The Mathematics Early Grade Teachers Need to Know—and What It Means to *Know* It

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*I am indebted to Larry Francis for several important suggestions.

I am honored to be addressing this audience. My charge is to describe, based on scientific evidence, the mathematics early grade teachers need to know.

No scientific evidence is needed for the assertion that all early grade teachers must know **whole numbers** and **fractions**. These are the cornerstones of the early grades mathematics curriculum.

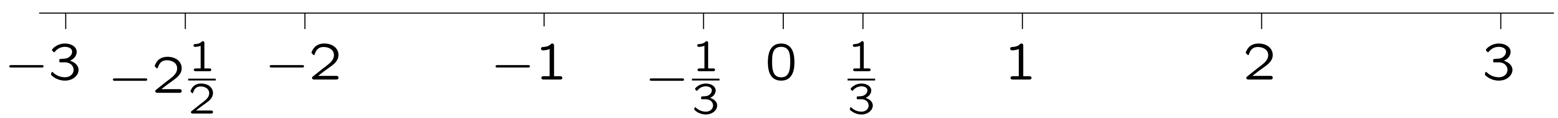
I will add four more topics: **negative numbers**, and *some* **geometry**, **algebra** and **probability**. Early grade *teachers* must know these too.

Of course, there are details to consider. *How much* whole number, fractions, negative numbers, geometry, algebra, and probablitiy should early grade teachers know?

I will only touch *lightly* on such details at the end, because you are not likely to find such a discussion to be very entertaining; an additional reason is that they are given in a separate article of mine (The mathematics K–12 teachers need to know).

I will make three general comments, however.

The first is that there is by now ample evidence of the efficacy of the number line for organizing the mathematical developments of whole numbers, fractions, and negative numbers.



See the U.S. National Mathematics Advisory Panel report, Chapters 3 and 4 on Conceptual Knowledge and Learning Processes:

http://www.ed.gov/about/bdscomm/list/mathpanel/report/conceptual-knowledge.pdf

http://www.ed.gov/about/bdscomm/list/mathpanel/report/learning-processes.pdf

All early grade teachers should learn to make effective use of the number line at every opportunity.

The second one is that, where *mathematical* content is concerned, the meaning of **scientific evidence** must be broadened.

In the context of education research, scientific evidence usually means data, i.e., consensus confers validity. A valid statement about "content" therefore has to be a summary of what is practiced by most nations, most teachers, most educators, etc.

Mathematics, however, is emphatically not a democratic product. No amount of common consensus can overrule the internal logical structure of mathematics.

One example: In the U.S., almost all state standards and almost all textbooks want the *algebra* of linear equations to be taught without any acknowledgement that similar triangles are involved.

This is mathematically scandalous.

So the scientific evidence for any statement about “content” must also take *mathematical considerations* into account. I am glad that in this seminar, I am among like-minded colleagues.

A third comment is that both data and mathematical considerations agree that whole numbers, fractions, and geometry are the foundational topics in the school mathematics of the early grades.

The only controversy may be whether every early grade teacher must know some negative numbers, algebra, and probability.

My answer is YES, because in mathematics, *what lies beyond a lesson shapes the lesson itself*. ("Mathematics is purposeful".)

For example, does a teacher tell second graders

you cannot subtract 5 from 3,

or does she say instead,

> *for now* we can only subtract a smaller number from a bigger number, but later on you will learn how to do subtraction between ANY two numbers.

A teacher has to say the latter, but cannot say it with conviction without having internalized the arithmetic of negative numbers.

Another example: *why are the commutative, associative, and distributive laws the popular objects of scorn among early grade students?*

Because if the teachers have never seen these laws used in substantive ways to explain the arithmetic of negative numbers and solve equations, they would not appreciate these laws' central importance and their lessons would betray their own disdain for these laws.

Every mathematics teacher needs to know the mathematics, *deeply,* beyond the level they teach. (Recommendation of the U.S. National Mathematics Advisory Panel.)

http://www.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf

Instead of discussing further *what mathematical topics* teachers should know, we have to first explain what it means to “know” something.

In mathematics education, *knowing* something means literally *knowing a fact by heart*. Mathematicians use the same word to mean a lot more. This confusion over the use of the same word in completely different ways has lead to colossal misunderstanding in the current educational discussion.

In mathematics, to say you **know** a fact means you know

what it says *precisely*,
what it says *intuitively*,
why it is true,
why it is worth knowing,
in what way it can be put to use,
the natural context in which it appears.

In short, *knowing* a fact means being able to tell the *whole story about this fact* rather than just a few sound bites.

All early grade teachers should *know*, in this sense, the above six topics (whole numbers, fractions, negative numbers, geometry, algebra, and probability).

Of course you think I am exaggerating.

Let us consider the preparation of a lesson on **place value**, and one on the **multiplication algorithm**. Two very elementary topics in whole numbers.

The teaching of *place value* is usually done by rote. Consider teaching children why

the 3 in 32 is 30, and the 2 is just 2.

The common emphasis is on pedagogy: *How* to impress students on the importance of the **place** (or **position**) of a digit. But ultimately, the message is simply:

This is place value. This is it.

Why is this bad for math education?

Mathematics is children's first encounter with quantitative information. If we cannot develop their curiosity from the beginning, there will be little chance of success in math education.

If a teacher lays down too many seemingly unreasonable rules with no explanation, students lose confidence in mathematics as a learnable subject and their curiosity will be snuffed out. They will stop learning.

"Place value" *is* unreasonable to children.

But place value can be explained. *Every teacher must explain it.*

A teacher should have the firm conviction that there is a way to explain *almost* everything in mathematics (i.e., except the axioms).

If a teacher consistently makes an effort to give explanations, she will be always asking herself why. Her students will also be asking why, and their curiosity will be kept alive.

Brief digression: In the US, school science education is beginning to draw *serious* attention.

The first requirement of a meaningful science education is that students *want* to know "why it works". Only then would it make sense to find an answer, and only then can science education begin.

Rare is the case that children with no curiosity about mathematics at the beginning will later redevelop a curiosity about mathematics or science.

So we must teach mathematics better.

Back to place value.

Before worrying about pedagogy (*how to teach* **it**), let us make sure we understand what **it** is. i.e., what is the reasoning underlying place value?

Know something first before worrying about how to explain it.

Content dictates pedagogy.

The fundamental feature of our numeral system is that

counting is done with only ten symbols:
0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The reason? Many numeral systems of the past used many more symbols and made computation almost impossible, e.g., Egyptian hieroglyphic numerals, Greek alphabetic numerals, Babylonian cuneiform numerals, etc.

Even the Roman numerals, which have *partial* place value, are too cumbersome to use for mathematics. *Compare:*

$3+2+4=9$	versus	$\mathrm{III}+\mathrm{II}+\mathrm{IV}=\mathrm{IX}$
$30+20+40=90$	versus	$\mathrm{XXX}+\mathrm{XX}+\mathrm{XL}=\mathrm{XC}$
$300+200+400=900$	versus	$\mathrm{CCC}+\mathrm{CC}+\mathrm{CD}=\mathrm{CM}$
$3\times 2=6$	versus	$\mathrm{III}\times\mathrm{II}=\mathrm{VI}$
$3\times 20=60$	versus	$\mathrm{III}\times\mathrm{XX}=\mathrm{LX}$
$3\times 200=600$	versus	$\mathrm{III}\times\mathrm{CC}=\mathrm{DC}$

Observe the symbolic simplicity on the left and contrast it with what is on the right. How to compute with Roman numerals?

Our numeral system, by using only ten symbols, achieves symbolic simplicity, but at a price. It is cognitively complex:

Limited to ten symbols, how to count beyond 9?

One way is to repeat the ten symbols ad infinitum and count one row after another:

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \rightarrow \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \rightarrow \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \rightarrow \\ \vdots & & & \vdots & & & \vdots & & & \vdots \end{array}$$

(Go through each row from left to right. Then repeat in the row below.)

The trouble is: there is no way to differentiate between

going five steps from upper left 0 and

going fifteen steps from upper left 0,

because both land on the symbol **5**.

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
⋮			⋮			⋮			⋮

The major breakthrough:

if we allow ourselves two places instead of one,

then we can differentiate among ten of these rows by putting these ten symbols 0, 1, 2, etc., in the place to the *left* of each of the ten numbers.

00	*01*	*02*	*03*	*04*	*05*	*06*	*07*	*08*	*09*
10	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*
20	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*
30	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*
⋮			⋮			⋮			⋮
⋮			⋮			⋮			⋮
90	*91*	*92*	*93*	*94*	*95*	*96*	*97*	*98*	*99*

The numbers in the first row, 00, 01, 02, 03, 04, 05, 06, 07, 08, 09, are nothing but the original one-digit numbers with a 0 attached to the left. For this reason, it is traditional to rewrite them *without* the zero to the left:

0	1	2	3	4	5	6	7	8	9
10	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*
20	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*
30	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*
⋮			⋮			⋮			⋮
⋮			⋮			⋮			⋮
90	*91*	*92*	*93*	*94*	*95*	*96*	*97*	*98*	*99*

Now why does the 3 in 32 mean **30**? Because the row beginning with 3 follows 3 rows of tens, i.e., after three rows of ten have been counted:

0	1	2	3	4	5	6	7	8	9
*1*0	*1*1	*1*2	*1*3	*1*4	*1*5	*1*6	*1*7	*1*8	*1*9
*2*0	*2*1	*2*2	*2*3	*2*4	*2*5	*2*6	*2*7	*2*8	*2*9
30	**3**1	**3**2	**3**3	**3**4	**3**5	**3**6	**3**7	**3**8	**3**9
*4*0	*4*1	*4*2	*4*3	*4*4	*4*5	*4*6	*4*7	*4*8	*4*9
⋮			⋮			⋮			⋮
⋮			⋮			⋮			⋮
*9*0	*9*1	*9*2	*9*3	*9*4	*9*5	*9*6	*9*7	*9*8	*9*9

Once we have done it with two places, we can repeat the reasoning and use three places, four places, . . . , *any number of places*.

This way, we can count to **any** number, no matter how large.

So the fact that the 3 in 32 means 30 is not an arbitrary decree. This is *dictated* by the need to count large numbers when there are only ten symbols .

(what it says *precisely*,
what it says *intuitively*,
why it is true,
why it is worth knowing,
in what way it can be put to use,
the natural context in which it appears.)

More importantly, the fact that the **value** of 3 depends on its **place** in the numeral simplifies computations, as we have seen, e.g., $300 + 200 + 400 = 900$, and as we will see in even more spectacular examples.

(what it says *precisely*,
what it says *intuitively*,
why it is true,
why it is worth knowing,
in what way it can be put to use,
the natural context in which it appears.)

Naturally, you don't teach all this to kindergarteners!

Teaching mathematics in the early grades is mainly a cumulative process. A teacher does what she can at a given moment, given children's cognitive development, but it should be possible to gradually convey to them a robust conception of what "place value" is by the end of the third grade.

More importantly, knowing that there is a reason for place value, a teacher would not drill first graders by rote, make them memorize that "the 3 in 32 is 30", and proceed to forget about it ever after.

For first graders, for example, she can hint at the reasoning by playing a game. Ask them:

> If they have to use only ten shapes (or ten colors) —analogs of the ten symbols 0, 1, . . . , 8, 9—to count, how would they count twenty objects?

It is not necessary that *they* come up with the correct (systematic) way using place value. What is important is that they begin to sense that there is *reasoning* lurking behind place value.

With second graders, for example, a teacher can play a different game: Using only five symbols 0, 1, 2, 3, 4 instead of ten, how would they count up to twenty-five? (It is the number 100 in base 5.) Try it also with three symbols, four symbols, . . .

The teacher may eventually have to show them how to do it. But even if the students don't completely understand her explanation, they will get the idea that *"there is a reason out there and I can find out one day"*.

This would be a vast improvement over *"I am here only to take orders, and I will never know why."*

A teacher can introduce to third and fourth graders some other numeral systems, such as the Roman numerals or the Greek *alphabetic numerals*, to show the complications when there is no place value.

The latter has been in use since about 4th century B.C. and is a decimal system.

$$1 = \alpha, \quad 2 = \beta, \quad 3 = \gamma, \quad 4 = \delta, \quad 5 = \epsilon, \quad 6 = \varsigma, \ldots$$
$$10 = \iota, \quad 20 = \kappa, \quad 30 = \lambda, \quad 40 = \mu, \quad 50 = \nu, \quad 60 = \xi, \ldots$$
$$100 = \rho, \; 200 = \sigma, \; 300 = \tau, \; 400 = \upsilon, \; 500 = \phi, \; 600 = \chi, \ldots$$

$$3 \times 2 = 6 \qquad \text{versus} \qquad \gamma \times \beta = \varsigma$$
$$3 \times 20 = 60 \qquad \text{versus} \qquad \gamma \times \kappa = \xi$$
$$3 \times 200 = 600 \qquad \text{versus} \qquad \gamma \times \sigma = \chi$$

The last group of multiplicative statements provide a natural segue to our next topic:

the multiplication algorithm.

Consider teaching the computation of 32×47.

$$
\begin{array}{rcccc}
 & & & 3 & 2 \\
 & & & 4 & 7 \\
\times & & {\scriptstyle 2} & {\scriptstyle 1} & \\
\hline
 & & 2 & 2 & 4 \\
 & 1 & 2 & 8 & \\
\hline
 & 1 & 5 & 0 & 4
\end{array}
$$

What do we tell third graders *why* they should learn this algorithm, and *why* the algorithm gives the right answer?

Here is the reason. Because 32×47 means

$$\underbrace{47 + 47 + \cdots + 47}_{\text{32 times}},$$

which do they prefer: do this addition, or find a shortcut?

(what it says *precisely*,
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why it is true,
why it is worth knowing,
in what way it can be put to use,
the natural context in which it appears.)

BRIEFLY: From their knowledge of *place value*,

$$\begin{aligned} 32 \times 47 &= 32 \times (40 + 7) \\ &= (32 \times 40) + (32 \times 7) \qquad \text{(dist. law)} \\ &= (\underline{32 \times 4}) \times 10 + (\underline{32 \times 7}) \end{aligned}$$

So do they know how to multiply a number by a **single-digit**? Now look at the first product, for example:

$$\begin{aligned} 32 \times 4 &= (30 + 2) \times 4 \\ &= (30 \times 4) + (2 \times 4) \qquad \text{(dist. law)} \\ &= (\underline{3 \times 4}) \times 10 + (\underline{2 \times 4}) \end{aligned}$$

Similarly,

$$32 \times 7 = (\underline{3 \times 7}) \times 10 + (\underline{2 \times 7})$$

The question then becomes: *do they know how to multiply a* **single digit** *by a* **single digit** (multiplication table)?

If they do, then,

$$32 \times 4 \;=\; 128 \qquad \text{and} \qquad 32 \times 7 \;=\; 224,$$

so that

$$32 \times 47 \;=\; (128 \times 10) + 224 \;=\; 224 + 1280$$

which is exactly the multiplication algorithm.

$$\begin{array}{rrrrr} & & & 3 & 2 \\ \times & & & 4 & 7 \\ \hline & & 2 & 2 & 4 \\ & 1 & 2 & 8 & 0 \\ \hline & 1 & 5 & 0 & 4 \end{array}$$

This says that if they know the multiplication table, then they can multiply *any* two numbers. This is the importance of the multiplication table.

When a teacher knows why the algorithm is true, she can tell her students why they must know the multiplication table.

(what it says *precisely*,
what it says *intuitively*,
why it is true,
why it is worth knowing,
in what way it can be put to use,
the natural context in which it appears.)

This algorithm should be put in context: the fact that the multiplication algorithm reduces the multiplication of *all numbers* to the multiplication of *single-digit numbers* is part of a general pattern.

What all four standard algorithms have in common is this:

If you can $+$, $-$, $\times$, *and* $\div$ *single-digit numbers,*
then you can $+$, $-$, $\times$, *and* $\div$ *all numbers,*
no matter how big.

(A very slight exception has to be made for $\div$.)

Therefore, learning the standard algorithms is much more than learning a set of rote skills. It is about learning

how to systematically break down a *complex* skill
into a sequence of *simple* skills.

Because many in mathematics education do not know this aspect of the standard algorithms, there has been a silly debate for twenty years about whether the standard algorithms should be taught.

Reducing the complex to the simple is an overriding theme in mathematics and science. This theme provides the right context for learning the standard algorithms.

Such content knowledge gives a teacher the confidence to teach the standard algorithms. Confidence is important.

> (what it says *precisely*,
> what it says *intuitively*,
> why it is true,
> why it is worth knowing,
> in what way it can be put to use,
> the natural context in which it appears.)

To summarize: Why should early grade teachers *know* mathematics in the sense described?

1. A teacher who *knows* mathematics does not impose rules without reason. She encourages her students to always ask why. When questions are routinely asked and routinely answered in the classroom, students take for granted that questioning the world around them is a way of life. *Their curiosity stays intact.*

2. The more a teacher knows about what she teaches, the more options she has on *how* to teach it, and the more she can convince students that mathematics is worth learning.

Content dictates pedagogy.

What we have just done is to look at the content knowledge needed for teaching the early grades from the perspective of mathematics itself. Every concept or skill in mathematics has always been developed as a response to questions such as these:

what does it say *precisely*?
what does it say *intuitively*?
why is it true?
why is it worth knowing?
in what way can it be put to use?
what is the natural context in which it appears?

“Learning” and “discovering” are two sides of the same coin.

With their curiosity intact, learners would ask the same questions about a new concept or new skill the same way a working mathematician would.

Knowing what basic questions must be answered in the process of developing new concepts or skills then gives us an excellent idea of what content knowledge a teacher must possess.

This too is *scientific evidence*.

Finally, let me touch on the issues of how to acquire this content knowledge and how much content knowledge is enough.

I will address the second one first: ideally, all early grade teachers should *know* the mathematics of the school curriculum up to introductory algebra.

This statement assumes that such a teacher may be called upon to teach any of the early grades and must therefore be ready for such a contingency.

If, in the unlikely event that a teacher gets to spend her whole life teaching the first grade, does she need to know that much mathematics?

Ideally, yes, but in practical terms, one may have to modify the content knowledge requirement to be three grades beyond the grade she teaches.

This discussion assumes that there is an common curriculum for the early grades. This is a big assumption!

Until last year*, the U.S. also did not have such a common curriculum. However, a set of Common Core State Standards for Mathematics (**CCSSM**) was released in 2010 and they will be implemented in 2014. At least 43 of the 50 states have agreed to implement the CCSSM.† (http://www.corestandards.org/)

CCSSM can serve as a useful reference for this discussion.

*2010.
†As of August, 2011.

The question of how elementary teachers can acquire this content knowledge is a sore subject in the U.S.

Most college textbooks written for elementary teachers emphasize pedagogy at the expense of mathematics. So a body of knowledge that is mathematically defective has been recycled for decades in the American schools.

Only recently is this critical problem of teaching teachers the knowledge they need beginning to receive proper attention. Maybe one or two of the latest books can be consulted.

For Brazil, a suggested stopgap measure is to translate the Singapore grades 1-6 textbooks, **Primary Mathematics**,

http://www.singaporemath.com/Primary_Mathematics_US_Ed_s/39.htm

or the Japanese grades 1-6 textbooks, **Mathematics for Elementary School**,

http://www.globaledresources.com/products/books/math_elementary_notfound/

In general, they give a *usable introduction* to what elementary teachers need to know.

I should also call your attention to some videos of Japanese lesson study and a Teaching Guide for grades K–6:

http://hrd.apec.org/index.php/Classroom_Videos_from_Lesson_Study

http://www.globaledresources.com/products/books/guide_arithmetic_1-6.html

These should answer some questions about pedagogy. Note, however, that the starting point of a lesson study is a basic mastery of the content. There is no escaping the fact that *sound content knowledge is the foundation of teaching.*