What is Mathematics Education?*

*Text of a plenary presentation at the 2007 NCTM Annual Meeting, March 23, 2007, in Atlanta. I wish to thank Tom Parker for his many insightful suggestions.

Mathematics education is

$$mathematical \quad engineering$$

I am not making an analogy. I am not using "engineering" as a metaphor.

Rather, I am giving a precise description of what mathematics education really is.

Engineering is the customization of abstract scientific principles to satisfy human needs.

Chemical engineering:

Chemistry $\longrightarrow$ the plexi-glass tanks in aquariums, the gas you pump into your car, shampoo, Lysol, . . .

Electrical engineering:

Electromagnetism $\longrightarrow$ computers, power point, iPod, lighting in this hall, motors, . . .

Striking example of electrical *engineering*:

In 1956, IBM launched the first computer with a hard disc drive. The hard drive **weighed over a ton and stored 5MB of data**.

Today's hard drives weigh only a few ounces and hold 100,000 times as much data.

These hard drives are built on the **same scientific principles**. But 50 years of continuous engineering have created refinements that make them enormously better adapted to the needs of consumers.

Mathematical engineering
(Mathematics education of K–12):

Abstract mathematics	$\longrightarrow$	mathematics that meets the needs of students and teachers in the K–12 classrooom.

This is the job of mathematics educators: How to engineer the abstract mathematics for use by students and teachers in K–12.

Products that engineering $cannot$ produce:

Mechanical engineering: perpetual motion machines, 100% efficient engines.

Chemical engineering: machines that extract oxygen from water without use of energy.

Mathematical engineering: $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$, mathematics that can be learned like whole language.

Engineering cannot cater to *any* human need no matter how scientifically absurd, any more than it should produce any product that is useless though scientifically correct.

Engineering must mediate between two extremes:

(1) **inviolable** scientific principles.

(2) **user-friendliness** of the final product.

What are the inviolable scientific principles in mathematical engineering?

Precision: Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.

Definitions: Bedrock of the mathematical structure (no definitions, no mathematics).

Reasoning: Lifeblood of mathematics; core of problem solving.

Coherence: Every concept and skill builds on previous knowledge and is part of an unfolding story.

Purposefulness: Mathematics is goal-oriented. It solves specific problems.

What mathematical engineers (i.e., mathematics educators) bring into the K–12 classroom must **respect** these

$five\ basic\ characteristics\ of\ mathematics$

Here are three examples that do not:

the teaching of fractions in grades 5–7,

the teaching of geometry in high school,

FASM.

Fractions:

$No\ definition.$ The statement "fractions have multiple representations" is meaningless.

$No\ reasoning.$ No definition, therefore no reasoning. E.g. *WHY* is $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$?

$No\ coherence.$ "Fractions are such different numbers from whole numbers!"

Is this the kind of criticisms only mathematicians enjoy, insisting that school mathematics be like the mathematics they do? NO. Students are not learning fractions because fractions are not being clearly explained.

Geometry

The mathematics: Both Euclid's version and Hilbert's reformulation go directly from *axioms* to *theorems* to *proofs*. Very user-unfriendly.

The engineering product (school geometry): Either it

(1) follows Euclid, and therefore does not address students' learning capacity, or
(2) has no definitions, no theorems, and no proofs, is just experimental geometry and therefore does not respect the basic characteristics of mathematics.

POOR engineering in both cases.

FASM (Fundamental Assumption of School Mathematics): *All the information about the arithmetic operations on fractions can be extrapolated to all real numbers.*

Ex. $\dfrac{3}{\pi}+\dfrac{\sqrt{2}}{5.1}=\dfrac{3\times 5.1+\sqrt{2}\times\pi}{\pi\times 5.1}$

School mathematics is the mathematics of rational numbers. Any excursion into irrational numbers depends on FASM.

The use of FASM in school mathematics is good engineering provided it is made explicit (the **precision** of mathematics). The fact that FASM is **never** mentioned in textbooks in schools or in college textbooks for teachers is POOR engineering.

The non-learning of fractions and geometry has been among the most notorious problems in school mathematics education. Almost nothing has been done about geometry. What has been done for fractions does not touch on the fundamental issue that the teaching of fractions violates all five basic characteristics of mathematics.

Mathematicians' input has been missing . . . while students' non-learning continues.

As of 2007, mathematical engineering urgently needs the close collaboration of mathematicians and educators.

The reasons that mathematical engineering is intrinsically bound to both mathematics **and** education:

1. The customization of mathematics begins with knowing the classroom needs at each grade level.

This requires knowledge of the school mathematics curriculum.

For example, what third graders need to know about $area$ is different from what tenth graders need to know about the same concept. In addition, even third graders need to know the concept of $length$ before taking up $area$, and they also need to know that the concept of area requires the designation of a $unit\ area$.

2. The varied nature of the needs requires the ability to devise more than one correct approach to a given topic.

This requires solid content knowledge.

For example, the meaning of reflection in the plane can be

- (a) taught by folding papers, or
- (b) defined by using perpendicular bisector of a segment, or
- (c) defined by use of coordinates.

(a) is appropriate for 5th graders, but not for 10th graders.

3. The nature of the need dictates the choice of the best approach among the alternatives.

This requires a deep knowledge of both pedagogy and mathematics: how to reach out to students on their own terms without sacrificing the basic characteristics of mathematics.

It is all too tempting to push aside these basic characteristics in the name of reaching out to students, i.e., it is easy to do defective engineering.

Ex. Define $\frac{2}{3} \times \frac{5}{8}$ to be "*$\frac{2}{3}$ of $\frac{5}{8}$ pounds of sugar*", without making precise what it means (*what does "of" mean, and what does sugar have to do with fractions??*). This violates **precision**.

No chemical engineer can function without knowing the fundamental principles of chemistry.

No electrical engineer can function without knowing the fundamental principles of electromagnetism.

No mathematical engineer can function without knowing the basic characteristics of mathematics.

But we have no mathematical engineers as of 2007. Mathematicians generally know mathematics, and educators generally know education.

School of engineering is not just a collection of physicists and marketing researchers in various aspects of life. *It consists of* **engineers** *who know both physics and the real world.*

The separation of mathematicians from educators in the U.S. of the recent past has led to a deterioration of the **mathematics** in mathematics education.

The deterioration: Mathematics that does not respect some or all of the basic characteristics of the discipline

- has crowded K-12 textbooks, old and new, and professional development materials,
- has been taught far too often in the school classroom,
- has infiltrated standardized assessments,
- has corrupted official curricular standards,
- has adversely affected mathematics education research, especially in fractions and algebra.

If electrical engineers produce defective products over several decades, departments of electrical engineering would have been shut down long ago.

The crisis in mathematics education is real.

The job of educators and mathematicians has been cut out for them:

SHORT TERM: **Work together to improve every aspect of mathematics education.**

LONG TERM: **Let there be mathematical engineers.**

Let us put aside professional prejudices, because we cannot afford to lose another generation of students. Let us work together. The recent **Focal Points** of NCTM is a first step among several of this kind of collaboration. But **many, many more steps** are needed to ensure a better mathematics education in our nation.

Wu's article on mathematical engineering :

How mathematicians can contribute to K-12 mathematics education, *Proceedings of International Congress of Mathematicians, Madrid 2006*, Volume III, European Mathematical Society, Zürich, 2006, 1676-1688.
http://math.berkeley.edu/~wu/ICMtalk.pdf