What do the Standards for Mathematical Practice mean?*

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*I am indebted to Larry Francis for many wonderful suggestions.

There is a common perception that the Common Core State Standards for Mathematics (CCSSM) are not fundamentally different from other sets of standards from the past. The perception is that the CCSSM may have reshuffled the order of presentation of the topics a little bit, but they deal with the *same kind of mathematics.*

However, the Standards for Mathematical Practice (MP) in the CCSSM are perceived to be a different kind of animal. They are said to represent the "vision of the common core".

Eight MP:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

In the Spring 2013 NCSM Newsletter, David Foster of SVMI was quoted as saying that the MP are "the best thing about the CCSSM". Furthermore,

"These are the verbs of mathematics—what students should be doing while engaged with mathematics. I believe the practices will be the most influential aspect of the CCSS; *their use* [emphasis mine] will shift the mathematical thinking from the teacher to the students. If fully enacted the practices will have a dramatic impact on student learning."

So the MP *are there to be used*. Just like that. *Post the MP around the classroom. Study them. Test students on them.* That would do it. No sweat at all.

Many people agree.

This is why one-day or two-day sessions on the MP are all over the country. This is the silver bullet that will supposedly unlock the secret to the implementation of the CCSSM.

See if your own experience resonates with some of the stories that have come my way:

Our curriculum director, xxxx xxxxx is still pushing (and I am sure she is not alone across the country) for the Mathematical Practices to be graded . . . yep, each one . . . on the report card . . .

Both xxxx and I had experience with teachers posting the MP in their classroom and defining them with their students.

[An 8th grade teacher] uses the MP as a teaching tool for the students. She often asks her students to read the poster [of the MP]. I asked her to define the MP and provide examples of how she uses it in class. Unfortunately, she was only knowledgeable about #1, 3, 7, and 8.

Our supervisors go around the schools with a checklist of Practice Standards to evaluate teachers.

All this may be all right, except that the following communication from a high school teacher is too jarring to ignore:

I will be implementing Common Core in the fall. I've had a fair amount of professional development on the common core, but the mathematics curriculum will not change until later.

The idea seems to have taken hold: **“Do” the MP first, and implement the CCSSM curriculum later.**

First of all, your textbook publishers have already done a superb job of that!

How to make a textbook

aligned to the Common Core?

Answer: Reprint the old textbook, but sprinkle lots of MP in margins and put on a jazzy new cover.

Let us go a step further and look at the following *vignette.*

A supervisor visits a fourth grade classroom where the teacher, Mr. Sherman, has just introduced the concept of **equivalent fractions** and explains that, for example, $\frac{2}{3} = \frac{8}{12}$ because,

$$\frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{4}{4} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}.$$

Then the following discussion takes place among students:

Carl: Hmmm, I seem to also get $\frac{2}{3} = \frac{10}{15}$ because

$$\frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{5}{5} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}$$

So $\frac{8}{12} = \frac{10}{15}$. Could this be right?

Bryant: I guess so. Yes, it must be right.

Abby: What does it mean to say $\frac{8}{12}$ is "equal to" $\frac{10}{15}$?

Diane: They name the same amount.

Abby: Same amount of *what?*

Diane: Anything!

Abby: Like a *chair?*

Bryant: C'mon, don't be silly. Use something nice, like the same amount of pizza.

Abby: Oh, I have to choose carefully *each time?*

Carl: You know, I have thought about it, and I don't know why $\frac{2}{3} \times \frac{5}{5} = \frac{2 \times 5}{3 \times 5}$.

Bryant: Look, you see 2 and 5 on top with $\times$ in between, and you multiply. The same with 3 and 5. You know how it is with whole numbers, right?

Carl: Is that how you do it? So $\frac{2}{3} + \frac{5}{5} = \frac{2+5}{3+5}$?

Diane: Oh, this is neat. Now we can add fractions too!

Abby: Are you sure? Mr. Sherman didn't say so. Why don't we wait?

Carl: You are right. We've to wait and see what Mr. Sherman has to say. But it sure looks like it.

The supervisor is very impressed.

First and foremost, the students are making sense of problems and persevere in solving them (**MP1**). They construct viable arguments and are surely critiquing the reasoning of others (**MP3**).

They learn to reason abstractly when Carl concludes that $\frac{8}{12} = \frac{10}{15}$. (**MP2**) They also attend to precision (**MP6**) when they enter into a careful discussion of the meaning of "naming the same amount" and the equality of two fractions.

The leap from $\frac{2}{3} \times \frac{5}{5} = \frac{2 \times 5}{3 \times 5}$ to $\frac{2}{3} + \frac{5}{5} = \frac{2+5}{3+5}$ is encouraging because they are making use of structure (**MP7**) to make a conjecture (**MP3**). The result may be wrong, but *it is the process that counts.*

Therefore, in one short discussion among students, the supervisor gets to witness 5 (five!) MP in action (**MP1, 2, 3, 6, 7**). This is a triumph for the Common Core.

What is not to like?

She writes a glowing report for Mr. Sherman.

But let us see what the CCSSM have to say about this.

Because the fact about *equivalent fractions* is foundational to almost everything in the development of fractions—including addition and multiplication—the explanation of this fact cannot make use of the multiplication of fractions. Mathematics does not tolerate circular reasoning.

This is the hierarchical structure of mathematics (**MP7**), and a teacher is obligated to make students aware of it. Apparently Mr. Sherman didn't.

A key step in his calculation,

$$\frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{4}{4} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$$

is easily seen to be all wrong.

His students have modeled their reasoning on *a piece of circular reasoning.*

The starting point of any reasoning is having precise *definitions* of the concepts.

MP6 actually says, "[Students] try to use clear definitions in discussion with others and in their own reasoning."

However, it does not appear that the students had any definition for a *fraction* or for *equal fractions*. Recall Abby's question: *What does it mean to say* $\frac{8}{12}$ *is "equal to"* $\frac{10}{15}$*?*

In grade 3 of the CCSSM, equivalent fractions are clearly defined to be the same point on the number line (Abby and Mr. Sherman missed this).

3.NF 3a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

This definition then opens the door to an explanation of equivalent fractions by subdivisions on the number line: No *fraction* multiplication is needed.

4.NF 1. Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size.

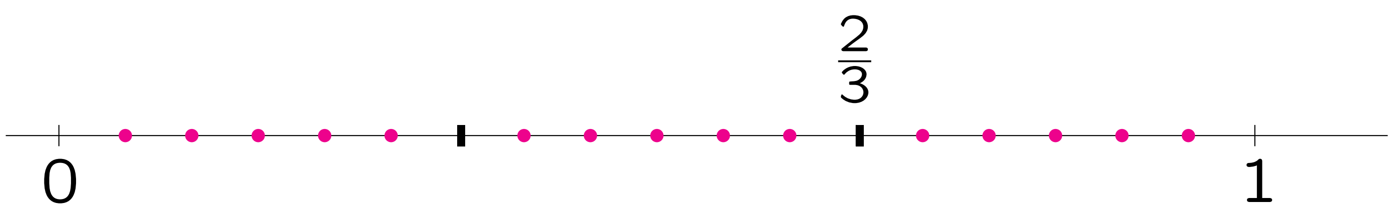
Let us explain Carl's claim: why $\frac{2}{3} = \frac{10}{15}$.

$\frac{2}{3}$ is the 2nd division point to the right of 0 when the unit segment $[0, 1]$ is divided into 3 segments of equal length:

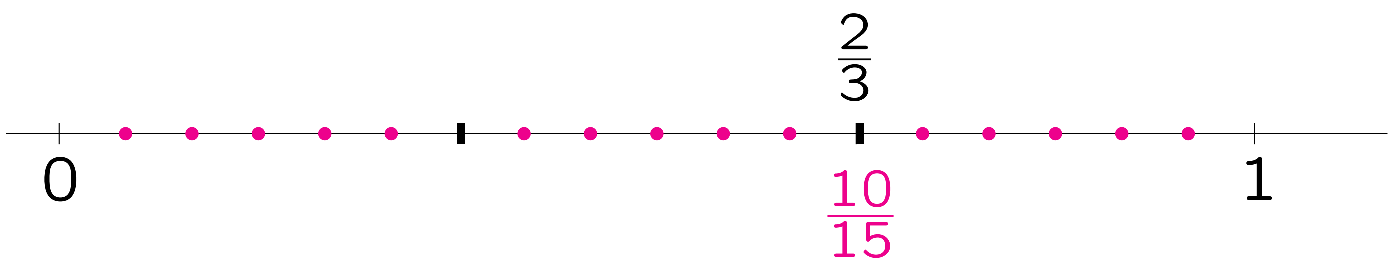
Similarly, $\frac{10}{15}$ is the 10th division point to the right of 0 when the unit segment $[0, 1]$ is divided into 15 segments of equal length.

We must show that the 2nd point to the right of 0 in the sequence of $\frac{1}{3}$'s is also the 10th point to the right of 0 in the sequence of $\frac{1}{15}$'s.

We divide each of the $\frac{1}{3}$'s into 5 *equal parts*, getting a division of $[0, 1]$ into $\frac{1}{15}$'s $(3 \times 5 = 15)$:



We see that the 2nd point to the right of 0 in the sequence of $\frac{1}{3}$'s is also the 10th point to the right of 0 in the sequence of $\frac{1}{15}$'s.



Mr. Sherman should have used this reasoning but he didn't.

Moreover, Mr. Sherman used *fraction* multiplication without saying what it means or why $\frac{2}{3} \times \frac{4}{4} = \frac{2\times 4}{3\times 4}$. Students were misled into making wild extrapolations of this computation to the addition of fractions. Bryant's spurious argument ("you see 2 and 5 on top with $\times$ in between, and you multiply") went unchallenged.

Mr. Sherman's class seems not to be accustomed to correct reasoning. **MP3** *becomes meaningless in this context.*

At the end, students decided to let Mr. Sherman tell them whether fractions can be added by adding the numerators and denominators.

But reasoning should be objective. Students who exemplify **MP3** do not wait for an authority figure to tell them what is right or what is wrong. They reason it through to decide for themselves what is right and what is wrong.

There *is* something good about the students' discussion, but in order to minimize distraction, let us just say that, *overall*, the students failed, in a pronounced way, to enact the MP.

The supervisor should have had little reason to be impressed.

You may begin to get the idea at this point that:

(1) The CCSSM are not about the *same kind of mathematics* of the past, at least it is not about the math in *standard school textbooks.*

(2) The MP do not provide the royal road to the implementation of the CCSSM.

You may study the MP very hard, you may discuss at length each and every word of the MP, and you may recite the MP as mantras as often as you want. But in order to *put them to good use*, teachers, students, and supervisors must know mathematics.

The supervisor's lapse in judgment (see *vignette*) is a good indicator that *when divorced from content knowledge, the MP have no substance.*

For example, if we want students to look for and make use of structure (**MP7**), they must know something about structure in mathematics in the first place. For example, the *hierarchical structure* that we mentioned above.

Students have to be routinely exposed to a *logical* development of mathematical concepts and skills for a long time before we can expect them to have any idea what hierarchical structure means.

Given the textbooks we have, and given the present level of classroom instruction, how likely is it that students could acquire this knowledge about hierarchical structure?

Without this knowledge, how can they look for structure and make use of it (**MP7**)?

We have two obstacles: the quality of textbooks and teachers' content knowledge. *We need to discuss both carefully* (but more of this later).

For now, let us continue the discussion of structure. An example of *structure* on a smaller scale is the concept of **division**. Do we teach students that this concept has the same structure for whole numbers, fractions, and *rational numbers* (i.e., fractions and negative fractions)?

Def: Let **N** stand for *whole numbers*, or *fractions*, or *rational numbers*. If a and b are numbers in **N**, and $b \neq 0$, then $\boldsymbol{a \div b}$ is the *unique* number c in **N** so that

$$a \;=\; c\,b$$

(We have to fudge a bit when **N** is the whole numbers.) In fact this definition remains valid even when **N** is the *real numbers* or the *complex numbers*.

What has this got to do with math learning?

Everything! When the division concept is shown to have the same *structure* throughout, it facilitates learning because students have less to memorize or to learn through the years.

To illustrate, *why* does $\dfrac{21}{-7}$ equal -3?

Note that $\frac{21}{-7}$ is the common way of writing $21 \div (-7)$.

By our definition, $\frac{21}{-7}$ is the unique rational number c so that $21 = c(-7)$. But we know $(-3)(-7) = 21$, so this -3 is the number c.

For the same reason (though slightly more subtle),

$$\frac{5}{-3} = \frac{-5}{3} = -\frac{5}{3}$$

You may be telling yourself that it is so much simpler to just *memorize* the rule: When dividing two numbers of opposite signs, the quotient is negative.

So just know two things: (**1**) *Teaching by rote* is the most effective way of killing math learning. The CCSSM ask for change:

7.NS 2c. Apply properties of operations as strategies to multiply and divide rational numbers.

(**2**) If students are not routinely exposed to such explanations as why $\dfrac{21}{-7}$ equals -3, how can they be expected to enact the MP?

Students will never know what it means to "reason abstractly" (**MP2**) or to "use definitions in reasoning" (**MP6**) if they don't get to see them in action.

So once more: *One can only get to know the MP by putting correct content knowledge into practice.*

Now is the time to return to the two obstacles: the quality of textbooks and teachers' content knowledge.

The mathematics in all the "CC-aligned" textbooks that I have seen violates most of the content standards in the CCSSM.

Most of our teachers do not possess the content knowledge to teach according to the CCSSM because we have collectively failed to provide them with this knowledge.

I will now explain why, if we can help teachers acquire the needed content knowledge, then we will solve *both* problems at the same time.

A representative of a major publisher once told me point-blank: *We only publish textbooks that teachers want to read.*

Just think about the "bottom line" of any business, and about how textbooks are ultimately adopted by teachers, and you will understand.

If we can help teachers see the *mathematical* flaws in the existing school textbooks, they will be confident enough to tell the publishers to clean up their act. Only then will publishers consider publishing better textbooks.

But not before.

We owe it to teachers to provide them with content-based PD to make up for lost time, *because:*

When they were in K–12 as students, they had to learn from textbooks with the same flaws as the textbooks in current use—CC-aligned or not. Their knowledge of school mathematics is thereby impaired, thanks to our collective negligence.

Let us call this body of flawed "knowledge" **TSM** (*Textbook School Mathematics*).

When teachers were in college, they did not learn any more about school mathematics, because colleges did not (and still do not) teach pre-service teachers how to rectify the errors in TSM. Our collective negligence again.

It therefore came to pass that our teachers' knowledge of school mathematics was (and is) essentially TSM.

TSM is too full of flaws to be able to support the learning or teaching of the CCSSM curriculum.

For example:

"Researchers have consistently found that teachers lack a deep conceptual understanding of fractions, and that teachers' mathematical content knowledge is positively correlated with students' mathematics achievement." (**What Works Clearinghouse**, *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*, 2010.)

The glaring omission in this extremely misleading statement is a *mea culpa* from What Works Clearinghouse: *The guilty party responsible for teachers' lack of understanding is the education establishment.*

The teachers need all the help we can give.

What they need the most, here and now, is

content-based professional development.

Part of the difficulty with in-service PD (professional development) is that it has a bad tradition, as reported in an Education Week Article of February 2013: *Teachers Say They Are Unprepared for Common Core*:

> "Due to resources, professional development is still the drive-by variety" in most districts, said the AFT's Ms. Dickinson.

In-service PD often means fun and games, new manipulatives, pedagogical strategies, and classroom projects that you can use the next day.

Other times, it means making teachers feel good about themselves, making them feel that *they already know math*, or that mathematics can be learned without hard work.

On the opposite end of the spectrum, there is PD that teaches *fun math* or even *good math* to teachers, e.g., taxicab geometry, finite geometry, Pick's theorem, field axioms, Gaussian integers, etc.

No mathematical knowledge is irrelevant, but for now, the need to address the bread-and-butter issues of the CCSSM is so urgent that we must push these aside and get on with our main task: teach the content imbedded in the CCSSM.

Now, many claim to be *concentrating on content*, but “content” means different things to different people.

For example, W. Gary Martin, Professor of Mathematics Education at Auburn University, was quoted in an Education Week article as saying, “We’re still heavily focused on more procedural goals, more about the content than building the understanding and reasoning abilities.”

To Martin, "content" is equated with procedural knowledge!

When I (or any professional mathematician) talk about "content-based PD", content means *procedural knowledge, understanding, and reasoning combined.* In order words, all that the MP are trying to promote.

Content-based PD, in this sense, is very rare. But this is what teachers truly need in order to implement the CCSSM!

If my presentation today serves any purpose, it would be to impress on you the fact that *the only way to enact the MP is through content-based PD.*

How to make such PD possible would be the topic of another two or three presentations. But there are a few documents on my homepage that can serve as a guide.

There are three areas in the school curriculum where the CCSSM have made the most substantial contributions: fractions, negative numbers, and the geometry of grade 8 and high school.

The following three documents on my homepage

http://math.berkeley.edu/~wu/

address fractions and geometry.

Teaching Fractions According to the Common Core Standards 2011 (For teachers and math educators), 88 pages.

Teaching Geometry According to the Common Core Standards 2013 (For teachers of grades 4-12 and math educators), 156 pages.

Teaching Geometry in Grade 8 and High School According to the Common Core Standards 2013 (For teachers of grades 8-12 and math educators), 201 pages.

These documents are long, but not detailed enough to serve as textbooks for PD.

However, they will give you a good idea of what *content* means and what your PD ought to be about. They give a *detailed outline* of what is important about fractions and geometry in the CCSSM curriculum.

The following two documents are about CCSSM rational numbers and algebra.

Pre-Algebra 2010 (Draft of textbook for teachers of grades 6-8), 358 pages.

Introduction to School Algebra 2010 (Draft of textbook for teachers of grades 6-8), 216 pages.

They can all be accessed for free. So I wish you good luck and happy reading.

The last two documents, in drastically revised form, will be published in two volumes in 2016 as *From Pre-Algebra to Algebra*. It will not be free, nor is the next reference, which gives a more detailed and more comprehensive exposition of the mathematics of K–6 than the preceding documents.

Hung-Hsi Wu, *Understanding Numbers in Elementary School Mathematics*, Amer. Math. Society, Providence, R.I., 2011.