The Mathematics K–12 Teachers Need to Know*

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K–12 math teachers have a dual obligation:

• They must teach mathematics that respects its basic characteristics.

• They must also address the needs of the school classroom, including students' diverse background and mathematical maturity.

> *In the* **New Math**, *for example, they ignored the second.*
>
> *Until very recently, there was a tendency to slight the first.*

Why emphasize that mathematics of the classroom must respect the basic characteristics of mathematics? *Because:*

- Pre-service or in-service professional development generally does not address this issue.

- Textbooks for teachers generally do not concern themselves with this issue.

- Mathematics education itself may slight the importance of the basic characteristics of mathematics when distracted by other concerns (equity, pedagogical strategies, cognitive developments, . . .)

What are the basic characteristics of mathematics? They are not easy to describe, but you'd know it when they are not there.

EXAMPLE. The way **real numbers** are taught in schools is contrary to the spirit of mathematics.

School mathematics is the mathematics of rational numbers. Any excursion into irrational numbers depends on pure extrapolation from the rationals.

E.g. $$\frac{3}{\pi} + \frac{\sqrt{2}}{5.1} = \frac{3 \times 5.1 + \sqrt{2} \times \pi}{\pi \times 5.1}$$

Implicitly, the computation invokes at every turn the

> **Fundamental Assumption of School Mathematics (FASM):** *All the information about arithmetic operations on rational numbers can be extrapolated to all real numbers.*

The use of FASM in school mathematics is good education provided it is made explicit. The fact that FASM is not mentioned in school textbooks or college textbooks for teachers renders the mathematics in those books defective. *It violates the characteristic of precision*, to be discussed next.

The basic characteristics of mathematics

Precision: Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.

Definitions: Bedrock of the mathematical structure. No definitions, no mathematics.

Reasoning: Lifeblood of mathematics. The engine that drives problem solving.

Coherence: Mathematics is a tapestry in which all the concepts and skills are interwoven.

Purposefulness: Mathematics is goal-oriented. It solves specific problems.

These characteristics are not independent of each other.

Students who want to be scientists, engineers, or mathematicians need to know mathematics that respects these basic characteristics.

All students need to know this kind of mathematics if school mathematics education is to live up to its educational potential: *to provide the best discipline of the mind in the school curriculum.*

Key Question: Why must our teachers know this kind of mathematics?

Trivial Answer: If teachers don't know it, then their students won't know it either.

Nontrivial Answer: Teachers who know this kind of mathematics can make themselves better understood, can win students' trust, and can open up mathematics to their students.

Students cannot not learn mathematics if the don't participate in the doing of mathematics.

They will not participate if they believe mathematics is one giant black box to which even their teachers do not have the key.

Teacher can hope to win their students' trust and inspire them to participate only if they can make transparent what they are talking about **(definitions and precision)**, and can explain why students should learn a skill or a concept **(reasoning and purposefulness)**.

I will discuss four examples to show how teachers who know the basic characteristics of mathematics can teach better.

Example 1. **Place value.**

Example 2. **Translations, rotations, reflections.**

Example 3. **The equal sign.**

Example 4. **Fractions, decimals, and percent**.

Example 1. **Place value.**

Consider the number

$$\underline{\underline{7}} \quad 3 \quad \underline{\underline{7}} \quad 5 \quad \underline{\underline{7}}$$

We **tell** students that the three 7's are different, but expect them to have *conceptual understanding* of place value. *The expected outcome is inconsistent with the input.*

Place value is offered as a rule, but it would help if teachers we can *explain* the reason for such a rule **(reasoning)**.

> *Place value is a consequence of the way we CHOOSE TO COUNT.*

We want to count using **only** ten symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. This decision forces us to use more than one position (place) to count to large numbers.

Illustrate with THREE symbols: 0, 1, 2. Counting stops after three steps. To continue, *one way* is to repeat the three symbols indefinitely:

0	1	2	
0	1	2	
0	1	2	etc.

But to keep track of the repetitions, we label each repetition by a symbol to the **left**:

00	**0**1	**0**2
10	**1**1	**1**2
20	**2**1	**2**2

Adding one symbol to the left of 0 1 2 allows us to count up to nine. Then we are stuck again. To keep going, we repeat these nine symbols indefinitely:

00	01	02	10	11	12	20	21	22	
00	01	02	10	11	12	20	21	22	
00	01	02	10	11	12	20	21	22	etc.

but again label each repetition by a symbol to the **left**:

000	**0**01	**0**02	**0**10	**0**11	**0**12	**0**20	**0**21	**0**22
100	**1**01	**1**02	**1**10	**1**11	**1**12	**1**20	**1**21	**1**22
200	**2**01	**2**02	**2**10	**2**11	**2**12	**2**20	**2**21	**2**22

The **convention** is to omit 0's on the left:

0	1	2	10	11	12	20	21	22
100	101	102	110	111	112	120	121	122
200	201	202	210	211	212	220	221	222

This way, students get to see the **origin of place value**: we use three places only after we have exhausted what we can do with two places. Thus the *2* in *2*01 stands not for 2, but the **third** round of counting the NINE two-digit numbers, i.e., the *2* in *2*01 signifies the beginning of the 18th number ($18 = 9 + 9$). Therefore, 201 is the 19th number ($9+9+1$).

In the same way, if we use ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, then

> the *3* in *3*24 signifies the 4th round of counting the 100 two-digit numbers and therefore stands for 300 ($= 100 + 100 + 100$), and 324 is the 300th-and-20th-and-4th number.

When teachers know the underlying reasoning of place value, this content knowledge opens up their pedagogical options. They can explain place value with greater conviction, and they can also allow their students to count with any number of symbols to **see place value for themselves.**

Knowing how to count is fundamental to the *teaching* of whole numbers.

E.g., A teacher who can *explain* the **definition** of **addition** as iterated counting (i.e., $412 + 735$ is the number we get by counting to 735 if we start with 412) can teach the addition algorithm as follows:

> We already know how to add **any** two numbers (just count), but it is hard work. The addition algorithm is a short-cut to add any two numbers by counting only *single digit* numbers.

The same holds for ALL standard algorithms. **This is why they should be taught.**

Example 2. **Translations, rotations, reflections.**

These **basic isometries** are usually taught as means to increase art appreciation: *Look for symmetries in designs! Look for symmetries in nature! Look for symmetries in tessellations!*

Are these basic isometries only good for fun and games, or do they possess mathematical substance yet to be unveiled?

A teacher who knows the **purposefulness** of mathematics and the importance of **definitions** would teach these basic isometries differently, because she knows *mathematically* what they are for.

She would define two figures to be **congruent**, NOT if they have the "same size and same shape", but if a translation, a rotation, and/or a reflection bring one on top of the other.

Congruence therefore becomes a tactile and learnable concept.

She would also use congruence to give correct definitions of **length, area, volume**, thereby exhibiting to her students the fundamental role of these basic isometries in mathematics **(coherence)**.

A teacher who understands the purposefulness of mathematics would always emphasize the reasons to learn a concept or skill. *Knowing the reasons facilitates students' learning process.*

For example, she would make clear, that

- students learn about rational exponents of numbers because they have to deal with exponential functions, and

- the importance of learning about axioms and proofs in geometry is not to do *pro forma* proofs of trivial statements, but to establish conviction about the truth of statements that are nontrivial.

Example of nontrivial Euclidean theorems that belong to every high school geometry course.

• *The three altitudes of a triangle meet at a point.*

• *The line segment joining the midpoints of two sides of a triangle is parallel to the third side, and is half the length of the third side.*

• *Given any three points A, B, C in the plane, then*

$$\textit{dist}(A,B) + \textit{dist}(B,C) \geq \textit{dist}(A,C),$$

and equality holds if and only if A, B, C are collinear and B is between A and C.

Example 3. **The equal sign.**

Education research in algebra has decided that students' defective understanding of the equal sign as

> *an announcement of the result of an arithmetic operation*

rather than as

> *expressing a relation*

is a major reason for their failure to achieve algebra.

It also decides that the notion of "equal" is complex and difficult for students to comprehend.

However, in mathematics, the concept of **equality** is a matter of **definition**. The notion of "equal" is unambiguous and *NOT* difficult to comprehend.

If teachers can emphasize the importance of definitions, and always define the equal sign in different contexts with **precision** and care, any misunderstanding of the equal sign would be the concern of professional development and not of education research.

Here is the list of definitions of $\boldsymbol{A} = \boldsymbol{B}$ that arises in school mathematics:

A and B are expressions in **whole numbers**: both count to same number, or same point on number line (e.g., $A = 2 + 5$, $B = 4 + 3$);

A and B are expressions in **fractions**: same point on number line (e.g., $\frac{1}{2} + \frac{1}{3} = 2 - 1\frac{1}{6}$);

A and B are expressions in **rational numbers**: same point on number line (e.g., $\frac{1}{3} - \frac{1}{2} = 2 - 2\frac{1}{6}$);

A and B are two **sets**: $A \subset B$ and $B \subset A$;

A and B are two **functions**: A and B have the same domain of definition, and $A(x) = B(x)$ for all x in their common domain;

A and B are two **abstract polynomials**: pairwise equality of the coefficients of the same power of the indeterminate;

$A = (a, a')$, $B = (b, b')$ are **ordered pairs** of numbers: $a = b$ and $b = b'$.

If teachers misuse the equal sign as announcement of an answer, or a "call to action", students will follow suit. We need teachers who are aware of the characteristics of **definitions** and **precision** in mathematics.

In particular, we need teachers who do not corrupt students' conception of the equal sign by writing, as textbooks do,

$$27 \div 4 = 6 \quad \text{remainder} \quad 3$$

This literally uses the equal sign as "an announcement of the result of an arithmetic operation."

We need teachers who write instead:

$$27 = (6 \times 4) + 3$$

Example 4. **Fractions, decimals,** and **percent**.

Here we focus on the teaching of these topics in **grades 5 and up.** This is where informal knowledge of fractions begins to give way to a formal presentation, and where students' drive to achieve algebra begins to take a serious turn.

Students are told:

> a **fraction** is a piece of pizza, part of a whole, a division, and a ratio;
>
> a **decimal** is a number obtained by counting hundreds, tens, ones, tenths, hundredths, thousandths, etc.;
>
> a **percent** is *part of a hundred.*

Students are also told to "reason mathematically" using these concepts to solve problems.

A teacher who knows the basic characteristics of mathematics would know that the foundation of mathematical reasoning is clear and correct **definitions**. She would recognize that

> this "definition" of a fraction has too many components, some of them don't make sense, e.g., what is a "ratio"? and how to multiply two pieces of pizza?
>
> if decimal and percent are as described, how to compute with them?
>
> if these are all supposed to be numbers, why are they all different? **(**issue of **coherence)**

The teacher would recognize the need of a serviceable definition of a fraction, e.g., **a point on the number line**, and the need to define decimals and percent as certain kinds of fractions:

> a **decimal** is any fraction with denominator equal to ten, hundred, thousand, etc., (so that 3.52 and 0.0067 are, by definition, $\frac{352}{100}$ and $\frac{67}{10000}$, respectively), and
>
> a **percent** is a fraction of the form $\frac{N}{100}$, where N is a fraction.

Such a teacher can now teach percent problems with ease.

E.g., *what percent of* 76 *is* 88*?*

A similar problem, *what fraction of* 76 *is* 88*?*, is done by writing down, if k is that fraction, $k \times 76 = 88$, so $k = \frac{88}{76} = \frac{22}{19}$.

Since percent is also a fraction, we do the original problem the same way: if N% of 76 is 88, then

$$\frac{N}{100} \times 76 = 88,$$

and $N = \frac{8800}{76} = 115\frac{15}{19}$. Thus, the *answer* is $115\frac{15}{19}$%.

Such a teacher can also teach the *multiplication algorithm* for decimals and the *place value* of decimals with ease.

For example, the multiplication of decimals is reduced to the multiplication of whole numbers **(coherence)**, *because*:

$$\begin{aligned} 2.6 \times 0.105 &= \frac{26}{10} \times \frac{105}{1000} \\ &= \frac{26 \times 105}{10 \times 1000} \\ &= \frac{2730}{10000} \\ &= 0.2730 \end{aligned}$$

She would teach the **place value** of decimals on the basis of the *place value of whole numbers* **(coherence)**:

$$
\begin{aligned}
3.712 &= \frac{3712}{1000} \\
&= \frac{3000 + 700 + 10 + 2}{1000} \\
&= \frac{3000}{1000} + \frac{700}{1000} + \frac{10}{1000} + \frac{2}{1000} \\
&= 3 + \frac{7}{10} + \frac{1}{100} + \frac{2}{1000}
\end{aligned}
$$

Conclusion: Teachers need more than specific pieces of skills or concepts to improve students' achievement in mathematics. They need change in their perception of mathematics as a discipline that embodies the five basic characteristics.

Such a change cannot be accomplished in two-day or three-day workshops. It requires sustained effort over a long period of time.