced by $19\frac{1}{2}$ can be done the same way using division. With this understood, one would then say that the division $61 \div 19\frac{1}{2}$ is also a *partitive division* in the sense that it gives the size of one part when "61 is divided into $19\frac{1}{2}$ equal parts".

There is nothing wrong with the analogy provided one understands the earlier reasoning given above that led to the answer of $3\frac{5}{39}$ gallons. However, if the analogy is used, one would have the difficult task of convincing sixth graders what it means to have one part when "*61 is divided into* $19\frac{1}{2}$ *equal parts*".

Note that one could go from the equation $61 = V \times 19\frac{1}{2}$ to $V = 61 \times \frac{2}{39}$ directly by multiplying both sides of the equation by $\frac{2}{39}$.

Arithmetic operations on (finite) decimals

We have already noted in the discussions on page 38 and page 49 of grade 5 that, although the explanations of the addition, subtraction, and multiplication algorithms for finite decimals given there are ostensibly only for decimals up to two decimal digits, they are in fact valid for any (finite) decimals. However, we did not discuss the division of finite decimals, and we do so now.

A common theme among the three algorithms for decimals ($+$, $-$, and $\times$, again, see page 38 and page 49) is that all such computations are reduced to computations of whole numbers. We now show that the same theme also prevails in division. To this end, we introduce a new notation for fraction division and the concept of a *complex fraction*. From now on, we will write a division of fractions, $\frac{m}{n} \div \frac{k}{\ell}$, as

$$\frac{\frac{\boldsymbol{m}}{\boldsymbol{n}}}{\frac{\boldsymbol{k}}{\boldsymbol{\ell}}}$$

and retire the division symbol "$\div$" altogether. A main reason for doing this is the fact already observed on page 41 that for whole numbers m and n, $m \div n = \frac{m}{n}$. There is thus no reason to use two symbols when one is enough, and since the fraction notation and the fraction concept are more fundamental, we use $\frac{m}{n}$.

A division between two fractions in this new notation for division is what is called a **complex fraction**. Thus if A and B are fractions and $B \neq 0$, then $\frac{A}{B}$ is a complex fraction. By definition, a complex fraction is just a fraction, but we single them out for a reason: often fractions come to us naturally as the division of one fraction by another. There will be many such examples in the next few pages. For a complex fraction $\frac{A}{B}$, A and B are called the **numerator** and **denominator**, respectively, of the complex fraction. This is consistent with the usual concept of "numerator" and "denominator" because an ordinary fraction is also a complex fraction: if m and n are whole numbers ($n \neq 0$), then

$$\frac{m}{n} = \frac{\frac{m}{1}}{\frac{n}{1}}.$$

What we need right away is the fact that the fundamental fact about equivalent fractions (page 19) is valid also for complex fractions in the sense that, if D is any nonzero fraction, then

$$\frac{DA}{DB} = \frac{A}{B} \tag{18}$$

The validity of equation (18) rests on the invert and multiply rule: if $A = \frac{a'}{a}$, $B = \frac{b'}{b}$, and $D = \frac{d'}{d}$, then,

$$\frac{DA}{DB} = \frac{\frac{a'd'}{da}}{\frac{d'b'}{db}} = \frac{a'd'db}{b'd'ad} = \frac{a'b}{ab'},$$

where the last step uses the fundamental fact about equivalent fractions. By invert-and-multiply again,

$$\frac{a'b}{ab'} = \frac{\frac{a'}{a}}{\frac{b'}{b}} = \frac{A}{B},$$

as claimed.

We can now show how to reduce the division of decimals to the division of whole numbers. It suffices to look at a few special cases because it will be clear that the reasoning is completely general. Consider, for example, $1.0027 \div 8.5$, which we now agree to write as $\frac{1.0027}{8.5}$. Recalling the definition of finite decimals (page 30), we have:

$$\begin{aligned} \frac{1.0027}{8.5} &= \frac{\frac{10027}{10^4}}{\frac{85}{10}} \\ &= \frac{\frac{10027}{10^4} \times 10^4}{\frac{85}{10} \times 10^4} \qquad \text{(equation (18))} \\ &= \frac{10027}{85000} \end{aligned}$$

Thus the division of 1.0027 by 8.5 is the same as the *whole number* division of 10027 by 85000 in the sense of page 41.

Another example: we claim that the division of 0.025 by 0.00007 is the same as the division of 2500 by 7, because

$$\begin{aligned} \frac{0.025}{0.00007} &= \frac{\frac{25}{10^3}}{\frac{7}{10^5}} \\ &= \frac{\frac{25}{10^3} \times 10^5}{\frac{7}{10^5} \times 10^5} \qquad \text{(equation (18))} \\ &= \frac{2500}{7} \end{aligned}$$

Now if we divide one finite decimal by another, we expect the quotient to be expressed as a finite decimal rather than a fraction. Unfortunately, the quotient will most often not be a finite decimal but an "infinite" one. We will touch on this rather complicated subject in the seventh grade.

Percent

A **percent** is, by definition, a complex fraction whose denominator is 100. The number $\frac{N}{100}$, where N is a fraction, is usually called ***N* percent**. Note that $\frac{N}{100}$ is usually written as $\boldsymbol{N\%}$. Thus "2 percent" means the fraction $\frac{2}{100}$, and "seven-and-a-half percent" means the complex fraction $7\frac{1}{2}/100$.

Students are usually told that "$N\%$ of something" can be interpreted as the totality of N parts when that something is divided into 100 equal parts. This is an intuitive description of what "percent" means, and we now explain how this intuitive description follows logically from our precise definition. For definiteness, consider the claim:

> *"$7\frac{1}{2}\%$ of 512 dollars" means the totality of $7\frac{1}{2}$ parts when we divide 512 dollars into 100 equal parts.*

We back up this claim as follows. Because $7\frac{1}{2}\%$ is a fraction, the definition of fraction multiplication on page 46 implies that

$$
\begin{aligned}
7\frac{1}{2}\% \text{ of } 512 \;=\; 7\frac{1}{2}\% \times 512 \;&=\; \frac{7\frac{1}{2}}{100} \times 512 \\
&=\; 7\frac{1}{2} \times \left(\frac{1}{100} \times 512\right)
\end{aligned}
$$

By the definition of fraction multiplication on page 46 again, $\frac{1}{100} \times 512$ is one part when 512 is divided into 100 equal parts. Therefore the product $7\frac{1}{2} \times (\frac{1}{100} \times 512)$ is

$$7\tfrac{1}{2} \textit{ copies of } \left(\frac{1}{100} \times 512\right),$$

which is therefore

$$7\tfrac{1}{2} \textit{ parts when 512 is divided into 100 equal parts.}$$

This is then the reason why one can make the above claim.

We have given an unambiguous definition of *percent* as a *complex* fraction. On the one hand, it is not just a fraction, but rather, a complex fraction with a denominator of 100. On the other hand, it is a number, and the advantage of knowing that it is a number is that we can now apply all we know about fractions to compute with percents. This realization makes it possible to treat all percent problems in an entirely *routine* fashion. We demonstrate this advantage by doing the four traditional types of problems dealing with percent.

(i) What is 45% of 70?

(ii) Express $\frac{5}{16}$ as a percent.

(iii) What percent of 70 is 45?

(iv) 70 is 45% of what number?

(i) By the definition of fraction multiplication on page 46 , the answer is

$$45\% \times 70 \;=\; \frac{45}{100} \times 70 \;=\; \frac{3150}{100} \;=\; 31.5$$

(ii) If $\frac{5}{16} = C\%$, then $\frac{5}{16} = C \times \frac{1}{100}$. Multiplying both sides by 100 gives

$$C = 100 \times \frac{5}{16} \;=\; 31\frac{1}{4}.$$

So $\frac{5}{16} = 31\frac{1}{4}\%$.

(iii) Let us say 45 is $N\%$ of 70. So by the definition of fraction multiplication on page 46, $45 = N\% \times 70$. Thus $45 = N \times \frac{1}{100} \times 70$, and $N = \frac{45\times 100}{70} = 64\frac{2}{7}$. The answer is therefore $64\frac{2}{7}\%$.

(iv) Let that number be N. Then we are given that $70 = 45\% \times N = N \times 45\%$, and therefore by the definition of division on page 60 ,

$$N \;=\; \frac{70}{45\%} \;=\; \frac{70}{\frac{45}{100}} \;=\; 70 \times \frac{100}{45} \;=\; \frac{7000}{45} \;=\; 153\frac{1}{3}$$

We call attention to the fact that students experience grave difficulties in learning about percent. There may be many reasons, but it is a fact that if students are not told what "percent" is other than some vague idea about "out of a 100", then they are not provided with the necessary tools for learning it. With the definition of percent as a complex fraction, we see that every step of each of the above solutions is based entirely on what we know about fraction multiplication and fraction division. There is no subtle reasoning, and there is no guesswork. There is thus reason to believe that subject of percent can now be routinely teachable.

Ratio

In a vague sense, a *ratio* arises from the *multiplicative comparison* of two numbers.

In order for the preceding sentence to have any mathematical meaning, a lot more needs to be said. Precisely, the **ratio** of A to B, where A and B are numbers, is the division $\frac{A}{B}$. Due to historical reasons, the ratio of A to B is also denoted by $A : B$ (read as ***A* to *B***); it is a strange notation, but you shouldn't let that bother you. Since the only numbers we know at this point are fractions, it is understood that A and B are fractions for now. It is traditional to say things like "the ratio of boys to girls is $3 : 5$", or "the ratio of circumference to diameter of a circle is π", but what is meant is that, respectively, "the ratio of *the number of* boys to *the number of* girls is $3 : 5$" and "the ratio of *the length of* the circumference to *the length of* the diameter is π".

In the standard literature, the precise meaning of "ratio" is rarely given, and even if some meaning is given, it is not clear *mathematically* what it is. We have just given a precise definition, to the effect that a ratio is a number obtained by the division of one number by another. Now the hard work begins: what does the ratio $A : B$ say about A and B, and how is this meaning of "the ratio of boys to girls is $3 : 5$" related to the intuitive, *but imprecise*, meaning of "to every 3 boys there are 5 girls"? Such a precise definition would amount to nothing if we cannot use it to answer these questions.

It is convenient to break up the discussion of ratio *roughly* into two cases: the *discrete* case of two whole numbers, and the *continuous* case of two arbitrary numbers. We will explain the terminology as we go along.

First the **discrete case**, by which we mean *roughly* that the objects we deal with have a "natural smallest unit". This is a vague statement to be sure, but precision will be seen to be unnecessary. For example, there are situations in which we wish to compare two numbers (quantities) that arise from counting, e.g., the number of wings to beaks in the bird house, the number of chairs to legs in a classroom, the number of men to women in an auditorium, etc. In each case, we stop with wings, birds, chairs, legs, etc., and under no circumstance would we raise any question about 'a third of a wing" or "three-sevenths of a chair". Therefore if we refer all measurements to the smallest units, whole numbers are the only numbers we need to use. This is in contrast with a discussion in the **continuous case** where there is no such thing as a "natural smallest unit". Take the comparison of lengths, for instance. There is no such thing as "the smallest unit of length we should use". On a large scale, we would use miles or kilometers, and for objects around us, we might use feet, inches, or tenths of inches. But if you are doing quantum physics, you would think nothing of using angstroms ($1/10^{10}$ of a meter). And so on. Therefore in discussions of the continuous case, we expect to see fractions, decimals, and even arbitrary numbers used. Therein lies the difference between the two cases.

Let us use a discrete example to explain why we want to compare multiplicatively if an additive comparison is so much simpler. For example, if A is the number of legs of all the chairs in the room and B is the number of those same chairs, the usual statement is that the ratio of legs to chairs is $4:1$. So that $\frac{A}{B} = \frac{4}{1}$, or, $A = 4B$. This then gives out the clear information that to each chair there are 4 legs. Now, what would $A - B$ be? The answer is that it all depends on what B is. If there are 10 chairs (so $B = 10$), then $A - B = 30$, but if there are 17 chairs (so $B = 17$), $A - B = 51$, or if there are 21 chairs, $A - B$ would be 63, etc. So giving the difference of $A - B$ fails to convey, in a clear-cut manner, the vital information that there are 4 legs to each chair regardless of what A and B may be. This is one reason why the multiplicative comparison is preferred.

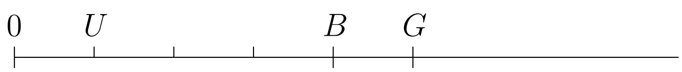
We next address the issue of the *intuitive* content of the ratio concept on the basis of the precise mathematical definition given above. Consider the statement: "In an auditorium, the ratio of boys to girls is $4:5$". Thus let the number of boys and girls be denoted by B and G, respectively. We are given that $\frac{B}{G} = \frac{4}{5}$. Keeping in mind that B and G are just whole numbers, we apply the cross-multiplication algorithm to conclude that the equality is equivalent to

$$\frac{B}{4} = \frac{G}{5}$$

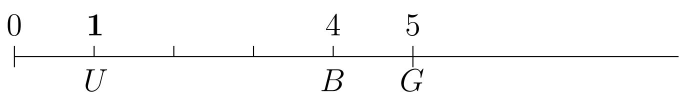
Let the common value of $\frac{B}{4}$ and $\frac{G}{5}$, which is a fraction, be denoted by U, then

$$B = 4U \quad \text{and} \quad G = 5U$$

Now we put B and G on the number line where the unit 1 is "one person". Then we get the following picture.



But this means that if we let U be the new unit on this number line, then $B = 4$ and $G = 5$, as shown:



Usually when B and G are given as specific whole numbers (e.g., 36 boys and 45 girls), it will always be the case that B is a multiple of 4 and G is a multiple of 5, so that U is in fact a whole number. In general, the fact that $\frac{4}{5}$ is a reduced fraction will also imply that U is a whole number.[13] Accepting this fact, we may conclude:

> *Suppose the ratio of the number of boys B to the number of girls G is $4:5$. Then there is a whole number U so that*
>
> $$B = 4U \quad \textit{and} \quad G = 5U$$
>
> *Thus the boys can be divided into 4 equal groups of U boys, and the girls can be divided into 5 equal groups of U girls.*

This is the first interpretation of *ratio* between two whole numbers that gives substance to the vague statement that "there are 4 boys to every 5 girls".

We can go a little further. The above analysis makes use of the partitive interpretation of division $\frac{B}{4}$ and $\frac{G}{5}$ for whole numbers (see page 41). We can also use the

[13]The proof of this fact requires something like the Euclidean Algorithm.

measurement interpretation for the analysis (see page 55). So again let $\frac{B}{4} = \frac{G}{5} = U$. Then

$$B \;=\; U \cdot 4 \quad \text{and} \quad G \;=\; U \cdot 5$$

This means: if we divide the boys into equal groups of 4 and if we divide the girls into equal groups of 5, then there is the same number of groups U in both the boys and the girls. Thus we have arrived at the second interpretation of *ratio* between two whole numbers that also gives substance to the vague statement that "there are 4 boys to every 5 girls":

> *Suppose the ratio of the number of boys B to the number of girls G is* $4:5$. *Then if we divide the boys into equal groups of 4 and divide the girls into equal groups of 5, the number of groups is the same among boys or girls.*

The following ratio problem is standard:

> *In a class of 27 students, the ratio of boys to girls is* $4:5$. *How many are boys and how many are girls?*

There are many ways to solve this problem, but we just give two.

First solution. By multiplying both sides of $\frac{B}{G} = \frac{4}{5}$ by the number G, we obtain $B = \frac{4}{5}G$. Substitute this value of B into $B + G = 27$ to get $\frac{4}{5}G + G = 27$, so that by the distributive law, $(\frac{4}{5} + 1)G = 27$. Thus $\frac{9}{5}\, G = 27$, and

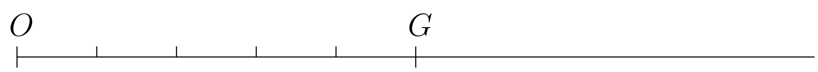
$$G = \frac{27}{\frac{9}{5}} = 27 \times \frac{5}{9} = 15$$

So $B = 27 - G = 12$.

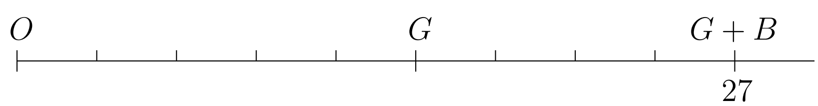
One should always double-check: $\frac{12}{15} = \frac{3 \times 4}{3 \times 5} = \frac{4}{5}$, which is consistent with the ratio of boys to girls being $4:5$.

Second solution. This method makes use of picture-drawing on the number line, and may be the most attractive of all four to a beginner, but before we describe it we should give it some perspective. While it is good to know about solutions that are accessible to picture-drawing, please keep in mind that the kind of symbolic manipulation in the preceding solution is so basic that you have to learn it all, no matter what. So learn about picture-drawing, but learn the other method too.

Again, we start by multiplying both sides of $\frac{B}{G} = \frac{4}{5}$ with G to get $B = \frac{4}{5} \times G$. Here is the critical step: we interpret this equality by making use of the definition of fraction multiplication (page 46). Therefore, $B = \frac{4}{5} \times G$ means B is the number of students in 4 of the parts when the girls are divided into 5 equal parts. Here is a pictorial representation: if the unit of the number line is 1 student, then G is a whole number on this number line and we can divide the segment $[0, G]$ into five equal parts.



Because the length of $[0, B]$ is the concatenation of 4 of the parts above, $G + B$, being the length of the concatenation of $[0, G]$ and $[0, B]$, is the concatenation of $5 + 4$ of these parts, as shown.



Now $G + B = 27$ and $[0, 27]$ is therefore divided into 9 equal parts. Each part then has length 3, and therefore $G = 5 \times 3 = 15$ and $B = 4 \times 3 = 12$, the same as before.

As a reminder, notice that every step of the preceding solution is based on precise definitions or facts that we have thoroughly explained.

Next, we consider the **continuous case** of ratios, so that the quantities A and B are now arbitrary numbers and are no longer just whole numbers. Although 6th graders are not expected to know *precisely* what is meant by "the ratio of circumference to diameter of a circle is π", they probably have heard about it. This is a typical example of the continuous case of ratios to keep in mind. In grade eight, the concept of *similar triangles* will be introduced, and for two triangles that are similar, the ratios of corresponding sides are the same number. A standard way of expressing this fact is that **the corresponding sides of similar triangles are proportional**. Accepting this fact, the following is a standard ratio problem:

> *Two triangles are similar and one side of the first triangle is 3 inches while the corresponding side of the second triangle is 4 inches. If another*

side of the first triangle is 5 inches, how long is the corresponding side in the second triangle?

Let x inches be the length of the side in the second triangle corresponding to the side of 5 inches. Then the sides being proportional means

$$\frac{3}{4} = \frac{5}{x}$$

The cross-multiplication algorithm implies that $3x = 4 \times 5$, and therefore $x = \frac{20}{3} = 6\frac{2}{3}$ inches.

There is a subtle point here. We used the cross-multiplication algorithm, but we have explained that algorithm only for fractions, i.e., only if $\frac{5}{x}$ is a fraction, i.e., only if x is a whole number. But we already know x is not a whole number but a fraction, so $\frac{5}{x}$ is a *complex fraction*. Therefore to do this problem correctly, the cross-multiplication algorithm for complex fractions should be explained first, in the following form:

Let A, B, C, D be fractions ($B \neq 0$ and $D \neq 0$). Then

$$\frac{A}{B} = \frac{C}{D} \quad \textit{implies} \quad AD = BC$$

and conversely,

$$AD = BC \quad \textit{implies} \quad \frac{A}{B} = \frac{C}{D}\,.$$

This is reminiscent of the extension of the fundamental fact about equivalent fractions to complex fractions in equation (18) on page 65.

In general, when it comes to considerations of percent, ratio, and rate, we need to extend all the standard operations from fractions to complex fractions.

Rate

There is at present a lot of confusion in the school classroom and in the education literature about what "rate" means and what exactly "rate" problems are. Therefore an appropriate first step in introducing the concept of *rate* in a school classroom would be to assure students that there is no need to worry about these questions, for the simple reason that there is no good answer to either of them and, moreover, even if we did, neither would promote mathematics learning. Rather, the key issue is that in applying the concept of fraction division to various (idealized) everyday situations, the reasoning underlying some of these applications (such as those related to speed, water flow, lawn mowing, etc.) turns out to be central to mathematics: the reasoning with linearity, the subject of more intense study in grades 7, 8 and beyond. We then see fit to refer to them for convenience as "rate problems". That is all. What we should concentrate on is the reasoning rather the vocabulary.

The discussion of the concept of rate should again be broken up into the discrete case and the continuous case. As in the discussion of ratio, **discrete** in this case means, roughly, that the objects we are dealing with have a "natural smallest unit". Here is a typical example.

> *If half a dozen ballpoint pens sell for $4.92, how much would 10 ballpoint pens cost?*

Observe first of all that if we measure money not by dollars but by cents, then only whole numbers need be used for this problem. We note that *it is implicitly assumed that all ballpoint pens under discussion cost the same amount.* This is a reasonable assumption that students can accept so that, although it is not made explicit, it is always understood. Now, we can solve the problem if we can find out how much each ballpoint pen costs. Let us say it is x dollars. Then we are given that $6x = 4.92$ dollars. So $x = 0.82$ dollars (note: this is a fourth grade problem: if a number when multiplied by 6 gives 492, then the number is 82, so the answer is 82 cents). Therefore 10 pens would cost $10x = 10 \times 0.82 = 8.2$ dollars.

The cost per ballpoint pen, $0.82 in this case, is called the **unit cost** or **unit rate**. For these discrete rate problems, once the unit rate is found, the rest is straightforward.

We now turn to the continuous case. The main purpose is to give a careful elucidation of Standard 6.RP3b:

> b. Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate,

how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?

Let us look at a problem in the continuous case that is similar to the ballpoint pen problem.

Ann walks briskly and covers 3 miles the first hour. How many miles does she cover in 84 minutes?

We imitate the discrete case by first asking: How many miles does Ann walk in a minute? Let us say she walks x miles in a minute. We are given that in 60 minutes (1 hour) she walks 3 miles. Therefore $60x = 3$ and $x = \frac{1}{20}$ miles per minute. In 84 minutes, Ann will walk

$$84x \;=\; 84 \times \frac{1}{20} \;=\; 4\frac{1}{5} \quad \text{miles}$$

There are serious issues with both the problem and the solution. While it is true that all ballpoint pens in the preceding discrete problem are commonly understood to cost the same (this can be ascribed to an abstraction of an almost universal experience), there is no reason to assume that Ann walks exactly the same number of miles *each minute*. It is not just a matter of the different ways people walk, but a *mathematical* problem must be up-front about its explicit assumptions, especially when the assumption involved is so out of the ordinary (does anyone ever walk exactly the same distance every minute?). The crucial observations are therefore:

(1) The problem should at least make explicit the assumption that Ann walks the same distance each minute.

(2) Without this assumption, this problem cannot be solved (e.g., what is to prevent Ann from taking a 30-minute rest after walking 64 minutes, for instance?)

We can say more. Even if we make the assumption that Ann walks the same distance every minute, can we do the following related problem?

Ann walks briskly and covers 3 miles the first hour and she walks the same distance every minute. How many miles does she cover in $84\frac{1}{3}$ *minutes?*

This problem is again unsolvable for the following reason. If she walks 3 miles in 60 minutes and she walks the same distance every minute, then of course she walks $\frac{1}{20}$ of a mile each minute. After 84 minutes, we can say unequivocally that she has walked

$$84 \times \frac{1}{20} \;=\; 4\,\frac{1}{5}\;\text{ miles}$$

However, what she does in the next 20 seconds ($\frac{1}{3}$ minute) after she has walked 84 minutes cannot be determined. Perhaps she takes a rest for 20 seconds before running in the remaining 40 seconds in order to cover the $\frac{1}{20}$ mile that she is supposed to cover in that minute. In that case, she would cover $4\frac{1}{5}$ miles in $84\frac{1}{3}$ minutes. Or perhaps she walks at a steady pace and covers $\frac{1}{60\times 20}$ of a mile each second and, in that case, she would cover

$$4\,\frac{1}{5} + \left(20 \times \frac{1}{60 \times 20}\right) \;=\; 4\,\frac{13}{60}\;\text{ miles}$$

in $84\frac{1}{3}$ minutes.

One could make the preceding problem solvable again by making the explicit assumption that *Ann walks the same distance every 20 seconds*, but clearly we would not be promoting the learning of useful mathematics if we have to make an *ad hoc* assumption each time for problems of this kind. To drive home the point, suppose we change the problem to the following:

> *Ann walks briskly and covers 3 miles the first hour and she walks the same distance every minute. How many miles does she cover in* 84.05 *minutes?*

Should we now change the assumption to: *Ann walks the same distance every* 0.01 *minute?* We are up against the insurmountable problem that there is no "natural smallest unit of time".

In a situation of this nature, the right assumption to make is to say that

> there is a number s so that in *any* time interval of t minutes' duration, Ann walks a distance of ts miles.

If this sounds strange, it shouldn't be because this is the precise meaning of "Ann walks at a **constant speed** of s miles per minute". Indeed, if in the preceding assumption, we let $t = 1$ minute, then we see that Ann walks a distance of $1 \times s$ miles in *any* time interval of 1 minute's duration. The number s is called the **unit rate**;

more precisely, it is s miles per minute. Of course the assumption of constant speed says much more. For example, what distance does Ann walk from 2 minutes (after she starts walking) to 2.24 minutes? The answer is $0.24\,s$ miles because, according to this assumption, Ann walks during this time interval

$$(2.24 - 2)s \;=\; 0.24\,s \quad \text{miles.}$$

It is often not realized that it takes something this sophisticated to make sense of "constant speed". It does not help that textbooks almost never give a definition of this concept.

Let us finally formulate the problem correctly and solve it correctly.

> *Ann walks briskly and covers 3 miles the first hour and she walks at a constant speed. How many miles does she cover in* 84.05 *minutes?*

Let us say Ann's constant speed is s miles per minute and she starts at time 0 (minute). In the time interval from 0 minutes to 60 minutes (1 hour), we are given that she walks 3 miles. But the assumption of constant speed also says that in 60 minutes, she walks $60s$ miles. Therefore

$$60s \;=\; 3, \quad \text{and therefore} \quad s \;=\; \frac{1}{20} \quad \text{miles per minute.}$$

Once more, by the assumption of constant speed, in 84.05 minutes, Ann walks $84.05\,s$ miles, which is then

$$84.05 \times \frac{1}{20} \;=\; 4.2025 \quad \text{miles.}$$

"Rate" is a much-feared topic in schools because it is usually taught incorrectly, which gives students the sinking feeling that, even if they can follow directions and get correct answers, they don't know what is going on. There seems to be no awareness of the fact that it is not possible to properly discuss the concept of rate without calculus. Therefore this concept is thrown around in middle school as if it were something all students already understood thoroughly. The disjunction between an unwarranted assumption and the cold reality of what students don't know inevitably brings about non-learning. Teachers (and of course textbooks and education documents) have to practice self-control and limit themselves to only discussing "constant rate". But as we have seen, even the latter is not an easy concept to get across to students.

We should mention that there is another way to approach constant rate, which may be a trifle more intuitive: First defines *average rate over a time interval* and then defines "constant rate" as the situation where the average rate over *any* time interval is always the same number. Either way, students need careful explanations.

7TH GRADE

Analyze proportional relationships and use them to solve real-world and mathematical problems. 7.RP

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks 1/2 mile in each 1/4-hour, compute the unit rate as the complex fraction (1/2)/(1/4) miles per hour, equivalently 2 miles per hour.

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2d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Unit rates

This is a continuation of the discussion on the concept of rate started in grade 6 (page 74). Consider the following problem:

> *If a person walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, compute the unit rate associated with the ratio $\frac{1}{2} : \frac{1}{4}$.*

By definition, the unit rate associated with $\frac{1}{2} : \frac{1}{4}$ is just the quotient,

$$\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{1}{2} \times \frac{4}{1} = 2 \quad \text{miles per hour.}$$

Now consider a related problem:

> *If Claudio walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, how far does he walk in $\frac{1}{8}$ hour?*

There is not enough information to do this problem. One possibility is that Claudio rests during the first $\frac{1}{8}$ of an hour and then walks briskly $\frac{1}{2}$ of a mile in the second $\frac{1}{8}$ of an hour. Then he repeats exactly the same walking pattern in each of the succeeding $(\frac{1}{4})$-hour intervals. It is not difficult to check that in each $\frac{1}{4}$-hour interval, indeed he walks exactly $\frac{1}{2}$ mile. Such being the case, if you look at how far he walks in the first $\frac{1}{8}$ hour, it is 0 miles. If however you look at how far he walks in the second, or fourth, or sixth $\frac{1}{8}$-hour interval, then he walks $\frac{1}{2}$ mile. Now if you want to know how far he walks in the $\frac{1}{8}$-hour interval from the time $\frac{1}{60}$ hour (after he starts) to the time $(\frac{1}{60} + \frac{1}{8})$ hour, then *we genuinely do not and cannot possibly know.*

On the other hand, the following problem is doable, and it illustrates the power, as well as the subtlety, of the concept of constant speed.

> *If a person walks at constant speed, and he walks $\frac{1}{2}$ mile in a $\frac{1}{4}$ hour interval, how far does he walk in $\frac{1}{8}$ hour?*

Let the constant speed be s mph (miles per hour). We are given that $\frac{1}{4}\,s = \frac{1}{2}$ mile. Therefore $s = 2$ mph. His constant speed is therefore 2 mph. Again by the definition of a constant speed of 2 mph, the distance he walks in $\frac{1}{8}$ hour is $\frac{1}{8} \times 2 = \frac{1}{4}$ mile.

Incidentally, it would be instructive to students to have "constant rate" developed in detail in at least one context other than speed. Take water flow, for instance. Here one measures the total volume of water coming out of the source, say a faucet, in a given time interval in terms of gallons (let us say). Then, by definition, the **rate of the water flow is constant** if there is a fixed number r so that the total volume of water coming out of the faucet in a time interval of length t minutes is equal to tr gallons. If $t = 1$ minute, then the definition says that the total volume of water coming out of the faucet in *every* 1-minute interval is r gallons. Thus in this case, the constant rate of this water flow is r gallons per minute.

Fractions and decimals

The question of whether, given a fraction such as $\frac{3}{8}$ or $\frac{2}{7}$, there is a finite decimal equal to it at first sounds mysterious. But when one recalls that a decimal is a fraction with 10^k as denominator for some positive integer k, then one realizes that this is quite reasonable. All it asks for is whether a fraction such as $\frac{3}{8}$ or $\frac{2}{7}$ can be written as a fraction whose denominator is 10^k for some positive integer k. If the denominator of the given fraction is a product of 2's or 5's or both, it is quite obvious how to do this. We simply make use of the fundamental fact about equivalent fractions and the fact that any power of 10 is the product of 2's and 5's to get it done. For example:

$$\begin{aligned}
\frac{1}{2} &= \frac{5\times 1}{5\times 2} = \frac{5}{10} = 0.5,\\
\frac{5}{4} &= \frac{5}{2^2} = \frac{5^2\times 5}{5^2\times 2^2} = \frac{125}{10^2} = 1.25,\\
\frac{3}{8} &= \frac{3}{2^3} = \frac{5^3\times 3}{5^3\times 2^3} = \frac{375}{10^3} = 0.375,\\
\frac{2}{5} &= \frac{2\times 2}{2\times 5} = \frac{4}{10} = 0.4,\\
\frac{34}{25} &= \frac{2^2\times 34}{2^2\times 5^2} = \frac{136}{100} = 1.36,\\
\frac{27}{125} &= \frac{27}{5^3} = \frac{2^3\times 27}{2^3\times 5^3} = \frac{216}{10^3} = 0.216, \quad \text{etc.}
\end{aligned}$$

In particular, we see that $\frac{3}{8} = 0.375$. However, this approach totally breaks down in case the fraction is $\frac{1}{3}$ or $\frac{2}{7}$, for example.

A fraction is said to be **reduced** if its numerator and denominator have no (whole number) common divisor except 1. Thus $\frac{34}{50}$ is not reduced because 2 divides both 34 and 50. It is known that a reduced fraction is equal to a finite decimal exactly when the denominator is a product of 2's or 5's or both. For some simple cases such as $\frac{1}{3}$, we can show directly that it can never be equal to a finite fraction, as follows. Assume that $\frac{1}{3}$ is equal to a finite decimal $\frac{n}{10^k}$, when n and k are positive integers. From $\frac{1}{3} = \frac{n}{10^k}$ and the cross-multiplication algorithm, we get $10^k = 3n$ and therefore

$$\underbrace{99\cdots 9}_{k} + 1 = 3n.$$

Multiply both sides by $\frac{1}{3}$ and use the distributive law, we get

$$\underbrace{33\cdots 3}_{k} + \frac{1}{3} = n,$$

which may be rewritten as

$$\frac{1}{3} = n - \underbrace{33\cdots 3}_{k}.$$

Now the right side is an integer, so $\frac{1}{3}$ is an integer. The assumption that $\frac{1}{3}$ is equal to a finite decimal therefore leads to the conclusion that $\frac{1}{3}$ is an integer. That assumption is thus wrong, and $\frac{1}{3}$ cannot be a finite decimal after all.

The recognition that some fractions cannot be equal to a finite decimals therefore forces us to change our original question. We now ask instead:

> *Given a fraction, can we find a finite decimal that is either equal to it or approximates it as closely as we wish?*

We now present a uniform method that answers this question completely. Precisely:

(i) It produces a finite decimal equal to a given fraction whose denominator divides a power of 10.

(ii) In general, it produces a finite decimal that approximates the given fraction as closely as we want, e.g., for $\frac{2}{7}$, it produces a finite decimal that differs from $\frac{2}{7}$ by less than $1/10^k$ for any chosen positive integer k.

This method will turn out to involve two essential ideas: the long division algorithm for whole numbers, and the product formula for fractions.

Consider problem ***(i)***. It suffices to illustrate the method with $\frac{3}{8}$. Because 8 divides 10^3, we rewrite $\frac{3}{8}$ as

$$\frac{3}{8} \;=\; \frac{10^3 \times 3}{10^3 \times 8} \;=\; \frac{10^3 \times 3}{8} \times \frac{1}{10^3},$$

where the last equality uses the product formula. Because $\frac{10^3}{8}$ is a whole number, $\frac{10^3 \times 3}{8}$ is a whole number—let us call it N for now—so that from the last expression, we see that $\frac{3}{8}$ is equal to

$$N \times \frac{1}{10^3} \;=\; \frac{N}{10^3}$$

and $N/10^3$ is by definition a decimal, as desired.

Let us work out the details. We have

$$\frac{10^3 \times 3}{8} \;=\; 3000 \div 8,$$

according to page 41. By the long division algorithm, we find out that $3000 \div 8 = 375$:

$$\frac{3 \times 10^3}{8} = \frac{3000}{8} = 375$$

$$\begin{array}{cccccc}
 & & & 3 & 7 & 5 \\ \cline{3-6}
8 &) & 3 & 0 & 0 & 0 \\
 & & 2 & 4 & & \\ \cline{3-6}
 & & & 6 & 0 & \\
 & & & 5 & 6 & \\ \cline{3-6}
 & & & & 4 & 0 \\
 & & & & 4 & 0 \\ \cline{3-6}
 & & & & & 0
\end{array}$$

Therefore,

$$\frac{3}{8} \;=\; \left(\frac{10^3}{8} \times 3\right) \times \frac{1}{10^3} = 375 \times \frac{1}{10^3} = \;\frac{375}{10^3}\,,$$

which by definition is the decimal 0.375.

There is an interesting observation to make. We said earlier that because 8 divides 10^3, we rewrote $\frac{3}{8}$ as

$$\frac{3}{8} \;=\; \frac{10^3 \times 3}{10^3 \times 8}.$$

The truth is that the exponent 3 in 10^3 could have been any whole number ≥ 3. If we had rewritten, say,

$$\frac{3}{8} \;=\; \frac{10^7 \times 3}{10^7 \times 8},$$

would it have made any difference? Of course not, because

$$\begin{aligned}
\frac{3}{8} \;=\; \frac{10^7 \times 3}{10^7 \times 8} \;&=\; \frac{10^7 \times 3}{8} \times \frac{1}{10^7} \\
&=\; \frac{(10^3 \times 3) \times 10^4}{8} \times \frac{1}{10^3 \times 10^4} \\
&=\; \frac{(10^3 \times 3)}{8} \times \frac{1}{10^3} \qquad\qquad \text{(equation (7) on page 49)} \\
&=\; 0.375
\end{aligned}$$

Next we tackle ***(ii)***. Again, we illustrate the general reasoning with a specific example, $\frac{2}{7}$. Let us approximate $\frac{2}{7}$ by a finite decimal that is within distance $1/10^7$ of $\frac{2}{7}$ (on the number line). Experience with $\frac{3}{8}$ above suggests that we multiply the numerator and denominator of $\frac{2}{7}$ by 10^7. Then as before, we have:

$$
\frac{2}{7} \;=\; \frac{10^7 \times 2}{10^7 \times 7} \;=\; \left(\frac{2 \times 10^7}{7}\right) \times \frac{1}{10^7} \;=\; \frac{20000000}{7} \times \frac{1}{10^7}
$$

We perform the long division of 20000000 by 7 and obtain the quotient 2857142 and remainder 6:

$$
20000000 = (2857142 \times 7) + 6
$$

$$
\begin{array}{cccccccccc}
 & & & 2 & 8 & 5 & 7 & 1 & 4 & 2 \\ \cline{3-10}
7 &) & \mathbf{2} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \mathbf{1} & \mathbf{4} & & & & & & \\ \cline{3-5}
 & & & 6 & 0 & & & & & \\
 & & & 5 & 6 & & & & & \\ \cline{3-6}
 & & & & 4 & 0 & & & & \\
 & & & & 3 & 5 & & & & \\ \cline{4-7}
 & & & & & 5 & 0 & & & \\
 & & & & & 4 & 9 & & & \\ \cline{5-8}
 & & & & & & 1 & 0 & & \\
 & & & & & & & 7 & & \\ \cline{6-9}
 & & & & & & & 3 & 0 & \\
 & & & & & & & 2 & 8 & \\ \cline{7-10}
 & & & & & & & & \mathbf{2} & \mathbf{0} \\
 & & & & & & & & \mathbf{1} & \mathbf{4} \\ \cline{8-10}
 & & & & & & & & & 6
\end{array}
$$

Therefore,

$$
\begin{aligned}
\frac{2}{7} &= \left(\frac{(2857142 \times 7) + 6}{7}\right) \times \frac{1}{10^7} \\
&= \left(2857142 + \frac{6}{7}\right) \times \frac{1}{10^7} \\
&= \frac{2857142}{10^7} + \left(\frac{6}{7} \times \frac{1}{10^7}\right) \qquad \text{(distributive law)}
\end{aligned}
$$

so that using the definition of decimal notation, we get:

$$
\frac{2}{7} \;=\; 0.2857142 + \left(\frac{6}{7} \times \frac{1}{10^7}\right)
$$

Since

$$
\frac{2}{7} - 0.2857142 \;=\; \left(\frac{6}{7} \times \frac{1}{10^7}\right),
$$

we see that the difference between the fraction $\frac{2}{7}$ and the finite decimal 0.2857142 is

$$\frac{6}{7} \times \frac{1}{10^7},$$

which is at most $1/10^7$ because $\frac{6}{7}$ is at most 1 (cf. the comment on fraction multiplication on page 46). Thus, the finite decimal 0.2857142 is the desired finite decimal; it is called **the 7-digit decimal expansion of $\mathbf{\frac{2}{7}}$**. Clearly, this reasoning is perfectly general and is independent of the fraction $\frac{2}{7}$ or the choice of the integer k in ***(ii)*** above.

Here is a key observation: *If we take more than 6 decimal digits, the 285714 block of digits will repeat, and this repeating phenomenon is true of all the finite decimals that approximate a fraction in this sense.*

Let us first explain why, for example, if we take a finite decimal of 18 decimal digits that differs from $\frac{2}{7}$ by no more than $1/10^{18}$, then it will be

$$0.285714 \;\; 285714 \;\; 285714$$

This is because the two-by-two boldface block $\begin{matrix} 2 & 0 \\ 1 & 4 \end{matrix}$ is seen to repeat in the preceding long division of 20000000 by 7. By the nature of long division, the digits in the quotient corresponding to—and following—this block must also repeat. Thus we see that the digit 2 in 285712 is being repeated.

To understand in general why this repeating behavior must show up each time we approximate a fraction by a finite decimal (in the sense described), we have to describe the long division algebraically. A routine inspection will reveal that the sequence of steps in the preceding long division of 20000000 by 7 is captured completely by the following sequence of divisions-with-remainder:

$$\left\{ \begin{array}{rcl} 20 & = & (2 \times 7) + 6 \\ 60 & = & (8 \times 7) + 4 \\ 40 & = & (5 \times 7) + 5 \\ 50 & = & (7 \times 7) + 1 \\ 10 & = & (1 \times 7) + 3 \\ 30 & = & (4 \times 7) + 2 \\ 20 & = & (2 \times 7) + 6 \end{array} \right. \qquad (19)$$

The *algorithmic nature* of this sequence of divisions-with-remainder should be pointed out: the dividend (first number) of each line is 10 times the remainder of the preceding one. Moreover, the quotient can be read off by going down the first number of the right side of each line, *vertically*: 2857142. There is another noteworthy feature of this sequence of division-with-remainders, however, and it is the fact that the divisor in each line is always 7. Therefore, the remainder (the last number on each line) of each such division-with-remainder can only be one of seven numbers: 0, 1, 2, 3, 4, 5, 6. This is because the remainder in a division-with-remainder must be a whole number smaller than the divisor. As a consequence, if we are presented with 8 remainders in such a sequence of divisions-with-remainder, then at least two of them must be the same. It follows that after at most seven such divisions-with-remainder, the remainder of the next division-with-remainder must be one of these seven. The minute there is such a repetition, the algorithm dictates that division-with-remainder in the next line has to be a repetition of the earlier division-with-remainder. Thus in the set of equations (19), the fact that the remainder of the last (and seventh) line, which is 6, repeats the reminder of the first line[14] dictates that the next division-with-remainder, were we to continue, will have to be exactly the same as the second line of the set of equations (19):

$$60 \;=\; (8 \times 7) + 4$$

This is why the digits in the quotient of 2×10^{18} divided by 7 must repeat. (We should add that, in the case at hand, not only does the remainder 6 of the last line of the set of equations (19) repeat the remainder of the first line, but by luck, the dividend of the last line (which is 20) already repeats the dividend of the first line. This is why we see the repetition of the two-by-two boldface block $\begin{matrix} 2 & 0 \\ 1 & 4 \end{matrix}$ as well as the repetition of the 2 in 285714.)

A little reflection will show that this reasoning does not depend on the specific numbers 2 and 7 and is perfectly general. We may therefore conclude that if we approximate a fraction $\frac{m}{n}$ by a decimal of at least $2n$ decimal digits, then we will see the phenomenon of a repeating block of digits. Informally we express this phenomenon by saying that "*a fraction is equal to an infinite repeating decimal*". In advanced

[14]In this particular case, the fact that the first number (which is 20) of the seventh line is exactly the same as the first number of first line is an accident; this is why we do not use that as the basis of our argument.

university mathematics classes, the meaning of an "infinite decimal" will be elucidated and then this phrase will acquire a precise meaning.