From arithmetic to algebra

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Why can't our students achieve introductory algebra?

This presentation specifically addresses only introductory algebra, which refers roughly to what is called **Algebra I** in the usual curriculum. Its main focus is on all students' access to the truly basic part of algebra that an average citizen needs in the high-tech age. The content of the traditional Algebra II course is on the whole more technical and is designed for future STEM students.

In place of Algebra II, future non-STEM would benefit more from a mathematics-culture course devoted, for example, to an understanding of probability and data, recently solved famous problems in mathematics, and history of mathematics.

At least three reasons for students' failure:

(A) Arithmetic is about computation of specific numbers.
Algebra is about what is true in general for **all** numbers, **all** whole numbers, **all** integers, etc.

Going from the specific to the general is a giant conceptual leap. **Students are not prepared by our curriculum for this leap.**

(B) They don't get the foundational skills needed for algebra.

(C) They are taught incorrect mathematics in algebra classes. *Garbage in, garbage out.*

These are not independent statements. They are inter-related.

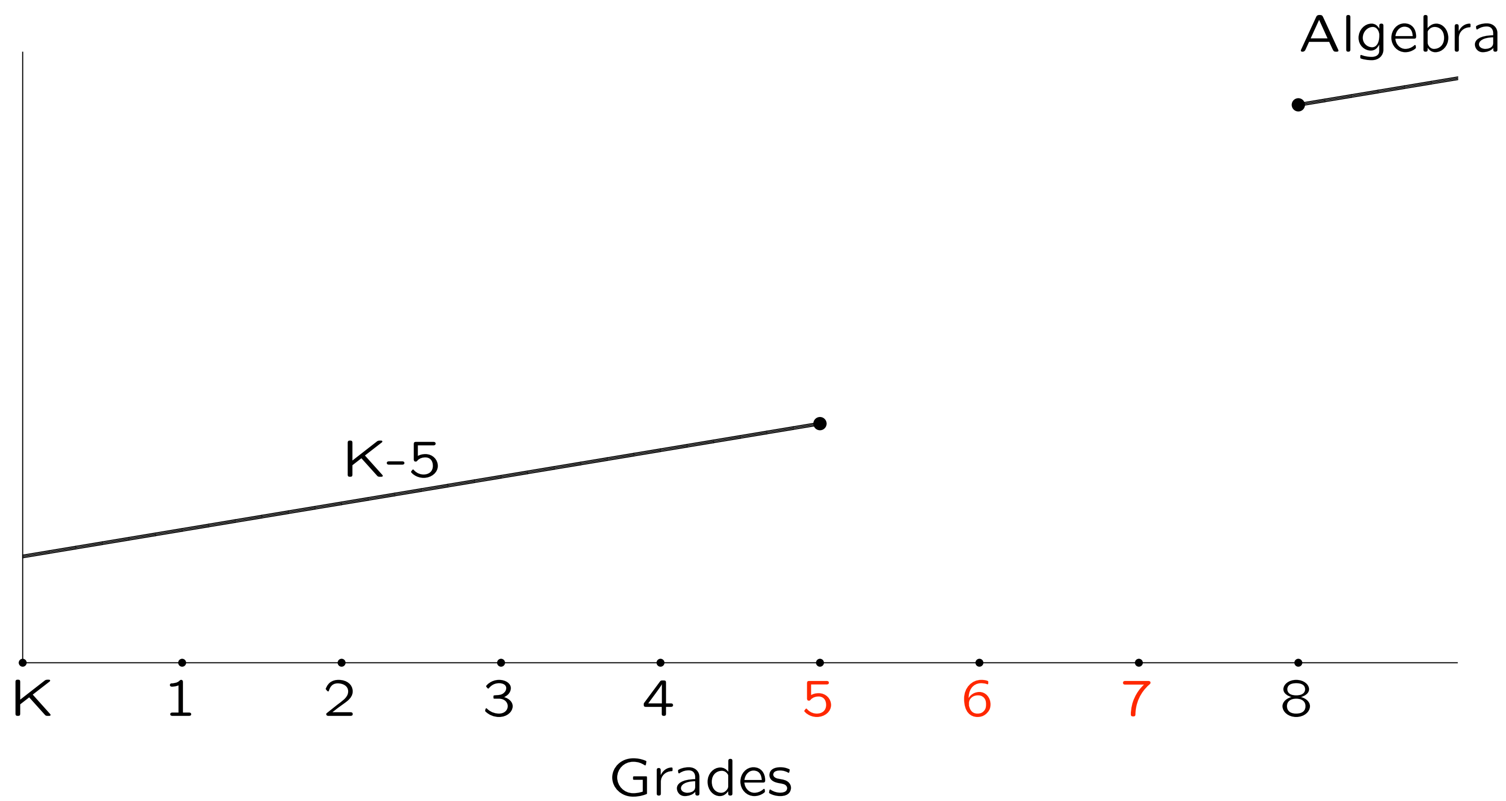
Consider (A) and (B):

The K–3 school math curriculum is *mainly* exploratory, and will be ignored in this presentation for simplicity.

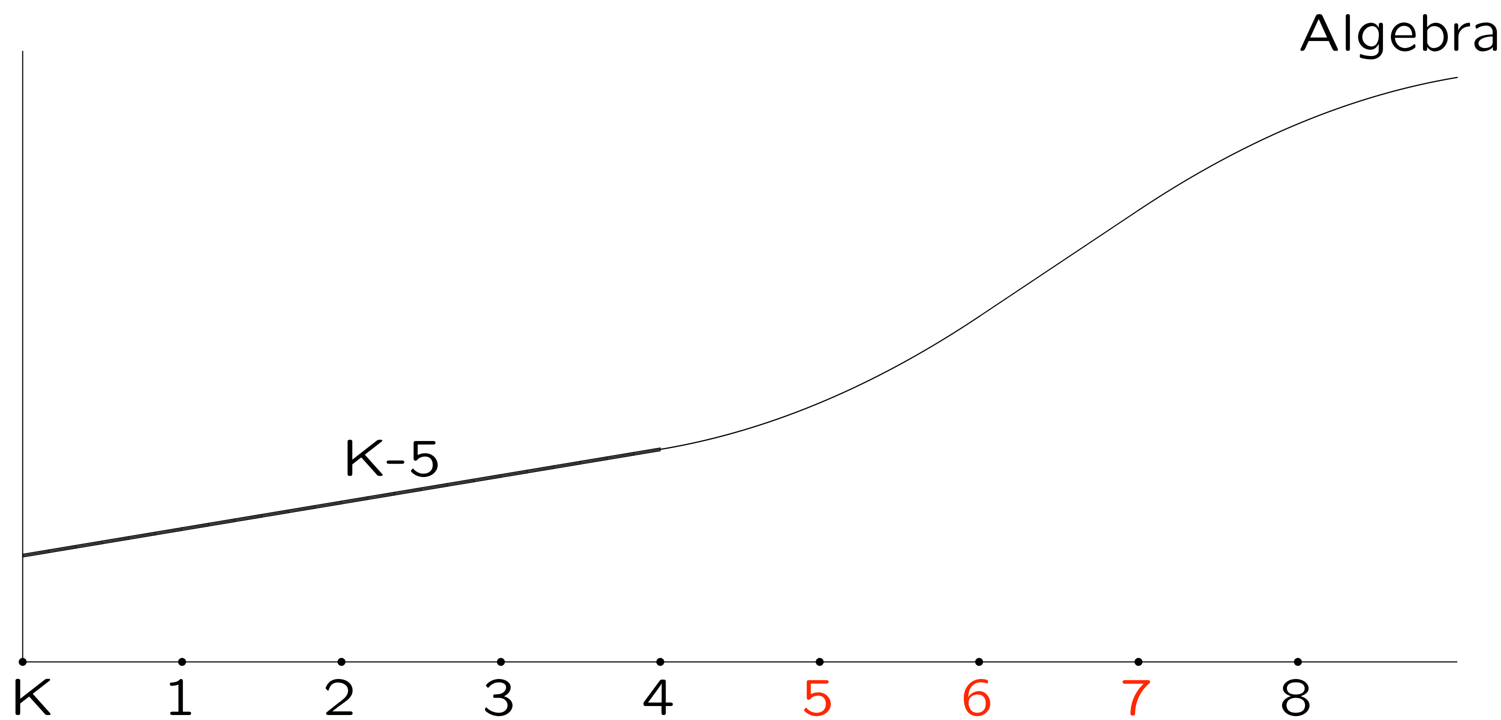
Grades 5–7 directly prepare students for algebra. Will focus on these grades.

Here, abstract mathematics appears in the form of *fractions*, *geometry*, and especially *negative fractions*. (If you have any doubts about why geometry is abstract, try defining a *polygon* correctly.)

Graphically, we can present the situation this way:



To go from grade 5 to grade 8, one **can** gradually elevate the level of sophistication to give students a smooth transition:



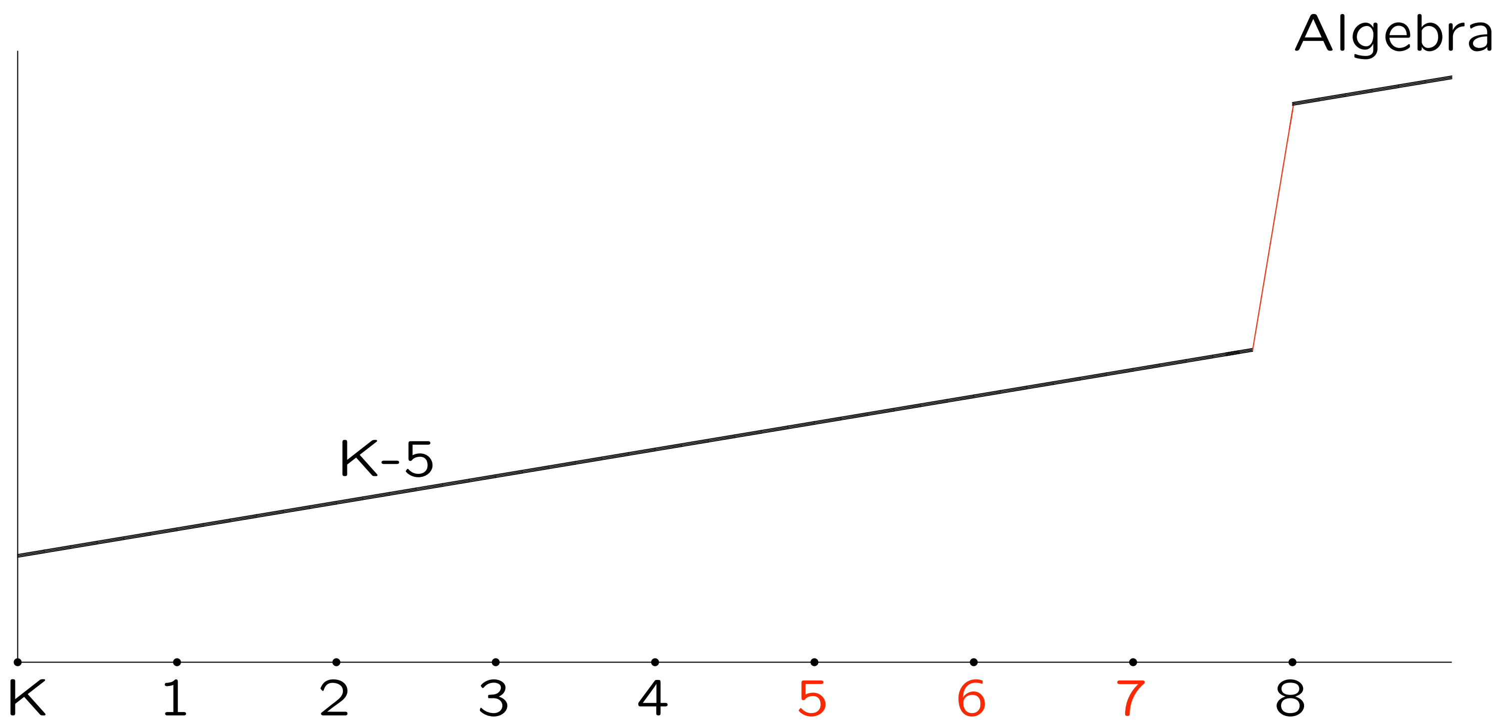
Grades 5–7 are about **fractions, negative numbers,** and **basic geometry** (area, length, congruence, and similarity).

Ample opportunity for the introduction of precision and abstraction to prepare students for algebra.

However, the current 5–7 curriculum chooses to dumb down the mathematics and **replace** precise reasoning and abstraction with hands-on activities, picture-drawings, analogies, and metaphors, with emphasis on "replace". (More on this later).

This is an **artificially depressed curriculum**.

Implicit curricular message If students cannot negotiate the steep climb to algebra in grade 8, *that is their problem!*



Focus on (A).

Arithmetic: computes with specific numbers.
Algebra: introduces concepts of generality and abstraction (they go hand-in-hand; cannot be separated).

A typical computation in arithmetic: students are asked to check

$$(1-3)\,(1+3+3^2+3^3+3^4) \;=\; 1-3^5$$

Or,

$$(1-\tfrac{1}{2})\,(1+\tfrac{1}{2}+(\tfrac{1}{2})^2+(\tfrac{1}{2})^3) \;=\; 1-(\tfrac{1}{2})^4$$

Or,

$$(1-\tfrac{2}{3})\,(1+\tfrac{2}{3}+(\tfrac{2}{3})^2+(\tfrac{2}{3})^3+(\tfrac{2}{3})^4+(\tfrac{2}{3})^5) \;=\; 1-(\tfrac{2}{3})^6$$

In algebra, the corresponding problem becomes: for **all** numbers x (positive or negative) and for **all** positive integers n, show

$$(1-x)(1+x+x^2+\cdots+x^{n-1}+x^n) \;=\; 1-x^{n+1}$$

The skills that lead to accurate computation of, say,

$$1+\tfrac{2}{3}+(\tfrac{2}{3})^2+(\tfrac{2}{3})^3+(\tfrac{2}{3})^4+(\tfrac{2}{3})^5$$

cease to be helpful, because the number x can now assume an infinite number of values.

What they learn in algebra is that, by broadening their narrow focus on stepwise numerical accuracy to an overall strategic accuracy in the use of the abstract *associative laws, commutative laws,* and the *distributive law*, they can arrive at the general statement **much more easily**.

Thus, in going from arithmetic to algebra, students must acquire a different mindset.

The concern with the *numerical value* of each computation,

$$1+\tfrac{2}{3}, \quad 1+\tfrac{2}{3}+(\tfrac{2}{3})^2, \quad 1+\tfrac{2}{3}+(\tfrac{2}{3})^2+(\tfrac{2}{3})^3, \quad \ldots$$

now yields to the concern with how to apply the abstract laws of operations correctly and judiciously.

This is a vastly different conceptual landscape.

Currently, students are not given the opportunity to prepare for this change of landscape in grades 5–7, e.g., the importance of the laws of operations is not sufficiently emphasized.

Next, consider the teaching of fraction addition.

Some typical additions:

$$\begin{aligned}
\frac{5}{6}+\frac{1}{8} &= \frac{20+3}{24} \\
\frac{11}{15}+\frac{3}{20} &= \frac{44+9}{60} \\
\frac{8}{21}+\frac{5}{14} &= \frac{16+15}{42}
\end{aligned}$$

Key ingredients:

only whole numbers appear in numerators and denominators, get LCD.

In algebra, add rational expressions, e.g., for a number x,

$$\frac{x^2-4}{2x-3} + \frac{7x}{x^4+1}$$

Now numerators and denominators are almost never whole numbers (e.g., $x = \frac{3}{11}$).

Students either put away what they learned about fraction addition and learn rational expressions as another rote skill,

or, they have to see each of x^2-4, $2x-3$, $7x$, x^4+1 as a number, and for **any** numbers A, B, C, D, learn to add like this by ignoring LCD:

$$\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}$$

Thus

$$\frac{x^2-4}{2x-3} + \frac{7x}{x^4+1} = \frac{(x^2-4)\,(x^4+1) \;+\; (7x)\,(2x-3)}{(2x-3)(x^4+1)}$$

This is part of the abstract thinking needed in algebra, but the usual way of teaching fraction addition does not promote such abstract thinking.

If students are taught fraction addition **correctly**, their learning curve in algebra would be far less steep.

Final example. Simple computations in arithmetic:

$$\begin{aligned}
2\,(4) \quad &- \; 3\,(5) &= \; ? \\
2\,(\tfrac{-1}{2}) \; &- \; 3\,(2) &= \; ? \\
2\,(\tfrac{1}{3}) \quad &- \; 3\,(\tfrac{23}{9}) &= \; ? \\
2\,(-4) \; &- \; 3\,(\tfrac{-1}{3}) &= \; ?
\end{aligned}$$

Their common answer is -7.

Algebra looks instead at **all** solutions of $2x - 3y = -7$. What properties does the set of **all** solutions of the equation possess?

This is a higher level of mathematical thinking.

How teaching fractions correctly would promote transition to algebra.

The concept of a fraction is itself an abstraction.

If we stop telling students that a fraction is a piece of pizza and, instead, introduce them to precise definition as a point on number line, they will learn to reason with definitions and acquire logical, abstract thinking in the process.

At present, fraction has NO definition, and therefore no reasoning is possible in teaching fractions. *Artificially depressed curriculum.*

With precise definition of fraction addition, students learn how to **prove**

$$\frac{a}{b} + \frac{c}{d} \;=\; \frac{ad + bc}{bd}$$

for **all** fractions $\frac{a}{b}$ and $\frac{c}{d}$. Students are exposed to use of symbols and generality. They also learn to ignore LCD even when a, b, c, d are whole numbers. Helps learning of rational expressions.

Similarly, with precise definition of multiplication of fractions, they learn how to **prove**

$$\frac{a}{b} \times \frac{c}{d} \;=\; \frac{ac}{bd}$$

for **all** fractions $\frac{a}{b}$ and $\frac{c}{d}$. This formula is a rote skill at present.

Another example of the benefit of teaching fractions correctly:

Theorem: *For all fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$,

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$

This is the **cross-multiplication algorithm.** When fractions are precisely defined, students learn to **prove** this theorem, and learn about a property true for **all** fractions.

Because a fraction has no definition at present, this theorem also degenerates into a rote skill.

To summarize: Fractions are the main topic of the math curriculum in grades 5–7, and therefore **naturally** interpolate between the concrete arithmetic of whole numbers and the abstract considerations of algebra.

If they are taught with mathematical integrity, they provide the needed gentle introduction to abstract thinking, use of symbolic notation, and concept of generality. This is one reason why teaching of fractions must be improved if students are to achieve eighth grade algebra.

This fact has nothing to do with educational research, and everything to do with a basic understanding of the structure of mathematics. Mathematics education must respect this structure.

To understand (B) and (C) (what we fail to teach students in grades 5-8), consider the set of all solutions of previous equation $2x - 3y = -7$.

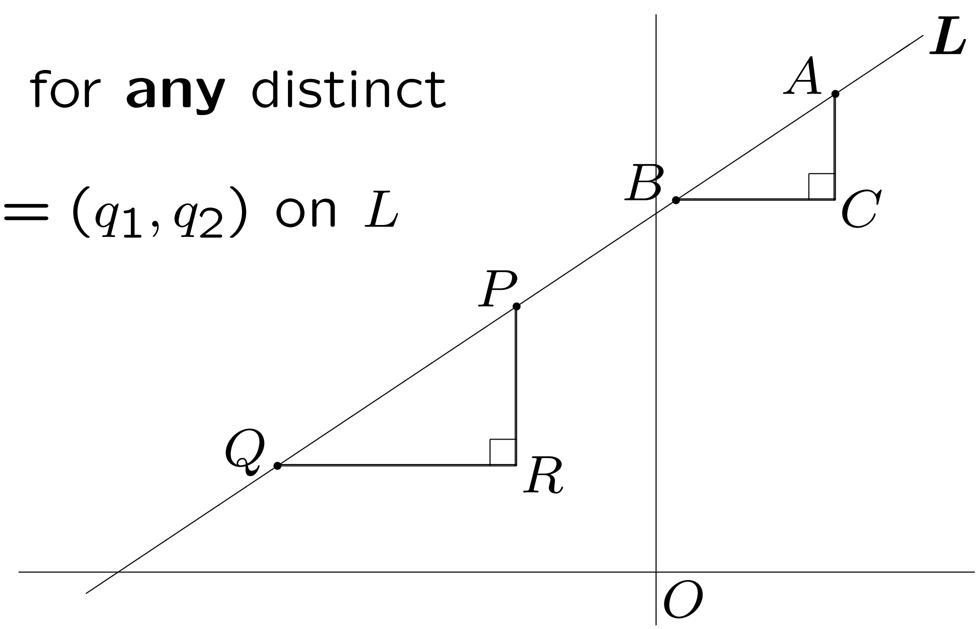
The set of all solution is a line. Algebra classes almost never explain **why**. First reason of this failure: *slope of a line is almost never correctly defined.*

Slope of L is $\dfrac{p_2 - q_2}{p_1 - q_1}$ for **any** distinct

$P = (p_1, p_2)$ and $Q = (q_1, q_2)$ on L

So **must prove**

$$\frac{p_2 - q_2}{p_1 - q_1} = \frac{a_2 - b_2}{a_1 - b_1}$$



Proof requires concept of similar triangles, *almost never adequately taught in grades 6–7*.

Why important to have a correct definition of slope:

(1) Eliminates conceptual sloppiness, promotes clear thinking.

(2) Makes possible the right way to think of slope: given line, can compute slope by choosing the two points that best suit your purpose.

(3) Eliminates need to memorize different forms of equation of a line: two-point form, point-slope form, slope intercept form, standard form. (*Anecdotes abound for this horror*)

Why graph of $2x - 3y = -7$ is a line (Same proof in general)

Step 1: Two lines with same slope, passing same point, are the same line.

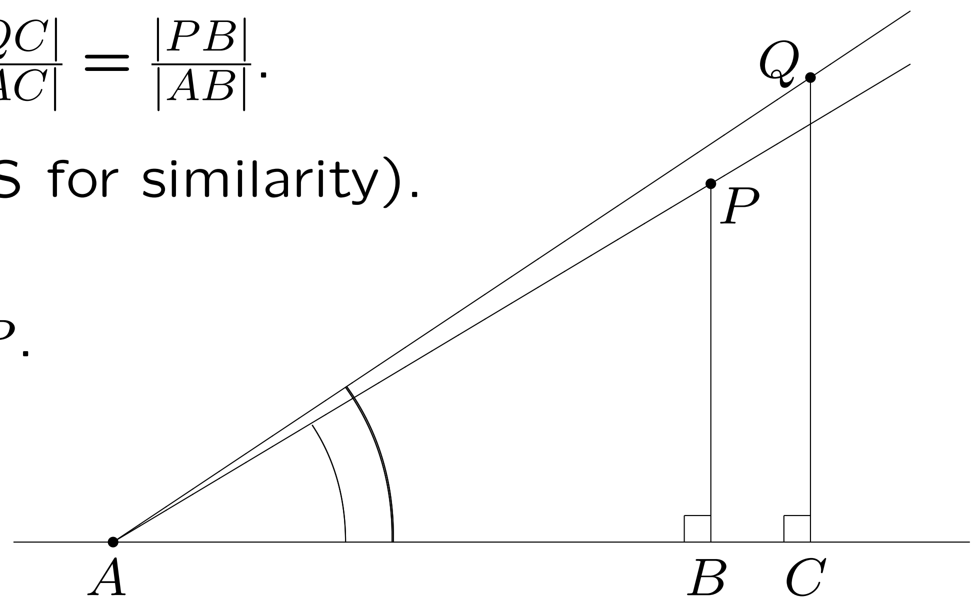
Let line AP, line AQ have same slope. Will prove lines coincide.

Same slope implies $\frac{|QC|}{|AC|} = \frac{|PB|}{|AB|}$.

$\triangle APB \sim \triangle AQC$ (SAS for similarity).

$|\angle PAB| = |\angle QAC|$.

So line $AQ =$ **line** AP.



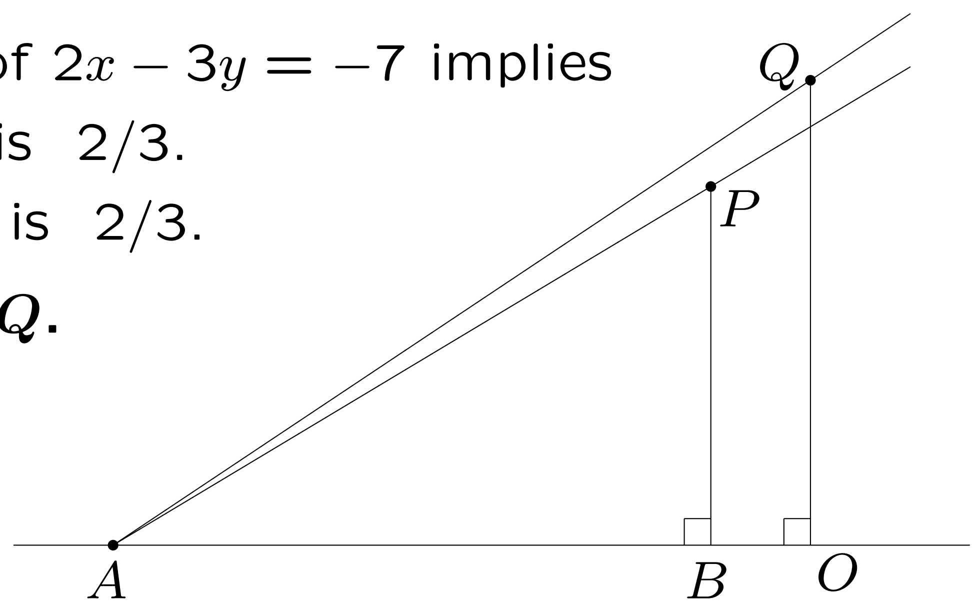
Step 2: Let $A = (a_1, a_2)$, $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be solutions of $2x - 3y = -7$. Suffices to prove line AP and line AQ coincide.

A, Q being solutions of $2x - 3y = -7$ implies

slope of line AQ is $2/3$.

Likewise, slope of AP is $2/3$.

So line $\boldsymbol{AP}$ = line $\boldsymbol{AQ}$.



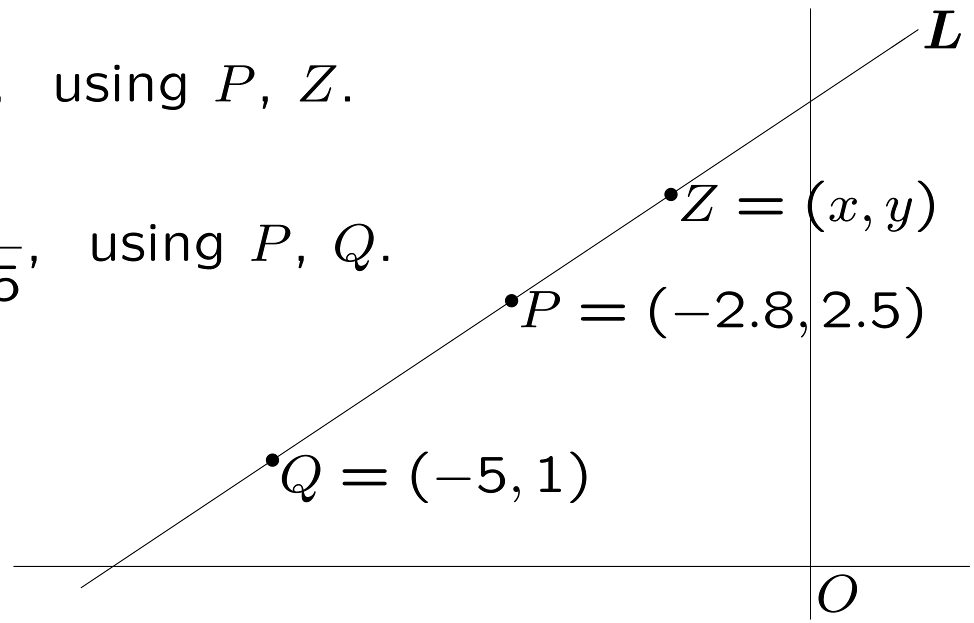
Example Find equation of line L passing through $P = (-2.8, 2.5)$ and $Q = (-5, 1)$.

Let $Z = (x, y)$ be an arbitrary point on L. What equation do these x and y satisfy?

Slope of L is $\dfrac{y - 2.5}{x + 2.8}$, using P, Z.

Slope is also $\dfrac{2.5 - 1}{-2.8 + 5}$, using P, Q.

So $\dfrac{y - 2.5}{x + 2.8} = \dfrac{1.5}{2.2}$.



Further discussion of the incorrect mathematics taught in algebra would center on the lack of definitions for central concepts (e.g., slope of line, graph of equation, half-planes, constant rate, etc.), lack of reasoning (e.g., why graph of linear inequality is half-plane), and the promotion of basic misconceptions. Will only give two examples of the last.

Proportional reasoning (PR). It is regarded as a unifying theme in middle school. It is about "understanding the underlying relationships in a proportional situation and working with these relationships". The problem with the teaching of PR lies in what constitutes this "understanding".

Example (taken from a standard assessment series) John's grandfather enjoys knitting. He can knit a scarf 30 inches in 10 hours. He knits for 2 hours each day.

a. How many inches can he knit in 1 hour?
b. How many days will it take Grandpa to knit a scarf 30 inches long?
c. How many inches long will the scarf be at the end of 2 days? Explain how you figured it out.
d. How many hours will it take Grandpa to knit a scarf 27 inches long? Explain your reasoning.

Without the assumption that Grandpa knits at a constant rate, this problem cannot be solved. At present, students are supposed to understand that, of course, Grandpa knits at a constant rate.

Have you ever seen anyone knit at a constant rate?

Worse: We do not teach *precise* definition of **constant rate**. We **withhold** the *necessary* assumption of constant rate that makes a problem solvable. And yet we expect students to solve these problems.

This is NOT how mathematics should be taught.

What we need is a more reasonable curriculum that:

(i) teaches the concept of constant rate in grades 6–7, and teaches how to use the *constancy* to solve problems; constant *speed*, for example, means *no matter what the time interval from* s *to* t *may be*,

$$\frac{\text{distance traveled from time } s \text{ to time } t}{t - s} = \text{fixed constant } v$$

(ii) teaches in algebra the relationship between constant rate and linear functions;

(iii) teaches in algebra that PR is about problems that can be modeled by linear functions without constant term ($f(t) = vt$);

(iv) always informs students explicitly which rate problems are modeled by linear functions (i.e., which problems are about constant rate);

(v) if it wants students to learn *when* PR is applicable, supplies numerical data and asks for a mathematical judgment (i.e., never give a problem that require students to guess what the assumptions are).

We cannot afford to confuse conceptual understanding with pure guesswork. PR problems related to rates must make explicit the assumption of constancy.

Concept of variable and the use of symbols

We are told that algebra begins with the use of "variables".

> Working with variables and equations is an important part of the middle-grade curriculum. Students' understanding of variable should be far beyond simply recognizing that letter can be used to stand for unknown numbers in equations.

So what is a "variable"? Here are two among many answers from textbooks.

I

Variable is a letter or other symbol that can be replaced by any number (or other object) from some set. A **sentence** in algebra is a grammatically correct set of numbers, variables, or operations that contains a verb. Any sentence using the verb $=$ (is equal to) is called an **equation**.

A sentence with a variable is called an **open sentence**. The sentence $m = \frac{s}{5}$ is an open sentence with two variables...

II

A **variable** is a quantity that changes or varies. You record your data for the variables in a table. Another way to display your data is in a coordinate graph. A **coordinate graph** is a way to show the relationship between two variables.

[To make a coordinate graph] in many cases you can determine which variable to assign to which axis by thinking about how the two variables are related. Does one variable depend on the other? If so, put the **dependent variable** on the y-axis and the **independent variable** on the x-axis. . .

SOME FACTS

(1) In mathematics, there is no *concept* called a "variable". No mathematical object ever "varies" or "changes". Moreover, "open sentence" is not a basic part of mathematics or symbolic logic literature, just as NBA Dunking Contest is not a basic part of learning how to play basketball.

A **suggested mental image** of a "varying quantity" should not be elevated to the status of a *concept* in mathematics proper.

(2) The concept of a "variable" is considered to be part of mathematics only in school mathematics and in 19th century mathematics literature. The term "variable" *is* used in mathematics *informally* to refer to symbols in equations or elements in the domain of a function.

(3) This misconception about "variables" is not limited to textbooks. In a 2008 Special Issue of the AERA journal, **Educational Researcher**, devoted to the NMP report, a commentator remarks that the report "completely ignores algebra as a preparation for calculus, which would entail strong emphasis on variable as varying magnitude, ... Even successful university calculus students have difficulty solving problems that depend on understanding ideas such as varying magnitude . . ."

When we as a community fail to teach students correct mathematics, they are bound to come back to haunt us with the same misconceptions and misunderstanding. This should be incentive enough for us to collectively do a better job.

(4) **The most basic aspect of the learning of algebra is the proper use of symbols.** Instead of defining what a "variable" is, mathematics requires that there be a precise description of what a symbol stands for each time it is employed. This is the basic protocol of the use of symbols.

(5) Asking students to interpret what $\sqrt{6x-5}$ means without saying what x is is not a productive method of teaching algebra, just as asking students to reply to "Is he 6 foot 5?" without providing any context is not a good way to teach English. *Who is 'he'?"*

Some examples of proper use of symbols

If $f(x)$ denotes the amount of food (in pounds) the first x horses eat per day, then x is an integer from 1 to 25 (25 horses are being studied). If $g(n)$ is the amount of food (in pounds) the first n toddlers eat per day, then n denotes an integer from 1 to 48 (a group of 48 toddlers are under observation).

We assert $g(x) < f(x)$ for all x from 1 to 25. The only way to check this is to verify, integer by integer, that $g(1) < f(1)$, $g(2) < f(2)$, ..., $g(25) < f(25)$. There is NO consideration of the "varying quantity" x.

Some examples (cont.).

"A linear equation in two variables $ax + by = c$" means precisely:

Let a, b, c be specific numbers. Consider all the ordered pair of numbers (x, y) so that $ax + by = c$.

With this understood, we then refer to a, b, c as **constants**, and x, y as the **variables**.

Some examples (cont.).

"Solving a quadratic equation $ax^2+bx+c=0$" means, precisely:

Let a, b, c be specific numbers. Find **all** *numbers x so that $ax^2+bx+c=0$.*

Such a number x is called a **solution** of the equation. *A priori*, we do not know how many such x there are. The most basic consequence of the *quadratic formula* is that there are exactly two solutions (when complex numbers are allowed and multiplicity is counted).

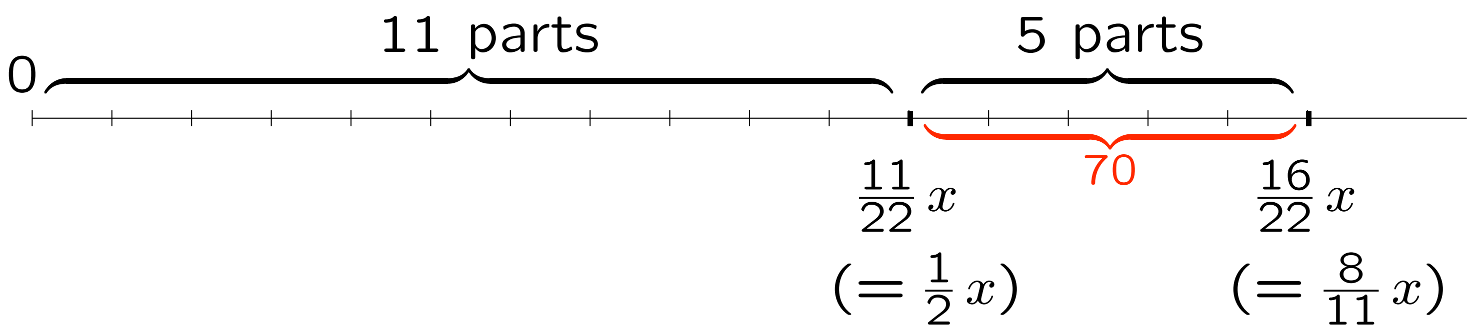
Will solve a word problem in detail, first by arithmetic, and then by algebra:

If $\frac{8}{11}$ of a number exceed half of that number by 70, what could the number be?

Arithmetic solution There may be more than one such number. Fix one of them and call it x.

To compare $\frac{8}{11}x$ with $\frac{1}{2}x$, rewrite them as $\frac{16}{22}x$ and $\frac{11}{22}x$. Now $\frac{16}{22}x$ is the totality of 16 parts when x is divided into 22 equal parts.

Use number line. Divided the segment $[0, x]$ into 22 equal parts, we have the following picture:



Given $\frac{8}{11}x - \frac{1}{2}x = 70.$ There are 5 equal parts in 70, so each part is 14, and since the segment $[0, \frac{1}{2}x]$ contains 11 of these parts, $\frac{1}{2}x$ is $11 \times 14 = 154$. Therefore $x = 308$.

So 308 is the ONLY possible number. Check:

$$\frac{8}{11} \times 308 - \frac{1}{2} \times 308 = 70.$$

The usual algebraic solution: Let x be the number. Given

$$\frac{8}{11}x - \frac{1}{2}x = 70$$

Solve this equation with variable x: $(\frac{8}{11} - \frac{1}{2})x = 70$. Thus

$$\frac{5}{22}x = 70, \quad \text{and} \quad x = \frac{22}{5}70 = 308$$

Does this make sense? What are the rules regarding computations with a variable? How do we know 308 is the solution of this problem, i.e., have we checked that $\frac{8}{11}$ of 308 exceeds half of 308 by 70?

In introductory algebra, there is no need to compute with a "variable", whatever that is.

Instead, we directly compute with numbers, as follows.

Suppose x is a number so that $\frac{8}{11}$ of it exceeds half of it by 70. This does not say we know there is such a number, only that if there is one, we will compute with it. So by assumption, x is a fixed number so that,

$$\frac{8}{11}x - \frac{1}{2}x = 70$$

This is now an equation between two **numbers**.

Can apply distributive law to numbers on left: $(\frac{8}{11} - \frac{1}{2})\,x = 70$, so that $\frac{5}{22}\,x = 70$. Multiply both $\frac{5}{22}\,x$ and 70 by $\frac{22}{5}$:

$$\frac{22}{5} \times \left(\frac{5}{22}\,x\right) \;=\; \frac{22}{5} \times 70$$

Apply associative law to left:

$$\left(\frac{22}{5} \times \frac{5}{22}\right) x \;=\; 308$$

We conclude $x = 308$.

(**This is what the associative, commutative, and distributive laws are about.**)

What have we accomplished?

Not much, merely that IF there is such a number x, then it has to be 308. We are *not* saying we have found a number so that $\frac{8}{11}$ of it exceeds half of it by 70.

This is the same as saying: **IF** there is a unicorn, we know it has a single horn. *Such a statement does not imply that there is a unicorn.*

Nevertheless, we can verify that the number 308 fits the bill:

$$\frac{8}{11} \times 308 - \frac{1}{2} \times 308 = 70 \quad \surd$$

Conclusions:

(1) The usual algebraic solution, when properly deconstructed, is NOT the result of a computation with a "variable", just a computation with ordinary numbers.

(2) The usual algebraic solution does not solve the equation in the sense of finding a solution, only that **if a solution is handed to us, we can find its specific value** (308 in this case).

In principle, the usual algebraic solution must be double-checked each time to verify that it is a solution. But we are saved from such tedium because of a general **Theorem** guarantees it once and for all:

Given fixed numbers a, b, c, d *with* $a \neq c$*, there is one and only one number* x *so that* $ax + b = cx + d$*. In fact,*

$$x = \frac{d - b}{a - c}$$

The proof is exactly the reasoning above.

If $a = \frac{8}{11}$, $b = 0$, $c = \frac{1}{2}$, and $d = 70$, we retrieve the preceding solution.

Summary:

We had no need of employing any heroic measures beyond a proper use of symbols to solve the problem. But we had to compute with numbers, especially fractions, to obtain the solution.

Preceding reasoning gives a clear indication why fluency in computations with fractions is a critical foundation for the learning of algebra.

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