Professional Development and *Textbook* **School Mathematics***

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*With the exception of the Appendix, this is the transcript of a presentation to the Committee on Education of the American Mathematics Society, on October 28, about the state of professional development in general, and my own effort along this line in particular.

The high-profile NAS volume, *Rising Above the Gathering Storm* (2007), envisions the end of American leadership in science and technology in the coming decades. It makes four recommendations for change, and the first is to **"Increase America's talent pool by vastly improving K-12 science and mathematics education."**

The recommended action of highest priority is to **place knowledgeable math and science teachers in the classroom.**

The question is **HOW**?

We have a serious content-knowledge deficit problem among math teachers in our nation.

This is not teacher-bashing. This is essential, professional diagnosis of a national crisis.

I have been teaching teachers since 2000, and I know firsthand how badly we have treated them: Thus far, we have not taught them what they need to know in order to carry out their basic duties.

Professional development (**PD**) for in-service math teachers is generally taken to be "feel-good sessions". Some believe that its main goal is to give teachers encouragement and sharpen their pedagogical skills.

Others believe that teachers should be exposed to *fun* mathematics (such as the Königsberg bridge problem or taxicab geometry), even in the face of their inability to deal with bread-and-butter issues such as how to teach fractions, why negative times negative is positive, what *similarity* means, or why the parallel postulate is important.

The better kind of in-service PD, which is experienced by a small percentage of teachers, does address topics of substance such as students' mathematical thinking, teacher-student communication, and refined teaching practices.

Somewhere, the issue of content knowledge *will* surface in such a discussion. But when content knowledge is only one of many things that clamor for teachers' attention, it will not be properly addressed—at least not in 2011—and I will explain why in a minute.

At present, pre-service PD is generally of two varieties:

For elementary teachers, professors are satisfied with embellishing the procedures in K–6 mathematics just a little bit. But do they teach what teachers truly need to know? What long division means? What a fraction is? No.

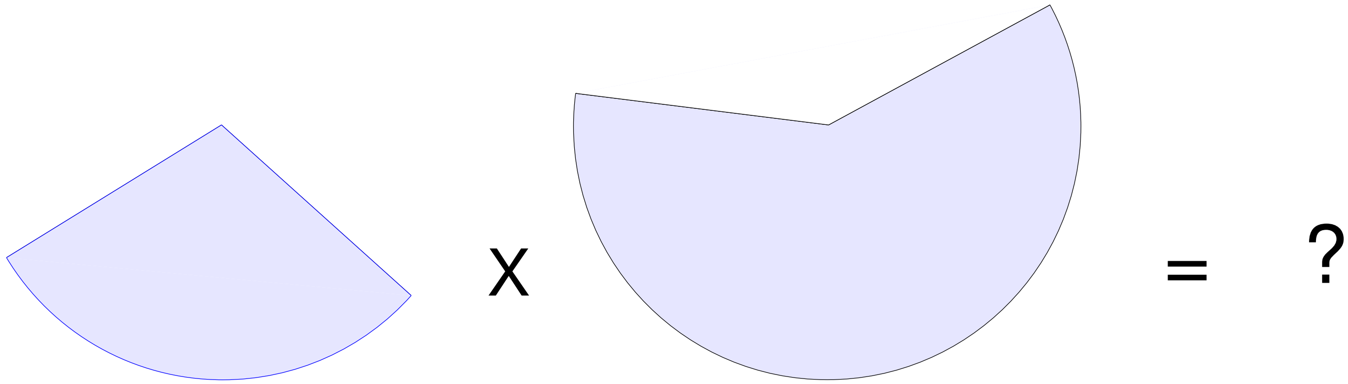
For high school teachers, we simply ask them to learn "advanced" mathematics because it is good for their souls. We firmly believe in the **Intellectual Trickle Down Theory:** By learning a lot of abstract mathematics, they will better understand elementary mathematics and become good school teachers. *Except that this theory doesn't work, not in theory and not in practice.*

Universities do not give teachers the mathematical knowledge they need to teach. *What teachers know about school mathematics is what they were taught as students in K–12.*

So what were they taught in K–12? This is a long story but it has to be abbreviated to just three sound bytes for the moment.

(1) Students are taught to write $27 \div 4 = 6 \ R \ 3$ instead of $27 = (6 \times 4) + 3$. So writing $69 \div 11 = 6 \ R \ 3$, we see that $27 \div 4 = 69 \div 11$.

(2) Students are taught that a fraction is like a piece of pie. So how to multiply two pieces of pie?



$$\frac{2}{7}\times\frac{3}{5} = ?$$

Because elementary teachers were taught mathematics only by analogies and metaphors, they also teach their own students only by analogies and metaphors. They want a more robust knowledge of mathematics but universities do not give it to them.

(3) A problem involving **"proportional reasoning":** A group of 8 people are going camping for three days and need to carry their own water. They read in a guide book that 12.5 liters are needed for a party of 5 persons for 1 day. How much water should they carry? (*NCTM Standards* (1989), page 83.)

This is meant to illustrate the importance of learning how to think *proportionally* (if one person drinks ℓ liters a day, then 5 persons drink $5\,\ell$ liters a day) except that *a crucial hypothesis is missing:* everybody drinks roughly the same amount each day.

Teachers should be teaching students how to make logical deductions from an assumption: *All people drink roughly the same amount each day* $\Rightarrow$ if one person drinks ℓ liters a day, then 5 persons drink $5\,\ell$ liters a day.

Instead, teachers are misled into emphasizing the need for ingenuity in making up hypotheses as they go along.

Existing textbooks and the education literature perpetuate innumerable myths such as $27 \div 4 = 6\ R\ 3$, multiplying two pieces of pie, and why students must acquire "proportional reasoning".

Right now, there is little awareness in the school culture or the mathematics education literature that such myths have no place in mathematics education.

This defective body of mathematical knowledge has taken hold in schools *and* in the mathematics education literature. This is the mathematics that has been embedded in standard textbooks for decades, and it defines our de facto **national mathematics curriculum**.

I call it *textbook* school mathematics (**TSM**).

"*We have found the enemy and he is us.*"†

†I am indebted to Larry Francis for the URL.

Teachers have no choice but teach TSM to their students **because their university education did not teach them anything better.** This is how TSM gets recycled in schools.

I do not believe any real progress can be made in school mathematics education until we eradicate TSM from school classrooms and from the mathematics education literature.

Helping teachers to replace their knowledge of TSM with a correct version of school mathematics has to be a primary obligation of every kind of PD as of 2011.

Universities must begin to take this obligation seriously.

Professional developers must also take this obligation seriously.

In this context, I want to bring up the volume published by CBMS in 2001, *The Mathematical Education of Teachers* (**MET**). This volume provides guidance to all mathematics departments on how to educate future teachers.

To the extent that MET seems oblivious to the defects inherent in TSM (it actually repeats a good many of them), this guidance may be misleading.

MET is being revised for a second edition. I believe you will be as eager as I to see the new MET declare war on TSM.

Now the *Common Core Standards* (CCSS) come along.

To a large extent, the CCSS begin to set the record straight. *Overall*, they succeed in cutting a path through the TSM jungle. There is hope that CCSS will one day banish these myths from schools.

PD must now confront TSM.

The need for content-based PD becomes urgent.

"Content" is easily said but much less easily done. *WHY?*

Because any content-based PD must make *foundational changes* in teachers' content knowledge. It has to help teachers replace the TSM they learned in 13 years of schooling with something that is mathematically sensible.

Foundational changes cannot be achieved in a few half-day sessions or even in a few days.

There is no shortcut to such a makeover.

In my personal experience, *when content knowledge is only one of many topics of concern in PD, it will not get the attention it deserves, nor will it inspire the needed effort to make such foundational changes.*

Perhaps this is only one man's speculation. But perhaps not.

For multiple reasons, we will put this conjecture to the test by looking at a notable recent example.

In 2007–2009, the U.S. Department of Education conducted a large scale study of the impact of PD on 7th grade teachers and students: *Middle School Mathematics Professional Development Impact Study. Findings after the Second Year of Implementation*. The focus of the study was on *fractions, decimals, percent, ratio, rate, and proportion*. And the result?

> *In summary, the study results indicate that after two years of implementation, the PD program did not have a statistically significant impact on teacher knowledge or on student achievement in rational numbers. (page 53)*

(The "rational numbers" here refers to nonnegative rational numbners, i.e., *fractions*.)

This is a devastating blow to the belief that the success of CCSS depends on effective PD to *increase teacher content knowledge*. **But is it?** What exactly was this intensive PD?

In the first year, teachers were given:

> *3 institute days of content instruction (18 hours)*
>
> *5 one-day follow-up seminars during the school year (30 hours)*
>
> *10 days of coaching (20 hours)*

A total of 68 hours; **48** hours of institute and seminars.

In the second year, they were given:

2 institute days of content instruction (12 hours)

3 one-day follow-up seminars during the school year (18 hours)

8 days of coaching (16 hours)

A total of 46 hours; **30** hours of institute and seminars.

On the positive side, such PD is far more than what most teachers get in a single year:

> A 2005–2006 national survey of teachers found that only 11 percent of elementary teachers and 22 percent of secondary math teachers participated in any PD lasting more than 24 hours.

On the other hand, *percent, ratio, rate, and proportion* are among the most feared topics in middle school mathematics.

Why fear? All of these topics come *after* the division of fractions. Since a fraction is a piece of pie, dividing two pieces of pie is *very* hard. So TSM has inspired the ditty: "ours is not to reason why, just invert and multiply". How does one proceed on such a shaky foundation?

More than that, the definition of **percent** in TSM is that it means "per hundred", or "out of a hundred". On this basis, try to explain

what percent is 9 out of 17?

As to **ratio**, TSM has definitions like these:

> A ratio is a comparison of any two quantities. A ratio may be used to convey an idea that cannot be expressed as a single number.
>
> A ratio is a way to describe a relationship between numbers. If there are 13 boys and 15 girls in a classroom, then the ratio of boys to girls is 13 to 15.

If so, what then is the ratio of the circumference to the diameter of a circle?

About **rate**, TSM has confounded generations of students and teachers by flaunting this concept without ever pointing out that *rate* has no definition in school mathematics. (One has to use calculus.)

School mathematics can only discuss **average rate**, but TSM hardly ever gives this concept a precise definition. Consequently, TSM completely messes up the concept of **constant rate**, which is the backbone of any discussion of *rate* in K–12.

Hence the fear of rate problems.

Thus, to improve teachers' knowledge of *percent, ratio, rate, and proportion*, effective PD must help teachers overcome *many* obstacles of a foundational nature.

First, the PD has to revamp what teachers learned from TSM about fractions (this is a tall order). Then it must introduce correct definitions of percent, ratio, and rate, *and help teachers reason with them*.

PD that fails to do this cannot hope to improve teacher knowledge, and therefore will not lead to improved student achievement.

In the two years of the *PD impact study*, teachers learned mathematics mainly in the institute and seminar days: 48 hours in the first year, and 30 hours in the second year.

Could $48 + 30$ hours spread over two years bring about the needed foundational changes even under the best of circumstances?

Not likely.

Not when the content instruction seemed unaware of the pervasive defects in textbooks (i.e., TSM) or the need for an overhaul. See Appendix B of the report.

Not when "the PD was not presented to teachers as an opportunity to improve their understanding of rational number content". (p. 21)

("Rational number content" is the way the report refers to *fractions, decimal, percent, ratio, rate, and proportion.*)

Indeed, in the PD of this two-year study,

> *the focus of the presentation in both years was on SK, and instruction in common knowledge of mathematics content (CK) was mainly implicit. (p. 21)*

SK is "the additional knowledge of rational numbers that may be useful for teaching rational number topics".

CK is "the knowledge of topics in rational numbers that students should ideally have after completing the seventh grade".

Therefore, on the face of the evidence, the only valid conclusion one can draw from this **PD Impact Study** may be this:

> *The intensive PD had no impact on teacher content knowledge or student achievement. There may have been a miscalculation in deciding* **how much focus must be placed on content** *in order to help teachers overcome TSM.*

What does this *two-year PD impact study* say about CCSS?

If Common Core hopes not to repeat the debacle of the **New Math**, then it must have knowledgeable teachers to implement the standards. This PD impact study sends the unmistakable signal that, to this end, one must rethink what PD ought to be.

But what one gathers from recent meetings and pronouncements on PD seems to be that *doing more of the same more vigorously will be good enough for CCSS.*

Can’t we learn from this PD impact study?

Can’t we learn from the New Math?

Shall we bear witness yet again to the sorry spectacle of math teachers mouthing words they do not understand?

Shall we resign ourselves to the fate of CCSS being just a passing fad, the same way that the New Math was just a passing fad?

Suppose we accept the need for “intensive” and “long term” PD focused on content. What does this mean?

The reason I am speaking to you today is of course to tell you my own interpretation of these terms.

I have been teaching summer institutes for elementary and middle school teachers every summer since 2000. The basic structure of these institutes has not varied.

1. Three weeks, 8 hours a day; daily homework assignments.

2. Five hours of lectures and discussions on mathematics; two hours of small group discussions of mathematics and homework.

3. Five Saturday follow-up sessions in the succeeding school year to discuss implementation, 6 hours each session.

This amounts to 120 hours of content instruction, and 30 hours of discussions of classroom implementation.

Teachers are paid $100 a day. (It is not enough.)

This then begs the question of what I mean by “content” (content is easily said).

I have taught three kinds of summer institutes, starting with whole numbers and ending (roughly) with Algebra I. (*See next slide.*) Together, they cover the mathematics of K–8.

Each institute is designed to be the prerequisite of the next, but I have had occasion to teach each of them by itself.

1. [Elementary] Whole number algorithms; arithmetic of fractions; elementary number theory (e.g., divisibility rules, Euclidean algorithm).

2. [Pre-algebra] Percent, ratio, rate; negative numbers; translation, rotation, and reflection; congruence; dilations and similarity; length and area.

3. [Beginning algebra] Use of symbols, expressions, equations, geometry of linear equations, functions and their graphs, linear functions, linear programming; laws of exponents; quadratic functions and their graphs.

For the elementary institute, see:

H. Wu, *Understanding Numbers in Elementary School Mathematics*, American Mathematical Society, 2011.
http://www.ams.org/bookstore-getitem/item=mbk-79

For the pre-algebra institute, see:

http://math.berkeley.edu/~wu/Pre-Algebra.pdf

For the algebra institute, see:

http://math.berkeley.edu/~wu/Algebrasummary.pdf

It may be worth pointing out that, although these three sets of materials were essentially written no later than 2006, their content and logical development mirror that of CCSS.

Moreover, the second document above, i.e.

http://math.berkeley.edu/~wu/Pre-Algebra.pdf

is identical to the reference cited on page 92 of CCSS as

Wu, H., "Lecture Notes for the 2009 Pre-Algebra Institute," September 15, 2009

The goal of these documents is always to systematically replace teachers' knowledge of TSM with a usable version of mathematics.

Altogether, these summer institutes retrace the mathematics of grades K–8 as it is taught in those grades, *but in a way that is appropriate for teachers*.

For example, I start with whole numbers, but I do *not* use the Peano axioms because one doesn't do that in the primary grades. I also do not babytalk the mathematics, because I deal with adults.

You would want to ask me about my success rate, of course. I too would like to know, but the federal funding agencies (e.g., NSF-EHR) did not see any merit in my comprehensive evaluation proposal. So that was that.

(The funding of these summer institutes has always been haphazard; I simply took whatever was offered to me. I never succeeded in obtaining federal or state grants.)

I teach 25 to 30 teachers a year on average, so it is a small output. Each year I ask for, and get, anonymous detailed evaluations from all the teachers. They are encouraging, but I have no data to make any claims. For an early example of such evaluations, see the Burmester-Wu article, Lessons from California (2001).

Do I think teachers learn all they need to learn in three weeks? *No,* but I believe they learn something substantial by the last Saturday follow-up session (nine months after the summer institute). I also believe they learn much more when they come back to take a second institute.

Let me mention in passing that I also did pre-service PD for high school teachers in the Berkeley math department from 2006 to 2010. Berkeley has a *Math Major with a Concentration on Teaching*, and at the heart of this major are three new courses for prospective high school teachers.[‡]

[‡]See the Appendix for more details.

Math 151: Fractions; rational numbers; elementary number theory; basic isometries and congruence; dilations and similarity; use of symbols and polynomials.

Math 152: Functions; linear functions of one and two variables; linear inequality and linear programming; quadratic, exponential and logarithmic functions; Euclidean geometry and discussion of axiomatic systems.

Math 153: Trigonometry and periodic functions; limits and LUB; decimal expansions of real numbers and fractions; length, area, and volume; continuous functions; differentiation and integration; logarithms and e^x; laws of exponents.

These three courses, established in 2006, develop the mathematics of (more or less) grades 6–12 carefully, grade by grade, in order to replace prospective teachers' knowledge of TSM.

Again, these courses closely parallel the scope and sequence of the CCSS curriculum from grade 6 to roughly high school geometry.

I am at present writing textbooks for them. (Optimistic date of completion: 2012.)

Appendix

I will give some details about the *Math Major with a Teaching Concentration* at the University of California, Berkeley. The requirements of this Concentration consist of satisfactory completions of the three new courses, Math 151–153, created specifically for this purpose (see below), in addition to two years of the usual lower division calculus sequence and the following courses:

55 **Discrete Mathematics** (sophomore level)

110 **Linear Algebra** (junior level; elementary linear algebra is part of the lower division calculus sequence)

113 **Introduction to Abstract Algebra** (junior level)

Any two out of {128A **Numerical Analysis** (junior-senior level), 130 **The Classical Geometries** (senior level), 135 **Introduction to the Theory of Sets** (senior level)},

160 **History of Mathematics** (senior level)

Statistics 20 **Introduction to Probability and Statistics** (sophomore level)

Students in this program will also be encouraged to take 104 **Introductory to Analysis** (junior level), 115 **Introduction to Number Theory**, and 185 **Introduction to Complex Analysis**. In some cases, we allow the substitution of 104 for 135 for the fulfillment of the major requirement. The total number of required upper division courses is the same as that of the normal requirement of a regular Math Major.

Content and rationale of Math 151–3

The following three courses directly address the mathematical needs of teachers in grades 7-12, but especially in grades 9-12. In a literal sense, these courses constitute "*Elementary Mathematics from an Advanced Standpoint in year 2006*".

Each course is 4 units, and meets four hours per week: three hours of lectures plus an addition hour of problem-solving. For prospective teachers, the latter is an absolute necessity.

Math 151

Prerequsites: Math 1A-1B, 53 (lower division calculus courses)

Development of the rational number system. Use the number line (real line), starting with the concept of "parts of a whole": fractions, decimals, and negative fractions. (The rational number system, not the real number system, is the backbone of K–12 mathematics. It is safe to say that all school teachers must know a mathematically correct version of the rational numbers that is also accessible to K–12 students if they hope to function effectively in the school environment.)

Basic number theory (Euclidean algorithm and the Fundamental Theorem of Arithmetic). Proof of the existence and uniqueness of the reduced form of a fraction and the characterization of fractions with a finite decimal expansion. (These facts are part of the bread and butter of the K–12 math curriculum.)

Geometry of the plane: congruence. The parallel postulate; transformations in the plane; rotations, reflections, and translations and their basic properties; isometry; fundamental theorem of isometries in the plane (proof postponed to Math 152); the concept of congruence; existence of rectangles and the setting up of coordinate systems. (TSM does not let on the fact that a great deal of geometry is needed to study the graphs of linear equations. The geometric discussion in Math 151 sets the record straight. The hope is that if enough pre-service teachers become better-informed, they can induce textbook publishers to clean up their act.)

Dilations and similarity. The concept of similarity; statement of the Fundamental Theorem of Similarity (proof postponed to Math 152); basic criteria of similarity for triangles. (These criteria are what is really needed to discuss the geometry of the line, but they cannot be done in the context of school mathematics without first discussing congruence.)

Proper use of symbols and linear equations. Standard identities and the finite geometric series; coordinatization of the plane; geometry of linear equations; algebraic formulation of parallelism and perpendicularity; simultaneous equations of two variables. (The emphasis is on the *correct* use of symbols; symbols are routinely abused in TSM, leading to the misconception that "variable" is a *mathematical* concept that must be mastered by all students in algebra.)

Math 152

Prerequisites: Math 151, Math 54 (sophomore calculus), *Math 113*

Linear functions. Linear inequalities in two variables and their graphs; linear programming. (The latter introduces a basic idea that has honest "real world" applications.)

Standard nonlinear functions. Quadratic functions and basic properties; polynomial functions and basic properties; applications to the geometry of conic sections; laws of exponents; exponential functions; inverse functions and logarithms. (This part not only treats the most basic information about the functions most commonly used in the school curriculum, but also aims at rectifying common misconceptions in TSM about what completing the square is for, how rational exponents are defined and what for, and how inverse functions are defined.)

Formal algebra. Polynomial forms (elements in $\mathbb{R}[x]$); basic theorems about roots of polynomials and factoring; complex numbers; the Fundamental Theorem of Algebra; quadratic polynomials with complex coefficients; binomial theorem and mathematical induction. (Polynomials were introduced in Math

151 as a polynomial functions; this is pedagogically convenient as well as mathematically sound because the ring of polynomial functions is isomorphic to $\mathbb{R}[x]$. But at some point, students—and teachers—have to face up to formal algebra, and this is the time.)

Euclidean plane geometry. Proofs of basic theorems in the plane on triangles and circles using as starting point the (assumed) properties of reflections, translations, and rotations; in particular, proofs of the Pythagorean theorem, the fundamental theorem of isometries in the plane, and the Fundamental Theorem of Similarity; theorems of Ceva and Menelaus; the Miquel point and the nine-point circle; construction of the regular pentagon; discussion of the classical construction problems. (The axiomatic approach to plane geometry used in TSM has proven to be pedagogically untenable. The present approach of using reflections, translations, and rotations in a less formal setting is more natural and gets to interesting theorems very rapidly.)

Axiomatic system. Informal discussion of the issue of the organization of geometric facts and axiomatization; general concept of an axiomatic system; intuitive discussion of hyperbolic geometry. (While the axiomatic approach may not serve as an effective introduction to plane geometry, school students should have some knowledge of an axiomatic system. Hence we take it up *after* much theorem-proving.)

Math 153

Prerequisites: Math 151, Math 54 (sophomore calculus), *Math 113*

Trigonometry. The trigonometric functions and similar triangles; basic identities; inverse trigonometric functions; De Moivre's formulas and conic sections revisited. (TSM is careless about the extension of the domain of definition of a trigonometric function, and trigonometric identities are claimed to hold for all numbers when in fact the proofs are only valid for angles up to 90 degrees. TSM also tends not to emphasize that the definitions of trigonometric functions in terms of the angles of a right triangle depend critically on the concept of similar triangles. An additional element worthy of note is that students need some persuasion as to why they have to learn the awkwardly defined inverse sine, cosine, and tangent functions.)

The concept of limit. The real line and the least upper bound axiom; basic theorems about limits; Existence of n-th root; convergence of infinite series. (This material is not easy for math majors, so a lot of examples and a careful deconstruction of the nonintuitive definition of convergence are given. For the existence of the positive n-th root of a positive number, special attention is given to the square root. This is the place to make future teachers aware of how nontrivial it is to have a square root.)

Decimal expansion of a number. Definition of infinite decimal; existence of the decimal expansion of a real number; why decimal expansion of a fraction repeats. (This is the first instance where this course sequence deviates from the school curriculum; the decimal expansion of a fraction is usually taught by rote in middle school. In that event, the focus is entirely on the fact that the decimal repeats. We need to make teachers see that this is the trivial part of the theorem; the difficult part is to understand that (i) the sequence of digits generated by long division can be interpreted as a number, and (ii) this number is equal to another number—the original fraction.)

Length, area and volume: basic definitions and formulas. Curves which have length and regions which have area. Second proof of the Pythagorean theorem using area.(This is the second instance where this course-sequence deviates from the school curriculum. Circumference of circles and area of disks are taught anywhere between the 7th and the 11th grades, but in the context of this course sequence, we need the concept of limits to give this discussion substance. Length and area, when done in a way that is simple enough for school use but still faithful to the basic spirit of mathematics, become very delicate issues. Note that for pedagogical reasons, π is defined as the area of the unit disk as this afford a direct numerical estimation of π.)

Calculus. Basic theorems of continuous functions; derivative and integral; Fundamental Theorem of Calculus; relationship with the concept of area and volume; logarithm defined in terms of integral; inverse functions and $\exp x$; proof of the general laws of exponents. (This is the most basic part of analysis tailored for use by teachers; the proof of the laws of exponents—where the exponents can be irrational— illustrates the power of abstraction.)