From Arithmetic to Algebra, Part 1: Algebra as Generalized Arithmetic*

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The school mathematics curriculum is an organic entity: it grows from the simple to the complex, from the concrete to the sophisticated and abstract. It cannot be otherwise because its main function is to provide guidance to students' first tentative steps in the learning of mathematics and, lest we forget, the logical structure of mathematics itself also grows from the simple to the complex.

Unfortunately, there are unnecessary discontinuities in the development of the school mathematics curriculum[1] that disrupt student learning, and one result of this disruption has led to fear and apprehension in the learning of algebra. In this short article, we will focus on the discontinuity from arithmetic to algebra that results from the failure to guide students from the concrete to the sophisticated and abstract. Here, "arithmetic" refers to whole numbers, finite decimals, and fractions, and "algebra" refers to *introductory* school algebra, i.e., rational numbers, the extensive use of symbolic notation, linear equations of one or two variables, and quadratic equations. We will omit any reference to the more advanced topics of school algebra such as exponential functions, logarithms, and the formal algebra of

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[1]We want to be explicit that we have in mind mainly the discontinuities in the American *school curriculum* rather than any inherent discontinuities in school mathematics itself.

polynomial forms (see Schmid-Wu, 2008, Wu, 2016, and Wu, *to appear*), and will, instead, focus on two of the major characteristics of school algebra: generality and abstraction. Rather than indulging in the rather futile and time-consuming exercise of explaining the meaning of the latter concepts, we trust that their meaning will emerge with clarity in the course of the ensuing discussion.

The main message of this article is that school algebra is just *generalized arithmetic.* Now, this is a slogan that probably means different things to different people. It therefore behooves us to explain—as precisely as possible—what *generalized arithmetic* means in the present context. Arithmetic is concerned with accurate computations with specific numbers. In the way school mathematics is usually taught, children's main concern in arithmetic is with the here and now, in the sense that students are content with the correctness of getting the answers to, e.g., $151 - 67$ or $\frac{5}{6} + \frac{3}{8}$ as 84 and $\frac{58}{48}$, respectively, but not much beyond that. Algebra is, however, students' first contact with mathematics proper, and it brings to students an awareness of "the big picture". It asks students to begin thinking about whether a given computation is part of a general phenomenon, or whether they can put a given computation in a broader context to understand it better. But for arithmetic students who have been happily computing away, they have to wonder *why* they should switch gears to learn this *generalized arithmetic.* Why bother, indeed?

Any mathematics curriculum that is serious about smoothing students' passage from arithmetic to algebra will have to answer the last question. What might the answer be?

We will discuss two topics in algebra, summing a *finite geometric series* and solving a linear equation, by way of answering this question. In the process, we also hope to clarify the nature of generalized arithmetic. We forewarn the reader that this discussion will involve some precise *mathematics*, something not commonly found in writings of this kind. We do so because it is necessary, and also because the heart of any discussion in *mathematics* education usually lies in the mathematics itself.

Suppose we ask eighth-graders to compute (calculators allowed)

$$1 + 3 + 3^2 + 3^3 + \cdots + 3^9 + 3^{10}$$

As arithmetic, the answer is 88,573, and that is all there is to it. But does this computation stand alone, or is this part of a general phenomenon? The eighth-graders should first take note of the unusual *pattern*, or *structure*, in this sum: the increasing powers of 3 (assuming they know $1 = 3^0$ and $3 = 3^1$). The first thought that should cross their minds is that they should be able to get the sum of increasing powers of 3 up to, not just 10, but any integer. In fact, why 3? Why not the sum of increasing powers of an arbitrary number up to any integer? Right away, this very question has left arithmetic behind because, trying to add the increasing powers of an *unknown number* up to some *unknown integer* is to give up any hope of computing with explicit numbers. In fact, we have to confront a more fundamental problem even before we get started: how to express this kind of *generality* (sum of increasing powers of an *unknown number* up to an *unknown integer*) in a concise way. At this point, we have to tell eighth-graders that, for more than a thousand years, humans struggled mightily with the issue of how to handle such *generality* until around 1600, when they finally found an efficient way to use symbols for this purpose. So carrying on this mathematical legacy, we let r be an arbitrary number and let n be an arbitrary positive integer and then ask for the sum of $1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n$. Of course, by giving up on getting explicit numbers at each step, the only way to do any computation is to rely on the associative and commutative laws of $+$ and $\times$, and the distributive law which are applicable to *all* numbers, known or unknown. Such a computation will finally allow eighth-graders get to see the importance of these laws of operations, perhaps for the first time.

Instead of plunging headlong into the new world of generality, it may be prudent to first go only partway and try to get the sum of $1 + 3 + 3^2 + 3^3 + \cdots + 3^{23} + 3^{24}$. This is a sum of 25 explicit numbers, so it looks like an ordinary arithmetic problem. But because getting this sum by brute force computation will require more patience than most eighth-graders can muster, it is clear that something more than ordinary arithmetic skills will be needed to get an answer. One way to proceed is the following. We look at this sum as one number regardless of the fact that we do not know what it is explicitly, and denote it by S. Right away we are capitalizing on the advantage of using symbols. This is a key step, because, at this point, we are no longer doing the arithmetic of old by insisting on an explicit answer for every computation at every step. Rather, we will compute with S simply *as a number.* That said, since S is the sum of increasing powers of 3, the idea of trying to multiply S by 3 almost suggests itself because, by applying the distributive law, we immediately get a sum that is "almost the same" as S itself. Precisely, we get:

$$\begin{aligned} 3S &= 3 \cdot 1 + 3 \cdot 3 + 3 \cdot 3^2 + \cdots + 3 \cdot 3^{23} + 3 \cdot 3^{24} \\ &= 3 + 3^2 + 3^3 + \cdots + 3^{24} + 3^{25} \\ &= (1 + 3 + 3^2 + 3^3 + \cdots + 3^{24}) - 1 + 3^{25} \\ &= (S - 1) + 3^{25} \end{aligned}$$

Therefore,

$$3S \;=\; S + 3^{25} - 1 \tag{1}$$

Now, still without knowing the exact value of S, we add $-S$ to both sides to get

$$(-S) + 3S \;=\; (-S) + S + 3^{25} - 1$$

After one more application of the distributive law on the left side, we get $(-S) + 3S = (-1+3)S = 2S$, so that

$$S \;=\; \frac{1}{2}\,(3^{25} - 1) \tag{2}$$

With the help of a calculator, we get $S = 423,644,304,721$, and we should not fail to point out to eighth-graders that the answer has been obtained without any sweat.

This is the first fruit of *abstract thinking:* an answer obtained not by brute force computations with explicit numbers but by injecting reasoning and *abstract pattern-recognition* into computations with numbers in general. For example, the decision to multiply S by 3 to get to equation (1)—which is as good as the final answer (2) itself—was prompted by the reasoning that the abstract pattern of a sum of powers of 3 will essentially repeat itself after this multiplication. In one sense, what we have done is just arithmetic because it is nothing more than the application of arithmetic operations on numbers, *and numbers only.* At the same time, it also goes beyond arithmetic because this is a computation with unknown numbers using only the laws of operations. We are doing *generalized arithmetic.*

What we have just learned is that, by stepping away from computations with explicit numbers and by trying to discern patterns on a larger scale, we have much to gain even in ordinary computations.

We can now return to our original problem of getting the sum of

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n \qquad \text{for any } r \text{ and any positive integer } n.$$

Observe that if $r = 1$, this sum is equal to $n + 1$ and is of no interest. So we assume $r \neq 1$ from now on. If we retrace the preceding reasoning with care, we can see without difficulty that it remains valid almost *verbatim* if we replace 3 by r and the exponent 24 by n. Thus, if $r \neq 1$, and if n is a positive integer, then

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n \;=\; \frac{r^{n+1} - 1}{r - 1} \tag{3}$$

This is the so-called *summation formula for a finite geometric series*. Equation (2) is the special case of equation (3) when $r = 3$ and $n = 24$.

To fully savor this kind of generality, suppose we want the following sum:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{27}} + \frac{1}{2^{28}}$$

We notice that this sum fits the left side of identity (3) with $r = \frac{1}{2}$, and $n = 28$. Hence,

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{27}} + \frac{1}{2^{28}} \;=\; \frac{(\frac{1}{2})^{29} - 1}{\frac{1}{2} - 1} \;=\; 2 - \frac{1}{2^{28}}$$

Now we can ask the same questions again about identity (3): is this part of a general phenomenon or is it an isolated result? Can we put it in a broader context so that we can understand it better? Indeed we can understand it better, but to get the "broader context" we will have to cast a wider net. All eighth-graders know the well-known identity $(x - y)(x + y) = x^2 - y^2$ for all numbers x and y. It is a simple exercise in the use of the distributive law to prove the following more general identity: for any positive integer n and for all numbers x and y,

$$(x - y)\,(x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + xy^{n-1} + y^n) \;=\; x^{n+1} - y^{n+1} \tag{4}$$

Thus $(x-y)(x+y) = x^2-y^2$ is a special case of identity (3) when $n = 1$. Moreover, identity (3) is also the special case of identity (4) when $x = r$, $y = 1$, and $r \neq 1$. In particular, identity (3) on summing a finite geometric series is now seen to be related to the mundane identity $(x - y)(x + y) = x^2 - y^2$.

More is true, however. Consider 64,339,280,491, which is a big number. It is not easy to tell whether 64,339,280,491 is a *prime*, i.e., a number divisible only by 1 and itself. But it happens to be equal to $31^7 - 4^7$, so we know from identity (4)—with $x = 35$, $y = 4$, and $n = 6$—that 31 divides it because,

$$64{,}339{,}280{,}491 \;=\; 31^7 - 4^7 \;=\; 31 \times (35^6 + 35^5 \cdot 4 + \cdots + 35 \cdot 4^5 + 4^6)$$

It is not a prime! Similar considerations show that if x and y are positive integers and $x - y > 1$, then for any $n \geq 1$. $x^n - y^n$ is never a prime. Identity (4) now opens up a completely new area for discussion about prime numbers (see Section 1.3 of Wu, 2016 for more details). What is noteworthy is that while we are still doing arithmetic, we discover that, by introducing symbols to represent numbers and by welcoming the concept of generality into our computations, the consideration of the summation of a finite geometric series is seen, via identity (4), to be related to the question of whether some whole numbers are primes. This is an example of the power of generality that eighth-graders can appreciate.

Next, consider the solutions of equations, a topic central to any school algebra curriculum and—in fact—to the historical development of algebra itself. We can make our point with a simple linear equation, $4x + 1 = 2x - 3$. In the usual presentation of school algebra, x is a *variable* and we solve the equation by the following symbolic manipulations:

Step A: $(-2x) + 4x + 1 \;=\; (-2x) + 2x - 3.$

Step B: $2x + 1 \;=\; -3.$

Step C: $2x + 1 + (-1) \;=\; -3 + (-1).$

Step D: $2x \;=\; -4.$

Step E: $x \;=\; -2.$

The answer of -2 is correct, but a little reflection would reveal that *these five steps make no sense whatsoever.* Consider Step A, for example. Since x is a quantity that *varies*, what does it mean to say that the two quantities that vary, $4x + 1$ and $2x - 3$, are somehow "equal"?[2] And how do $4x + 1$ and $2x - 3$ stay being "equal" after the varying quantity $-2x$ has been added to both? Moreover, the passage from Step A to Step B requires that we apply the distributive law and the associative law of addition to both $(-2x) + 4x + 1$ and $(-2x) + 2x - 3$. Now students learned in arithmetic that these laws of operation are

[2]For example, since x can vary, x may be equal to 0. In that case, $4x + 1 = 1$ and $2x - 3 = -3$. How can 1 be equal to -3?

applicable to numbers, but where is it *explained* that these laws are equally applicable to *quantities that vary*? And so on. Such a drastic turn of events, from explicit computations to unknowable computations, can be disorienting to students. Is mathematics supposed to teach students how to reason, or is it supposed to encourage rote-learning?

This is but one of many blatant examples of how the usual school curriculum disrupts students' transition from arithmetic to algebra. The main culprit is the cult of "variables" and its inevitable subsequent abuses. We can rectify this unwarranted disruption by restoring the idea of algebra as *generalized arithmetic*, as follows.

First, we have to come to terms with *what an "equation" is.* The meaning of the **equation** $4x+1 = 2x-3$ is that it is a *question*: is there a *number* x so that $4x+1 = 2x-3$? Such a number x is then called a **solution** of $4x + 1 = 2x - 3$. Note that no "variable" is involved. Just numbers. The way to *solve* this equation is to assume that there is a solution x. Remember: this x is *now* a single number. So we have the equality of two numbers, $4x + 1 = 2x - 3$, and Steps A to Step E now become five successive statements about numbers, each step being a consequence of the preceding one. At the end, what we get is this:

If there is a solution x of $4x + 1 = 2x - 3$, this solution has to be -2.

This does *not* say that -2 is a solution of $4x+1 = 2x-3$. To prove that, we must substitute $x = -2$ into $4x + 1 = 2x - 3$ to check that the two sides are indeed equal, which is easily accomplished. In this light, once Steps A to E are properly interpreted as computations with numbers, known and unknown, they are actually correct! Thus, solving equation (*any* equation, as it turns out) is part of *generalized arithmetic.*

We emphasize once again that no "variable"[3] is involved in the solution. (See Section 3.1 of Wu, 2016 for a more comprehensive discussion of these issues.)

We hope the preceding discussion has given some indication of why we say that school algebra is *generalized arithmetic:* it is arithmetic on a more abstract level. It is incumbent

[3] Let it be noted that *variable* is not a mathematical concept, period. If students are taught to use symbols correctly, they will have no need to worry about what a "variable" means. See Section 1.1 of Wu, 2016.

on the school curriculum to help students bridge the gap between arithmetic and algebra by introducing abstraction into the arithmetic curriculum when feasible, by making use of symbols when the occasion calls for it, and, above all, by making reasoning a daily routine in arithmetic because all of mathematics—at *any* level—is built on reasoning. The failure of the usual school curriculum in the U.S.[4] to meet these basic requirements accounts for the discontinuity in the school curriculum from arithmetic to algebra that is so detrimental to the learning of algebra.

We can go into more detail. In the usual presentation of school mathematics in the U.S.,[5] whole-number arithmetic is taught with a strong focus on computational accuracy and little, if any, attention is paid to the reasoning in the standard algorithms. For fractions, it almost goes without saying that few if any definitions are offered for the main concepts (including the concept of a fraction), and the absence of definitions then makes it impossible to provide reasoning for any computational algorithms in fractions. Teaching fractions has been mostly about hands-on activities, analogies, and story-telling. The *concrete* completely displaces the *abstract*. But this strategy is not working. It cannot even get students to compute explicit numbers accurately. One item in the 2011 TIMSS asked about the computation of $\frac{1}{3} - \frac{1}{4}$, and only 29.1% of U.S. eighth-graders got it right. Sadly, the general situation worldwide is not much better: only 37.1% of eighth-graders got it right (see Askey, 2013). Such a strategy of banishing abstraction from arithmetic is designed to delude students into thinking that they can win battles and skirmishes in their march through fractions without having to do any abstract thinking. Unfortunately, it is this strategy of avoidance that causes students to lose the war by the time they get to algebra, where they get shell-shocked when confronted with abstraction, generality, and the extensive use of symbols.

References

[1] Askey, R. A. (2013). TIMSS fraction item. Retrieved from:
`http://tinyurl.com/zll8a7u`

[4] As of 2018, most schools in the U.S. are trying to adjust to a new curriculum mandated by the Common Core Standards (see Common Core 2010). The outcome is uncertain.

[5] Keep in mind the preceding footnote.

[2] Common Core (2010). *Common Core State Standards for Mathematics*. Retrieved from: http://www.corestandards.org/Math/

[3] Schmid, W. and Wu, H. (2008) The major topics of school algebra (March 31, 2008). https://math.berkeley.edu/~wu/NMPalgebra7.pdf

[4] Wu, H. (2016). *Teaching School Mathematics: Algebra*. Providence, RI: American Mathematical Society. *Its* ***Index*** *is available at:* http://tinyurl.com/haho2v6

[5] Wu, H. (to appear). *Rational Numbers to Linear Equations, Algebra and Geometry,* and *Pre-Calculus, Calculus, and Beyond.*

From Arithmetic to Algebra, Part 2: How to Teach Arithmetic Better*

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In Part 1, we pointed out that a main focus of arithmetic in the school curriculum is on accurate computations with specific numbers. While introductory school algebra is also concerned with computations with numbers, it computes with known and unknown numbers alike—relying only on the laws of operations (associative and commutative laws of $+$ and $\times$ and the distributive law)—and begins to look for abstract patterns in numbers that are true for numbers in general. Now in Part 2, we will consider students' difficulty in making the transition from arithmetic to algebra and, more importantly, how to deal with this difficulty.

The usual school curriculum fails to address this difficulty. Some educators became aware of the difficulty of this transition and have come to advocate the introduction of "algebraic thinking" in the elementary grades, e.g., Blanton, 2018, Kaput, 2008, and Kieran, 2004. Their intentions cannot be faulted, but as in all things in mathematics education, good intentions are not enough because the devil lurks in the details. Given the fact that the present school algebra curriculum in the U.S. is very seriously flawed (cf. Wu, 2016b, especially Sections 1.1, 2.1, 4.3, 5.1, 7.2, 8.4, and 10.4),[1] one has to first find out what

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[1]The same can be said about the American school mathematics curriculum as a whole at the moment; cf. Section 2.3 and Appendix 2 of Wu, 2018. However, it would be naive to assume that U.S. is the only country that is struggling with school mathematics education.

this "algebraic thinking" is all about in the context of such a defective curriculum because, "What we think algebra is has a huge bearing on how we approach it" (Kaput, 2008, p. 8). In addition, the general recommendations for achieving this goal usually involve pedagogical embellishments and the introduction of new elements in arithmetic instruction (such as thought-provoking problems) while seemingly leaving unperturbed the existing defective arithmetic curriculum.

Our belief is that a simpler approach, one that sharply focusses on teaching *correct* arithmetic in elementary school would be more effective in bringing about improvement. One should not mistake this belief to mean that we are pursuing the teaching of correct arithmetic as an end in itself (although there is no denying that it is a laudable goal). The virtue of correct arithmetic is that it provides the right platform on which to launch algebra, as the following recommendations (1)-(7) will make this point abundantly clear. Correct arithmetic also happens to be *more learnable* than defective arithmetic, and it goes without saying that better-informed arithmetic students will be in a better position to learn algebra. As for the matter of the increased learnability of *correct mathematics,* there is now something close to a consensus in the U.S. (see, e.g., Common Core, 2010) that, for example, giving a fraction a precise definition as a point on the number line (Jensen, 2003 and Wu, 2001) is more pedagogically effective than making believe that a fraction is simultaneously a piece of pizza, a division, and a ratio. Moreover, defining finite decimals *correctly* as "decimal fractions" (Wu, 2011, or Wu, 2001) makes the comparing of decimals and the four arithmetic algorithms for decimals completely transparent, and transparency is clearly a prerequisite for learnability.

We will eschew generalities in the following discussion. In particular, we will not engage in making *broad* suggestions on how to reconceptualize parts of the arithmetic curriculum. We will, instead, get down to the fundamentals by suggesting specific changes in the *mathematics* taught in the arithmetic curriculum, and the specificity is made possible by the ability to reference the following six volumes by chapter and verse: Wu, 2011, 2016a, 2016b, and *to appear.* (We may add that it is precisely with the goal of achieving a wholesale change in the *mathematical content* of school mathematics that these six volumes have been written.[2]) That said, we will now indicate several key improvements that we believe should be made in the teaching of arithmetic to smooth students' transition from arithmetic to algebra.

For those who do not have access to the books of Wu, 2011, Wu, 2016a, and 2016b, we will make an effort to also reference their early drafts that are free and online: Wu, 2000, 2001, 2010a, and 2010b. Understandably, the expositions in the latter are less polished than the later versions in the books.

(1) *Mathematical* explanations (in addition to pictures and analogies) should be given for the standard algorithms in a grade-appropriate manner, including the emphasis on the importance of the associative and commutative laws of addition for the standard algorithms for addition and subtraction (e.g., as explained on page 68 of Wu, 2011 or pp. 46 and 49 of Wu, 2000), and the crucial role played by the distributive law in the multiplication algorithm and the long division algorithm (e.g., page 85 of Wu, 2011 or pp. 62, 66, and 82 of Wu, 2000). When students get to know the reasoning behind the algorithms, they can make better sense of the algorithms as well as these laws of operations (e.g., as explained on Chapter 2 of Wu, 2011 or Section 2 of Wu, 2000). At present, one reason these laws are not taken seriously by many teachers and students is that the existing curriculum does not put them to use in a mathematically substantive way to make any lasting impression on students. It therefore comes to pass that these laws are regarded as nothing more than things to memorize for acing standardized tests.

(2) Even in arithmetic, students can begin to learn about abstractions and structure. Indeed, the overriding theme of the four standard algorithms is that *a knowledge of single-digit computations empowers us to compute with any whole numbers, no matter how large* (see Chapter 3 of Wu, 2011, and it is of course repeated ad nauseam all through Chapters 4-7, *loc. cit.*; also see pp. 38-40 of Wu, 2000). If we remind elementary students of this fact, and do it often enough, then they would not only understand the reason these algorithms

[2]Precisely, the goal is to eradicate what we call Textbook School Mathematics (TSM) from K-12 mathematics education altogether (see Section 2.3 and Appendix 2 of Wu, 2018 for an explanation of TSM), and the above-mentioned six volumes show how this could be done by giving a complete and *coherent* exposition of the mathematics of the K-12 curriculum that is grade-appropriate, *and equally importantly,* mathematically correct.

are worth learning, but also become more familiar with abstract thinking and less likely to be shocked in their confrontation with algebra. (In fact, if this overriding theme were forcefully brought out by teachers in their teaching, they might be more successful in persuading students to memorize the multiplication table.)

(3) In the same vein, the overriding importance of the theorem on equivalent fractions (Chapter 13 of Wu, 2011, or Section 1.3 of Wu, 2016a; see also Section 3 of Wu, 2001) should be impressed on students, including its direct impact on the comparison (ordering) of fractions (pp. 31-35 of Wu, 2016a or Section 5 of Wu, 2001), the addition and subtraction of fractions (Section 1.4 of Wu, 2016a or Section 6 of Wu, 2001), and the multiplication and division of fractions (Sections 1.5 and 1.6 of Wu, 2016a or pp. 33-37 and 72 of Wu, 2010a). Without this understanding, students do not see equivalent fractions as the abstract unifying theme that connects all the above diverse skills. Rather, they think of fractions as a fragmentary subject and come to believe that the only reason for having this theorem is for simplifying fractions.

(4) Finite decimals should be defined as a special class of fractions (the *decimal fractions*) and taught as such (see Section 12.3 of Wu, 2011 or pp. 20-22 of Wu, 2010a). This approach is both historically and pedagogically correct (*loc. cit.*), and it is only from this vantage point that the four arithmetic operations—especially multiplication—on finite decimals can be made transparent and therefore more learnable (see Section 14.2, p. 256, p. 269, and Section 18.4 of Wu, 2011; see also pp. 48-49, 65-66, and 76-79 of Wu, 2010a). This is another opportunity for students to appreciate mathematical abstraction and structure when two kinds of seemingly different numbers are revealed to be basically one and the same.

(5) The parallel between the arithmetic operations on whole numbers and those on fractions should be stressed (pp. 173-174, 221, 262, and 284-286 in Wu, 2011, or pp. 46, 63, and 81-82 of Wu, 2001). The fact that this parallel enhances the learnability of fractions is too obvious for comment. Less obvious but no less important is the fact that, by emphasizing this parallel, we can reinforce students' appreciation for abstraction and structure: what they learn in whole numbers will help them learn fractions because these are similar topics. It used to be believed (*perhaps* less so in the last few years with the advent of the Common Core State Standards for Mathematics (see Common Core, 2010)) that "fractions are such different numbers from whole numbers", and this false belief has naturally hampered

student learning in fractions.

(6) The teaching of fractions should respect—in a grade-appropriate manner—the abstraction and generality that are inherent in the subject. If we were to make an effort to state the various basic facts about fractions concisely, then students would naturally (and gradually) learn about generality and become familiar with the use of symbols. There are ways to do this without sacrificing mathematical correctness, e.g., Jensen, 2003, or Part 2 of Wu, 2011.[3] For example, the theorem on equivalent fractions is the statement that for *any* fraction $\frac{m}{n}$ and for *every* nonzero whole number c,

$$\frac{m}{n} = \frac{cm}{cn}$$

Likewise, the cross-multiplication algorithm (one of the most important skills in fractions) is not "the butterfly" but the statement that, for *any* two fractions $\frac{m}{n}$ and $\frac{k}{\ell}$, $\frac{m}{n} = \frac{k}{\ell}$ is equivalent to $m\ell = nk$ (in the process, of course students will also learn what the phrase "is equivalent to" means). The formula for the addition of two *arbitrary* fractions $\frac{m}{n}$ and $\frac{k}{\ell}$ is

$$\frac{m}{n} + \frac{k}{\ell} = \frac{m\ell + kn}{n\ell}$$

And so on. Such exposure to generality and the use of symbols smooths students' passage to algebra.

(7) Finally, *we should strive to provide reasoning for every claim in arithmetic.* The need for doing this can be easily understood when one realizes that, whereas computations with specific numbers in arithmetic can be accomplished (at least on a superficial level) by the memorization and execution of rote skills, it is much more difficult to learn algebra by rote memorization because—as we pointed out about generalized arithmetic—the key issues in algebra are about the truth of statements concerning numbers *in general* (most often for *all* numbers). Reasoning becomes the only vehicle for navigating the terrain of algebra and we are therefore obligated to acclimate students to algebra by exposing them to the use of reasoning from day one. For example, every statement in Wu, 2011, or Wu, 2016a and 2016b is supported by reasoning.

[3] The treatment of fractions in Chapter 1 of Wu, 2016a, is slightly more sophisticated, but we have to mention it because it is one that happened to serve as a blueprint for the Common Core Standards.

We should also single out a special topic in arithmetic for discussion, as it points to a pervasive curricular defect which is most likely not limited to the U.S. Problems involving *constant speed* or *constant rate* not only are a staple of arithmetic, but also occupy a singular position of prominence in the general discussion of arithmetic (even in popular culture!). It is time to rectify past errors in the teaching of *constant speed* and get it right for students by *defining* this concept correctly. Such a definition is absolutely essential, for two reasons. The first is that without a definition, students are forced to solve problems about speed or work *entirely by rote*. For example, a prototypical problem such as "If Helen walks $2\frac{1}{2}$ miles in an hour, how far does she walk in $1\frac{1}{3}$ hours?", simply *cannot* be solved (except by rote) without the assumption that Helen always walks at the same constant speed (see, e.g., the fairly detailed discussion of this common misconception in Section 7.2 of Wu, 2016b; one can also consult pp. 96-99 of Wu, 2010a). A second reason is that a precise definition of constant speed or rate will inevitably include the use of abstraction and symbolic notation. One possible definition of a motion in *constant speed* is that, with a unit of time (e.g., hour) and a unit of distance (e.g., mile) chosen, a motion is said to be traveling at a **constant speed of *s* mph** if in *any* time interval $[t, t']$,

$$\frac{\text{distance traveled in the time interval } [t, t']}{t' - t} \;=\; s$$

(see pp. 111-112 of Wu, 2016a). The weight of this definition is of course on the fact that the quotient is equal to s for *any* t and t' so that $0 \le t < t'$. Exposing students to such a definition will further enhance students' ability to learn algebra.

There is at least one other consideration that has an important bearing on students' learning of introductory algebra: the need to infuse the geometric concept of similarity into the teaching of *slope* and the graphs of linear equations in two variables. Students' confusion over slope in the U.S. is well-known. However, since this issue falls outside our main concern here, which is the transition from arithmetic to algebra, we will simply refer readers to the existing literature: Section 4.3 of Wu, 2016b or pp. 57-62 of Wu, 2010b.

We have made an effort to explain in some detail the meaning of introductory algebra as *generalized arithmetic* and how we might improve the arithmetic curriculum to facilitate students' transition from arithmetic to algebra. We do not pretend that achieving such improvement will be easy as it involves sustained professional development for teachers and the creation of reasonable textbooks for students. But we *must* try.

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