Learnable and unlearnable school mathematics*

Hung-Hsi Wu

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When the subject of math comes up in a conversation, it is rarely passed over with indifference. Typically, it provokes either fear and loathing, or a defensive posture of dismissiveness that borders on contempt. Since mathematics—along with English Language Arts (ELA)—is the only subject taught in every grade from kindergarten to grade 12, it is nothing short of scandalous that we continue to subject all students to a thirteen-year ritual that is feared, detested, and devalued by so many. What is the cause of this strange phenomenon and how can we do better?

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The answers to the twin questions—what's wrong with K-12 mathematics and how we should go about fixing it—are the concerns of the present article. Some believe that the solution to the school mathematics education conundrum lies in teaching math differently or teaching it better (e.g., Richards, 2020 and Green, 2014, respectively). Teaching clearly matters, but if the "mathematics" that is taught is *unlearnable*[1] and most likely also unteachable, then we cannot count on making changes in teaching to get us out of this educational impasse. It turns out that the "mathematics" taught in American schools in more or less the last five decades is unlearnable regardless of how it is taught. Our first task is to give a brief description of this "mathematics", and explain the urgent need to root it out from our schools. Clearly, we have no right to subject every student to thirteen years of unlearnable mathematics. Then we suggest at least one way to make school mathematics *learnable* again. Finally, we take an unflinching look at the long road that lies ahead before we can make learnable mathematics a reality in school classrooms as of 2021.

Characteristics of "learnable mathematics"

Unlearnable school mathematics comes in all shapes and forms, but for the past five decades (essentially the period beginning right after the New Math), it is the body of "mathematical knowledge" encoded in almost all the standard school mathematics textbooks[2] that will be the focus of our attention. We will refer to this as **TSM**, Textbook School Mathematics (see Wu, 2011b). Before we can explain why TSM is unlearnable, it

[1]By this we mean "unlearnable to most students".
[2]There *may be* one or two exceptions among the smaller publishers in the last couple of years.

is necessary to first describe what we consider to be the main characteristics of *learnable school mathematics*. We have to first accept that *school mathematics* (i.e., the mathematics of K-12) is not a part of the mathematics used by professional mathematicians; for lack of a better name, we call the latter *university mathematics*. School mathematics is a version of university mathematics that has been *engineered* or *customized* for consumption by K-12 students (Wu, 2006). For example, whereas a fraction in university mathematics is an equivalence class of ordered pairs of nonnegative integers (under the equivalence relation of the cross-multiplication algorithm), a fraction in elementary school will have to be just a certain point on the number line (see pp. 33–34 following, and also Wu, 1998). Then, by **learnable school mathematics**, we mean school mathematics that possesses the following five characteristics[3] (observe that they overlap with each other).

(i) Clear definitions. Each concept is precisely *defined* so as to make clear to students, *in a grade-appropriate manner*, exactly what they need to know about the concept for the understanding of the topic at hand and for subsequent mathematical developments. This is what the *transparency* of mathematics is all about: everything is up front and no information is withheld from learners. This transparency builds learners' trust in mathematics and the belief that they are not wasting their time trying to learn it.

(ii) Logical reasoning. Every claim is supported by *logical reasoning* as to why it is true (though some of the reasoning may be postponed to a

[3]These are the *Fundamental Principles of Mathematics* (Wu, 2011b) in a different guise.

later date) so that students never have to believe anything without a logical explanation. The fact that mathematics is built up logical-step by logical-step further reinforces mathematical transparency as well as learners' trust in mathematics.

(iii) Precise language. The language used is *precise* so that no guessing is necessary for the understanding of the mathematical message. Transparency again.

(iv) Coherence. The diverse concepts and skills together form a *coherent* narrative, so that they can be seen as part of a general pattern. This facilitates mathematics learning by making it a continuous, incremental process.

(v) Purposefulness. The *purpose* for learning each concept or skill is consistently made clear. The learner is therefore always aware of *why* something is worth learning.

In what follows, we will make frequent references to these **five characteristics of learnable school mathematics**.

A mathematical exposition that fails to observe any one of these characteristics is obviously less than learnable. TSM routinely violates *all* them, but it would take many volumes to detail all of TSM's sins in this regard (the six volumes, Wu, 2011a, 2016a, 2016b, 2020a, 2020b, and 2020c expose most of them). In the next section, will limit ourselves to the discussion of a confusion that TSM creates in students' learning of numbers and operations in grades K-7 (whole numbers, fractions, mixed numbers, decimals, and negative numbers). We will also make some passing comments on TSM's concept of a "variable" (see page 24), its non-definition of the *slope* of a line (see page 25), and its generally incoherent geometry curriculum in secondary school (see page 26).

Some examples of TSM

The first obstacle that children face in learning about whole numbers is the concept of **place value**. Children are taught, for example, that the digit 4 in the number 432 stands for 400, the digit 3 stands for 30, and the digit 2 stands for 2 itself. It is well-known that many children have difficultly learning this convention because they consider it arbitrary and unreasonable, and rightly so. Since children generally find it hard to articulate their skepticism or doubts, it is incumbent on the curriculum to anticipate their questions by explaining what the convention is about and why it is worth learning (cf. characteristic (v)). But TSM simply suggests that students be drilled to memorize place value by rote. This makes a bad first impression on students because it leads them to believe that *math is just lots of memorization and drills.* Next comes the cascade of the four (whole number) standard algorithms, taught essentially without reasoning or purpose (see the preceding characteristics (ii) and (v)). Children have their hands full just learning to execute the algorithms correctly (think "regroup", "carry", and "borrow"), so their traversal of the algorithms consists mainly of assiduous attempts to master a sequence of seemingly unrelated (and, to them, unreasonable) rote skills. This is especially true of the multi-digit multiplication and long division algorithms. Children most likely wonder *why* they are made to learn all this by rote, and what this collection of tricks might be good for, especially when calculators are available. One manifestation of this discontent is the reluctance by many students around grade 3 to memorize the multiplication table. Some in mathematics education have accordingly preached the crowd-pleasing but cognitively ill-advised sermon that "times tables are unnecessary" (Barshay, 2015). We will come back to the multiplication table in the later section on **How we can do better** (see pp. 33 ff.). For now, we turn to TSM's treatment of the long division algorithm.

TSM says "multiplication and division are inverse operations". This is a prime example of the imprecision of language in TSM (cf. characteristic (iii) above). As it stands, this sentence makes no sense"[4], but it turns out that what TSM intends to say is that if we have a whole number in the form of $(m \times n)$, where m and n are whole numbers and $n \neq 0$, then it makes sense to **divide the whole number $(m \times n)$ by n**, and the definition is

$$(m \times n) \div n = m \quad \text{(for all whole numbers } m \text{ and } n,\ n \neq 0) \tag{1}$$

One therefore sees in (1) that if m is multiplied by n, then one can "reverse" the multiplication by the use of division by n to restore m to itself. We can make more sense of this definition with some specific numbers. Let $m = 18$ and $n = 4$. Then, since $18 \times 4 = 72$, (1) says $72 \div 4 = 18$.

For a later need, we recall the two interpretations of the division $72 \div 4 = 18$. First, let us agree to the convention that the multiplication 18×4 means adding 18 copies of

[4]Multiplication converts *a pair of whole numbers* such as 3 and 7 to a single number 3×7, so that the *inverse operation* of multiplication would have to convert a single whole number to a pair of whole numbers. But the division operation does not do that.

4; then 4×18 means adding 4 copies of 18. Now, with $72 = 18 \times 4$, the **measurement** interpretation of $72 \div 4 = 18$ is that when 72 objects are divided into equal groups each consisting of 4 objects, there are $(72 \div 4)$ such equal groups. On the other hand, with $72 = 4 \times 18$, the **partitive** interpretation of $72 \div 4 = 18$ is that when 72 objects are divided into 4 equal groups, the number of objects in each equal group is $(72 \div 4)$.

Implicit in the definition of whole number division in (1) is the fact that if a whole number such as 73 is not a multiple of 4, then the division $73 \div 4$ makes no sense since there is no whole number that can be used to express the result. Nevertheless, TSM says one can expand the measurement interpretation of $72 \div 4$ and interpret $73 \div 4$ to mean: how many groups of 4 objects are there among 73 objects?[5] Since $73 - 72 = 1$ and $72 = 18 \times 4$, the answer is clearly 18 groups, with 1 object left over. (In this case, we happen to know that $72 \div 4 = 18$ ahead of time.) In general, TSM tells students that, to get the answer for this kind of problem, do the long division algorithm by drawing a "division house" and compute. So, for $73 \div 4$, we have:

$$\begin{array}{r@{\,}r@{\;}l}
 & & 1\;\;8 \\ \cline{3-3}
4 &) & 7\;\;3 \\
 & & 4 \\ \cline{3-3}
 & & 3\;\;3 \\
 & & 3\;\;2 \\ \cline{3-3}
 & & \;\;\;\;1
\end{array} \qquad (2)$$

Now taking note of the number 18 on the roof of the house and the number 1 in the basement, TSM tells students to write $73 \div 4 = 18\ R1$ to show that dividing 73 by 4 gets 18 equal groups of 4's in 73, with remainder 1.

Why is this approach to the division of 73 by 4 unlearnable? First, the meaning of

[5] What TSM tries to say is "what is the *maximum* number of groups of 4 objects among 73 objects? Here is another example of TSM's lack of concern for precision.

the concept of division, "$\div$", has changed from "the inverse operation to multiplication" to something that is not. Now, what are students supposed to believe about the meaning of division? This is how students lose their faith in mathematics (and their teachers): they no longer know what (or whom) they can trust. (See characteristic (i) above.) Moreover, something like $73 \div 4 = 18\ R1$ is confounding to many elementary students because "$18\ R1$" is clearly *two* numbers, namely, 18 and 1. What then is this mysterious object "$73 \div 4$" and how can it be *equal* to two numbers with an "R" in between? (Cf. (iii) above.) Even if students cannot articulate precisely their feeling of unease, many sense that something is not right. Finally, even more confounding to students' intuition is how the "division house" in (2)—a mystery even to most adults—could lead to the correct answer of 18 equal groups of 4 with 1 left over. Many students might want to look inside the black box that is the "division house", but they have probably given up trying at this point.

This sloppy way of using the equal sign "$=$" in $73 \div 4 = 18\ R1$ turns out to be a recurrent theme in TSM: it does not matter whether students actually know what "$73 \div 4$" is, but if upon seeing "$73 \div 4$" they know *what to do* next—in this case, draw the "division house" and compute to get "$18\ R1$"—then TSM considers its mission accomplished. We will return to this point several times below (e.g., pp. 9, 13, 14, 15, and 23).

The next topic that elementary students have to confront is fractions. Elementary students' despair over fractions has been immortalized in the Peanuts cartoons (Google *Peanuts and fractions*), and it is easy to understand this despair. Perhaps more than any topic in school mathematics, TSM's treatment of fractions is long on telling students *what to do* but short on telling them *what is it* that they are doing or explaining *why* what they are doing is correct (cf. characteristics (i) and (ii) above). Consider the well-known limerick, "Ours is not to reason why, just invert and multiply", that comes from the following formula that TSM wants students to memorize: if $\frac{2}{5}$ and $\frac{3}{4}$ are two arbitrary fractions, then here is how to do division:

$$\frac{2}{5} \div \frac{3}{4} \;=\; \frac{2}{5} \boxed{\times \frac{4}{3}} \quad \left(= \frac{8}{15}\right) \tag{3}$$

So what is the meaning of $\frac{2}{5} \div \frac{3}{4}$? One answer in TSM is that this asks for how many $\frac{3}{4}$'s there are in $\frac{2}{5}$. Now, students can understand "how many 3's there are in 15" by counting, but they are lost trying to figure out how to get a fraction out of "how many $\frac{3}{4}$'s there are in $\frac{2}{5}$". TSM provides no explanation, but it does provide the recipe in (3) with, as usual, some pleasing metaphors. So instead of explaining (3) directly, students are asked to consider something simpler such as $\frac{2}{5} \div \frac{1}{10}$. There are clearly two $\frac{1}{10}$'s in $\frac{1}{5}$ (because $\frac{1}{5} = \frac{1}{10} + \frac{1}{10}$), so there are four $\frac{1}{10}$'s in $\frac{2}{5}$. Therefore $\frac{2}{5} \div \frac{1}{10} = 4$, so that

$$\frac{2}{5} \div \frac{1}{10} \;=\; \frac{2}{5} \boxed{\times \frac{10}{1}} \quad \left(= \frac{20}{5} = 4\right)$$

TSM now says, *in a similar vein*, (3) should be correct. Here then is a case where students expect a straightforward answer but get, instead, a mixture of metaphor and non sequitur. At this point, students sadly conclude that there is no point asking questions anymore, "just invert and multiply". Hence the limerick, and the loathing.

The message of (3) is therefore: no need to know what $\frac{2}{5} \div \frac{3}{4}$ actually means, because you now know *what to do with it.* Also observe that this carries the same implicit message as "$73 \div 4 = 18\ R1$" that we mentioned on the preceding page.

As a matter of fact, TSM does worse than staying mum about "invert and multiply": it even manages to stay evasive about what a fraction *is*. First, TSM tells students that a *fraction* is "part of a whole". To make it more relatable to students, it usually settles for the analogy of a piece of pie or pizza as a way to think about a fraction. Now, the use of analogy to explain an abstract concept has its place in mathematics textbooks, but since mathematics ultimately demands *precise* answers, a precise definition *in addition to a vague analogy* is what students need. After all, fractions are "numbers" that students have to add, subtract, multiply, and divide. It is hard for them to imagine adding, multiplying, etc., pieces of pizza or "parts of a whole" (cf. characteristic (i) above). For example, how do pieces of pizza help to explain equation (3))? As for "part of a whole", TSM requires students to do at least three things to come to grips with this concept: decide on each occasion what is in fact "the whole", divide it "equally" into the prescribed number of parts, and finally take a certain number of these parts. A fraction therefore becomes a *very* prickly and cumbersome object. How can students be expected to handle fractions with ease? For example, consider a typical so-called "rate problem" such as the following:

> Trina walks $2\frac{1}{2}$ miles in 52 minutes. How many miles does she walk in 70 minutes?

What should the "whole" be in this situation? Is it 1 mile? Or is it 1 minute? And where do the pizzas come in? Clearly students who ponder this will be overwhelmed.

Beyond "part of a whole" or a piece of pizza, TSM says a fraction "is also a division", i.e., a fraction such as $\frac{5}{3}$ is also a division, $5 \div 3$. You will recall from our preceding discussion of the division of whole numbers that students learn $5 \div 3$ is equal to the two numbers 1 $R2$. It turns out that TSM does not want students to think about 1 $R2$ here. But why not? TSM is not in the business of answering such questions. Instead, TSM simply tells students as a matter of course that they *should know from the partitive interpretation of whole-number division* that $5 \div 3$ means the size of one part when the totality of 5 "wholes" is divided into 3 equal parts. Needless to say, students have been taught nothing of the sort. What they were taught about the partitive division of $5 \div 3$ was that if 5 objects are divided into 3 equal groups,[6] $5 \div 3$ is the number of objects in one group. *What TSM teaches about a fraction being a division is thus a total obliteration of the need for definitions, precision, and coherence* (characteristics (i), (iii), and (iv)). After all that, it remains to point out that TSM fails to explain *why* a fraction as "part of a whole" suddenly becomes this strange kind of division. *No reasoning.* If students had any trust in TSM to begin with, surely there won't be any left after such a discussion about fraction division.

Let us take a closer look at the latest meaning of $5 \div 3$. TSM in fact frowns on fractions such as $\frac{5}{3}$ because the visualization of "part of a whole" as a piece of pizza now requires the drawing of *two* pizzas to represent $5 \div 3$, which is cumbersome:

One shudders at the thought of the number of pizzas needed to represent, for example, $\frac{47}{3}$. For this reason, TSM replaces what is known as **improper fractions**, i.e., fractions

[6] *Which is of course impossible.*

whose numerator is greater than or equal to their denominator, such as $\frac{5}{3}$, by **mixed numbers** such as $1\frac{2}{3}$, to be read as **1 and** $\mathbf{\frac{2}{3}}$. The latter is obviously a direct verbal transcription of the preceding picture: 1 pizza and $\frac{2}{3}$ of a pizza. In general, TSM gives the following rule for converting an improper fraction $\frac{39}{5}$ to a mixed number: do the long division to get $39 \div 5 = 7\ R4$. Then from the right side, we get the whole numbers 7 and 4 and use these to get the mixed number $7\frac{4}{5}$ (notice that the denominator 5 remains unchanged), i.e.,

$$\frac{39}{5} = 7\,\frac{4}{5} \tag{4}$$

On the other hand, to convert a mixed number such as $7\frac{4}{5}$ to an improper fraction, TSM says to do it this way:

$$7\,\frac{4}{5} = \frac{(7 \times 5) + 4}{5} = \frac{39}{5} \tag{5}$$

In any case, in addition to learning about fractions, students must now also learn about yet another "different" kind of number—the mixed numbers—and how to convert between improper fractions and mixed numbers using the prescribed rules, rules that again remain largely opaque to them. This lack of *coherence* (see (iv) above) and lack of transparency (see (ii) above) can not help but add to students' cognitive load.

TSM calls a fraction whose numerator is smaller than its denominator a **proper** fraction. Basically, TSM doesn't like seeing improper fractions and wants every improper fraction converted to a mixed number. This implicit (but overwhelming) preference for *proper* fractions in TSM then leads to students' mistaken belief that fractions as "small" numbers (smaller than 1). This handicaps students' conceptual understanding of fractions in general because, soon, they will need to conceptualize fractions as the collection of *all* the numbers $\frac{m}{n}$ where m, n are *arbitrary* whole numbers with $n \neq 0$.

Let us turn now to the arithmetic of fractions. TSM says that to calculate the sum, $\frac{7}{8} + \frac{5}{6}$, first get the **least common denominator (LCD)** of the denominators 8 and 6, which is 24. Then because $24 = 3 \times 8$ and $24 = 4 \times 6$, TSM tells students to make use of *equivalent fractions* to get the answer:

$$\frac{7}{8} + \frac{5}{6} \;=\; \frac{3 \times 7}{3 \times 8} + \frac{4 \times 5}{4 \times 6} \;=\; \frac{21}{24} + \frac{20}{24} \;=\; \frac{41}{24} \tag{6}$$

And, of course, the answer is immediately changed to the mixed number $1\frac{17}{24}$.

Notice that (6) does not explain what the sum $\frac{7}{8} + \frac{5}{6}$ is, only that if students want to get the answer to the sum $\frac{7}{8} + \frac{5}{6}$ (whatever that is), they should change the sum to that of two fractions with their LCD as the denominator:

$$\frac{3 \times 7}{3 \times 8} + \frac{4 \times 5}{4 \times 6} \tag{7}$$

Now, the answer to such an addition is very simple: add the numerators and keep the denominator intact to get $\frac{41}{24}$.

Formula (6) is baffling to students. They learn from whole numbers that adding is putting things together, so they expect more of the same when they come to adding fractions. But the manipulations in (6), especially the use of LCD, suggest nothing about "putting things together". So they wonder about *why can't adding fractions be like adding whole numbers?* Elementary students are unlikely to be able to articulate this thought, but there is no doubt that this question—or some variation thereof—is on the mind of every single one of them. Because TSM does not answer this question, students reluctantly come to the belief that fractions are, indeed, *a different kind of* *number.* Students must now *re-learn* addition all over again and, getting no explanations for things like equation (6), they have no choice but to forge ahead by keeping their questions to themselves while trying to memorize the new procedures by rote.

To survive in math classes, they learn to forget about reasoning and to suppress the urge to make sense of what they do (see characteristics (ii) and (iv) above, respectively).

TSM's teaching of fraction subtraction is similar to that of addition, with one critical difference. As in the case of subtraction among whole numbers, one can only subtract a "smaller" fraction from a "bigger" fraction. Thus, before writing down $\frac{7}{8} - \frac{5}{6}$, we must make sure that $\frac{7}{8} > \frac{5}{6}$. But what does it mean that $\frac{7}{8}$ is "bigger" than $\frac{5}{6}$? Once again, TSM resorts to its usual trick of telling students *what to do to get the answer* without giving a definition of "bigger than" between fractions. For example, TSM suggests using equivalent fractions to change both fractions to fractions with the same denominator, and tells students that the resulting fraction with the larger numerator is the larger fraction. Going one step further, TSM does not explain the meaning of subtraction between fractions but once again just tells students what to do to get the right answer for fraction subtraction. Here is the analog of (6) for subtraction:

$$\frac{7}{8} - \frac{5}{6} = \frac{3 \times 7}{3 \times 8} - \frac{4 \times 5}{4 \times 6} = \frac{21}{24} - \frac{20}{24} = \frac{1}{24} \tag{8}$$

Next, the multiplication of fractions. Consider the product $\frac{2}{5} \times \frac{3}{4}$. TSM says that this means $\frac{2}{5}$ *of* $\frac{3}{4}$ of a pizza. To find out what $\frac{2}{5} \times \frac{3}{4}$ is, first consider a simpler example: what is $\frac{2}{1} \times \frac{3}{4}$? In other words, what is 2 servings of $\frac{3}{4}$ of a pizza? This is obviously equal to $\frac{6}{4}$ of a pizza because $\frac{3}{4} + \frac{3}{4} = \frac{3+3}{4} = \frac{6}{4}$. So $\frac{2}{1} \times \frac{3}{4} = \frac{6}{4}$. Since $\frac{6}{4} = \frac{2 \times 3}{1 \times 4}$, we get

$$\frac{2}{1} \times \frac{3}{4} \;=\; \frac{2 \times 3}{1 \times 4}$$

So, at least in this case, multiplication of fractions is achieved simply by multiplying the numerators and the denominators. We can look at another example: $\frac{1}{3} \times \frac{3}{4}$. By the definition of multiplication, we want to know how much is $\frac{1}{3}$ *of* $\frac{3}{4}$ of a pizza. When $\frac{3}{4}$ of a piece of pizza is divided into 3 equal parts, clearly one part is $\frac{1}{4}$ of a pizza. Therefore $\frac{1}{3}$ *of* $\frac{3}{4}$ of a pizza is $\frac{1}{4}$ of a pizza. By the definition of multiplication, we get, $\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$. By equivalent fractions, $\frac{1}{4} = \frac{1 \times 3}{3 \times 4}$, so

$$\frac{1}{3} \times \frac{3}{4} \;=\; \frac{1 \times 3}{3 \times 4}$$

Again, the product of two fractions in this case is the fraction obtained by multiplying the respective numerators and denominators.

With these two simple examples at hand, TSM says we can now believe that the multiplication algorithm for fractions in general shares this simple pattern:

$$\frac{2}{5} \times \frac{3}{4} \;=\; \frac{2 \times 3}{5 \times 4} \tag{9}$$

If TSM could show that formula (9) is a consequence of the definition of multiplication, students would have a coherent narrative connecting the definition of multiplication to the right side of the formula, and this is what normal mathematics learning should be about. But since TSM fails to do that, students' learning is limited to the rote memorization of the definition of fraction multiplication and (9). Nothing about reasoning.

Equations (6) and (9), when viewed together, raise a question that has intrigued and bedeviled generations of school students: if multiplying fractions is as simple as in (9), why shouldn't adding fractions be just as simple? In other words, they *should get*

$$\frac{5}{6}+\frac{7}{8}=\frac{5+7}{6+8} \tag{10}$$

All that TSM can say is that, *despite evidence to the contrary,* the addition of fractions *is* putting things together. Since equation (10) would have us believe that putting $\frac{1}{2}$ and $\frac{1}{2}$ together gives us $\frac{2}{4}$, which is $\frac{1}{2}$ and not 1, (10) must be false. In view of students' general distrust of the addition algorithm (6) on page 13, this argument against (10) is only mildly convincing at best. The two formulas (6) and (9) therefore remain seemingly random and arbitrary rules and equation (10) has continued to be a siren song tempting students into typical and predictable errors such as (10). As late as the 1990's, some calculus students at major universities were known to add rational expressions in this "straightforward" manner:

$$\frac{7}{x-2}+\frac{x^3-x}{x^2+5} = \frac{7+(x^3-x)}{(x-2)+(x^2+5)}$$

Since we have already discussed the division of fractions, let us pause and reflect on why students cannot learn fractions from TSM. TSM does not level with students by telling them exactly what a fraction is, what it means for one fraction to be bigger than another, what the arithmetic operations on fractions are, and how these operations are related to the corresponding operations on whole numbers. Consequently, learning fractions becomes a venture in trying to learn how to work with *a completely new kind of numbers* without the benefit of getting to know what they are or where they come from. All students can do is commit random and arbitrary rules to rote memorization for the sole purpose of getting answers. Students are forced to fly blind; fear and loathing inevitably ensue.

Now recall that TSM also introduces mixed numbers to avoid dealing with so-called improper fractions. Thus TSM has to also discuss the arithmetic operations on mixed numbers separately. To this end, every mixed number has to be first converted to an improper fraction. After the result of the arithmetic operation on the improper fractions has been obtained, it has to be converted back to a mixed number if necessary. For example,

$$5\frac{3}{4} \times 4\frac{1}{4} \;=\; \frac{23}{4} \times \frac{17}{4} \;=\; \frac{391}{16} \;=\; 24\,\frac{7}{16} \tag{11}$$

We will return to this product of mixed numbers later on.

In TSM, yet another kind of numbers—the (finite) decimals—are introduced alongside the teaching of fractions as numbers that are "extensions" of whole numbers, i.e., if

$$43 \;=\; 4 \text{ tens and } 3 \text{ ones},$$

then for a typical decimal,

$$43.57 \;=\; 4 \text{ tens and } 3 \text{ ones } \textit{and } 5 \text{ tenths } \textit{and } 7 \text{ hundredths} \tag{12}$$

The use of the word "and" in this intuitive fashion is reminiscent of the same use in TSM's definition of a mixed number above equation (4) on page 12. TSM also tells students that decimals are related to fractions, e.g., $43.57 = \frac{4357}{100}$. If this fact is related to the preceding displayed equation (12), TSM doesn't say. The arithmetic of decimals? No problem: for addition and subtraction, line up the decimals on top of each other by their decimal points and add vertically as in the case of whole numbers and then insert the decimal point back into the final answer. For multiplication such as 0.23×4.5, first ignore the decimal points and multiply the whole numbers 23 and 45 to get 1035. Where to put the decimal point in 1035? TSM says 0.23×4.5 should have 3 **decimal digits**, i.e., 3 digits to the right of the decimal point because, since there are 2 decimal digits in 0.23 and 1 decimal digit in 4.1, the product must have $2+1 = 3$ decimal digits. Therefore the final answer must be $0.23 \times 4.5 = 1.035$. No reasoning is offered. The division of decimals makes use of an analogous rote procedure involving the "division house" which we will not go into here for lack of space. What matters is that the arithmetic of decimals is a collection of rote procedures related to those of whole numbers.

In summary, what students learn from TSM thus far is that there are *four* kinds of numbers, namely, whole numbers, fractions, mixed numbers, and (finite) decimals. The arithmetic operations on these four different kinds of numbers appear to be different and—at least from students' perspective—are given by four separate sets of inscrutable rules. Students have a pretty solid intuitive grasp of what whole numbers are because they can always fall back on counting on their fingers. But fractions as "parts-of-a-whole" is a bridge too far for many students: they cannot conjure up any usable mental image for "part-of-a-whole", and who can divide two "parts-of-a-whole" anyway? Students' main difficulties with decimals and mixed numbers are similar in that TSM treats a decimal or a mixed number as not one object but a composite object with multiple components. For example, many believe that "4.79 is greater than 4.8 because 79 is way more than 8" (see Griffin, 2016). One reason for this confusion is that, whereas students are usually capable of comparing *one single object* to another, comparing "4-

and-7-tenths-and-9-hundredths" with "4-and-8-tenths" is once again a bridge too far. Who knows how to compare such complicated contraptions? Students' (and teachers') discomfort with mixed numbers can be inferred from the following passage in the NCTM Standards (National Council of Teachers of Mathematics, 1989):

> This is not to suggest, however, that valuable instruction time should be devoted to exercises like $\frac{17}{24} + \frac{5}{18}$ or $5\frac{3}{4} \times 4\frac{1}{4}$, which are much harder to visualize and unlikely to occur in real-life situations. (p. 96)

The product $5\frac{3}{4} \times 4\frac{1}{4}$ is the one computed in equation (11) on page 17. While it is short, it does involve converting both mixed numbers using an arcane rule into improper fractions (which are strangers to students in the first place), multiplying them, and then converting the result back to mixed numbers using another arcane rule. That is a lot of *rote* mental gymnastics for such a short computation!

There are two more major topics in school students' introduction to real numbers: negative numbers and irrational numbers. TSM's treatment of irrational numbers is so flawed that it would take a separate article to discuss it in any detail (but see pp. 103-118 in Wu, 2020c, for a general overview). We will briefly discuss the introduction of negative numbers instead.

As with fractions, mixed numbers, and decimals, TSM only gives intuitive discussions of what negative numbers are in lieu of giving a definition. Thus, (-3) is "like" 3 degrees below 0. It is therefore to be expected that TSM will botch the arithmetic operations on **rational numbers** (i.e., the fractions and negative fractions), and indeed it does. TSM does not present these operations as *natural extensions* of those on fractions but chooses instead to lay down yet another set of arbitrary rules. Due to the lack of space, we will discuss only the one aspect of the multiplication of rational numbers that has intrigued every learner young and old: why negative times negative is positive. Consider a simple example: most students would prefer $(-3) \times (-2)$ to be equal to (-6) because, after all, $(-3) + (-2)$ is equal to (-5). So how does TSM explain $(-3) \times (-2) = 6$?

A typical TSM explanation goes like this. First of all, the fact that $4 \times (-2) = -8$ is no mystery because $4 \times (-2)$ is just adding 4 copies of (-2), and therefore $4 \times (-2) = (-2) + (-2) + (-2) + (-2) = -8$. Similarly,

$$3 \times (-2) = -6, \quad 2 \times (-2) = -4, \quad 1 \times (-2) = -2, \quad 0 \times (-2) = 0$$

TSM then suggests looking for a *pattern* in the sequence of products $n \times (-2)$ as n runs through the whole numbers in descending order. Let us start with $n = 4$.

$$\begin{array}{rcr} 4 \times (-2) & = & -8 \\ 3 \times (-2) & = & -6 \\ 2 \times (-2) & = & -4 \\ 1 \times (-2) & = & -2 \\ 0 \times (-2) & = & 0 \end{array}$$

TSM points out that each number on the right *increases by* 2 as we go from each equality to the one below it, i.e., the right side goes from -8, to -6, to -4, to -2, and finally to 0. Now, once a pattern is identified for $n \times (-2)$ as n runs through all whole numbers, TSM says *it must persist when n runs through the negative integers as well.* So going down this sequence of products $n \times (-2)$ beyond $n = 0$, the integer n now assumes the values -1, -2, -3, ... Keeping in mind the pattern of "increasing by 2 as n goes from

0 to -1, to -2, to -3, ...", we get

$$\begin{array}{rcll} 1 \times (-2) & = & -2 & \\ 0 \times (-2) & = & 0 & (= \ 2 + (-2)) \\ (-1) \times (-2) & = & 2 & (= \ 2 + 0) \\ (-2) \times (-2) & = & 4 & (= \ 2 + 2) \\ (-3) \times (-2) & = & 6 & (= \ 2 + 4) \end{array}$$

This is why $(-3) \times (-2) = 6$, according to TSM.

Notice that this explanation shares a common characteristic with those of two earlier formulas (3) and (9) (on pp. 9 and 15, respectively), namely, each is nothing more than an extrapolation from a small number of simple examples. Even among those few who have not yet given up hope of learning some valid reasoning from TSM, most can see through the absurdity of insisting that a fact that is true for whole numbers must also be true for negative integers. In this case, the absurdity explodes in a rather spectacular fashion. Indeed, a striking fact about rational numbers is that, given an inequality between two rational numbers such as $2 \leq 5$, while it is true that multiplying both sides of the inequality by a positive number or 0 preserves the inequality (e.g., $3 \times 2 \leq 3 \times 5$ because $6 \leq 15$), multiplying both sides by a negative number *reverses* the inequality (e.g.,

$$(-3) \times 2 \ \geq \ (-3) \times 5$$

because $\mathbf{-6 \geq -15}$). But according to TSM's reasoning for $(-3) \times (-2) = 6$, the fact that $n \times 2 \leq n \times 5$ for *all whole numbers* n is enough to ensure that the same inequality will persist even when n is a negative integer. Thus, letting $n = -3$, TSM says we should get

$$(-3) \times 2 \ \leq \ (-3) \times 5\,,$$

which simplifies to the absurd statement that $-6 \leq -15$. Here then is a situation where TSM implicitly teaches students something that is blatantly false. This gives a new meaning to the *unlearnability* of TSM.

Mathematics is *hierarchical* in the sense that each assertion is built on facts that have been established before. Such being the case, one would expect that the mathematical misinformation that TSM feeds elementary students will come back to haunt them in their mathematics learning in middle and high school. In at least one case, this expectation has been vindicated by education research: the misinformation about the equal sign "=" implicitly given by "equalities" such as "$73 \div 4 = 18\ R1$" turns out to hamper students' learning of algebra. Now, the equal sign "=" is of course ubiquitous in school mathematics and its meaning cannot be simpler: for two numbers A and B, the equality "$A = B$" means we *know* what A is and we *know* what B is, and we can explain why A and B are the same number. For example, $(4 \times 15) + 27 = 101 - 14$ because, after a simple computation, both $(4 \times 15) + 27$ and $101 - 14$ are seen to be equal to 87. But education researchers were startled to discover that many elementary students lack this basic understanding of the equal sign: these students can only think of the symbol "=" as a "do something signal" (see, e.g., Falkner, Levi, and Carpenter, 1999). The researchers concluded that "a lack of such understanding is one of the major stumbling blocks for students when they move from arithmetic to algebra" (*loc. cit.*, page 234).

It is wrong to blame students for "a lack of such understanding" about the equal sign; they *acquire this lack of understanding from TSM.* As we took pains to emphasize in connection with the "equality" "$73 \div 4 = 18\ R1$" and the formulas (3), (6), (8), and

(9),[7] none of these five equalities says that the numbers on both sides of the equal sign are equal. In fact, TSM cannot explain what "$73 \div 4$" or "$\frac{2}{5} \div \frac{3}{4}$" is, for example. Rather, TSM merely uses "=" to indicate that if you divide 73 by 4 you will *get* 18 with remainder 4 (in the case of the former) and if you divide $\frac{2}{5}$ by $\frac{3}{4}$, you will *get* $\frac{2}{5} \times \frac{4}{3}$ (in the case of the latter). Many years of exposure to such abuse of the equal sign will certainly cement students' misconception that, at the sight of the symbol "=", they have to "do something" regardless of whether they know what they are doing or not. Unsurprisingly, it eventually comes to pass that students who have been handicapped by having to memorize the myriad rote procedures of unlearnable arithmetic have trouble learning algebra.

What TSM does is create artificial obstacles in students' learning path that make their quest for learning rational numbers (and later, algebra) unnecessarily difficult. Thus, TSM makes believe that the rational numbers are a complicated entity that consists of at least five different kinds of numbers—whole numbers, fractions, mixed numbers, (finite) decimals, and negative fractions—and *each has its own method for doing arithmetic operations.* Moreover, TSM neither gives precise definitions for any of these numbers nor provides the reasoning for their arithmetic algorithms. Many elementary students find the artificial obstacles and the need to memorize without understanding too onerous to overcome. This is a big reason for the widespread fear and loathing of mathematics. The only way out of this morass is to uproot TSM from schools and replace it with learnable mathematics.

[7] And of course also in *numerous* other instances that we have not had room to discuss here.

When we turn our attention to the secondary mathematics curriculum, we should not expect TSM to suddenly become more learnable. In fact, the increased mathematical sophistication opens new avenues for incompetence, and TSM seems to have availed itself of all of them. Due to the lack of space, we will only touch lightly on three topics:

(1) *The concept of a "variable".* This is synonymous with the use of symbols in algebra, and students' struggle with the use of symbols is well-known. According to TSM, understanding the concept of a **variable** is crucial to the study of algebra. But what is the precise *definition* of a "variable"? (See item (i) of the characteristics of learnable mathematics.) There are various answers to the last question, and here are four of them:

- A symbol is a *variable*.
- A *variable* is a letter used to represent one or more numbers.
- A *variable* is a quantity that changes or varies.
- *Variable* is a letter or other symbol that can be replaced by any number (or other object) from some set. ... A sentence with a *variable* is called an *open sentence*, and it is called *open* because its truth cannot be determined until the variable is replaced by values,

It is doubtful that any of these "definitions" enlightens students about what a variable is or might be. For example, consider the solution of an equation $3x - 1 = 5x$. If x "varies", what does the equality mean? Does it say the two sides remain equal even when x varies? At this point, students know better than to ask such questions; they have learned to just go through the motions and crank out an answer:

$$3x - 1 = 5x \implies -1 = 5x - 3x \implies -1 = 2x \implies x = -\tfrac{1}{2}$$

(2) *The slope of a line in the coordinate plane.* The concept of slope is absolutely fundamental to students' understanding of linear equations in two variables, a central topic in introductory algebra. The voluminous literature on students' misconceptions—or lack of understanding—of slope testifies to their difficulty with this concept. Educators seem to have missed the obvious *mathematical* reason for this difficulty: TSM feels no need to give a precise definition of the **slope** of a line L in the coordinate plane and chooses instead to present it *intuitively* as "rise-over-run". Consequently, many students end up not knowing for sure that slope is even a ratio (rather than a pair of numbers), and even those who do may not know that they can choose ***any*** *two points on* L to measure the "rise" and the "run". On top of all that, TSM does not explain to students why it is worth their effort to learn this concept (cf. characteristic (v) of learnable mathematics). It should therefore come as no surprise that in an Algebra I class,

> [the] most difficult problems for students were those requiring identification of the slope of a line from its graph. That difficulty persisted across grade levels and types of mathematics course (e.g., Algebra I, Advanced Algebra, Geometry)". (Postelnicu and Greenes,

2012)

(3) *Geometry.* The high school geometry course has long been either the butt of jokes or a horror story in school math education. Outside of the high school geometry course, TSM gives almost no definitions and no proofs, but *in this one course, definitions are suddenly taken with the utmost seriousness and every assertion*—no matter how trivial or how boring—*must be proved.* How can students navigate this hodgepodge of definitions-axioms-theorems that sticks out like a sore thumb in the K-12 curriculum? They mostly can't, and this is the reason the geometry course is a joke. More can be said. Students are told in middle school that two plane figures are **congruent** (respectively, **similar**) if they are the same shape and the same size (respectively, the same shape but not necessarily the same size). Yet, in the TSM high school geometry course, all that mumbo jumbo about shape and size is forgotten and only the congruence and similarity of *triangles* are discussed. Students are not even given the definitions of the congruence and similarity of circles and parabolas[8] in that course. When then does TSM expect students to learn about such fundamental concepts as congruence and similarity?

Why you should be concerned about TSM

The preceding brief and selective survey of TSM is meant to give at least a hint of why TSM is unlearnable and why it inspires such fear, loathing, and contempt in the general

[8] The congruence and similarity of parabolas are needed for a coherent discussion of quadratic functions of one variable. See Chapter 2 of Wu, 2020b, especially the discussion on page 73 and Theorem 2.11 on page 88.

public. The need to eradicate and replace TSM should by now be obvious to all. Because school mathematics education is also the foundation of the supply chain that delivers the nation's future mathematicians and scientists, the high-profile volume *Rising Above the Gathering Storm* (National Research Council, 2007) duly takes note of this fact. This volume was written at the behest of Congress to "ensure that the United States maintains its leadership in science and engineering" to compete successfully in the 21st century. To this end, the volume's first recommendation is to "Increase America's talent pool by vastly improving K-12 science and mathematics education." It goes without saying that such improvement requires TSM to be uprooted from the school classroom and replaced with learnable mathematics.

But there are also deeper reasons for the need to eradicate TSM.

At a time when the equity issue has come to the fore in discussions of school mathematics education, what seems to be generally overlooked is the bleak prospect of winning the war on inequity only to expose all students "equitably" to TSM. TSM is the ultimate inequity enforcer because, while some students' families can afford private tutors or some other ways to help them make sense of the mathematics of their grade *despite* TSM, most underprivileged students will be denied access to this kind of help. Algebra is indeed a "civil rights issue",[9] but not if the only kind of algebra available in schools is TSM algebra in which the equal sign means "do something", "variable" is a major concept (even if it is vague and elusive), the slope of a line is something about a kind of "rise-over-run", and laws of exponents are merely "number facts" (for the last, see

[9]See `https://en.wikipedia.org/wiki/Algebra_Project`.

the bottom of page 193 in Wu, 2016b). Enabling minority students to gain access to TSM algebra should never be the devoutly wished-for goal of school mathematics education. We must strive to bring *learnable algebra*—in fact, *learnable mathematics*—to all students. *That* should be what equity is all about.

A less obvious sin of TSM is what it has done to academic research on student learning. So long as TSM dominates school mathematics classrooms, the pedagogical research by educators will continue to be an exercise in what may be called "refuse reclamation": how to make unlearnable mathematics somewhat palatable to students. This has been, and continues to be a monumental waste of human resources because such scholarship should have been lavished on making *learnable school mathematics* more accessible, more engaging, and more insightful.

But the greatest sin of TSM is without a doubt its apparent efficacy in snuffing out children's curiosity. People who only know mathematics as TSM may be shocked that there could be any relationship between curiosity and school mathematics, but curiosity is the engine that drives all scientific and mathematical discoveries. There is a charming story told by the Nobel Prize-winning physicist I.I. Rabi. When Rabi was young, his mother, unlike other mothers in his neighborhood, did not ask her son, "Did you learn anything in school today?" but rather, "Izzy, did you ask a good question today?" Rabi said, "That difference—asking good questions—made me become a scientist." (See `https://en.wikiquote.org/wiki/Isidor_Isaac_Rabi`)

Mathematics, by its very definition, is nothing more than a logical organization of responses to questions about numbers and geometry raised through the ages. It is exceedingly difficult for students to learn any mathematics if they do not feel something of the urgency of wanting to know *why* or at least understand *the need to know why* (see (ii) and (v) of the characteristics of learnable mathematics). What is a fraction, what is a (finite) decimal, and why should we be interested in learning about them? Why do fractions have to be multiplied this way but added that way? What is the point of learning the standard algorithms? Why is the product of two negative numbers positive? And so on. To all these questions mathematics also provides direct answers that are at once complete and accessible to students (see the following section on **How we can do better**). When students routinely experience the satisfaction of having their questions answered, the positive feedback emboldens them to *routinely* ask more questions and get more answers. This is how learnable school mathematics provides the ideal environment for nurturing students' curiosity.

Unfortunately, when TSM consistently imposes concepts on students without telling them what the concepts are and why the concepts are worth knowing (e.g., fraction, decimal, slope, etc.) and, worse, why students have to perform the skills in the particular way they are told (e.g., the long division algorithm, the addition algorithm for fractions, etc.), many students become resigned to Lord Tennyson's lamentation:[10] "Theirs not to reason why. Theirs but to do . . . " as they are told and memorize everything that comes their way. After a thirteen-year immersion in such a reasoning-free culture, some will become convinced that many things in this world, especially in mathematics, are simply incomprehensible, and they should not bother about asking questions and simply move

[10]In his poem, "The Charge of the Light Brigade".

on with their lives. It therefore comes to pass that, in the era of high-tech when we need to keep our sense of curiosity alive for our very survival, TSM contributes mightily to quashing it instead.

In principle, school science education should also do its share to keep students' curiosity alive, but answers to science questions are usually less straightforward. For example, one can try to explain to students that "sunrise" is better understood in terms of the earth's rotation around its north-south axis in the course of its orbit around the sun, but such an explanation will implicitly draw on students' *faith* in humans' hard-earned understanding of the solar system through the millennia as well as their belief in the authority of the textbook or the teacher. But no such faith is needed for the understanding of mathematical reasoning. Moreover, unlike mathematics, the school science curriculum consists of many short pieces strung together rather than a single continuous strand running through all thirteen years. It also seems to be the case that school science education has its own unresolved TSM-type issues. For all these reasons, school mathematics education deserves special attention.

How we can do better

In years past, advocating the eradication of TSM would consist mainly of criticisms and a few slogans about what to do next, and even serious attempts at a reform would put forth little more than a new set of standards. There has never been a detailed and systematic account of *how* to achieve the desired goal. Two such major attempts in the last thirty or so years—even if TSM was not explicitly mentioned—are well-known: the 1989 NCTM reform (National Council of Teachers of Mathematics, 1989) and the 2010 Common Core Mathematics Standards (CCSSM, 2010). The present article is a deliberate break from this long tradition of what may be called "educational-improvements-by-decree". There are now available six volumes, comprising some 2500 pages, that give a complete presentation of the K-12 mathematics curriculum, describing in detail *how to replace TSM with learnable school mathematics*: Wu, 2011a, 2016a, 2016b, 2020a, 2020b, and 2020c. From the vantage point of these six volumes, we will revisit some of the high points of the unlearnability of TSM as presented earlier and give a brief indication of *one way* (perhaps among many) *to do better* in each case.

First of all, the place value concept in our **numeral system** (i.e., system of numeration) is not the result of a capricious decree by someone from on high. Rather, it is a very effective response to the human need for the ease of counting to any number—no matter how large—and the need for easy recording and computation. These needs are met by limiting the numeral system to the use of *only* ten symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, together with the systematic application of the concept of place value (see Wu, 2011a, Sections 1.1-1.2). The efficacy of the concept of place value for the purpose of counting to arbitrarily large numbers can be easily demonstrated to second-graders, for example, by showing them how to count with the use of, not *ten*, but only three or even two symbols such as $\{0,\ 1,\ 2\}$ and $\{0,\ 1\}$, respectively (see pp. 13-15 and Chapter 11 of Wu, 2011a).[11] The whole idea is to make students see from the beginning that, far from a set of arbitrary rules, everything in mathematics is done with a purpose in

[11] This experiment has been tried in one school with success.

mind, even if that purpose is mostly hidden in TSM. (See characteristic (v) on page 4.) There is an extended discussion of this circle of ideas about place value, with references to other historical numeral systems to provide contrast, in slides 13-30 of Wu, 2014, and of course also on pp. 5-22 of Wu, 2011a.

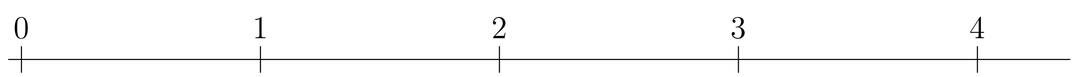
Next, the four standard algorithms for whole numbers. They are usually taught as four separate rote skills without any reasoning. However, students should not only learn the precise definition of each operation and the reasoning for each of the algorithms so that they can make sense of them (see Chapters 4-7 in Wu, 2011a), but also be made aware of—and be periodically reminded of—the overriding leitmotif that *knowing how to do an arithmetic operation with* ***single-digit*** *numbers enables them to do this operation with* ***any*** *two numbers* (see Chapter 3 in Wu, 2011a). Therefore, there are at least two reasons for students to learn the standard algorithms:

> ***(a)*** They get to experience the coherence of mathematics (see characteristic (iv) on page 4) from the beginning of their mathematical journey, the fact that these four disparate skills are actually variations on the same theme.
>
> ***(b)*** They get to learn a major guiding principle of mathematics and the sciences: reduce the complex to the simple—computations with *all* whole numbers of no matter how many digits boil down to computations with *single-digit* numbers.

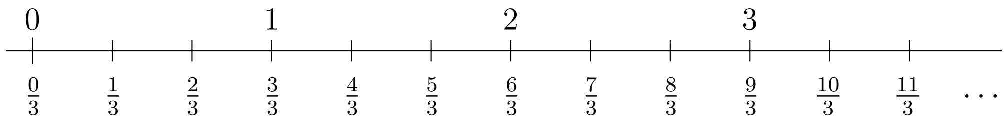
Indeed, one can dramatize (*b*) by asking third grade students to compute 12×47. *By definition,* this means they have to add 12 copies of 47 (this is why precise definitions are important; see characteristic (i) on page 3). Now show them the multiplication algorithm that immediately yields $12 \times 47 = 564$, and ask which of the two calculations they prefer. Of course, in this case, knowing the multiplications of single-digit numbers means knowing the multiplication table by heart. Through experiences like this, students get to *understand* why memorizing the multiplication table, far from being "unnecessary" (Barshay 2015), is important for their mathematics learning in the same way that memorizing the alphabet is important for their learning how to read or write. Compare slides 31-39 in Wu, 2014.

A discussion of one way to teach the long division algorithm, especially the issues surrounding the "division house" (2), is given in Wu, 2020e. We will touch on the concept of whole-number division again when we come to the division of fractions below.

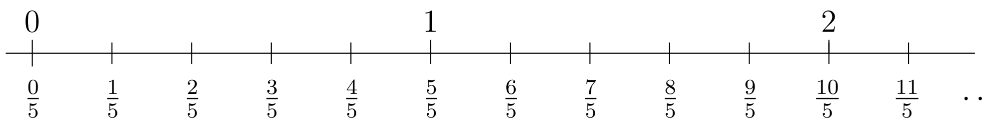
The whole numbers should be placed on the number line by the second grade (cf. page 20 of CCSSM, 2010) so that students get used to *identifying* the whole numbers with a sequence of equi-spaced points to the right of a point designated as 0.



Once students are comfortable with using the number line, they are ready for an introduction to fractions. (See Wu, 1998 and page 24 of CCSSM, 2010.) Intuitively, fractions are the points on the number line consisting of the whole numbers together with other points obtained by dividing the segments between consecutive whole numbers into **equal parts**, i.e., segments *of equal lengths.* For example, we have the **sequence of thirds** as the following sequence of points when the segments $[0, 1]$, $[1, 2]$, $[2, 3]$, $\ldots$ are each divided into 3 parts of equal length:



Note that by our convention, $\frac{0}{3}$ is just 0. In the usual nomenclature, these are the **fractions with denominator equal to 3**. In a similar fashion, we have the following **sequence of fifths**; these are the **fractions with denominator equal to 5**:



By letting the denominator be any nonzero whole number n, we obtain analogously an infinite sequence of points on the number line which we call the **sequence of $n$ths**. **Fractions** are, by definition, the collection of all the points in all these sequences. (See Sections 12.1 and 12.2 in Wu, 2011a or Section 1.1 in Wu, 2016a; both are based on Wu, 1998.)

Observe the similarity of the sequence of $n$ths (for any nonzero whole number n) with the sequence of whole numbers.

Now that fractions have been precisely defined, it is easy to see that a fraction is now just *a point on the number line* belonging to a sequence of $n$ths for some nonzero whole number n. This definition of a fraction is *conceptually* far simpler and *far more usable* than "part-of-a-whole". Intuitively, the number line, together with its sequences of $n$ths, serves to anchor students' conception of a fraction in the same way their fingers serve to anchor their conception of a whole number.

One may ask: does all that simplicity amount to anything? Does it advance students' mathematical mastery of fractions? Two short answers: yes and yes. For example, we have pointed that TSM does not explain what it means for two fractions to be "equal" or for one fraction to be "less than" another. However, our present definition of a fraction allows us to define unequivocally that two fractions are **equal** if they are the same point on the number line, and that a fraction $\frac{a}{b}$ is **less than** another fraction $\frac{m}{n}$ if $\frac{a}{b}$ is to the left of $\frac{m}{n}$ on the number line (see Wu, 2016a, p. 15). Referring to the preceding pictures, we see that $3 = \frac{9}{3}$ and $2 = \frac{10}{5}$; also $\frac{5}{3}$ is less than $\frac{11}{5}$ because $\frac{5}{3}$ is to the left of 2 ($= \frac{6}{3}$) and $\frac{11}{5}$ is to the right of 2 ($= \frac{10}{5}$) and, *a fortiori*, to the right of $\frac{5}{3}$. This seemingly abstract definition of a fraction can even *include* the interpretation of a fraction as a piece of pizza. For example, how can $\frac{5}{3}$—in the present definition—stand for 5 pieces of pizza when the pizza is divided into 3 "equal pieces"? For an answer, let the **unit** 1 on the number line stand for the *area* of a single pizza (note that the unit is *not* "a pizza" but "the *area* of a pizza"). Then an equal division of each segment between consecutive whole numbers (in terms of length) corresponds to an equal division of the pizza (in terms of area), and each segment between consecutive points in the sequence of thirds stands for a third of a pizza. Since $\frac{5}{3}$ is the 5th point (starting with $\frac{1}{3}$) in the sequence of thirds, it stands for 5 pieces each of which has an area of one-third of a single pizza. (This is a more precise way of saying "5 pieces of pizza when the pizza is divided into 3 equal pieces".) In general, this definition is adequate for all real-world applications of fractions when we let the unit 1 represent an appropriate measurable quantity such as the *weight of a bucket of water* or the *volume of an apple.* See (B) on page 12 of Wu, 2016a.

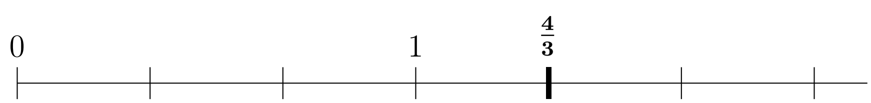
We explicitly point out that, using such a precise definition of a fraction, it is now possible to *prove* that a fraction "is also a division". The key point is the need to first *extend* the definition of whole number division $m \div n$ for *all* whole numbers m and n with $n \neq 0$—*when fractions are available*—before we give the proof that, e.g., $\frac{5}{3} = 5 \div 3$. See Section 15.2 of Wu, 2011a or Section 1.2 of Wu, 2016a.

The central fact about fractions is the following **theorem on equivalent fractions**: *Given any fraction, say $\frac{4}{3}$. If we multiply its numerator and denominator by the same nonzero whole number, say 5, then we get an equal fraction, i.e.,*

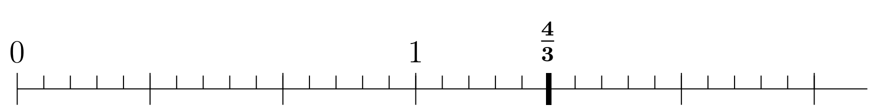
$$\frac{4}{3} = \frac{5 \times 4}{5 \times 3} \tag{13}$$

See Wu, 2016a, pp. 28-29. (In the older literature, two equal fractions were referred to as **equivalent fractions**.)

The proof of (13) is simplicity itself. We can easily locate $\frac{4}{3}$ on the number line as the 4th point in the sequence of thirds:



Next, we divide each segment between consecutive points in the sequence of thirds into 5 parts of equal length. This then means that each segment between consecutive whole numbers has now been divided into 3×5 parts of equal length, i.e., into 15 equal parts. So we now have a sequence of 15ths.



Therefore $\frac{4}{3}$, which is the 4th point in the sequence of thirds, is also the 20th point in the sequence of 15ths ($20 = 5 \times 4$). In other words,

$$\frac{4}{3} = \frac{20}{15} = \frac{5 \times 4}{5 \times 3},$$

which is (13). The reasoning is in fact completely general, so that

$$\frac{m}{n} = \frac{c \times m}{c \times n}$$

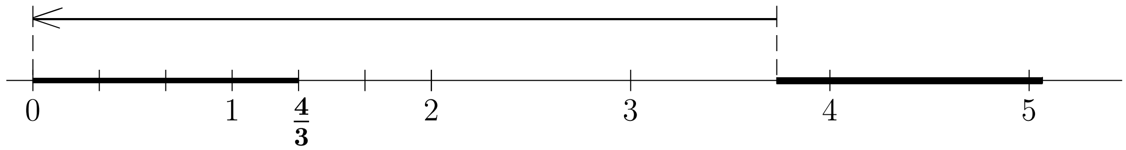
for all nonzero whole numbers m, n, and c.

In TSM, this theorem is a rote skill needed mainly for simplifying fractions. In the setting of learnable school mathematics, this theorem is the glue that holds together all aspects of fractions—as we now proceed to demonstrate—and serves to remind us of the coherence of mathematics (see characteristic (iv) on page 4).

The next topic is the arithmetic operations of fractions. Beyond the pivotal role that the theorem on equivalent fractions plays throughout the discussion, we call attention to another kind of coherence at work: the tight relationship between the arithmetic of whole numbers and that of fractions.

To define the four arithmetic operations on fractions, we have to first introduce the concept of the "length" of a segment. Given a segment on the number line, slide the segment until its left endpoint is at 0; let its right endpoint rest at some point A. Suppose A is a fraction.[12] Then we say the segment has **length** A. For example, the right thickened segment below has length $\frac{4}{3}$ because, after sliding its left endpoint to 0, its right endpoint "A" in this case is $\frac{4}{3}$:

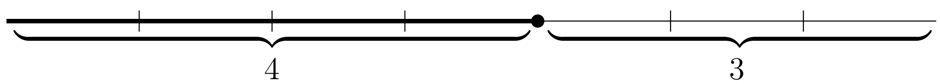
[12] When we have the real numbers at our disposal, the same definition of length will make sense regardless of whether A is a fraction or not.



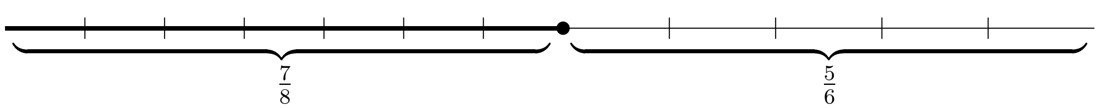
It is customary to identify a fraction $\frac{m}{n}$ *with any segment that has length* $\frac{m}{n}$.

We will now *define* the four arithmetic operations on fractions by hewing closely to those on whole numbers. Although specific numbers will be used in the following discussion to avoid an excessive reliance on the symbolic notation, please note that the definitions and the reasoning are valid in general.

Addition. For whole numbers, $4+3$ is the total length of the **concatenation** of a segment of length 4 and one of length 3 (i.e., the placement of the two segments end-to-end on a straight line):

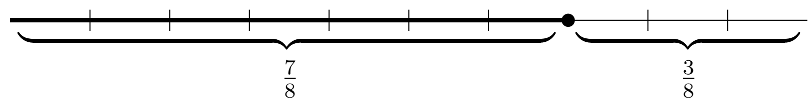


Similarly, we *define* the sum $\boldsymbol{\frac{7}{8}+\frac{5}{6}}$ to be the total length of the concatenation of a segment of length $\frac{7}{8}$ and one of length $\frac{5}{6}$ (see Wu, 2016a, pp. 43-44):



Two quick observations at this point: this definition shows that the addition of fractions—like that of whole numbers—is just "putting things together", and if these are two fractions with the same denominator, e.g., $\frac{7}{8}+\frac{3}{8}$, then this sum is *the same* as whole number addition, namely, 7 of the $\frac{1}{8}$'s together with 3 of the $\frac{1}{8}$'s make $(7+3)$ of the $\frac{1}{8}$'s. So,

$$\frac{7}{8} + \frac{3}{8} = \frac{7+3}{8} \quad \left(= \frac{10}{8}\right)$$

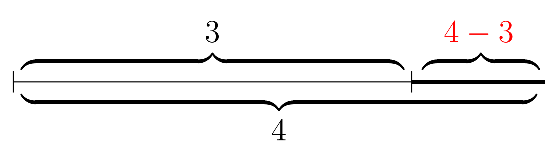


In general, the theorem on equivalent fractions reduces the task of finding the sum of *any* two fractions to that of finding the sum of two fractions with the same denominator, and the easiest-to-obtain common denominator for this purpose is the product of the given denominators. With this in mind, we have:

$$\frac{7}{8} + \frac{5}{6} = \frac{6 \times 7}{6 \times 8} + \frac{8 \times 5}{8 \times 6} = \frac{42}{48} + \frac{40}{48} = \frac{82}{48} \tag{14}$$

See p. 46 of Wu, 2016a. This then is the conceptually correct version of fraction addition to teach to students. This explains that the sum in (7) on page 13 is *unnecessary* for the purpose of addition, and it also explains why equation (10) on page 16 is wrong.

Subtraction. For whole numbers, $4 - 3$ is the length of the segment that remains after a segment of length 3 has been removed from one end of a segment of length 4.

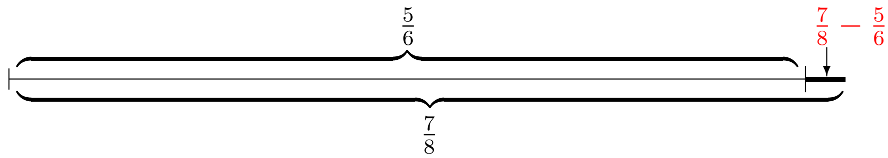


Before we can define the difference $\frac{7}{8} - \frac{5}{6}$, we have to make sure that $\frac{7}{8} > \frac{5}{6}$. We use the theorem on equivalent fractions to change both fractions to fractions with the same denominator (Wu, 2016a, pp. 31-33):

$$\frac{7}{8} = \frac{6 \times 7}{6 \times 8} = \frac{42}{48} \quad \text{and} \quad \frac{5}{6} = \frac{8 \times 5}{8 \times 6} = \frac{40}{48} \tag{15}$$

We now see that $\frac{7}{8}$ and $\frac{5}{6}$ are, respectively, the 42nd and 40th members in the *same* sequence of 48ths. Since the 40th member of the sequence is to the left of the 42nd member of the sequence, we get $\frac{5}{6} < \frac{7}{8}$, *by the definition of "less than"*.

Now we can *define* the difference $\mathbf{\frac{7}{8}} - \mathbf{\frac{5}{6}}$ by imitating the subtraction of whole numbers: it is the length of the segment that remains after a segment of length $\frac{5}{6}$ has been removed from one end from a segment of length $\frac{7}{8}$ (Wu, 2016a, pp. 51-52):



The difference is seen to be $\frac{2}{48}$ as in the case of addition above.

Multiplication. The product 2×7 among whole numbers means the sum of 2 copies of 7. Analogously, we *define* the product $\mathbf{\frac{2}{5}} \times \mathbf{\frac{3}{4}}$ to be the sum of 2 copies of one of the parts when the segment of length $\frac{3}{4}$ is divided into 5 equal parts. (Therefore the product of the fractions $\frac{2}{1} \times \frac{3}{1}$, *according to this definition*, is exactly the product of the whole numbers 2×3.) See Wu, 2016a, pp. 56-58.

To establish the credibility of this definition, let us quickly prove (9) on page 15, i.e., $\frac{2}{5} \times \frac{3}{4} = \frac{2 \times 3}{5 \times 4}$. (See Wu, 2016a, pp. 59-60.) By the theorem on equivalent fractions, we have $\frac{3}{4} = \frac{5 \times 3}{5 \times 4}$ so that

$$\frac{3}{4} = \frac{3}{5 \times 4} + \frac{3}{5 \times 4} + \frac{3}{5 \times 4} + \frac{3}{5 \times 4} + \frac{3}{5 \times 4}$$

Therefore if $\frac{3}{4}$ is divided into 5 equal parts, one part is $\frac{3}{5 \times 4}$. The "sum of 2 copies of one of these parts" is therefore

$$\frac{3}{5 \times 4} + \frac{3}{5 \times 4}$$

By the definition of multiplication, we get

$$\frac{2}{5} \times \frac{3}{4} \;=\; \frac{3}{5 \times 4} + \frac{3}{5 \times 4} \;=\; \frac{2 \times 3}{5 \times 4}$$

At this point, we have *explained* why the multiplication of fractions is computationally so simple while the addition of fractions is slightly more complicated, but not unreasonably so. Both are at least understandable now and therefore learnable.

We mention in passing that the associative and commutative laws can now be proved for fraction addition and multiplication, as can the distributive law for fractions.

Division. Recall that we write $72 \div 4$ *only because* we know ahead of time that 72 is a **whole number multiple** of 4 in the sense that $72 = 18 \times 4$. Then, by definition, $72 \div 4 = 18$. See equation (1) on page 6.

We copy this definition word-for-word for fraction division: if we know that $\frac{2}{5}$ is a **fractional multiple of** $\boldsymbol{\frac{3}{4}}$ in the sense that $\frac{2}{5} = A \times \frac{3}{4}$ for some fraction A, then we *define* $\boldsymbol{\frac{2}{5} \div \frac{3}{4} = A}$. (See Wu, 2016a, pp. 72-75.)

This seems to be a very restrictive definition because how can we find out whether or not $\frac{2}{5}$ is a fractional multiple of $\frac{3}{4}$, or, more to the point, whether or not there is an A so that $\frac{2}{5} = A \times \frac{3}{4}$? But it turns out that such an A always exists. In fact,

$$\frac{2}{5} \;=\; \left(\frac{2}{5} \times \frac{4}{3}\right) \times \frac{3}{4} \tag{16}$$

In the present case, the A in "$\frac{2}{5} = A \times \frac{3}{4}$" is therefore $\frac{2}{5} \times \frac{4}{3}$,[13] which means, by the definition of division, that equation (3) on page 9 is correct, i.e.,

$$\frac{2}{5} \div \frac{3}{4} \;=\; \frac{2}{5} \times \frac{4}{3}$$

[13]Due to the lack of space, we will leave out the simple proof that there is *only one such* "A".

We hasten to point out that equation (16) is trivially true because

$$\frac{2}{5} \;=\; \frac{2}{5} \times 1 \;=\; \frac{2}{5} \times \left(\frac{4}{3} \times \frac{3}{4}\right)$$

so that, by the associative law of multiplication, we obtain

$$\frac{2}{5} \;=\; \left(\frac{2}{5} \times \frac{4}{3}\right) \times \frac{3}{4},$$

which is (16). Note that this reasoning shows in fact that, given *any* nonzero fraction B, *every* fraction is a fractional multiple of B (see Lemma 1.7 on p. 75 of Wu, 2016a). Therefore the preceding definition of fraction division is in fact applicable to the division of any two fractions $C \div B$, $B \neq 0$.

It is time to also clear up the old business about "$73 \div 4 = 18\ R1$". We emphasize that, *without invoking fractions*, one cannot write $73 \div 4$ because 73 is not a whole number multiple of 4. The correct way to express the **division-with-remainder** of 73 by 4 in terms of (only) whole numbers is *not* the nonsensical "$73 \div 4 = 18\ R1$" in TSM, but

$$73 \;=\; (18 \times 4) + 1 \tag{17}$$

where 18 is called the **quotient**, and 1 is by definition the **remainder**. In general, given a whole number a and a *nonzero* whole number b, the **division-with-remainder** of a by b is the expression of a in terms of b, so that

$$a \;=\; (q \times b) + r \quad \text{where } q \text{ and } r \text{ are whole numbers and } \; r < b$$

The whole numbers q and r are, respectively, the **quotient** and the **remainder** of this division-with-remainder. The significance of the inequality $r < b$ is that it guarantees the quotient q to be the *largest* whole number q so that $q \times b \leq a$ (Section 7.2 of Wu, 2011a). Indeed, the next whole number $q+1$ satisfies $(q+1) \times b > a$ because, using $r < b$, we get

$$(q+1) \times b \;=\; (q \times b) + b \;>\; (q \times b) + r \;=\; a$$

Thus the general definition puts equation (17) in the proper perspective, in the following sense. We now know that 18 is the largest multiple of 4 that does not exceed 73 and that, although it is true that $73 = (17 \times 4) + 5$, the latter is not the division-with-remainder of 73 by 4 because the inequality $5 < 4$ is false.

For the reason that the division house in (2) on page 7 leads to equation (17) above, see Wu, 2020e.

Before leaving fractions, let us tie up a few loose ends. First of all, contrary to what is taught in TSM, the concept of a *mixed number* such as $7\frac{4}{5}$ cannot be introduced before the addition of fractions has been defined because the definition of a mixed number requires the use of fraction addition (recall characteristic (iv) on page 4):

$$\mathbf{7\,\frac{4}{5}} \;\stackrel{\text{def}}{=}\; \mathbf{7 + \frac{4}{5}}$$

See pp. 50-51 of Wu, 2016a.

When TSM calls this mixed number "7 *and* $\frac{4}{5}$", it is committing the mathematical sin of intentionally hiding fraction addition with the use of "and". This is a blatant attempt to obfuscate rather than to educate. That is TSM for you. In any case, if we perform the simple addition, then we get

$$7 + \frac{4}{5} \;=\; \frac{7 \times 5}{5} + \frac{4}{5} \;=\; \frac{39}{5}$$

This is equation (5) on page 12, and the derivation just given does not call for any rote-memorization. On the other hand, if an improper fraction $\frac{39}{5}$ is given, then we obtain a mixed number by the use of the division-with-remainder of 39 by 5 and also by making use of fraction addition, as follows:

$$\frac{39}{5} = \frac{(7 \times 5) + 4}{5} = \frac{7 \times 5}{5} + \frac{4}{5} = 7 + \frac{4}{5} = 7\,\frac{4}{5}$$

This is equation (4) on page 12, and again no rote-memorization is needed. We therefore see that, contrary to the impression created by TSM, there is nothing at all arbitrary or mysterious about the conversion of a mixed number to an improper fraction, and vice versa, because it is nothing more than a routine computation using fraction addition and division-with-remainder. Incidentally, the multiplication of mixed numbers in equation (11) on page 17 can now be done *directly* using the distributive law without first converting to improper fractions:

$$\begin{aligned} 5\frac{3}{4} \times 4\frac{1}{4} &= (5 + \frac{3}{4})\,(4 + \frac{1}{4}) = (5 \times 4) + (5 \times \frac{1}{4}) + (\frac{3}{4} \times 4) + (\frac{3}{4} \times \frac{1}{4}) \\ &= 20 + \frac{5}{4} + 3 + \frac{3}{16} = 23 + \frac{23}{16} = 23 + (1 + \frac{7}{16}) = 24\,\frac{7}{16} \end{aligned}$$

Next, we need the correct definition of a (*finite*) *decimal*: a finite decimal is a fraction whose denominator is *a power of* 10, i.e., numbers of the form 10^n for some whole number n (by definition, $10^0 = 1$). For example, by definition,

$$43.57 = \frac{4357}{100}\,,$$

where number of 0's in the denominator on the right corresponds to the number of *decimal digits* (the number of digits to the right of the decimal point in 43.57) in 43.57.

Similarly, by definition,

$$0.1234 \;=\; \frac{1234}{10000}$$

Once this is understood, we begin to understand equation (12) on page 17, i.e.,

$$43.57 \;=\; \frac{(4\times 10^3)+(3\times 10^2)+(5\times 10^1)+(7\times 10^0)}{10^2} \;=\; 40+3+\frac{5}{10}+\frac{7}{100}$$

The right side is the precise meaning of "4 tens and 3 ones and 5 tenths and 7 hundredths" in equation (12). Thus the definition of a decimal in (12) exhibits two typical mathematical sins of TSM: using the word "and" to camouflage fraction addition, and making believe that (finite) decimals are *another* kind of number. Where there was obfuscation in TSM, learnable school mathematics brings transparency.

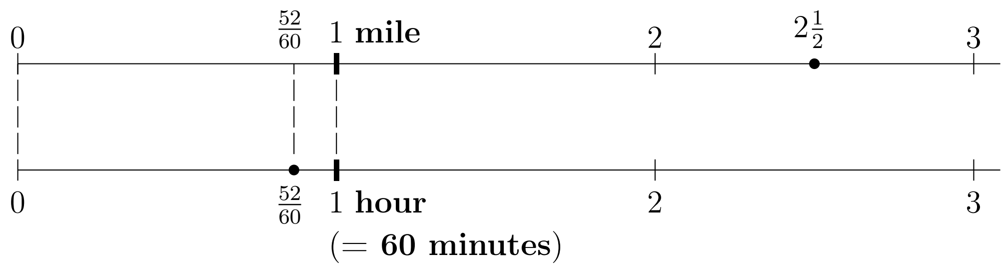
Finally, we take a second look at the "rate problem" on page 10:

> Trina walks $2\frac{1}{2}$ miles in 52 minutes. How many miles does she walk in 70 minutes?

We are immediately confronted with this issue: what is the "whole"? Is it "1 hour" or "1 mile"? As illustration, consider the computation of the *average speed* (Wu, 2011a, pp. 292-293) of Trina's walk in the first 52 minutes. By definition, it is the division

$$\frac{2\frac{1}{2}}{\frac{52}{60}} \quad \text{mph} \tag{18}$$

The problem here is that $2\frac{1}{2}$ belongs to the number line whose unit is "1 mile" and $\frac{52}{60}$ belongs to another number line whose unit is "1 hour", but we can only divide two fractions on the *same* number line. We get around this problem the obvious way by *identifying the two number lines* through the identification of the two 0's and the two units, "1 mile" and "1 hour", respectively, so that 52 minutes is identified with the fraction $\frac{52}{60}$ on the number line whose unit is "1 mile":



See p. 344 of Wu, 2011a. Once this is done, the division in (18) becomes possible: it is a division on the number line whose unit is "1 mile".

The most striking feature of this problem from TSM is, however, the fact that *it has no solution as it stands* until some assumption such as "Trina walks at a constant speed" is made explicit. See pp. 108-112 in Wu, 2016a for a detailed explanation (we should add that this reference has a discussion of why TSM's treatments of *ratio, percent* and *rate* are unlearnable, and why there is no mathematical justification for so-called "proportional reasoning"; see also Section 11.5.1 of Wu, 2021).

To summarize, it is easy to see that the learning of numbers up to this point can be made so much easier than what TSM makes it out to be: mixed numbers and (finite) decimals are simply certain kinds of fractions. Since whole numbers are also fractions and since the arithmetic of whole numbers is *so* similar to the arithmetic of fractions, all that students have to learn about numbers up to grade 6 is just *fractions*, not fractions as parts-of-a-whole or pieces of pizza, but fractions as certain points on the number line. *Numbers can be learnable after all.*

We will now briefly comment on the remaining concerns that were raised earlier.

The reason for "negative times negative is positive" is that we insist on the distributive law being valid for both positive and negative numbers. The argument is short but mathematically somewhat sophisticated. For middle school students, this would require a careful discussion of negative numbers as points on the number line to the *left* of 0 and how to *extend* the addition and multiplication of fractions to the addition and multiplication of **rational numbers**—the collection of all (positive) fractions and negative fractions. This is done in Chapter 2 of Wu, 2016a. Note that the *short* argument featuring the distributive law that proves $(-x)(-y) = xy$ for all numbers x and y is given on page 171, *loc. cit.*, but a full explanation of this fact that is suitable for middle school students spans pp. 164-171. As for the teaching of "variables" in algebra (page 24), the most important point to bear in mind is that *"variable" is a word used in mathematics for convenience but is* not *itself a mathematical concept.* Once students get to know the *basic protocol in the use of symbols* (Section 1.1 in Wu, 2016b), they will never be troubled by the so-called *meaning* of "variables" again. For example, this *basic protocol* specifies that we must be aware of the meaning of each symbol being used. So when faced with the equation $3x - 1 = 5x$ on page 24, we should know that the computations on page 25 are actually carried out *under the assumption* that this x is a solution of the equation and is therefore a fixed number. In particular, *this x does not vary.* See pp. 324-330 in Wu, 2020a.

Next, the teaching of slope (page 25). TSM does not explain the reason for introducing the concept of the *slope* of a line in the coordinate plane (see characteristic (v) on page 4) and, worse, does not give a correct definition of the concept. TSM seems to be content to teach *slope* by rote without a definition. The *purpose* for defining slope and its correct *definition* are addressed in detail in Section 4.3 of Wu, 2016b. As far as learnable school mathematics is concerned, the high point of the discussion of slope is undoubtedly the explanation of why the graph of a linear equation in two variables is indeed a line (which then explains the terminology of "linear" equations); knowing the proof of this theorem then helps students to write down with ease the equation of any line that satisfies standard prescribed geometric conditions (see Sections 4.4 and 4.6, respectively, *loc. cit.*).

Finally, a few comments on why the TSM geometry curriculum in secondary school needs restructuring (see page 26). The most notorious aspect of the problem—the fact that the high school geometry course is the *only* place in K-12 where definitions, theorems, and proofs are honored—would be automatically taken care of by teaching learnable school mathematics all through K-12. Indeed, since learnable mathematics emphasizes definitions, reasoning, and precision everywhere (see characteristics (i), (ii), and (iii) on pp. 3 ff.), it creates an environment in which such a high school geometry course would flourish. The remaining two issues of concern are these:

> (G1) Since linear equations in two variables are a staple in middle school, middle school algebra must teach *slope* in a way that is learnable. As mentioned above, this requires at least an informal but well-rounded discussion of similar triangles, a topic that is exclusively reserved in TSM for the high school course in geometry.
>
> (G2) TSM talks about *congruence* and *similarity* of geometric figures with-

> out ever giving a definition of these concepts except for triangles in the high school geometry course. Precise definitions of these general concepts require the introduction of transformations in the plane into K-12, and transformations have been unfortunately firmly associated in K-12 with recreational activities but not with serious mathematics.

These two divergent issues unexpectedly converge to a solution, as follows. Let middle school students be exposed to isometries and dilations, *intuitively,* via hands-on activities using transparencies. Using this informal knowledge, one can give an informal proof of the *angle-angle criterion* for triangle similarity that is needed for the definition of *slope*. This is accomplished in Chapter 4 of Wu, 2016a. Such a curricular change eases the teaching of *slope* in middle school while simultaneously building an intuitive foundation for the teaching of the general concepts of congruence and similarity in the plane as rigorous mathematics in high school. This then addresses the concern of (G1).

The issue raised in (G2) is resolved by making substantial modifications of Euclid's axiomatic system. The first modification is to allow the foundation of plane geometry to rest on transformations rather than on an abstract undefined notion of *triangle congruence*, and the second modification is to relax the requirement that the starting point for plane geometry be a minimum collection of *logically independent* axioms. It is perfectly acceptable in K-12 to start the geometric discussion with a sufficiently large number of assumptions to obviate the need for proving most of the crushingly boring but logically indispensable foundational theorems. *Pedagogically, there is nothing wrong with employing assumptions that overlap each other.* For this new development, see Chapters

4 and 5 of Wu, 2020a, and Chapters 6 and 7 of Wu, 2020b. It remains to point out that in Chapter 8 of Wu, 2020b, one can find a brief survey of the historical evolution of the foundations of geometry that puts the new foundation in the proper perspective .

A few thoughts on implementation

If the preceding discussion of unlearnable school mathematics in general, and Textbook School Mathematics (TSM) in particular, has been at all successful, it should make the mathematical and educational communities ashamed of themselves for allowing students of at least the past half century to be tormented and miseducated for thirteen years by such a conglomeration of mathematical illiteracies and nonsense. It is beyond debate that the number one priority of school mathematics education in 2021 has to be the replacement of TSM in school classrooms by learnable school mathematics.

It was mentioned earlier that the two major proposed reforms of the past thirty years—the 1989 NCTM reform and the 2010 Common Core Mathematics Standards—did try to do exactly that. Although neither reform used the term "TSM" (which was introduced only in 2011) or spelled out what "learnable mathematics" would look like beyond a few slogans, the intent was unmistakable. The fact that we are here advocating the same thing should give us pause, because history has taught us that the road to hell in mathematics education is paved with good intentions. Unless our good intentions this time around are bolstered with a well-thought-out plan of implementation, they too will go down in failure. We therefore propose that the following three steps be taken:

(1) Make available a *detailed* exposition of learnable school mathematics.

(2) Provide a long-term program, with the necessary funding, for the massive inservice and preservice professional development of teachers.

(3) Provide informed guidance on writing mathematically correct student textbooks.

It is easy to see that items (2) and (3) cannot go forward without first having item (1) firmly in place. The critical need for item (1) stems from the fact that, for those who have been exposed mainly to TSM (which is nearly everyone), *learnable mathematics* may as well be some fairy tale written in a defunct language. Teachers' and educators' long experience with TSM has given them no hint of why having precise definitions of concepts is important, what coherence means, what conceptual understanding is all about, or what valid reasoning looks like. This kind of knowledge cannot be acquired in a few fun-filled professional development sessions, no matter how charming or skillful the presenters may be. Such acquisition requires a long-term immersion in an environment of learnable school mathematics. It is for the purpose of creating this environment that our six volumes on learnable school mathematics—Wu, 2011a, 2016a, ..., 2020c—can hope to fill a void.

Those six volumes show at least how *definitions are used for reasoning* and, more importantly, how—for both mathematical integrity and student learnability—any reasoning about a concept *has* to be based entirely on its definition and *only* on its definition. They also show how a coherent presentation of school mathematics can facilitate learning for both teachers and students. For example, learning the arithmetic operations on fractions becomes so much easier when learners can fall back on their knowledge of the same on whole numbers or, later on, learning the arithmetic operations on rational numbers (fractions and negative fractions) becomes "more of the same" because these volumes remind them how things went when they transitioned from whole numbers to fractions. And still later on, in high school, how lengths of curves, areas of planar regions, and volumes of solids can be taught in almost the same way because they are conceptually identical (see Chapter 5 of Wu, 2016a and Chapters 4 and 5 of Wu, 2020c). And so on.

But we need to add a word of caution: even with item (1) in place, the road leading to items (2) and (3) is still a long and forbidding one. For the purpose of preservice professional development, very likely the schools of education *and* the departments of mathematics in universities across the land will look askance—perhaps for entirely different reasons—at any request for courses on the *mathematics of K-12* for preservice teachers. There are very few signs that such courses will be readily available on university campuses anytime soon.

For inservice teachers, there is an obstacle at the outset: the well-known 2011 IES impact study which purports to show that content-intensive inservice professional development is unlikely to raise student learning (Garet et al., 2011). Fortunately, there are solid reasons to believe that the impact study's failure to take into account the ruinous effects that TSM has had on school mathematics invalidate its depressing conclusion (see pp. 275-284 of Wu, 2021). The situation will further clarify when more voices are heard on the important issue of content-based inservice professional development. In addition, we feel compelled to point out that helping inservice teachers revamp their knowledge base in mid-career will be a *major* undertaking (see pp. 263-271 in Wu, *loc.*

cit.). Compounding the problem further, no large-scale program for inservice professional development will go very far without some financial incentive that encourages this intellectual (and even philosophical) make-over. Therefore, substantial state and federal government funding will be essential (cf. p. 272 of Wu, *loc. cit.*).

Finally, the textbook issue in item (3) would seem to be more susceptible to a solution because it only takes a small group of writers to get the textbooks written. This is true, but the fact remains that there are few in the education establishment whose conception of mathematics has not been compromised by TSM. It also appears to be the case that the field of school-mathematics textbook evaluation puts no more emphasis on mathematical content than does the rest of mathematics education. While the overall picture may not be optimistic, nevertheless—as we mentioned at the beginning—there is a chance that one or two recent entries in the textbooks sweepstakes may be able to improve on TSM.

Many attempts at reforming school mathematics education, big and small, statewide and federal, have come and gone in the past few decades. They all seem content to put forth a set of standards but then let these three basic items, (1)–(3), twist in the wind. Not surprisingly, the results were disheartening. It is inexcusable that we in the mathematics education community who have been entrusted with the sacred duty of educating the young would drop the ball so complacently and simply allow students to be taught unlearnable school mathematics. It is time to put an end to this shameful betrayal.

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Department of Mathematics
University of California at Berkeley