Elliptic curves with large analytic order of the Tate-Shafarevich group

Andrzej Dąbrowski and Mariusz Wodzicki

0. Introduction

Let E be an elliptic curve over $\mathbb{Q}$ of conductor $N = N(E)$ and let $L(E,s) = \sum_{n=1}^{\infty} a_n n^{-s}$ denote the corresponding L-series.

Let $Ш(E)$ be the Tate-Shafarevich group of E (conjecturally finite), $E(\mathbb{Q})$ be the group of global points, and R be the regulator (with respect to the Néron-Tate height pairing). Finally, let ω be the real period, and $c_\infty = \omega$ or 2ω (according to whether $E(\mathbb{R})$ is connected or not), and let c_{fin} denote the fudge factor (the Tamagawa number) of E.

The L-series $L(E,s)$ converges for $\operatorname{Re} s > 3/2$. The modularity conjecture (completely settled by Wiles-Taylor-Diamond-Breuil-Conrad [BCDT]) implies that $L(E,s)$ has analytic continuation to an entire function.

Conjecture 1 (Birch and Swinnerton-Dyer). *L-function $L(E,s)$ has a zero of order $r = \text{rank}\, E(\mathbb{Q})$ at $s = 1$, and*

$$\lim_{s\to 1} \frac{L(E,s)}{(s-1)^r} = \frac{c_\infty(E) c_{\text{fin}}(E) R(E) |Ш(E)|}{|E(\mathbb{Q})_{\text{tors}}|^2}. \tag{1}$$

It has been known for a long time that there are elliptic curves with arbitrarily large values of $|Ш(E)|$ (Cassels [Ca] was the first one to show this by considering quadratic twists by many different primes). Kramer [Kr] finds semistable elliptic curves with discriminant $m(16m+1)$ and $|Ш(E)| \geq 2^{2k-2}$, where k is the number of prime factors of $16m+1$. Their methods give at best families with $|Ш(E)| \gg N^{\frac{c}{\log\log N}}$ for some positive constant. Assuming the Birch and Swinnerton-Dyer conjecture, Mai and Murty [MM] have shown that for any elliptic curve E, and infinitely many square-free d,

$$\lim_{d} \frac{N(E_d)^{\frac{1}{4}-\epsilon}}{|Ш(E_d)|} = 0.$$

It was proved only in 1987 that there is any elliptic curve over $\mathbb{Q}$ for which group $Ш(E)$ is finite (Rubin [Ru], Kolyvagin [Ko], Kato). A few years later, Goldfeld and Szpiro [GS], and Mai and Murty [Ra], proposed the following general conjecture:

$$|Ш(E)| \ll N(E)^{1/2+\epsilon}, \qquad N \to \infty. \tag{2}$$

The Goldfeld-Szpiro-Mai-Murty conjecture, (2), holds if the Birch and Swinnerton-Dyer conjecture holds true for all rank zero quadratic twists E_d of E. Assuming additionally the Lindelöf conjecture, $L(E_d, 1) \ll N(E_d)^\epsilon$, one then easily obtains that

$$|Ш(E_d)| \ll N(E_d)^{1/4+\epsilon}.$$

In general, for elliptic curves that satisfy the Birch and Swinnerton-Dyer conjecture, Goldfeld and Szpiro [GS] show that the Goldfeld-Szpiro-Mai-Murty conjecture is equivalent to the Szpiro conjecture:

$$|\Delta(E)| \ll N(E)^{6+\epsilon}$$

where $\Delta(E)$ denotes the discriminant of the minimal model of E.

It is conjectured [dW] that 6 in the exponent of the conductor is the smallest possible number for which the above asymptotic bound holds.

Conjecture 2 (de Weger). *For any $\epsilon > 0$ there exist infinitely many elliptic curves over $\mathbb{Q}$ with*

$$|Ш(E)| \gg N(E)^{1/2-\epsilon}.$$

De Weger demonstrates [dW] that the above conjecture follows from the following three conjectures: the Birch and Swinnerton-Dyer conjecture in the rank zero case, the Szpiro conjecture, and the Riemann hypothesis for certain Rankin-Selberg zeta functions associated to certain modular forms of weight $\frac{3}{2}$.

He gives eleven examples of elliptic curves with the Goldfeld-Szpiro ratio

$$GS(E) := \frac{|Ш(E)|}{\sqrt{N(E)}} \geq 1,$$

the largest value being 6.893.... Nitaj [Ni] produces 47 other examples with $GS(E) \geq 1$, the largest value of $GS(E)$ being 42.265. Note that curves with $GS(E) > 1$ were already known from Cremona's tables [Cr].

As far as the order of the Tate-Shafarevich group is concerned, De Weger [dW] gives an example of an elliptic curve with $|Ш(E)| = 224^2$, Rose [Ro] gives another example of an elliptic curve with $|Ш(E)| = 635^2$, and the curve with the largest known order of that group:

$$|Ш(E)| = 1832^2$$

was produced by Nitaj [Ni]. For the family of cubic twists considered by Zagier and Kramarz [ZK]

$$E_d : \quad x^3 + y^3 = d \qquad (d \text{ cubic-free})$$

the largest value of $|Ш(E_d)|$ does not exceed 21^2 when $d \leq 70000$ (In this case, standard conjectures imply that $|Ш(E_d)| \ll N(E_d)^{1/3+\epsilon}$).

Under assumption of the Birch and Swinnerton-Dyer conjecture, they compute $|Ш(E)|$ for a rank zero elliptic curve E by evaluating $L(E,1)$ with sufficient accuracy. Subsequently, we shall refer to this number the *analytic order* of the Tate-Shafarevich group of E, and we will continue denoting it $|Ш(E)|$. This is unlikely to cause confusion.

For quadratic twists of a given curve one can calculate the analytic order of Tate-Shafarevich group using a well known theorem due to Waldspurger [Wa] combined with purely combinatorial methods. Consider, for example, the family

$$E'_d : \quad y^2 = x^3 - d^2 x \qquad (d \geq 1 \text{ an odd square-free integer})$$

of congruent number elliptic curves. Let

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad \Theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

and

$$\eta(8z)\eta(16z)\Theta(2z) =: \sum_{n=1}^{\infty} a(n) q^n.$$

When curve E'_d is of rank zero then, assuming as usual the Birch and Swinnerton-Dyer conjecture, we have (see [Tu]):

$$|Ш(E'_d)| = \left(\frac{a(d)}{\tau(d)} \right)^2 \tag{3}$$

where $\tau(d)$ denotes the number of divisors of d. (Coefficients $a(d)$ can also be calculated using the formula in [On].) Conjecturally

$$|Ш(E'_d)| \ll N(E'_d)^{1/4+\epsilon},$$

hence the sequence of curves E'_d (and, more generally, families of quadratic twists) is not a likely candidate to produce curves with large Goldfeld-Szpiro ratios.

The primary aim of this article is to present results of our search for curves with exceptionally large analytic orders of the Tate-Shafarevich group. We exhibit 134 examples of rank zero elliptic curves with $|Ш(E)| > 1832^2$ which was the largest known value for any explicit curve. Our record is a curve with

$$|Ш(E)| = 63,408^2.$$

We consider the family

$$E(n,p): \quad y^2 = x(x+p)(x+p-4\cdot 3^{2n+1})$$

and its isogenous curves for $p \in (\mathbb{Z}\setminus 0)\cap[-1000,1000]$ and $n \leq 19$. Compared to previously published results, we faced working with curves of very big conductor. Big conductor means a very slow convergence of the approximation to $L(E,1)$. The main difficulty was to design a successful search strategy for curves with exceptionally large Goldfeld-Szpiro ratios which translated into large analytic orders of the Tate-Shafarevich group.

Our explorations brought also a number of unplanned discoveries: examples of curves of rank zero with the value of $L(E,1)$ much smaller, or much bigger, than in any previously known example (see Tables 6 and 5 below). One particularly remarkable example involves a pair of non-isogeneous curves whose values of $L(E,1)$ coincide in the first 11 digits after the point! (see formula (23)).

Details of the computations, tables and related remarks are contained in section 3. Section 2 contains some heuristics concerning an (explicit) approach to conjecture 2 via our families of elliptic curves.

The actual calculations were carried by the second author in the Summer and the Fall 2002 on a variety of computers, almost all of them located in the Department of Mathematics in Berkeley. Supplemental computations were conducted also in 2003 and the Summer 2004.

The results were reported by M.W. at the conference *Geometric Methods in Algebra and Number Theory* which took place in December 2003 in Miami.

1. The Shimura correspondence and a theorem of Waldspurger

The Shimura correspondence is a map relating certain half-integral weight cusp forms to modular forms of even weight.

Theorem. [Sh] *Let $k \geq 3$ be an odd integer, N a positive integer divisible by 4, and χ a character modulo N. Suppose that $f \in S_{k/2}(N,\chi)$ is an eigenform for all $T(p^2)$ with corresponding eigenvalues λ_p. Define a function $g(z) = \sum b_n q^n$ by the identity*

$$\sum_{n=1}^{\infty} b_n q^n = \prod_p [1-\lambda_p p^{-s} + \chi(p)^2 p^{k-2-2s}]^{-1}. \tag{4}$$

Then $g \in M_{k-1}(N/2, \chi^2)$. In particular, g is an eigenform for all $T(p)$, also having the eigenvalues λ_p.

The connection between modular forms and L-series is provided by a theorem of Waldspurger (below we state a special case of his result).

Theorem. [Wa; Theorem 1] *Let Φ be a cusp form of weight 2, level divisible by 16, and trivial character which is the image of a form f of weight 3/2 and quadratic character χ under the Shimura correspondence. Then there is a function $A(t)$ from squarefree integers to $\mathbb{C}$ such that (i) $A(t)^2 = L(\Phi \otimes \chi^{-1}\chi_{-t}, 1)/\pi$, and (ii) for each positive integer N, there is a finite set C of functions $c(n)$ such that $\{\sum A(n^{sf})c(n)q^n | c(n) \in C\}$ spans the space of cusp forms of weight 3/2, level N, and character χ which correspond to Φ by the Shimura map.*

In constructing weight 3/2 modular forms one can use two methods. The first is to construct weight 1 cusp forms (using representation theory, say) and multiply these by theta series. The second is to use ternary quadratic forms to construct modular forms directly (Shimura).

Let E be an elliptic curve over $\mathbb{Q}$, of conductor N; let f denote the corresponding modular form of weight 2. Below (Examples) we give examples, when modular form F of weight 3/2 and level N' $(N|N')$ with respect to $\Gamma_0(N')$ which is an eigenform of the Hecke operators T_{p^2} and is mapped to f under the Shimura map, can be constructed explicitly. In general case one can show, using [Ko1], that such F exists for E with odd conductor.

We have the following result of Waldspurger (Corollary 2 of Theorem 1 in [Wa]).

Corollary. *If d and d' are square-free natural numbers prime to N such that $dd' \in \prod_{p|N} \mathbb{Q}_p/\mathbb{Q}_p^2$, and if $a(d)$ resp. $a(d')$ are d-th resp. d'-th Fourier coefficients of F, then*

(5)
$$\frac{\sqrt{d}L(E_d,1)}{a(d)^2} = \frac{\sqrt{d'}L(E_{d'},1)}{a(d')^2}.$$

Fix d_0 such that $L(E_0,1)\cdot a(d_0) \neq 0$. Take d such that $dd_0 \in \prod_{p|N} \mathbb{Q}_p/\mathbb{Q}_p^2$. Then Corollary combined with the Birch and Swinnerton-Dyer conjecture implies

(6)
$$a(d) \asymp |Ш(E_d)| \cdot \prod_{p|6Nd} c_p.$$

The work of Goldfeld and Viola [GV] implies (conjectural) average value of $|Ш(E_d)|$ in the rank zero case. For example, take $E = X_0(11)$. Then for $d = p$, primes satisfying $p \equiv 3 (\mathrm{mod}\ 4)$, we obtain

(7)
$$\text{the average value of } |Ш(E_d)| \ \sim\ A_E d^{1/2},$$

where A_E is an explicit constant depending only on E.

Kohnen and Zagier presented in [KZ],[Ko2] elementary proof of a version of Waldspurger's theorem for the case of modular forms on congruence subgroups. Their result gives explicitly the constant of proportionality between $a(d)^2$'s and central critical values of the quadratic twists of modular L-series. This allows, in particular, to deduce some results about the distribution of $L(E_d, 1)$ and partially confirm the above conjecture of Goldfeld and Viola.

The average value of $L(E_d, 1)$ is established in the work of Murty and Murty [MuM], and a good error term is derived in a paper of Iwaniec [Iw].

Examples

Notations. If $q := e^{2\pi iz}$, $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$, $\Theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$, $\Theta_t(z) = \Theta(tz) = \sum_{n\in\mathbb{Z}} q^{tn^2}$. $\chi_t := (\frac{t}{\cdot})$ shall denote Kronecker's character for $\mathbb{Q}(\sqrt{t})$.

Example 1. [Tu] Let $E : y^2 = x^3 - x$; let $\Omega = \Omega_E = 2.622...$ denote the real period. For any square-free integer d consider the quadratic twist $E_d : y^2 = x^3 - d^2x$.

Let $f_i(z) := \eta(8z)\eta(16z)\Theta(2^i z)$ $(i = 1, 2)$. It turns out that $f_1(z) = \sum a_1(n)q^n \in S_{3/2}(128, \chi_0)$ and $f_2(z) = \sum a_2(n)q^n \in S_{3/2}(128, \chi_2)$ are mapped to $\Phi(z) = \eta^2(4z)\eta^2(8z)$ (the cusp form associated to E) by the Shimura correspondence. If $d \geq 1$ is an odd square-free integer, then Tunnell [Tu] proved that

$$L(E_{id}, 1) = \frac{2^{i-1}a_i(d)^2\Omega}{4\sqrt{2^{i-1}d}}. \tag{8}$$

Therefore assuming the Birch and Swinnerton-Dyer conjecture, E_{id} has $\mathbb{Q}$-rank zero iff $a_i(d) \neq 0$. In addition, if $a_i(d) \neq 0$, then

$$|Ш(E_{id})| = \left(\frac{a_i(d)}{\tau(d)}\right)^2, \tag{9}$$

where $\tau(d)$ denotes the number of divisors of d.

One can check that

$$\eta(8z)\eta(16z) = (\Theta(z) - \Theta(4z))(\Theta(32z) - \frac{1}{2}\Theta(8z)).$$

Hence

$$f_1(z) = \sum_{x,y,z\in\mathbb{Z}} q^{2x^2+y^2+32z^2} - \frac{1}{2}\sum_{x,y,z\in\mathbb{Z}} q^{2x^2+y^2+8z^2}$$

and

$$f_2(z) = \sum_{x,y,z\in\mathbb{Z}} q^{4x^2+y^2+32z^2} - \frac{1}{2}\sum_{x,y,z\in\mathbb{Z}} q^{4x^2+y^2+8z^2}.$$

Consequently, for odd d,

$a_i(d) =$
$|\{(x,y,z)\in\mathbb{Z}^3 : d = 2ix^2+y^2+32z^2\}| - \frac{1}{2}|\{(x,y,z)\in\mathbb{Z}^3 : d = 2ix^2+y^2+8z^2\}|.$

Assuming the Birch and Swinnerton-Dyer conjecture, if E_d has rank zero, then Ono [On] proves that $\sqrt{|Ш(E_d)|}$ is a simple explicit finite linear combination of $\sqrt{|Ш(E_{d'})|}$, where $1 \le d' \le d$.

Example 2. [Fr1],[Fr2],[Fe] Let $E : y^2 = x^3 - 1$. For any square-free positive integer $d \equiv 1(\mathrm{mod}\ 4$, $(d,6) = 1$, consider the quadratic twist $E_d : y^2 = x^3 - d^3$.

Let $a(d)$ denote the d-th Fourier coefficient of $\eta^2(12z)\Theta(z)$. Frey proves the following result

$$L(E_d,1) = \begin{cases} 0 & \text{if } d \equiv 3(\mathrm{mod}\ 4) \\ a(d)^2 \frac{L(E,1)}{\sqrt{d}} & \text{if } d \equiv 1(\mathrm{mod}\ 24) \\ \left(\frac{a(d)}{a(13)}\right)^2 \sqrt{\frac{13}{d}} L(E,1) & \text{if } d \equiv 13(\mathrm{mod}\ 24) \\ \left(\frac{a(d)}{a(5)}\right)^2 \sqrt{\frac{5}{d}} L(E,1) & \text{if } d \equiv 5(\mathrm{mod}\ 24) \\ \left(\frac{a(d)}{a(17)}\right)^2 \sqrt{\frac{17}{d}} L(E,1) & \text{if } d \equiv 17(\mathrm{mod}\ 24) \end{cases} \tag{10}$$

One can show that $a(d) = \frac{1}{2}\sum(-1)^n$, where the sum is taken over all $m,n,k \in \mathbb{Z}$ satisfying $m^2+n^2+k^2 = d$, $3 \not| m$, $3|n$ and $2 \not| m+n$.

Let $l = l_1 + [\frac{l_2+1}{2}]$, where $l_i = |\{p|d : p \equiv i(\mathrm{mod}\ 3)\}|$. Assuming the Birch and Swinnerton-Dyer conjecture, we obtain

$$|Ш(E_d)| = \left(\frac{a(d)}{2^l}\right)^2, \tag{11}$$

if $a(d) \neq 0$.

Example 3. [Le] Let $E : y^2 = x^3+21x^2+112x$ be the elliptic curve with complex multiplication by the ring of integers in $\mathbb{Q}(\sqrt{-7})$. Let Ω be the real period of E; let Ω_d be the real period of E_d. Let ϕ be the cusp form associated to E. Then $\Phi = \phi \otimes \chi_2 \in S_2(2^6 7^2, \chi_1)$.

Let $g = g_1 + ... + g_6$ and $h = h_1 + ... + h_6$, where

$$g_1 = \sum [q^{(14m+1)^2+(14n)^2} - q^{(14m+7)^2+(14n+6)^2}]$$

$$g_2 = \sum [q^{(14m+3)^2+(14n)^2} - q^{(14m+7)^2+(14n+4)^2}]$$

$$g_3 = \sum [q^{(14m+5)^2+(14n)^2} - q^{(14m+7)^2+(14n+2)^2}]$$

$$g_4 = \sum [q^{(14m+1)^2+(14n+2)^2} - q^{(14m+5)^2+(14n+6)^2}]$$

$$g_5 = \sum [q^{(14m+3)^2+(14n+6)^2} - q^{(14m+1)^2+(14n+4)^2}]$$

$$g_6 = \sum [q^{(14m+5)^2+(14n+4)^2} - q^{(14m+3)^2+(14n+2)^2}]$$

$$h_1 = \sum [q^{(7m+1)^2+(2(7n)^2} - q^{(7m)^2+2(7n+2)^2}]$$

$$h_2 = \sum [q^{(7m+3)^2+2(7n)^2} - q^{(7m)^2+2(7n+1)^2}]$$

$$h_3 = \sum [q^{7m+1)^2+2(7n+1)^2} - q^{(7m+3)^2+2(7n+2)^2}]$$

$$h_4 = \sum [q^{(7m+2)^2+2(7n)^2} - q^{(7m)^2+2(7n+3)^2}]$$

$$h_5 = \sum [q^{(7m+2)^2+2(7n+2)^2} - q^{(7m+1)^2+2(7n+3)^2}]$$

$$h_6 = \sum [q^{(7m+3)^2+2(7n+3)^2} - q^{(7m+2)^2+2(7n+1)^2}],$$

and all sums are taken over all $m, n \in \mathbb{Z}$. One checks that $g\Theta_{28} \in S_{3/2}(784, \chi_7)$, $h\Theta_{14} \in S_{3/2}(392, \chi_7)$, $h\Theta_7 \in S_{3/2}(392, \chi_{14})$.

Let $\sqrt{\chi_2}$ be the Dirichlet character of conductor 16 which is defined by $\sqrt{\chi_2}(3) = i = \sqrt{\chi_2}(5)$. It turns out that the cusp form $\phi \otimes \chi_2$ is the image under the Shimura correspondence of each of the forms $g\Theta_{28} \otimes \sqrt{\chi_2}$, $h\Theta_{14} \otimes \sqrt{\chi_2}$, and $h\Theta_7 \otimes \sqrt{\chi_2}$.

Let $g\Theta_{28} = \sum a_n q^n$ and $h\Theta_{14} = \sum b_n q^n$. Let d be a positive square-free integer, prime to 7. Lehman proves the following results:

(1) If $d \equiv 1 (\text{mod } 4)$, then

$$L(E_d, 1) = \begin{cases} \frac{1}{2}\Omega_d a_d^2 & \text{if } (d/7) = 1 \\ \Omega_d a_d^2 & \text{if } (d/7) = -1 \end{cases} \tag{12}$$

(2) If $d \equiv 2, 3 (\text{mod } 4)$, then

$$L(E_d, 1) = \begin{cases} 2\Omega_d b_d^2 & \text{if } (d/7) = 1 \\ 4\Omega_d b_d^2 & \text{if } (d/7) = -1 \end{cases} \tag{13}$$

Therefore, assuming the Birch and Swinnerton-Dyer conjecture, we obtain (for d positive, square-free and prime to 7):

if $d \equiv 1 (\text{mod } 4)$ and $a_d \neq 0$, then $|Ш(E_d)| = \frac{a_d^2}{4^l}$, if $d\equiv 2,3 \pmod 4$ and $b_d\neq 0$, then $|Ш(E_d)|=\frac{b_d^2}{4^l}$.

Here $l=l(d)$ is defined as follows. Let l_1 be the number of odd prime divisors $p|d$ such that $(p/7)=1$, and let l_2 be the number of prime divisors $p|d$ such that $(p/7)=-1$. Define l to be $l_1+\frac{1}{2}l_2$ if l_2 is even, and to be $l_1+\frac{1}{2}(l_2-1)$ if l_2 is odd.

Example 4. [Ne] Let $E: x^3+y^3=1$ denote the Fermat's curve; let $\Omega=\frac{\Gamma(1/3)^3}{2\pi}$ denote the real period. L-series of E equals

$$L(E,s)=\sum_{\alpha\equiv 1(3)}\alpha(N\alpha)^{-s},$$

where α runs through all elements of the Eisenstein ring $\mathbb{Z}[\rho]$ congruent to 1 modulo 3. It is well known that $L(E,s)=L(f,s)$ for $f\in S_2(\Gamma_0(27),\chi_0)$.

Let $F_-(z)=\eta(6z)\eta(18z)\Theta(3z)$ and $F_+(z)=\eta(6z)\eta(18z)\Theta(9z)$. Then $F_-\in S_{3/2}(108,\chi_0)$ and $F_+\in S_{3/2}(108,\chi_{-3})$ are mapped, under Shimura's correspondence, to f. Let $F_\pm(z)=\sum_{n=1}^\infty c(\pm n)q^n$. Put

$$c'(n)=c(n)\times\begin{cases} e(n), & \text{for } n<0\\ e(-3n), & \text{for } n>0,\end{cases}$$

where

$$e(n)=\begin{cases} 1, & \text{for } n\not\equiv 5 \pmod 8\\ 1/3, & \text{for } n\equiv 5 \pmod 8).\end{cases}$$

Let d be a square-free integer prime to 3. Nekovár proves, that

$$L(E_d,1)=\Omega\Delta^{-1/2}c'(d)^2\times\begin{cases} 1, & \text{for } d<0\\ \sqrt{3}, & \text{for } d>0,\end{cases} \tag{14}$$

where Δ is the conductor of $\left(\frac{d}{\cdot}\right)$. Therefore, assuming the Birch and Swinnerton-Dyer conjecture, if $c(d)\neq 0$, then

$$|Ш(E_d)|=\left(\frac{t(d)c'(d)}{2^{\beta(d)}}\right)^2, \tag{15}$$

where $t(d)=|E_d(\mathbb{Q})_{\mathrm{tors}}|$, and

$$\beta(d)=\#\{p|d: p=x^2+27y^2\}+\frac{1}{2}\#\{p|d: p\equiv 2 \pmod 3\}.$$

Example 5. Let $E: y^2+y=x^3-x$ be the curve of conductor 11. The corresponding cusp form f equals $q\eta(z)^2\eta(11z)^2$. The weight 3/2 form $F=\sum_{n=1}^\infty a(n)q^n=\frac{1}{2}(\theta_1(q)-\theta_2(q))$ is mapped to f by the Shimura correspondence, where

$$\theta_1(q)=\sum q^{x^2+11y^2+11z^2},$$

(summation over $(x, y, z) \in \mathbb{Z}^3$ satisfying $x \equiv y (\text{mod } 2)$) and

$$\theta_2(q) = \sum q^{(x^2+11y^2+33z^2)/3}.$$

(summation over $(x, y, z) \in \mathbb{Z}^3$ satisfying $x \equiv y (\text{mod } 3)$ and $y \equiv z (\text{mod } 2)$)

One proves, that

$$L(E_d, 1) = \frac{Ka(|d|)}{\sqrt{|d|}}, \quad d < 0, d \equiv 2, 6, 7, 8, 10 (\text{mod } 11) \tag{16}$$

where $K = 2.917633233876991...$ It turns out that $K = \sqrt{3}\Omega$, where Ω is the period of the elliptic curve $y^2 + y = x^3 - 3x - 5$. Therefore, assuming the Birch and Swinnerton-Dyer conjecture, if $a(|d|) \neq 0$, then

$$|Ш(E_d)| = \frac{|E_d(\mathbb{Q})_{\text{tors}}|^2 a(|d|)^2}{c(E_d)}. \tag{17}$$

Remarks concerning representations of integers by ternary quadratic forms

Let $r(n; a, b, c)$ denote the number of representations on n by diagonal ternary quadratic form $ax^2 + by^2 + cz^2$. Let $L(s, -n)$ denote the Dirichlet L-series attached to the quadratic character $\left(\frac{\cdot}{-n}\right)$. Let $h(-n)$ denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-n})$.

According to Gauß, we have $r(n; 1, 1, 1) = 12h(-n)$, for positive $n \equiv 1 (\text{mod } 4)$. By using Lomadze's method [Lo] one can prove, for example, the following results

(i) Assume n is odd, square-free positive integer. Then

$$r(n; 1, 2, 8) = \frac{8\sqrt{n}}{\pi} L(1, -n); \tag{18}$$

(ii) For any square-free $n \equiv 1, 3 (\text{mod } 8)$, such that $L(E_n, 1) \neq 0$, (here $E : y^2 = x^3 - x$) we have

$$r(n; 1, 2, 32) \sim 4h(-n) \tag{19}$$

as n tends to infinity.

It is a classical problem to find an asymptotic formula for the number of integral points (in the region) on the ellipsoid $q(x_1, ..., x_k) = n$ as $n \to \infty$, where q is a positive definite integral quadratic form. It is well known, that

$$\sum_{n \leq x} r(n; 1, 1, 1) = \frac{4}{3}\pi x^{\frac{3}{2}} + O(x). \tag{20}$$

The general asymptotic formula (including positive ternary quadratic forms) is discussed in [DS-P].

2. An (explicit) approach to Conjecture 2

Fix a prime r of the type $4k+3$. Let $E(r,n;p,q)$ denote the elliptic curve $y^2 = x(x+p)(x-q)$, with $p+q = 4r^{2n+1}$, where p is an odd prime and q is a positive square-free integer. (Chen [Ch] showed that every sufficiently large even number is the sum of a prime and a natural number which has at most two prime factors.)

Standard descent methods lead to the following result.

Proposition 0. *Assume $p < q$ are odd primes, with $p \equiv 5 (mod\, 8)$. Then $E(r,n;p,q)$ have $\mathbb{Q}$-rank zero.*

Proposition 0 strongly supports the following expectation.

Conjecture A. *Fix a prime r of the type $4k+3$. For any $\epsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that $E(r,n;p,q)$ has $\mathbb{Q}$-rank zero, where $4r^{2n+1} = p+q$ with p a prime and q a positive square-free integer satisfying $p \ll q^{\epsilon}$.*

Proposition 1. *Assume $p < q$, with q having at most two prime factors. Then we have*

(i)
$$c_\infty(E(r,n;p,q)) = \frac{\pi}{r^{n+1/2} \cdot AGM(1, \sqrt{q/(p+q)})}$$

and

(ii)
$$c_{\text{fin}}(E(r,n;p,q)) = 2c_2c_qc_r,$$

where

$$c_2 = \begin{cases} 2 & if\; p \equiv 1 (mod\, 4) \\ 4 & if\; p \equiv 3 (mod\, 4), \end{cases}, \qquad c_r = \begin{cases} 2(2n+1) & if\; \left(\frac{-p}{r}\right) = 1 \\ 4 & if\; \left(\frac{-p}{r}\right) = -1, \end{cases}$$

and

$$c_q = \begin{cases} 2 & if\; q\; is\; a\; prime \\ 4 & if\; q\; is\; a\; product\; of\; two\; primes\; ; \end{cases}$$

The conductor is given by the formula

(iii)
$$N(E(r,n;p,q)) = 2^{f_2} pr \cdot rad(q),$$

where

$$f_2 = \begin{cases} 3 & if\; p \equiv 1 (mod\, 4) \\ 4 & if\; p \equiv 3 (mod\, 4)\; . \end{cases}$$

Proof. Use [Ni], Propositions 2.1, 3.1 and 3.2. □

Proposition 2. *Fix a prime r of the type $4k+3$. Assume the Birch and Swinnerton -Dyer conjecture holds true for the family $E(r;n,p,q)$. Then*

$$|Ш(E(r;n,p,q))| \ll N(E(r;n,p,q))^{1/2+\epsilon}.$$

Proof. It is easy to see that the Szpiro conjecture holds true for our family (combine $\Delta(E(r;n,p,q)) = 2^8 p^2 q^2 r^2$ with Proposition 1(iii)). Now use the main result from [GS]. □

Conjecture B. *For a fixed prime r of the type $4k+3$ and $\epsilon > 0$ there exist an infinite sequence $\{(n,p,q)\}$ satisfying Conjecture A and*

$$L(E(r,n;p,q),1) \gg N(E(r,n;p,q))^{-\epsilon}.$$

"Proof" of the conjecture 2. Assume the conjecture of Birch and Swinnerton-Dyer and hypothesis B hold true. Then we have, using proposition 1,

$$|Ш(E(r,n;p,q))| \gg c_{\text{fin}}^{-1} r^{n+(1/2)} L(E(r,n;p,q),1) \gg N(E(r,n;p,q))^{(1/2)-\epsilon}.$$

3. Examples of elliptic curves with large $|Ш(E)|$

In this section we compute the analytic order of $Ш(E)$, i.e., the quantity

$$|Ш(E)| = \frac{L(E,1)\cdot |E(\mathbb{Q})_{\text{tors}}|^2}{c_\infty(E) c_{\text{fin}}(E)}, \tag{21}$$

for certain special curves of rank zero. We use the following approximation of $L(E,1)$ (within an error of size 10^{-k}) [Co]:

$$S_m = 2\sum_{n=1}^{m} \frac{a_n}{n} e^{\frac{-2\pi n}{\sqrt{N}}}, \tag{22}$$

with $m \geq \frac{\sqrt{N}}{2\pi}(2\log 2 + k\log 10 - \log(1-\exp(-2\pi/\sqrt{N}))$.

Consider the family

$$E(n,p): \quad y^2 = x(x+p)(x+p-4\cdot 3^{2n+1}),$$

with $(n,p) \in \mathbb{N} \times (\mathbb{Z} \setminus \{0\})$. Any member of the family admits three isogenous (over $\mathbb{Q}$) curves $E_i(n,p)$ $(i = 2,3,4)$:

$$E_2(n,p): \quad y^2 = x^3 + 4(2 \cdot 3^{2n+1} - p)x^2 + 16 \cdot 3^{4n+2}x,$$

$$E_3(n,p): \quad y^2 = x^3 + 2(4 \cdot 3^{2n+1} + p)x^2 + (4 \cdot 3^{2n+1} - p)^2 x,$$

and

$$E_4(n,p): \quad y^2 = x^3 + 2(p - 8 \cdot 3^{2n+1})x^2 + p^2 x.$$

Isogenous curves do have the same L-series and ranks but may have different torsion subgroups, periods, Tamagawa numbers and orders of $Ш(E)$. In our situation (of 2-isogenies) the order of $Ш(E)$ can only change by a power of 2.

3.1. Tables.

(n,p)	$N(n,p)$	$\lvert Ш \rvert$	$\lvert Ш_2 \rvert$	$\lvert Ш_3 \rvert$	$\lvert Ш_4 \rvert$
$(11,336)$	15816054028824	529^2	1058^2	529^2	1058^2
$(11,301)$	5440722586421136	576^2	1152^2	576^2	288^2
$(11,865)$	15635299103673360	617^2	1234^2	617^2	617^2
$(11,-489)$	1473152464197864	680^2	680^2	1360^2	680^2
$(11,163)$	1473152461647240	346^2	1384^2	173^2	1384^2
$(12,63)$	63264216170568	554^2	1108^2	554^2	1108^2
(12.24)	81339706505952	603^2	1206^2	603^2	1206^2
$(12,22)$	143157883450560	416^2	1664^2	416^2	1664^4
$(12,262)$	42622006206125760	468^2	1872^2	234^2	1872^2
$(12,-605)$	4473683858657640	1031^2	1031^2	1031^2	2062^2
$(12,-56)$	569377945555104	1049^2	1049^2	2098^2	1049^2
$(12,934)$	151942571712321216	512^2	2048^2	256^2	2048^2
$(12,694)$	112899512607942336	576^2	2304^2	288^2	2304^2
$(12,382)*$	62143535763983040	648^2	2592^2	324^2	2592^2
$(12,-257)$	20904304573762872	1545^2	1545^2	3090^2	3090^2
$(12,466)$	75808606453660608	1435^2	5740^2	1435^2	5740^2
$(13,136)$	264786704158368	258^2	1032^2	258^2	1032^2
$(13,-69)$	16837319246889384	516^2	516^2	258^2	1032^2
$(13,60)$	457535849098320	552^2	1104^2	276^2	552^2
$(13,96)$	10765549390536	588^2	1176^2	294^2	588^2
$(13,876)$	835002924582096	340^2	1360^2	85^2	1360^2
$(13,-672)$	1281100377506040	389^2	1556^2	389^2	778^2
$(13,544)$	3111243773819208	929^2	1858^2	929^2	929^2
$(13,928)$	5307415849389480	470^2	1880^2	470^2	940^2
$(13,73)$	610744996281840	494^2	1964^2	247^2	988^2
$(13,-42)$	20497606039673280	502^2	2008^2	251^2	2008^2
$(13,-160)$	915071698203240	1079^2	1079^2	2158^2	1079^2
$(13,-3)$	1464114717117648	2364^2	2364^2	1182^2	2364^2
$(13,66)$	32210523776515392	618^2	2472^2	309^2	2472^2
$(13,-125)$	3660286792808760	639^2	1278^2	639^2	2556^2
$(13,-17)$	12444975095505720	348^2	1392^2	348^2	2784^2
$(13,-5)$	3660286792794360	1583^2	1583^2	1583^2	3166^2
$(13,708)$	21595692076981920	812^2	3248^2	406^2	1624^2

Table 1. Elliptic curves $E(n,p)$ ($11 \le n \le 13$; $-1000 \le p \le 1000$) with $\max_{1\le i\le 4} \lvert Ш_i \rvert \ge 10^6$.

(n,p)	$N(n,p)$	$\|Ш\|$	$\|Ш_2\|$	$\|Ш_3\|$	$\|Ш_4\|$
$(14,-212)$	21824460002049648	560^2	560^2	560^2	1120^2
$(14,-3)$	775119556121040	588^2	1176^2	294^2	1176^2
$(14,96)$	167129056756616	306^2	1224^2	153^2	612^2
$(14,-948)$	2033174929441680	312^2	1248^2	156^2	624^2
$(14,528)$	18118419624294264	356^2	1424^2	356^2	712^2
$(14,-800)$	8235645283809960	390^2	1560^2	195^2	1560^2
$(14,268)$	726037150017264	858^2	1716^2	429^2	1716^2
$(14,-281)$	15300603799975032	253^2	1012^2	253^2	2024^2
$(14,652)$	33560254531348080	268^2	2144^2	67^2	2144^2
$(14,-12)$	3294258113514528	1077^2	2154^2	1077^2	2154^2
$(14,240)$	8235645283778760	1184^2	2368^2	592^2	1184^2
$(14,100)$	16471290567565920	1186^2	2372^2	593^2	2372^2
$(14,-596)$	61355557364338608	598^2	1196^2	598^2	2392^2
$(14,-11)$	144947356994638704	1806^2	3612^2	903^2	3612^2
$(14,-33)$	72473678497325160	1002^2	2004^2	1002^2	4008^2
$(14,-672)$	11529903397328568	2310^2	4620^2	2310^2	2310^2
$(14,12)$	205891132094640	564^2	2256^2	282^2	4512^2
$(15,-852)$	8222777088032880	562^2	1124^2	281^2	1124^2
$(15,-12)$	5929664604325920	576^2	1152^2	288^2	1152^2
$(15,-1)$	59296646043258936	162^2	648^2	81^2	1296^2
$(15,-80)$	74120807554076040	679^2	1358^2	679^2	1358^2
$(15,-84)$	3242785330490832	775^2	1650^2	775^2	1650^2
$(15,88)$	130452621295164960	1232^2	2464^2	1232^2	2464^2
$(15,172)$	2489995878769488	1258^2	2516^2	629^2	1258^2
$(15,-96)$	14824161510815304	1434^2	2838^2	717^2	2838^2
$(15,-48)$	14824161510815016	3057^2	3057^2	3057^2	3057^2
$(15,12)$	336912761609424	240^2	1920^2	60^2	3840^2
$(15,-240)$	74120807554080840	965^2	3860^2	965^2	3860^2
$(15,-212)$	280600200026160	498^2	1992^2	249^2	3984^2
$(15,375)$	26953020928749960	1143^2	4572^2	1143^2	4572^2
$(15,60)$	37060403777035920	2299^2	4598^2	2299^2	2299^2
$(15,-248)$	141399694410862368	1185^2	4740^2	1185^2	4740^2
$(15,-6)$	237186584173036224	3705^2	3705^2	3705^2	7410^2
$(15,1)$	118593292086517776	4032^2	8064^2	2016^2	8064^2
$(15,-116)$	107475170953411824	2368^2	4736^2	2368^2	9472^2

Table 2. Examples of elliptic curves $E(n,p)$ ($n = 14, 15,\ -1000 \le p \le 1000$)

with $\max_{1\le i\le 4} |Ш_i| \ge 10^6$.

$(16,-8)$	106733962877866080	891^2	891^2	891^2	1782^2
$(16,92)$	61372028654772720	1064^2	2128^2	532^2	2128^2
$(16,588)$	116740271897662896	549^2	2196^2	549^2	1098^2
$(16,624)$	102025111574427912	1100^2	2200^2	550^2	1100^2
$(16,-408)$	72579094756950240	1863^2	3726^2	3726^2	3726^2
$(16,300)$	166771816996663440	1018^2	4072^2	509^2	4072^2
$(16,12)$	2084647712458320	792^2	3168^2	396^2	6336^2
$(16,-96)$	133417453597333128	3804^2	7608^2	1902^2	7608^2
$(16,592)$	17950711938549720	2221^2	8884^2	2221^2	4442^2
$(16,48)$	7021971241964856	4608^2	9216^2	2304^2	9216^2
$(16,-32)$	133417453597332744	5463^2	10926^2	5463^2	10926^2
$(16,268)$	279342793469411664	2916^2	11664^2	1458^2	11664^2
$(16,472)$	186310763603371680	3119^2	12476^2	3119^2	12476^2
$(16,-33)$	234814718331305640	3717^2	7437^2	3717^2	14868^2
$(17,-404)$	118434048164038608	3246^2	6492^2	1623^2	12948^2
$(17,-68)$	10206435200195943696	8284^2	33136^2	4142^2	33136^2
$(19,-32)$	19452264734491086120	31704^2	63408^2	31704^2	63408^2

Table 3. Examples of elliptic curves $E(n,p)$ ($16 \le n \le 19$; $-1000 \le p \le 1000$) with $\max_{1\le i\le 4} |Ш_i| \ge 10^6$.

3.2. Values of the Goldfeld-Szpiro ratio.

Let $GS(E)$ denote the ratio $|Ш(E)|/\sqrt{N(E)}$. In [dW] and [Ni] together there are 58 examples of elliptic curves of conductor less than 10^{10} with $GS(E)$ greater than 1 (the largest value being 42.265...). The largest values of $GS(E)$ we observed for our curves are as follows:

E	$\lvert Ш(E)\rvert$	$GS(E)$
$E_2(9, 544)$	344^2	1.20290...
$E_{2,4}(16, 48)$	9216^2	1.01357...
$E_2(10, 204)$	504^2	0.98366...
$E_{2,4}(19, -32)$	63408^2	0.91159...
$E_2(11, 160)$	322^2	0.57131...
$E_4(17, -404)$	12984^2	0.48986...
$E_4(16, -33)$	14868^2	0.45618...
$E_{2,4}(16, 472)$	12476^2	0.36060...
$E_{2,4}(17, -68)$	33136^2	0.34368...
$E_{2,4}(16, -32)$	10926^2	0.32682...
$E_{2,4}(16, 268)$	11664^2	0.25741...

Table 4. Elliptic curves $E_i(n, p)$ $(9 \le n \le 19,\ -1000 \le p \le 1000;\ 1 \le i \le 4)$ with the largest $\lvert Ш(E)\rvert$. Notation $E_{i,j}(n, p)$ means that the given values of $\lvert Ш(E)\rvert$ and $GS(E)$ are shared by curves the isogeneous curves $E_i(n, p)$ and $E_j(n, p)$.

3.3. Small non-zero and large values of $L(E, 1)$.

Let $S_k(N)$ denote the linear space of holomorphic cusp forms of weight k and level N. Let $H_k^+(N) \subset S_k(N)$ be the subset of newforms with $\epsilon = 1$. Iwaniec and Sarnak [IS] proved that the percentage of f's in $H_k^+(N)$ for which $L(f, \frac{k}{2}) \ge (\log N)^{-2}$ is at least 50 as $N \to \infty$, with technical assumption $\phi(N) \sim N$ (N squarefree).

Note that $L(E(7, p), 1) \ge (\log N(7, p))^{-2}$ for any $-1000 \le p \le 1000)$ in a rank zero case. It may not be true in general:

$$L(E(8, -131), 1) = 0.0002764516... < 0.0012048710... = (\log N(8, -131))^{-2},$$

$$L(E(9, 160), 1) = 0.0007372044... < 0.0015186182... = (\log N(9, 160))^{-2},$$

$$L(E(10, 142), 1) = 0.0002457384... < 0.0009026601... = (\log N(10, 142))^{-2},$$

$$L(E(11, 168), 1) = 0.0003276464... < 0.0009902333... = (\log N(11, 168))^{-2},$$

$$L(E(12, 800), 1) = 0.0001706491... < 0.0009613138... = (\log N(12, 800))^{-2}.$$

Here are some examples of large and small values of $L(E,1)$:

E	$L(E,1)$
$E(11,-733)$	88.203561907255071...
$E(13,-160)$	71.523635814751843...
$E(12,466)$	56.224807584564927...
$E(7,-433)$	36.275918867296195...
$E(10,687)$	30.274774697662334...
$E(9,767)$	29.638568367562609...
$E(9,-93)$	28.032198538875886...
$E(10,-837)$	28.032198538875886...

Table 5. Elliptic curves $E(n,p)$ ($n \leq 19$, $-1000 \leq p \leq 1000$) with the largest values of $L(E,1)$ known to us.

E	$L(E,1)$
$E(12,800)$	0.0001706491750110...
$E(10,142)$	0.0002457348122099...
$E(11,168)$	0.0003276464160384...
$E(14,672)$	0.0006067526222261...
$E(9,160)$	0.0007372044423472...
$E(10,-534)$	0.0009829392448696...
$E(10,408)$	0.0009829392504019...

Table 6. Elliptic curves $E(n,p)$ ($n \leq 19$, $-1000 \leq p \leq 1000$) with the smallest positive values of $L(E,1)$ known to us.

Note that

(23) $$L(E(10,408)) - L(E(10,-534)) = 0.0000000000553237117... .$$

This is the smallest gap between the values of $L(E,1)$ of two elliptic curves which is known to us. The analytic orders of the Tate-Shafarevich group are 2, 4, 1, 4 for the isogeneous curves $E_i(10,408)$, and 2, 8, 8, 8 for $E_i(10,-534)$ where i as usual takes values from 1 to 4.

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A. Dąbrowski: University of Szczecin, Institute of Mathematics, ul. Wielkopolska 15, 70-451 Szczecin, Poland; E-mail: dabrowsk@sus.univ.szczecin.pl

and

M. Wodzicki: Department of Mathematics, Evans Hall, University of California, Berkeley, CA 94720-3840 USA; E-mail: wodzicki@math.berkeley.edu