Homological properties of rings of functional-analytic type

(C*-algebras/cyclic homology/algebraic K-theory)

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ABSTRACT Strong flatness properties are established for a large class of functional-analytic rings including all C*-algebras. This is later used to prove that all those rings satisfy excision in Hochschild and in cyclic homology over almost arbitrary rings of coefficients and that, for stable C*-algebras, the Hochschild and cyclic homology groups defined over an arbitrary coefficient ring r C C of complex numbers (e.g., r = Z or Q) vanish in all dimensions.

In this note we study homological properties of rings of functional-analytic type. Rings and algebras are not assumed to have unit unless stated.

Definition: A ring A is said to be right universally flat if, for every unital ring R containing A as a right ideal, A is a flat right R-module.

Left universally flat rings are defined in a similar way. A ring will be referred to as universally flat if it is both left and right universally flat. Since A is left universally flat if and only if the opposite ring A° is right universally flat, we shall usually consider only one of the two properties.

With every k-algebra structure k → HomA,k(A, A) on A, where k denotes a commutative unital ring, one can associate the Hochschild homology groups HH*(A/k) and the cyclic homology groups HC*(A/k) of the k-algebra A, cf., e.g., section 4 of ref. 1.

Our first theorem establishes a connection between the universal flatness and the excision in Hochschild and in cyclic homology (cf. ref. 2).

THEOREM 1. Any left or right universally flat ring A satisfies excision in HH*(A/k) and HC*(A/k) for all flat k-algebra structures on A and arbitrary commutative unital ring k.

Warning: No nonzero ring has the property that every k-algebra structure on it is flat, not even a universally flat ring. A model example: k0 = Z[e]/(e²) and A is a Z[e]/(e²)-algebra via the augmentation map Z[e]/(e²) → Z. Then Tor°°p(A, Z) = A for any q ≥ 0; here A is an arbitrary ring. Thus the flatness of the k-algebra structure in the formulation of Theorem 1 is an essential restriction.

Theorem 1 says that for any flat k-algebra structure on A all pure extensions [cf. ref. 2 (p. 591)] in the category of k-algebras

\[ A \rightarrowtail R \rightarrowtail S \]

induce natural long exact sequences in the cyclic homology of k-algebras

\[ \ldots \rightarrow HC_q+1(S/k) \rightarrow HC_q(S/k) \rightarrow HC_q(A/k) \rightarrow \rightarrow HC_p(R/k) \rightarrow HC_p(S/k) \rightarrow \rightarrow \ldots \]

and similar long exact sequences in the Hochschild homology HH*(A/k) of k-algebras.

Because of the connection between the universal flatness and excision, it is important to determine how large is the class of right universally flat rings. Propositions 1 and 2 below demonstrate that it contains, e.g., all rings having a left unit or, more generally, a "local" left unit (see below).

PROPOSITION 1. For any ring A, the following conditions are equivalent: (a) for every unital ring R containing A as a right ideal, the cyclic right R-module R/A is projective; (b) there exists a unital ring R0 containing A as a right ideal such that the right R0-module R0/A is projective; (c) A has a left unit, i.e., eA = a for some e ∈ A and all a ∈ A.

PROPOSITION 2. For any ring A, the following conditions are equivalent: (a) for every unital ring R containing A as a right ideal, the cyclic right R-module R/A is flat; (b) there exists a unital ring R0 containing A as a right ideal such that the right R0-module R0/A is flat; (c) A has a "local" left unit, i.e., for every finite collection \{a1, ..., an\} ⊂ A, there exists e ∈ A such that ea = ai, i = 1, ..., n

COROLLARY 1. A ring having a "local" left unit is right universally flat. Similarly, a ring having a "local" right unit is left universally flat.

One also has the following.

PROPOSITION 3. A ring A that can be embedded into a unital simple ring as a left ideal is right universally flat.

COROLLARY 2. Every left ideal in the ring of differential operators on a smooth affine algebraic variety over a perfect field k is right universally flat.

The ring of differential operators is simple. Our next proposition provides a very useful general criterion of right universal flatness.

Notation: The right annihilator in A of an element x ∈ A will be denoted r(x): = \{a ∈ A | xa = 0\}.

PROPOSITION 4. Let us assume that A satisfies the following condition:

for every finite collection \{a1, ..., an\} ⊂ A there exists a unital ring B containing A as a left ideal, a finite partition of unity \(\psi_1 + \ldots + \psi_l = 1\), where

(Φ) \{\psi_1, ..., \psi_l\} ⊂ B, and elements \(\rho_i, \sigma, b_i \in A\), such that

\[ \psi \rho_i = \rho_i \sigma b_i \quad (i = 1, ..., n; j = 1, ..., l) \]

and \(r(\rho_i \sigma) = r(\sigma), j = 1, ..., l\).

Then A is right universally flat.

The proof uses a criterion of flatness due to H. Cartan and S. Eilenberg [example 6 in chapter VI of ref. 3 (p. 123)]. By considering the opposite ring Aоп, it is easy to formulate the corresponding right factorization property (Φн) that would similarly imply the left universal flatness of A.

Proposition 7 is then used to establish the universal flatness (i.e., both left and right universal flatness) of a number of important operator and function rings.

THEOREM 2. The class of universally flat rings includes: (a) every C*-algebra (real or complex), (b) the convolution algebra L^{1}(G), for any locally compact group G; (c) the ring...
The key steps in the proof of Theorem 4 are Theorems 1 and 2 above.

Theorem 4 has, e.g., the following application. Let $C^*_a(B)$ denote the space of mappings $\phi: B \times \ldots \times B \to C$ $(q+1)$ times, which are assumed to be only additive in each variable (in particular, they are not assumed to be continuous or C-multilinear). The standard Hochschild coboundary homomorphism operates on $C^*_a(B)$ according to the formula

$$d\phi(b_0, \ldots, b_{q+1}) = \sum_{i=0}^{q} (-1)^i \phi(b_0, \ldots, b_i b_{i+1}, \ldots, b_{q+1}) + (-1)^{q+1} \phi(b_{q+1}, b_0, \ldots, b_q),$$

for all $\phi \in C^*_a(B)$, and $0 \in C^*_a(B)$.

Corollary 5. For every stable $C^*$-algebra $B$, one has

$$H^{*+}_a(B) = H^{*+}_a(B) = 0.$$

Karoubi's conjecture. In the 1970s Karoubi (10) conjectured that the canonical comparison map between the algebraic and topological $K$-groups $K^a(B) \to K^b(B)$ is an isomorphism for any stable $C^*$-algebra. This conjecture since then has been partially established in dimensions $\leq 2$ and it is known that the conjecture would follow if someone would prove that $C^*$-algebras satisfy excision in algebraic $K$-theory (cf., e.g., refs. 11 and 12).

In ref. 2 the following implication was proved:

A ring $A$ satisfies $A \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfies excision in $K^b$ $\Rightarrow$ $A \otimes_{\mathbb{Q}} \mathbb{Q}$ satisfies excision in $H^{*+}_a$.

The converse is, for instance, given by the algebraic $K$-functor $K^a$.

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