MOSCOW MATHEMATICAL JOURNAL
Volume 2, Number 4, October–December 2002, Pages 769–798

VESTIGIA INVESTIGANDA

MARIUSZ WODZICKI

Dedicated to Yuri Ivanovich on the occasion of his 65th birthday

Abstract. Trace functionals on ideals in the algebra $\mathscr{B}(H)$ of bounded operators on a separable Hilbert space H are constructed and studied.

2000 Math. Subj. Class. Primary: 47L20; Secondary: 46L80, 81R60.
Key words and phrases. Operator ideals, exotic traces, renormalization.

ΜΕΤ'ΙΧΝΙΑ ΒΑΙΝΕ ΘΕΟΙΟ[1]

An operator trace on a Hilbert space H is a partially defined linear function of an operator such that

$$\tau(AT) = \tau(TA)$$

for *every* operator T in the domain of τ and *every* bounded operator A on H. No such nonzero function can be defined on the whole algebra $\mathscr{B}(H)$ of bounded operators on H due to the fact that $\mathscr{B}(H)$ coincides with its own commutator space $[\mathscr{B}(H), \mathscr{B}(H)]$ (more precisely, any operator $A \in \mathscr{B}(H)$ is a sum of two commutators; see [24]). The condition of linearity suggests that the domain of an operator trace should be a vector subspace of $\mathscr{B}(H)$, while the ability to form products AT and TA leads us to assume that the domain of a trace is a two-sided ideal in $\mathscr{B}(H)$.

In the present article, we investigate traces on arbitrary ideals in the algebra $\mathscr{B}(H)$. The support of the ordinary trace Tr is the Schatten ideal $\mathscr{L}_1$ of nuclear operators (also called the ideal of trace class operators). For a positive compact non-nuclear operator T, the sequence of partial sums

$$\sigma_n(T) := \sigma_n(\lambda(T)) = \sum_{i=1}^{n} \lambda_i(T)$$

diverges to ∞. The attitude of a "modern physicist" is to combat divergences of all kinds by the process of so-called "renormalization". The renormalization may involve subtracting counterterms or dividing by them. We shall employ the latter type of renormalization: for a fixed ideal J and a chosen positive sequence α (the "counterterm"), we consider the limits $\lim \sigma(T)/\alpha$, $T \in J$, as functions, possibly

Received May 25, 2002.
Supported in part by NSF Grant DMS 97–07965.

[1] *he was advancing in pursuit of traces of the Goddess* [30], 5.193.

taking infinite values, on the set of "infinite" positive integers $\mathbb{N}_\infty = \beta\mathbb{Z}_+ \setminus \mathbb{Z}_+$. Here $\beta\mathbb{Z}_+$ denotes, as usual, the universal compactification of the set of positive integers, known as the Stone–Čech compactification of $\mathbb{Z}_+$.

If a point $p \in \mathbb{N}_\infty$ satisfies the double requirement that the correspondence

$$T \mapsto \lim_p \frac{\sigma(T)}{\alpha} \tag{1}$$

be additive with respect to an operator T and that the values $\lim_p \sigma(T)/\alpha$ of the limit be finite for all T belonging to the cone of positive operators J_+, then (1) defines a trace on J (which may happen to be zero if α grows too rapidly). So, it is important to determine the following two sets: the set $\boldsymbol{A}_\alpha(J)$ of points $p \in \mathbb{N}_\infty$ for which the correspondence (1) is additive on the positive cone J_+, and the set $\boldsymbol{F}_\alpha(J)$ of $p \in \mathbb{N}_\infty$ for which it is finite. The latter depends directly on the *characteristic* set $\Sigma(J)$ of the ideal J ($\Sigma(J)$ is formed by the monotonically arranged sequences of eigenvalues $\lambda(T)$ of $T \in J_+$).

On the other hand, the additivity set $\boldsymbol{A}_\alpha(J)$ is defined by a *transcendental* condition, which reflects the transcendental nature of the correspondence between an operator T and its sequence of eigenvalues $\lambda(T)$. In view of this, the discovery that not only $\boldsymbol{F}_\alpha(J)$ but also $\boldsymbol{A}_\alpha(J)$ admits a purely spectral description, is rather surprising.

The first important result of the present article (Theorem 3.4 below) states that

For every positive sequence α and an ideal $J \subsetneq \mathscr{B}(H)$,

$$\boldsymbol{A}_\alpha(J) = \left\{ p \in \mathbb{N}_\infty \colon \lim_p \frac{\lambda}{\alpha\omega} = 0 \textit{ for all } \lambda \in \Sigma(J) \right\}. \tag{2}$$

Throughout, ω denotes the harmonic sequence $\omega = (1, \frac{1}{2}, \frac{1}{3}, \dots)$. The nonemptiness of the zero set on the right-hand side of (2) is often easy to verify. In fact, our next task is to give another description of this zero set in the most interesting case when α is concave, i.e., when α itself is the sequence of partial sums $\sigma(\pi)$ of some monotonic sequence $\pi \searrow 0$. The limits $\lim_p \sigma(T)/\sigma(\pi)$ are finite for all $p \in \mathbb{N}_\infty$ when T belongs to the principal ideal (π) generated by π. We prove that the additivity set coincides for this ideal with the set of *slow variation* $\mathbf{sv}(\sigma(\pi))$ of the sequence $\sigma(\pi)$; see Theorem 3.18 and the definition in 1.16 below. So, only when $\mathbf{sv}(\sigma(\pi)) = \varnothing$, does the method of multiplicative renormalization of the divergence of the ordinary trace Tr fail to produce a trace on the principal ideal (π). Remarkably, in [18, Theorem 5.16], we proved that this happens exactly when $(\pi) = [\mathscr{B}(H), (\pi)]$, in other words, when no nonzero trace exists on (π).

The equality of an ideal J with its commutator space $[\mathscr{B}(H), J]$ sets an obvious limitation on any attempt to construct a trace on J. No nonzero trace exists on such an ideal. Other limitations arise when we seek a *positive* trace; see Lemma 2.15 and Remark 2.17 below.

In the opposite case, when $\mathbf{sv}(\sigma(\pi)) = \mathbb{N}_\infty$, the method is totally successful: at *every* point $p \in \mathbb{N}_\infty$ the limit $\lim_p \sigma(\cdot)/\sigma(\pi)$ produces a positive nonzero trace on (π). This case corresponds to the sequence $\sigma(\pi)$ being slowly varying (in the classical sense). It was under this hypothesis that J. Dixmier gave the historically first construction of *exotic* traces on some operator ideals [17]. Actually, he constructed his traces on a slightly larger ideal than (π), which is defined by the requirement that $\lim_p \sigma(T)/\sigma(\pi) < \infty$ for all $T \in J_+$ *and all* $p \in \mathbb{N}_\infty$. We prove, without Dixmier's assumption, that the additivity set is nonempty *for this larger ideal* exactly when there exists any trace on it at all (see Theorem 3.20).

All traces discussed so far have the following strong positivity property:

$$\text{if} \quad S \prec T, \quad \text{then} \quad \tau(S) \le \tau(T) \qquad (S,\, T \in J_+), \tag{3}$$

where $S \prec T$ means that $\sigma(S) \le \sigma(T)$ term by term. In Section 4, we give a characterization of the class of $\prec$-positive traces (i. e., traces satisfying (3)) on a given ideal (Theorem 4.2).

In the nuclear case, i. e., when $J \subseteq \mathscr{L}_1$, construction (1) gives $\lim_p 1/\alpha$ multiplied by the ordinary trace Tr. There is no divergence here in need of renormalization. But the success with the multiplicative renormalization induced us to investigate the renormalization of the *convergence to* 0 of the sequence of remainders $\sigma_\infty(T) := \sigma_\infty(\lambda(T))$, where

$$\sigma_{n,\infty}(\lambda) := \sum_{i=n+1}^{\infty} \lambda_i \qquad (\lambda \in \ell_1).$$

It is noteworthy that the theory involved in the renormalization

$$T \mapsto \lim \frac{\sigma_\infty(T)}{\alpha} \qquad (T \in J_+) \tag{4}$$

is in most aspects parallel to that for (1) (there are certain complications though: the sequence of remainders $\sigma_\infty(\pi)$ need not satisfy the $\Delta_{\frac{1}{2}}$-condition (9), while $\sigma(\pi)$ always does, and this is frequently very useful). Equality (2) and other results of Section 3 have their counterparts for (4) (see Section 5), including the following interesting fact:

> *For* $\alpha = \sigma_\infty(\pi)$ *and* $J = (\pi)$*, the method of renormalization* (4) *produces nonzero traces if and only if the commutator space* $[\mathscr{B}(H),\, J]$ *is* properly *contained in the kernel* J^0 *of the usual trace* $\mathrm{Tr}\colon J \to \mathbb{C}$.

In other words, the method fails to produce nonzero traces on a principal ideal (π) only when the ordinary trace is the unique (up to a multiplicative constant) trace on (π).

These nonzero traces are strictly exotic: all of them are positive but none is $\prec$-positive. The two constructions discussed above turned out to be the proverbial "tip of the iceberg". Shortly afterwards (Autumn 1978), the author was able to show that the correspondence

$$T \mapsto \lim_p \frac{\sigma(T;\, \ell,\, u)}{\log u - \log \ell}, \tag{5}$$

where $\sigma(T;\, \ell,\, u) := \sum_{i=l_n}^{u_n} \lambda(T)$ and $\ell = (\ell_1,\, \ell_2,\, \dots)$ and $u = (u_1,\, u_2,\, \dots)$ are arbitrary sequences of positive integers subject only to the conditions

$$\ell < u \quad \text{and} \quad \lim u/\ell = \infty$$

defines a nonzero positive trace on the ideal $(\omega) = \{T \in \mathscr{K} : \lambda(|T|) = O(\omega)\}$ at *every* point $p \in \mathbb{N}_\infty$. The ideal (ω) and its powers (ω^s) are, implicitly, the most studied operator ideals today. This is so due to the close connections with the calculi of various algebras of pseudodifferential(like) operators, the noncommutative residue, and the extent of the influence of Alain Connes' pioneering work ([8]–[16], [29], [23], [38], [37]).

This brings us to the multiplicative renormalization of the double sequence of *interval sums* $d\sigma(T) = d\sigma(\lambda(T))$, where $d\sigma_{mn}(\lambda) := \sum_{i=m+1}^{n} \lambda_i$ is indexed by pairs of integers $0 \leq m < n$. Let P denote the set of such pairs. We consider the limits

$$\lim \frac{d\sigma(T)}{\alpha} \qquad (T \in J_+) \tag{6}$$

as continuous functions on the compact space $\beta P \setminus P$ and we introduce the maximal compact subspace $\boldsymbol{P} \subset \beta P \setminus P$ (and its obvious variant for $J \subseteq \mathscr{L}_1$) on which the values of (6) can possibly produce traces. Then we prove (Theorem 6.4) that for $\alpha = d\sigma(\pi)$ and $J = (\pi)$, the correspondence

$$T \mapsto \lim_q \frac{d\sigma(T)}{d\sigma(\pi)} \qquad (T \in (\pi)_+)$$

defines a positive trace at *every* point q of this maximal subspace of $\beta P \setminus P$ if π/ω is slowly varying. All of this is done in Section 6. The last section is independent of the previous material and provides an alternative approach to the proof of the characterization of the commutator space $[\mathscr{B}(H), J]$ for an arbitrary ideal, which is one of the main results of the article [18] (Theorem 5.6). This approach is based on proving that the quotient $J/[\mathscr{B}(H), J]$ is canonically isomorphic to the vector space

$$K(\Sigma(J)/{\approx_{\Sigma(J)}}) \otimes_{\mathbb{R}} \mathbb{C},$$

where $\approx_{\Sigma(J)}$ is a simple and explicit equivalence relation on the characteristic set $\Sigma(J)$ and K denotes the group completion functor which associates a monoid M with its "reflection" in the category of groups; see Section 7 for details.

The first two sections serve as a reference for the rest of the article and therefore should be viewed in this light. Section 2 contains a notable result, however. Any ideal in $\mathscr{B}(H)$ is equipped with a canonical nondegenerate positive structure (J, J_+). While it is true that an operator $T = X + iY$, where X and Y are selfadjoint, belongs to $[\mathscr{B}(H), J]$ if and only if both its "real" and "imaginary" parts do, the positive and the negative parts of a compact selfadjoint operator X usually do not belong to $[\mathscr{B}(H), J]$ when X does. In view of this, the following result (Theorem 2.9) may seem to be rather surprising:

$J/[\mathscr{B}(H), J]$ *inherits a* nondegenerate *positive structure from* J.

Historically, Jacques Dixmier appears to be the first one to discover that, beyond the ordinary trace, there is a realm of "exotic" traces [17]. His work became very widely known after Alain Connes linked it to a plethora of problems in Noncommutative Geometry and Quantum Field Theory (see the references cited above). In his thesis [34], followed by the two articles [35] and [36], Gary Weiss proved that there are many nonequivalent traces on $\mathscr{L}_1$. Independently, this result was obtained in 1981 by Tadeusz Figiel and Stanisław Kwapień (unpublished; see the final remark in [31]). Apparently, Nigel Kalton was the first to realize that there existed exotic positive traces on certain principal ideals of *nuclear* operators (the ordinary trace is the only positive or continuous trace on $\mathscr{L}_1$; see Corollary 2.16 below). This was later rediscovered by Albeverio, Guido, Ponosov and Scarlatti [1]–[3], who seem to have been influenced by an original article of Várga [33], whose construction gives a subclass of the traces of Section 3 via a different approach.

The current study complements and is a sequel to the article [18], where an exhaustive description of the commutator structure of ideals in $\mathscr{B}(H)$ is given in its numerous aspects. We refer the reader to that article for additional motivation and references.

1. Preliminaries about sequences

1.1. Multiplication by a real number $t \in (0, \infty)$ induces the map

$$t_\bullet\colon \mathbb{Z}_+ \to \mathbb{Z}_+, \qquad n \mapsto \lceil tn \rceil, \tag{7}$$

where $\lceil x \rceil := -[-x]$. The fact that (7) does not constitute an action of the multiplicative group $\mathbb{R}_+^*$ poses certain problems in comparison with the case of functions on $(0, \infty)$ (cf., e. g., [5]), though it becomes an action when restricted to the submonoids $\mathbb{Z}_+^\times$ and $(\mathbb{Z}_+^{-1})^\times$, where $\mathbb{Z}_+^{-1} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. We have $(1/\ell)_\bullet \circ \ell_\bullet = \mathrm{id}_{\mathbb{Z}_+} \neq \ell_\bullet \circ (1/\ell)_\bullet$ for $\ell \in \{2, 3, \dots\}$.

1.2. The pseudo-action (7) induces linear endomorphisms $t^\bullet := (t_\bullet)^*$ of the vector space $\mathbb{C}^{\mathbb{Z}_+}$ of $\mathbb{Z}_+$-indexed sequences

$$(t^\bullet \alpha)_n = \alpha_{\lceil tn \rceil} \qquad (\alpha \in \mathbb{C}^{\mathbb{Z}_+}). \tag{8}$$

We shall frequently use the notation

$$D_\ell \alpha = (1/\ell)^\bullet \alpha \qquad (\ell \in \mathbb{Z}_+).$$

Recall that a positive sequence $\alpha \in (0, \infty)^{\mathbb{Z}_+}$ satisfies the Δ_t-*condition* for some $t > 0$ if

$$\sup \frac{t^\bullet \alpha}{\alpha} < \infty. \tag{9}$$

1.3. Besides (8), we shall consider the following operations on $\mathbb{C}^{\mathbb{Z}_+}$:

a) *the sequence of partial sums*

$$\alpha \mapsto \sigma(\alpha), \qquad \sigma_n(\alpha) := \alpha_1 + \dots + \alpha_n;$$

b) *the arithmetic mean sequence*

$$\alpha \mapsto \alpha_a := \sigma(\alpha)\omega,$$

where ω will always denote the harmonic sequence

$$\omega = (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots);$$

c) *the difference sequence*

$$\alpha \mapsto \Delta\alpha, \qquad \Delta_n \alpha := \alpha_n - \alpha_{n-1}.$$

(It will be convenient to extend any $\mathbb{Z}_+$-indexed sequence to $\mathbb{N}$ by setting $\alpha_0 = 0$.)

1.4. On the space ℓ_1 of summable sequences, we also have the following two operations:

a$'$) *the sequence of remainders*

$$\alpha \mapsto \sigma_\infty(\alpha), \qquad \sigma_{n,\infty}(\alpha) := \sum_{i=n+1}^{\infty} \alpha_i$$

and

b$'$) *the sequence of "arithmetic means at infinity"*

$$\alpha \mapsto \alpha_{a,\infty}(\alpha) := \omega\sigma_\infty(\alpha).$$

1.5. The set of nonnegative monotonic sequences $\lambda \in c_0$ will be denoted $c_0^{\star}$. The *internal direct sum* $\lambda \oplus \mu$ of two sequences in $c_0^{\star}$ is defined as the monotonic rearrangement of the sequence

$$(\lambda_1, \mu_1, \lambda_2, \mu_2, \dots).$$

We record here for future reference several closely related inequalities involving the above operations. Below, λ denotes an arbitrary sequence from $c_0^{\star}$, m and n are positive integers such that $m \le n$, and $0 < s \le t < \infty$ are real numbers.

1.6. $$(n-m)\lambda_n \le \sigma_n(\lambda) - \sigma_m(\lambda) \le (n-m)\lambda_m\,.$$

Replacing n by $\lceil tn \rceil$ and m by $\lceil sn \rceil$ in 1.6, we obtain

1.7. $$((t^\bullet - s^\bullet)\omega^{-1})t^\bullet\lambda \le (t^\bullet - s^\bullet)\sigma(\lambda) \le ((t^\bullet - s^\bullet)\omega^{-1})s^\bullet\lambda.$$

Note that $(t^\bullet - s^\bullet)\omega^{-1} = (t-s)\omega^{-1}$ if $s, t \in \mathbb{Z}_+$. In this case, inequality 1.7 takes the form

1.7^bis^. $$(t-s)t^\bullet\lambda \le (t\,t^\bullet - s\,s^\bullet)\lambda_a \le (t-s)s^\bullet\lambda.$$

1.8. $$0 \le \left(\frac{s^\bullet\omega}{t^\bullet\omega} - 1\right)\frac{t^\bullet\lambda}{s^\bullet(\lambda_a)} \le \frac{t^\bullet\sigma(\lambda)}{s^\bullet\sigma(\lambda)} - 1 \le \left(\frac{s^\bullet\omega}{t^\bullet\omega} - 1\right)\frac{s^\bullet\lambda}{s^\bullet(\lambda_a)}.$$

1.9. $$0 \le \left(1 - \frac{t^\bullet\omega}{s^\bullet\omega}\right)\frac{t^\bullet\lambda}{s^\bullet(\lambda_a)} \le 1 - \frac{s^\bullet\sigma(\lambda)}{t^\bullet\sigma(\lambda)} \le \left(1 - \frac{t^\bullet\omega}{s^\bullet\omega}\right)\frac{s^\bullet\lambda}{t^\bullet(\lambda_a)}.$$

If $\lambda \in \ell_1^{\star} := \ell_1 \cap c_0^{\star}$, then we also have

1.10. $$(n-m)\lambda_n \le \sigma_{m,\infty}(\lambda) - \sigma_{n,\infty}(\lambda) \le (n-m)\lambda_m.$$

1.11. $$\left(\frac{s^\bullet\omega}{t^\bullet\omega} - 1\right)\frac{t^\bullet\lambda}{s^\bullet(\lambda_{a,\infty})} \le 1 - \frac{t^\bullet\sigma_\infty(\lambda)}{s^\bullet\sigma_\infty(\lambda)} \le \left(\frac{s^\bullet\omega}{t^\bullet\omega} - 1\right)\frac{s^\bullet\lambda}{s^\bullet(\lambda_{a,\infty})}.$$

1.12. $$\left(1 - \frac{t^\bullet\omega}{s^\bullet\omega}\right)\frac{t^\bullet\lambda}{t^\bullet(\lambda_{a,\infty})} \le \frac{s^\bullet\sigma_\infty(\lambda)}{t^\bullet\sigma_\infty(\lambda)} - 1 \le \left(1 - \frac{t^\bullet\omega}{s^\bullet\omega}\right)\frac{s^\bullet\lambda}{t^\bullet(\lambda_{a,\infty})}.$$

1.13. Any function $\alpha\colon \Gamma \to \mathbb{C}$ on a set Γ has a unique extension to a continuous function $\beta\Gamma \to \overline{\mathbb{C}}$, where $\overline{\mathbb{C}} = \overline{\mathbb{R}} + i\overline{\mathbb{R}}$ and $\overline{\mathbb{R}} = [-\infty, \infty]$. We shall denote its restriction to $\beta\Gamma \setminus \Gamma$ by $\lim \alpha$. The value at a point $p \in \beta\Gamma \setminus \Gamma$ will be denoted $\lim_p \alpha$; it can be calculated as follows. For any base $\mathcal{B}$ of the ultrafilter p, there exists precisely one point $v \in \overline{\mathbb{C}}$ which is a cluster point of $\alpha(E) \subseteq \overline{\mathbb{C}}$ for each $E \in \mathcal{B}$. This is the value of $\lim_p \alpha$ (cf. [6, Ch. I, § 7.2]).

1.14. The level sets $\{p \in \beta\Gamma \setminus \Gamma \colon \lim_p \alpha = v\}$, where $v \in \overline{\mathbb{C}}$, will be denoted by $Z_v(\alpha)$ or $Z(\alpha)$, if $v = 0$. They are compact subspaces of $\beta\Gamma \setminus \Gamma$.

1.15. As usual, for $\alpha, \alpha' \in \mathbb{C}^\Gamma$ we shall write $\alpha \asymp \alpha'$ if $|\alpha'| \le K|\alpha| \le K'|\alpha'|$ for some constants $K, K' > 0$.

For a sequence $\alpha \in (\mathbb{C}^*)^{\mathbb{Z}_+}$, we define the set of *slow variation* $\mathbf{sv}(\alpha)$ as the following subset of $\mathbb{N}_\infty := \beta\mathbb{Z}_+ \setminus \mathbb{Z}_+ = \beta\mathbb{N} \setminus \mathbb{N}$:

$$\mathbf{sv}(\alpha) := \bigcap_{0<t<\infty} Z_1\left(\frac{t^\bullet \alpha}{\alpha}\right).$$

1.16. The case $\mathbf{sv}(\alpha) = \mathbb{N}_\infty$ corresponds to the classical definition of a *slowly varying sequence*. The following characterization theorem combines a number of subtle results (cf. Section 1.9 of the book [4] and the references therein, in particular, [5]).

Note that, because we use $\lceil x \rceil$ instead of $[x]$ in Definition 1.1, it is not entirely obvious that our definition of a slowly varying sequence is the same as in [28], [5], and [4]. That in fact it is, follows from the observation (used also in the proof of the implication (a) ⇒ (b) below) that, for any given *irrational* $t > 1$,

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{\alpha_{\lceil \frac{1}{t}\lceil tn\rceil\rceil}}{\alpha_n} = \frac{\alpha_{\lceil \frac{1}{t}\lceil tn\rceil\rceil}}{\alpha_{\lceil tn\rceil}}\;\frac{\alpha_{\lceil tn\rceil}}{\alpha_n} \to 1$$

as $n \to \infty$ (compare with the remark of de Haan and Balkema mentioned in the footnote to Section 1.9 of [4], p. 52).

Theorem 1.17. *For any sequence* $\alpha \in (0, \infty)^{\mathbb{Z}_+}$, *the following conditions are equivalent*:

(a) α *is slowly varying*;
(b) *the function* $x \mapsto \alpha_{\lceil x\rceil}$ *on* $(0, \infty)$ *is slowly varying*;
(c) $\alpha = \gamma e^{\sigma(\delta\omega)}$ *for some sequence* $\gamma \in (0, \infty)^{\mathbb{Z}_+}$ *which converges to a limit* $c > 0$ *and some real-valued sequence* $\delta \in c_0$;
(d) $\alpha \sim \alpha'$, *where* α' *is a sequence whose difference sequence* $\beta = \Delta\alpha'$ *has the property*

$$|\beta| = o(\beta_a); \tag{10}$$

(e) $\lim \alpha\omega^s = \begin{cases} 0 & \text{for every } s > 0, \\ \infty & \text{for every } s < 0; \end{cases}$

(f) *for any* $0 < s < t < \infty$ *and* $\varepsilon > 0$, *there exists an* N *such that*

$$\left|\frac{\alpha_m}{\alpha_n} - 1\right| < \varepsilon \tag{11}$$

for all integers m *and* n *such that* $\lceil sn\rceil \le m \le \lceil tn\rceil$ *and* $n \ge N$. □

We close this section with the following theorem describing the sets of slow variation for the sequence of partial sums $\sigma(\lambda)$ and, when $\lambda \in \ell_1^\star$, for the sequence of remainders $\sigma_\infty(\lambda)$ of a sequence $\lambda \in c_0^\star$.

Theorem 1.18. *For any nonzero sequence* $\lambda \in c_0^{\star}$,

$$\mathbf{sv}(\sigma(\lambda)) = \bigcap_{m=1}^{\infty} Z\left(\frac{D_m\lambda}{\lambda_a}\right)$$

and, when λ *is summable,*

$$\mathbf{sv}(\sigma_\infty(\lambda)) = \bigcap_{m=1}^{\infty} Z\left(\frac{D_m\lambda}{\lambda_{a,\infty}}\right).$$

Proof. The double inequality 1.8 with $s = 1$ gives us the following inequality between functions on $\mathbb{N}_\infty$:

$$0 \le \lim \frac{t^\bullet\sigma(\lambda)}{\sigma(\lambda)} - 1 \le (t-1)\lim \frac{\lambda}{\lambda_a} \qquad (t > 1),$$

whereas inequality 1.9 with $t = 1$ gives us the inequality

$$0 \le 1 - \lim \frac{s^\bullet\sigma(\lambda)}{\sigma(\lambda)} \le (1-s)\lim \frac{s^\bullet\lambda}{\lambda_a} \qquad (0 < s < 1). \tag{12}$$

Combined together, they produce the inclusion

$$\bigcap_{m=1}^{\infty} Z\left(\frac{D_m\lambda}{\lambda_a}\right) = \bigcap_{s>0} Z\left(\frac{s^\bullet\lambda}{\lambda_a}\right) \subseteq \mathbf{sv}(\sigma(\lambda)). \tag{13}$$

Conversely, inequality 1.8 with $t = 1/m$ gives us the inequality

$$0 \le \left(\frac{1}{ms} - 1\right)\frac{D_m\lambda}{s^\bullet(\lambda_a)} \le \frac{(1/m)^\bullet\sigma(\lambda)}{s^\bullet\sigma(\lambda)} - 1 \qquad \left(0 < s < \frac{1}{m}\right),$$

which, combined with the fact that

$$s^\bullet(\lambda_a) \asymp \lambda_a \tag{14}$$

for all $s > 0$, implies that $\mathbf{sv}(\sigma(\lambda))$ is contained in the intersection of the zero sets $Z\big(\frac{D_m\lambda}{\lambda_a}\big)$.

Suppose now that $\lambda \in \ell_1^{\star}$. In a similar vein, inequality 1.11 with $s = 1$ gives

$$0 \le 1 - \lim \frac{t^\bullet\sigma_\infty(\lambda)}{\sigma_\infty(\lambda)} \le (t-1)\lim \frac{\lambda}{\lambda_{a,\infty}} \qquad (t > 1), \tag{15}$$

whereas inequality 1.12 with $t = 1$ produces the inequality

$$0 \le \frac{s^\bullet\sigma_\infty(\lambda)}{\sigma_\infty(\lambda)} - 1 \le (1-s)\lim \frac{s^\bullet\lambda}{\lambda_{a,\infty}} \qquad (0 < s < 1). \tag{16}$$

Together, (15) and (16) imply the inclusion

$$\bigcap_{m=1}^{\infty} Z\left(\frac{D_m\lambda}{\lambda_{a,\infty}}\right) = \bigcap_{s>0} Z\left(\frac{s^\bullet\lambda}{\lambda_{a,\infty}}\right) \subseteq \mathbf{sv}(\sigma_\infty(\lambda)). \tag{17}$$

Inequality 1.11 with $t = 1/m$ gives the inequality

$$\left(\frac{1}{ms} - 1\right)\frac{D_m\lambda}{s^\bullet\lambda_{a,\infty}} \le 1 - \frac{(1/m)^\bullet\sigma_\infty(\lambda)}{s^\bullet\sigma(\lambda)} \qquad \left(0 < s < \frac{1}{m}\right). \tag{18}$$

The obvious analog of (14), $s^{\bullet}(\lambda_{a,\infty}) \asymp \lambda_{a,\infty}$, is *false* in general. However,

$$\lim \frac{s^{\bullet}\lambda_{a,\infty}}{\lambda_{a,\infty}} = \frac{1}{s} \qquad (0 < s < \infty)$$

on the set $\mathbf{sv}(\sigma_\infty(\lambda))$. In conjunction with (18), this implies the reverse inclusion in (17). $\square$

2. Preliminaries about traces on operator ideals

The purpose of this section is to prepare the ground for the sections that follow. The lack of clear exposition of some of the most basic aspects of operator ideal traces results in an inevitable verbosity, for which the author requests the reader's forgiveness.

2.1. For any ideal $J \subsetneq \mathscr{B}(H)$, the set

$$\Sigma(J) := \{s(T) \colon T \in J\} \subseteq c_0^{\star} \tag{19}$$

of the monotonically arranged sequences of singular numbers of operators $T \in J$ has the property:

$$\text{if} \quad \lambda = O(\mu \oplus \nu) \quad \text{for} \quad \mu,\, \nu \in \Sigma(J), \quad \text{then} \quad \lambda \in \Sigma(J), \tag{ChS}$$

which characterizes such sets, i. e., any subset $\Sigma \subseteq c_0^{\star}$ which satisfies (ChS) is of the form $\Sigma = \Sigma(J)$ for a unique ideal J. Set (19) is called the *characteristic set* of the ideal J.

2.2. For any sequence $\pi \in c_0^{\star}$, the union $\mathcal{O}_\pi := \bigcup_{m=1}^{\infty} \mathcal{O}_{\pi,m}$ of sets

$$\mathcal{O}_{\pi,m} := \{\lambda \in c_0^{\star} \colon \lambda = O(D_m \pi)\}$$

is the smallest characteristic set containing π. We shall denote the associated ideal by (π).

2.3. Any ideal J generated by finitely many compact operators $T_1, \ldots, T_\ell$ is principal (i. e., singly generated). Indeed, $J = (\pi)$, where $\pi = s(T_1) + \cdots + s(T_\ell)$.

2.4. Every characteristic set $\Sigma \subseteq c_0$ is a *semimodule* over the *semifield* $[0, \infty)$ (the more appropriate name "semi-vector space" seems to be too inconvenient to use. The theory of semimodules over semirings is fairly well developed; see, e. g., [25] and the references therein). In particular, linear maps between semimodules constitute morphisms in the category of semimodules.

2.5. Let V be a complex vector space. A linear map $\tau \colon J \to V$ will be called a V-valued *trace* on J if $\tau(AT) = \tau(TA)$ for all $A \in \mathscr{B}(H)$ and $T \in J$. It is convenient to allow vector-valued traces. The quotient map

$$\boldsymbol{\tau} \colon J \to J/[\mathscr{B}(H),\, J] \tag{20}$$

where $[\mathscr{B}(H),\, J]$ denotes the commutator space

$$[\mathscr{B}(H),\, J] := \left\{ \sum_{i=1}^{m} [A_i,\, T_i] \colon A_i \in \mathscr{B}(H),\, T_i \in J \right\}$$

is tautologically a universal trace; i. e., the V-valued traces on J are in a one-to-one correspondence with the linear maps $J/[\mathscr{B}(H),\, J] \to V$.

2.6. Real traces. Recall that a *real* structure on V is an isomorphism $\rho\colon V \to \overline{V}$ of the vector space V with its complex conjugate space $\overline{V}$ such that $\bar{\rho} \circ \rho = \mathrm{id}_V$. Equivalently, it is a choice of a real vector subspace $V_{\mathbb{R}} \subseteq V$ such that $V = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}$ ($V_{\mathbb{R}}$ coincides with the fixed points of ρ). In this case, $V_{\mathbb{R}}$ is called the *real* part of V, and a trace $\tau\colon J \to V$ is said to be *real* if $\tau(T^*) = \tau(T)^\rho$ for $T \in J$, or, equivalently, if τ maps the Hermitian part J_h of J to $V_{\mathbb{R}}$.

Since $T \in [\mathscr{B}(H),\, J]$ if and only if $T^* \in [\mathscr{B}(H),\, J]$, the quotient $J/[\mathscr{B}(H),\, J]$ inherits a real structure from J, and the tautological trace $\boldsymbol{\tau}$ defined by (20) is real.

2.7. Positive traces. A choice of a *cone* V_+ (i. e., a $[0,\, \infty)$-semimodule) of V such that $V_{\mathbb{R}} = V_+ - V_+$ will be called a *positive structure* on a real vector space $(V,\, \rho)$. In particular, V_+ determines $V_{\mathbb{R}}$ and, *a fortiori*, the involution ρ. By abuse of language, we shall call the pair $(V,\, V_+)$ a *positive* vector space, and linear maps preserving positive cones will be referred to as *positive* linear maps.

A positive space $(V,\, V_+)$ will be said to be *nondegenerate* if $V_+ \cap (-V_+) = \{0\}$.

2.8. Examples. (a) $(\mathbb{C},\, [0,\, \infty))$.

(b) For every $\mathbb{C}$-vector space V, the correspondence $v \otimes \bar{w} \mapsto w \otimes \bar{v}$ defines a canonical *real* structure on $V \otimes_{\mathbb{C}} \overline{V}$, and

$$(V \otimes_{\mathbb{C}} \overline{V})_+ := \left\{ \sum_{i=1}^{m} v_i \otimes \bar{v}_i \colon v_i \in V \right\}$$

defines a nondegenerate positive structure on it.

(c) (*A variant of the previous example.*) If V is a Banach space then the completed injective tensor product $V \check{\otimes}_\varepsilon \overline{V}$ is canonically a positive Banach space.

In the case of a Hilbert space, $\overline{V}$ is canonically isomorphic to V^* and the positive structure on the space $\mathscr{K} = H \check{\otimes}_\varepsilon H^*$ of compact operators thus obtained coincides with the usual one.

(d) Let C be any $[0,\, \infty)$-semimodule. The group completion KC (see Section 7.1 below) is canonically a real vector space, and $V := KC \otimes_{\mathbb{R}} \mathbb{C}$ becomes a positive vector space with V_+ being the image of C. The map $C \to V$ is injective precisely if the additive monoid of C is cancellative. The obtained positive vector space is nondegenerate precisely when no nonzero $v \in C$ has an additive inverse. We shall call $KC \otimes_{\mathbb{R}} \mathbb{C}$ the positive vector space associated with the semimodule C. This construction is discussed in a slightly greater detail in Section 7.

(e) For any ideal $J \subsetneq \mathscr{B}(H)$, the quotient $J/[\mathscr{B}(H),\, J]$ inherits the positive structure from J:

$$(J/[\mathscr{B}(H),\, J])_+ := J_+/J_+ \cap [\mathscr{B}(H),\, J].$$

It is not at all clear that this structure is nondegenerate. However, it is.

Theorem 2.9. *For every ideal $J \subseteq \mathscr{B}(H)$, the positive structure on $J/[\mathscr{B}(H),\, J]$ is nondegenerate.*

Proof. Suppose that $T_1 = -T_2 + K$ for some positive operators $T_1, T_2 \in J$ and $K \in [\mathscr{B}(H), J]$. Then $T := T_1 + T_2 \in [\mathscr{B}(H), J]$. By Theorem 5.11(i) of [18], the principal ideal (T) generated by T is contained in $[\mathscr{B}(H), J]$ in view of positivity of T. Since $\lambda(T_i) \leq \lambda(T)$ for $i = 1, 2$, the operators T_i belong to (T) and, hence, to $[\mathscr{B}(H), J]$. $\square$

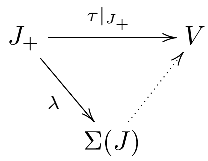
Theorem 5.11 used in the proof is deduced from the main result of [18]. No simple proof of Theorem 2.9 is known to the author.

The tautological trace (20) is, of course, a universal positive trace on J. Even when $J/[\mathscr{B}(H), J] \neq 0$, there may be no scalar (i. e., $\mathbb{C}$-valued) positive trace on J.

2.10. Every trace $\tau\colon J \to V$ is uniquely determined by its restriction to the positive cone J_+. Since any $T \in J_+$ equals $UD(\lambda(T))U^*$, where $D(\lambda)$ denotes the diagonal operator having a sequence λ on its diagonal and U is a suitable partial isometry, we have

$$T - D(\lambda(T)) = [U, D(\lambda(T))U^*] \in [\mathscr{B}(H), J]$$

and $\tau(T) = \tau(D(\lambda(T)))$. In particular, $\tau|_{J_+}$ factors through the characteristic set $\Sigma(J)$



where $\lambda\colon J_+ \to \Sigma(J)$ denotes the map that associates a positive operator with its sequence of eigenvalues.

In practice, the map $\Sigma(J) \to V$ is the restriction of τ to the semimodule of diagonal operators $D(\Sigma(J)) := \{D(\lambda)\colon \lambda \in \Sigma(J)\}$ which identifies naturally with $\Sigma(J)$. Guided by this remark and the desire to keep notation simple, we usually denote a trace $J \to V$ and the corresponding semimodule map $\Sigma(J) \to V$ by the same symbol.

Lemma 2.11. *There is a natural bijection*

$$\left\{ \begin{array}{c} V\text{-}\textit{valued} \\ \textit{traces on } J \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{semimodule maps } \tau\colon \Sigma(J) \to V \\ \textit{such that } \tau \circ \lambda\colon J_+ \to V \textit{ is additive} \end{array} \right\}. \qquad \square$$

2.12. The requirement that $\tau \circ \lambda$ be additive implies that τ is $\oplus$-additive, i. e., that $\tau(\lambda \oplus \mu) = \tau(\lambda) + \tau(\mu)$.

2.13. If V is equipped with a real structure, then real traces correspond to semimodule maps $\Sigma(J) \to V_{\mathbb{R}}$.

2.14. Positive traces require a little more care, since $\lambda \leq \mu$ in $\Sigma(J)$ does not imply that $\mu - \lambda \in \Sigma(J)$. Instead, we have to consider *monotonic* (i. e., order preserving) maps $\Sigma(J) \to V_+$. Thus, for a positive vector space (V, V_+), there is a natural bijection

$$\left\{ \begin{array}{c} \text{positive } V\text{-valued} \\ \text{traces on } J \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{monotonic semimodule maps} \\ \tau\colon \Sigma(J) \to V \text{ such that} \\ \tau \circ \lambda\colon J_+ \to V \text{ is additive} \end{array} \right\}.$$

For any characteristic set $\Sigma \subseteq c_0^{\star}$, the set

$$\Sigma^{(o)} := \{\lambda \in c_0^{\star} \colon \lambda = o(\mu) \text{ for some } \mu \in \Sigma\}$$

is characteristic and coincides with the product set $c_0^{\star}\Sigma$.

Lemma 2.15. *Let f be a monotonic homogeneous and $\oplus$-additive function $f\colon \Sigma \to [0, \infty)$ on a characteristic set Σ such that $\Sigma = \Sigma^{(o)}$. If $\Sigma \subseteq \ell_1^{\star}$, then $f = c\sigma$, where $c = f(\mathbb{1})$ is the value of f on the sequence*

$$\mathbb{1} := (1, 0, 0, \dots).$$

If $\Sigma \nsubseteq \ell_1^{\star}$, then f vanishes identically.

Proof. For any sequence $\alpha \in \mathbb{C}^{\mathbb{Z}_+}$, let $\alpha[k]_n := \alpha_{n-k}$ if $n-k > 0$ (otherwise put 0). If $\lambda \in \Sigma^{(o)}$, then $\lambda = \alpha\mu$ for some positive sequence $\alpha \in c_0$.

Suppose that f vanishes on $c_f^{\star} := \{\lambda \in c_0^{\star} \colon \lambda_n = 0 \text{ for } n \gg 0\}$. The monotonicity of f implies that

$$\begin{aligned} f(\lambda[-\ell]) &= f(\alpha[-\ell]\,\mu[-\ell]) \le \|\alpha[-\ell]\|_\infty\, f(\mu[-\ell]) \\ &\le \|\alpha[-\ell]\|\, f(\mu) \to 0 \quad \text{as} \quad \ell \to \infty. \end{aligned}$$

On the other hand, the $\oplus$-additivity of f combined with the vanishing of f on sequences with finite support shows that $f(\lambda) = f(\lambda[-\ell])$ for all $\ell \in \mathbb{Z}_+$. Hence f vanishes identically on $\Sigma^{(o)} = \Sigma$.

In the general case, by using all the three properties of f, we obtain the inequality

$$\sigma_n(\lambda)f(\mathbb{1}) = f(\lambda_1\mathbb{1} \oplus \cdots \oplus \lambda_n\mathbb{1}) \le f(\lambda) \qquad (\lambda \in \Sigma)$$

(which is an equality if λ has finite support), which implies that $f(\mathbb{1}) = 0$ if $\Sigma \nsubseteq \ell_1^{\star}$ and that $g := f - f(\mathbb{1})\sigma$ is a positive $\oplus$-additive function on Σ. Since g vanishes on $c_f^{\star}$, it must vanish identically on Σ. □

For an ideal $J \subseteq \mathscr{B}(H)$, let $J^{(o)}$ denote the ideal whose characteristic set equals $\Sigma(J)^{(o)}$. We have $J = \mathscr{K} J$, of course.

Corollary 2.16. *The ordinary trace* Tr *is the only* (*up to a multiple*) *positive trace on any ideal of nuclear operators $J \subseteq \mathscr{L}_1$ such that $J = J^{(o)}$, in particular, on $J = \mathscr{L}_1$. Moreover, any positive trace $\tau\colon J \to \mathbb{C}$ vanishes on $J^{(o)}$ if $J \nsubseteq \mathscr{L}_1$.* □

Remark 2.17. According to Theorem 7.3 below or to the chronologically earlier Theorem 5.11(ii) proven in [18], an ideal $J \subsetneq \mathscr{B}(H)$ admits a nonzero trace if and only if $\lambda_a \notin \Sigma(J)$ for at least one $\lambda \in \Sigma(J)$. In particular, for every sequence π such that $\pi \not\asymp \pi_a$, there are sequences $\mu = o(\pi)$ such that $\mu_a \neq O(\pi)$, which means that the ideal $(\pi)^{(o)}$ supports scalar nonzero traces but, in view of Lemma 2.15, not a single one is positive.

For future reference, we close this section with four different forms of an important inequality relating the sequences of eigenvalues of positive compact operators S, T, and $S+T$ to each other. The notation $\sigma(S)$ for $\sigma(\lambda(S))$, etc., is used throughout.

2.18. $$\sigma(S+T) \le \sigma(S) + \sigma(T) \le 2^{\bullet}\sigma(S+T).$$

2.19. $$\lambda(S+T)_a \le \lambda(S)_a + \lambda(T)_a \le 2(2^{\bullet}(\lambda(S+T)_a)).$$

2.20. $$0 \le \sigma(S) + \sigma(T) - \sigma(S+T) \le (2^\bullet - 1^\bullet)\sigma(S+T) \le \frac{\lambda(S+T)}{\omega}.$$

2.21. $$0 \le (\lambda(S) + \lambda(T))_a - \lambda(S+T)_a \le \lambda(S+T).$$

The double inequality 2.18 follows immediately from the min-max characterization of eigenvalues of a positive operator (which is essentially due to Ernest Fischer [20]).

3. The renormalization of the sequence of partial sums

3.1. Let us fix an arbitrary positive sequence α and an ideal $J \subsetneq \mathscr{B}(H)$. The correspondence

$$T \mapsto \lim \frac{\sigma(T)}{\alpha}$$

defines a unitary invariant map $J_+ \to C(\mathbb{N}_\infty, [0, \infty])$ which needs be neither additive nor finite, so we associate it with the following two subsets of $\mathbb{N}_\infty$:

the *additivity set*

$$\boldsymbol{A}_\alpha(J) := \bigcap_{S,T \in J_+} \left\{ p \in \mathbb{N}_\infty \colon \lim_p \frac{\sigma(S) + \sigma(T) - \sigma(S+T)}{\alpha} = 0 \right\} \tag{21}$$

and the *finiteness set*

$$\boldsymbol{F}_\alpha(J) := \bigcap_{\lambda \in \Sigma(J)} \left\{ p \in \mathbb{N}_\infty \colon \lim_p \frac{\sigma(\lambda)}{\alpha} < \infty \right\} = \bigcap_{\lambda \in \Sigma(J)} \left(\mathbb{N}_\infty \setminus Z_\infty \left(\frac{\sigma(\lambda)}{\alpha} \right) \right). \tag{22}$$

The additivity set is always compact, whereas $\boldsymbol{F}_\alpha(J)$ is the intersection of cozero sets.

3.2. Local Marcinkiewicz ideals. Recall that the Marcinkiewicz ideal $\mathscr{M}(\psi)$ associated with a sequence $\psi \in (0, \infty)^{\mathbb{Z}_+}$ consists of all compact operators whose sequence of singular numbers belongs to the set

$$m^\star(\psi) := \{\lambda \in c_0^\star \colon \|\lambda\|_{m(\psi)} < \infty\}, \tag{23}$$

where $\|\lambda\|_{m(\psi)} := \sup(\lambda_a \psi)$. Equipped with $\| \ \|_{m(\psi)}$, $\mathscr{M}(\psi)$ is a symmetrically normed ideal (cf. [18, Sections 4.4 and 2.25]).

We can define a local analog of (23), since, for *any* subset $X \subseteq \mathbb{N}_\infty$, the set

$$m^\star(\psi;\, X) := \{\lambda \in c_0^\star \colon \lim_p (\lambda_a \psi) < \infty \text{ for any } p \in X\}$$

is characteristic (put $m^\star(\psi;\, \varnothing) = c_0^\star$). We shall call the corresponding ideal $\mathscr{M}(\psi;\, X)$ the *local Marcinkiewicz ideal* (*associated with a subset* $X \subseteq \mathbb{N}_\infty$). The system of rearrangement invariant seminorms

$$\|T\|_{\mathscr{M}(\psi), K} := \sup_{p \in K} \lim_p (s(T)_a \psi)$$

(K being an arbitrary nonempty compact subset of X) makes it a complete locally convex ideal (a Banach ideal, if X is closed).

Every ideal $J \subsetneq \mathscr{B}(H)$ whose finiteness set contains X, i. e., such that

$$\boldsymbol{F}_\alpha(J) \supseteq X,$$

is contained in $\mathscr{M}(\omega/\alpha;\, X)$, and $\mathscr{M}(\omega/\alpha;\, X)$ is the largest among such ideals.

3.3. The restriction of $\lim \sigma(\cdot)/\alpha$ to the set

$$\boldsymbol{T}_\alpha(J) := \boldsymbol{A}_\alpha(J) \cap \boldsymbol{F}_\alpha(J) \tag{24}$$

defines a trace functional on J:

$$\tau_\alpha^J \colon J_+ \to C(\boldsymbol{T}_\alpha(J),\, [0,\, \infty)), \qquad T \mapsto \left(\lim \frac{\sigma(T)}{\alpha}\right)\Big|_{\boldsymbol{T}_\alpha(J)} \tag{25}$$

provided $\boldsymbol{T}_\alpha(J) \neq \varnothing$. For this reason, (24) will be called the *trace* set (of τ_α^J). The trace τ_α^J is $\prec$-positive (see definition (32) below).

The finiteness set $\boldsymbol{F}_\alpha(J)$ is defined directly in terms of the sequence α and the characteristic set $\Sigma(J)$, while the definition of the additivity set $\boldsymbol{A}_\alpha(J)$ is entirely "transcendental". It is, therefore, rather remarkable that the additivity set admits a purely spectral description too.

Theorem 3.4. *For any sequence $\alpha \in (0,\, \infty)^{\mathbb{Z}_+}$ and any ideal $J \subsetneq \mathscr{B}(H)$,*

$$\boldsymbol{A}_\alpha(J) = \bigcap_{\lambda \in \Sigma(J)} Z\left(\frac{\lambda}{\alpha\omega}\right) = \left\{p \in \mathbb{N}_\infty \colon \lim_p \frac{\lambda}{\alpha\omega} = 0 \text{ for any } \lambda \in \Sigma(J)\right\}. \tag{26}$$

3.5. The map $c_0^{\star} \to \mathscr{Z} := \{\text{closed subsets of } \mathbb{N}_\infty\}$ given by the correspondence

$$\lambda \mapsto Z\left(\frac{\lambda}{\alpha\omega}\right)$$

is a morphism of directed sets $(c_0^{\star},\, \leq) \to (\mathscr{Z},\, \supseteq)$.

Since each characteristic subset of $c_0^{\star}$ is directed by the relation $\leq$ and since $\mathbb{N}_\infty$ is compact, we infer that

$$\bigcap_{\lambda \in \Sigma(J)} Z\left(\frac{\lambda}{\alpha\omega}\right) = \varnothing$$

if and only if

$$\alpha\omega = O(\mu) \quad \text{for some} \quad \mu \in \Sigma(J). \tag{27}$$

Condition (27) implies and, when $\alpha\omega$ is equivalent to a monotonic sequence, is equivalent to the simple condition

$$\begin{pmatrix} \alpha & & & \\ & \frac{\alpha_2}{2} & & \\ & & \frac{\alpha_3}{3} & \\ & & & \ddots \end{pmatrix} \in J.$$

Thus we obtain the following corollary of Theorem 3.4.

Corollary 3.6. *For a sequence $\alpha \in (0,\, \infty)^{\mathbb{Z}_+}$ and an ideal $J \subsetneq \mathscr{B}(H)$, the additivity set $\boldsymbol{A}_\alpha(J)$ is empty if and only if condition (27) holds.* □

The assertion of Theorem 3.4 is no less interesting when α is taken to be the constant sequence of 1's; the set $\boldsymbol{A}_1(J)$ is the additivity set of the ordinary, unrenormalized, trace Tr.

Corollary 3.7. *For any ideal $J \subsetneq \mathscr{B}(H)$, the following conditions are equivalent*:

(a) *J is contained in the ideal*

$$(\omega)^{(o)} = \{T \in \mathscr{K} : s(T) = o(\omega)\}; \tag{28}$$

(b) $\boldsymbol{A}_1(J) = \mathbb{N}_\infty$ *, i. e., the ordinary trace* Tr*, when considered on J, is "everywhere" additive.* □

Recall that (28) does not admit any nonzero positive trace (see the second assertion of Corollary 2.16 above).

Corollary 3.8. *For any ideal $J \subsetneq \mathscr{B}(H)$, the following conditions are equivalent*:

(a) $(\omega) \nsubseteq J$;

(b) $\boldsymbol{A}_1(J) \neq \varnothing$;

(c) *there exists a trace $\tau \colon J \to \mathbb{C}$ which extends the ordinary trace* Tr *from $J \cap \mathscr{L}_1$.* □

The equivalence of conditions (a) and (c) is established in Corollary 7.5 below. Note that, in accordance with Corollary 2.16, no such extension can be positive, except in the trivial case $J \subseteq \mathscr{L}_1$.

3.9. For any sequence $\psi \in (0, \infty)^{\mathbb{Z}_+}$ and a subset $X \subseteq \mathbb{N}_\infty$, the set $z^\star(\psi; X) := \bigcap_{m=1}^\infty z^\star_m(\psi; X)$, where

$$z^\star_m(\psi; X) := \{\lambda \in c^\star_0 \colon \lim_p (D_m\lambda)\psi = 0 \textit{ for any } p \in X\},$$

is characteristic (we set $z^\star(\psi; \varnothing) = c^\star_0$). The corresponding ideal, denoted $\mathscr{Z}(\psi; X)$, is the largest among the ideals J such that

$$\bigcap_{\lambda \in J} Z(\lambda\psi) \supseteq X.$$

This leads to the following corollary of Theorem 3.4.

Corollary 3.10. *For any sequence $\alpha \in (0, \infty)^{\mathbb{Z}_+}$ and a subset $X \subseteq \mathbb{N}_\infty$, the ideal $\mathscr{Z}\left(\frac{1}{\alpha\omega}; X\right)$ is the largest among the ideals J such that*

$$\boldsymbol{A}_\alpha(J) \supseteq X.$$ □

3.11. For a principal ideal $J = (\mu)$, where $\mu \in c^\star_0$, the obvious inclusions

$$Z\left(\frac{\mu}{\alpha\omega}\right) \supseteq Z\left(\frac{D_2\mu}{\alpha\omega}\right) \supseteq Z\left(\frac{D_3\mu}{\alpha\omega}\right) \supseteq \cdots$$

combined with the equality

$$\bigcap_{\lambda \in (\mu)} Z\left(\frac{\lambda}{\alpha\omega}\right) = \bigcap_{m=1}^\infty Z\left(\frac{D_m\mu}{\alpha\omega}\right)$$

yield the following assertion.

Corollary 3.12. *If a sequence $\mu \in c^\star_0$ satisfies the $\Delta_{\frac{1}{2}}$-condition (9), then*

$$\boldsymbol{A}_\alpha((\mu)) = Z\left(\frac{\mu}{\alpha\omega}\right).$$ □

The proof of Theorem 3.4 will be split into several steps. The first step is a direct consequence of inequality 1.8 for $s = 1$.

Lemma 3.13. *For any integer $\ell > 1$ and $\lambda \in c_0^{\star}$,*

$$Z\left(\frac{\lambda}{\alpha\omega}\right) \subseteq Z\left(\frac{(\ell^{\bullet}-1)\sigma(\lambda)}{\alpha}\right) \subseteq Z\left(\frac{\ell^{\bullet}\lambda}{\alpha\omega}\right). \tag{29}$$

□

Proposition 3.14. *For any integer $\ell > 1$ and characteristic set $\Sigma \subseteq c_0^{\star}$,*

$$\bigcap_{\lambda\in\Sigma} Z\left(\frac{(\ell^{\bullet}-1)\sigma(\lambda)}{\alpha}\right) = \bigcap_{\lambda\in\Sigma} Z\left(\frac{\lambda}{\alpha\omega}\right).$$

Proof. Lemma 3.13 combined with the identity $D_\ell \circ \ell^{\bullet} = \mathrm{id}$ produces the inclusion

$$Z\left(\frac{(\ell^{\bullet}-1)\sigma(D_l\lambda)}{\alpha}\right) \subseteq Z\left(\frac{l^{\bullet}(D_l\lambda)}{\alpha\omega}\right) = Z\left(\frac{\lambda}{\alpha\omega}\right),$$

which implies the inclusion

$$\bigcap_{\lambda\in\Sigma} Z\left(\frac{(\ell^{\bullet}-1)\sigma(\lambda)}{\alpha}\right) \subseteq \bigcap_{\lambda\in\Sigma} Z\left(\frac{\lambda}{\alpha\omega}\right),$$

since $D_\ell\lambda \in \Sigma$ whenever $\lambda \in \Sigma$. The reverse inclusion follows directly from (29). □

The assertion of the next lemma is a direct consequence of inequality 2.20.

Lemma 3.15. *For any positive compact operators S and T,*

$$Z\left(\frac{(2^{\bullet}-1^{\bullet})\sigma(S+T)}{\alpha}\right) \subseteq Z\left(\frac{\sigma(S)+\sigma(T)-\sigma(S+T)}{\alpha}\right).$$

□

Lemma 3.16. *For any ideal $J \subsetneq \mathscr{B}(H)$,*

$$\bigcap_{\lambda\in\Sigma(J)} Z\left(\frac{\lambda}{\alpha\omega}\right) \subseteq \boldsymbol{A}_\alpha(J). \tag{30}$$

Proof.

$$\begin{aligned}\bigcap_{\lambda\in\Sigma(J)} Z\left(\frac{\lambda}{\alpha\omega}\right) &\overset{\text{Prop. 3.14}}{=} \bigcap_{\lambda\in\Sigma(J)} Z\left(\frac{(2^{\bullet}-1^{\bullet})\sigma(T)}{\alpha}\right)\\ &\overset{\text{Lem. 3.15}}{\subseteq} \bigcap_{S,T\in J_+} Z\left(\frac{\sigma(S)+\sigma(T)-\sigma(S+T)}{\alpha}\right) = \boldsymbol{A}_\alpha(J).\end{aligned}$$

□

Consider the projection P on $H = \ell_2(\mathbb{Z}_+)$ given by

$$P(e_i) = \begin{cases} e_i & \text{if } i \text{ is odd,}\\ 0 & \text{if } i \text{ is even}\end{cases}$$

and, for a given sequence $\lambda \in c_0^{\star}$, set

$$S = P\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots\end{pmatrix}P \quad\text{and}\quad T = (1-P)\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots\end{pmatrix}(1-P). \tag{31}$$

We have $\sigma(S)+\sigma(T) = 2^{\bullet}\sigma(\lambda)$ and $\sigma(S+T) = \sigma(\lambda)$. Hence

$$(2^{\bullet}-1^{\bullet})\sigma(\lambda) = \sigma(S)+\sigma(T)-\sigma(S+T)$$

and, accordingly,

$$Z\left(\frac{(2^\bullet - 1^\bullet)\sigma(\lambda)}{\alpha}\right) = Z\left(\frac{\sigma(S)+\sigma(T)-\sigma(S+T)}{\alpha}\right),$$

where the operators S and T are as in (31). Combined with Lemma 3.13, this yields the reverse of inclusion (30) and completes the proof of Theorem 3.4.

3.17. Renormalization with a concave sequence α. The sequence of partial sums $\sigma(T)$ is concave, so it is natural to focus attention on concave renormalizing sequences. Each such sequence is of the form $\alpha = \sigma(\pi)$ for a unique $\pi \in c_0^\star$. Consider the principal ideal $(\pi) := \{T \in \mathscr{K} : s(T) \in \mathscr{O}_\pi\}$, where

$$\mathscr{O}_\pi := \{\lambda \in c_0^\star \colon \lambda = O(D_m \pi) \text{ for some } m \in \mathbb{Z}_+\}$$

denotes the smallest characteristic set containing the sequence π. The finiteness set $\boldsymbol{F}_{\sigma(\pi)}((\pi))$ is the whole space $\mathbb{N}_\infty$. It turns out that the additivity set $\boldsymbol{A}_{\sigma(\pi)}((\pi))$ coincides with the slow-variation set of $\sigma(\pi)$.

Theorem 3.18. *For any nonzero $\pi \in c_0^\star$,*

$$\boldsymbol{A}_{\sigma(\pi)}((\pi)) = \mathbf{sv}(\sigma(\pi)). \qquad \square$$

This theorem is proved by combining Theorem 3.4 with the assertion of Theorem 1.19.

In the particularly interesting case when $D_2\pi \asymp \pi$, i.e., when π satisfies the $\Delta_{\frac{1}{2}}$-condition (9), inequality 1.8 shows that

$$\frac{t^\bullet \sigma(\pi)}{s^\bullet \sigma(\pi)} - 1 \asymp \frac{\pi}{\pi_a}$$

for any $0 < s < t < \infty$. Hence we have the following corollary.

Corollary 3.19. *For any nonzero sequence $\pi \in c_0^\star$ satisfying the $\Delta_{\frac{1}{2}}$-condition and any pair of distinct positive real numbers s and t,*

$$\boldsymbol{A}_{\sigma(\pi)}((\pi)) = Z\left(\frac{\pi}{\pi_a}\right) = Z_1\left(\frac{t^\bullet \sigma(\pi)}{s^\bullet \sigma(\pi)}\right) = \mathbf{sv}(\sigma(\pi)). \qquad \square$$

The following theorem combines certain results of [18] with the results of this section.

Theorem 3.20. *For any nonzero sequence $\pi \in c_0^\star$, the following conditions are equivalent*:

(a) $\boldsymbol{A}_{\sigma(\pi)}((\pi)) = \varnothing$;
(b) $\boldsymbol{A}_{\sigma(\pi)}(\mathscr{M}(1/\pi_a)) = \varnothing$;
(c) *no nonzero trace exists on the principal ideal* (π);
(d) *no nonzero trace exists on the Marcinkiewicz ideal* $\mathscr{M}(1/\pi_a)$;
(e) $\pi \asymp \pi_a$.

Proof. The implication (e) ⇒ (a) follows from Theorem 3.4. If the additivity set of the Marcinkiewicz ideal $\mathscr{M}(1/\pi_a)$ is empty, then there exists a $\mu \in c_0^\star$ such that $\mu_a = O(\pi_a)$ and $\pi_a = O(\mu)$ (the latter so by Corollary 3.6). It follows that $\mu \asymp \mu_a \asymp \pi_a$ and therefore

$$(\pi_a)_a \asymp (\mu)_a \asymp \pi_a.$$

By Theorem 3.8 of [18], the condition $(\pi_a)_a \asymp \pi_a$ implies that $\pi_a \asymp \pi$. Finally, the equivalence of the last three conditions is guaranteed by Theorems 5.16 and 5.22 combined with Proposition 2.26 (all three in [18]). □

3.21. The theorem just proven implies and simultaneously improves one of the two main results in [33] (Theorem IRR). It is particularly notable that the multiplicative renormalization method fails to produce a trace on the ideal (π) or on the Marcinkiewicz ideal $\mathscr{M}(1/\pi_a)$ *only* if no nonzero trace exists on either of these ideals. Besides, being $\prec$-positive, the trace $\tau^J_{\sigma(\pi)}$ is continuous with respect to the Marcinkiewicz norm (see (3.4) above).

4. The $\prec$-positive traces

The multiplicatively renormalized trace $\tau^J_\alpha \colon J_+ \to C(\boldsymbol{T}_\alpha(J), [0, \infty))$ discussed in the previous section has a strong positivity property built into it:

$$\sigma(S) \le \sigma(T) \quad \text{implies} \quad \tau^J_\alpha(S) \le \tau^J_\alpha(T). \tag{32}$$

In other words, τ^J_α is $\prec$-positive. This is so because $\tau^J_\alpha(T)$ is a monotonic function of the arithmetic mean sequence $\lambda(T)_a$ rather than just $\lambda(T)$.

Lemma 4.1. *Any $\prec$-positive trace $\tau \colon J_+ \to [0, \infty]$ is of the form*

$$\tau(T) = \varphi(\lambda(T)_a) \tag{33}$$

for a unique monotonic linear map $\varphi \colon a(\Sigma(J)) \to [0, \infty]$. □

Indeed, if $\lambda(S)_a = \lambda(T)_a$, then S and T have the same sequences of eigenvalues.

Note that the semimodule $a(\Sigma(J))$ is invariant under the action of the multiplicative monoid of positive integers $\mathbb{Z}^\times_+$, because $\ell^\bullet(\lambda_a)$ coincides with μ_a for the sequence of "interval" means

$$\mu_n := \frac{\lambda_{(\ell-1)n+1} + \cdots + \lambda_{\ell n}}{\ell},$$

which belongs to $\Sigma(J)$ when λ does.

The following theorem completely characterizes the $\prec$-positive traces on a given ideal.

Theorem 4.2. *Let (V, V_+) be a positive vector space, and let $\varphi \colon a(\Sigma(J)) \to V_+$ be a monotonic linear map. The following conditions are equivalent*:

(a) *formula* (33) *defines a trace on J*;
(b) *$\varphi \circ \ell^\bullet = \frac{1}{\ell}\varphi$ for every integer $\ell \ge 1$*;
(b′) *$\varphi \circ \ell_0^\bullet = \frac{1}{\ell_0}\varphi$ for some integer $\ell_0 \ge 2$*;
(c) *φ is constant on the equivalence classes of the following relation on $a(\Sigma(J))$*:

$$\eta \sim_{\Sigma(J)} \zeta \quad \textit{if} \quad |\eta - \zeta| = O(\lambda) \textit{ for some } \lambda \in \Sigma(J).$$

Proof. The implication (b′) ⇒ (a) (for $\ell_0 = 2$) is a consequence of inequality 2.19. Let $\eta = \lambda_a \in a(\Sigma(J))$, and let S and T be the operators defined in (31). Then the equalities

$$2\varphi(2^\bullet\lambda_a) = \varphi(\lambda(S)_a + \lambda(T)_a) = \varphi(\lambda(S)_a) + \varphi(\lambda(T)_a)$$

and

$$\varphi(\lambda_a) = \varphi(\lambda(S+T)_a)$$

demonstrate the equivalence of conditions (a) and (b′) for $\ell_0 = 2$.

Suppose that (b′) holds for a particular ℓ_0. By iteration, we obtain $\varphi \circ (\ell_0^m)^\bullet = (1/\ell_0^m)\varphi$, and the double inequality

$$\eta \le \ell(\ell^\bullet \eta) \le \ell_0^m((\ell_0^m)^\bullet \eta) \qquad (\eta \in a(\Sigma(J))),$$

where $m = \lceil \log \ell \rceil / \log \ell_0$, give the implication (b′) ⇒ (b) in view of the monotonicity of φ.

Inequality 1.7^{bis} with $s = 1$ and $t = \ell$ gives the implication (c) ⇒ (b). When applied to the sequence $D_\ell \lambda$, the same inequality aided by the identity $\ell^\bullet \circ D_\ell = \mathrm{id}$ produces the inequality

$$\frac{1}{\ell-1}(D_\ell\lambda)_a + \lambda \le \frac{\ell}{\ell-1}\,\ell^\bullet((D_\ell\lambda)_a).$$

Suppose that $|\eta - \zeta| \le \lambda$ for some η, $\zeta \in a(\Sigma(J))$ and $\lambda \in \Sigma(J)$. Then, by adding $(\ell-1)^{-1}(D_\ell\lambda)_a$ to both sides of the inequality $\eta \le \zeta + \lambda$, we obtain the inequality

$$\eta + \left(\frac{D_\ell\lambda}{\ell-1}\right)_a \le \zeta + \left(\frac{D_\ell\lambda}{\ell-1}\right)_a + \lambda \le \zeta + \ell\left(\ell^\bullet\left(\left(\frac{D_\ell\lambda}{\ell-1}\right)_a\right)\right). \tag{34}$$

If φ is a monotonic additive map satisfying the condition (b′) for $\ell_0 = \ell$, then (34) results in the inequality

$$\varphi(\eta) + v \le \varphi(\zeta) + v,$$

which holds in V_+ for

$$v = \varphi\left(\left(\frac{D_\ell\lambda}{\ell-1}\right)_a\right) = \varphi\left(\ell\left(\ell^\bullet\left(\left(\frac{D_\ell\lambda}{\ell-1}\right)_a\right)\right)\right).$$

By exchanging the roles of η and ζ, we obtain the reverse of inequality (34). Thus, $\varphi(\eta) + v = \varphi(\zeta) + v$. But the monoid V_+, being embedded in the Abelian group V, is cancellative. Hence $\varphi(\eta) = \varphi(\zeta)$. This proves the implication (b′) ⇒ (c) and completes the proof of Theorem 4.2. □

Remark 4.3. If φ is a monotonic linear map $\Xi \to V_+$ defined on some semimodule Ξ containing both $a(\Sigma(J))$ and $\Sigma(J)$, then condition (c) is clearly equivalent to the following simpler condition

(c′) φ *vanishes* on $\Sigma(J)$.

5. The renormalization of the sequence of remainders

5.1. In this section, we shall consider exclusively ideals $J \subsetneq \mathscr{B}(H)$ contained in the ideal of nuclear operators $\mathscr{L}_1$. Construction (25) produces in this case essentially the ordinary trace Tr. More precisely, τ_α^J multiplies $\mathrm{Tr}\, T$ by the function $\lim 1/\alpha$ restricted to $\boldsymbol{T}_\alpha(J)$, which coincides with the set of points $p \in \mathbb{N}_\infty$ where $\lim \alpha$ does not vanish:

$$\tau_\alpha^J : T \mapsto (\mathrm{Tr}\, T)(\lim 1/\alpha)|_{\boldsymbol{T}_\alpha(J)}$$

In particular, the image of τ_α^J consists of scalar multiples of the function $\lim 1/\alpha$.

We still have another spectral invariant of $T \in J_+$ at our disposal: the sequence of remainders $\sigma_\infty(T) := \sigma_\infty(\lambda(T))$. We shall renormalize its convergence to 0 by investigating the correspondence

$$J_+ \to C(\mathbb{N}_\infty, [0, \infty]), \qquad T \mapsto \lim \frac{\sigma_\infty(T)}{\alpha}.$$

By replacing the sequence $\sigma(T)$ with $\sigma_\infty(T)$ in the definitions (21), (22), and (24), we obtain the corresponding additivity set $\boldsymbol{A}_{\alpha,\infty}(J)$, the finiteness set $\boldsymbol{F}_{\alpha,\infty}(J)$, and the trace set $\boldsymbol{T}_{\alpha,\infty}(J) = \boldsymbol{A}_{\alpha,\infty}(J) \cap \boldsymbol{F}_{\alpha,\infty}(J)$. If $T_{\alpha,\infty}(J)$ is nonempty, then the correspondence

$$\tau^J_{\alpha,\infty} \colon T \mapsto \lim \frac{\sigma_\infty(T)}{\alpha}\bigg|_{\boldsymbol{T}_{\alpha,\infty}(J)}$$

defines a positive vector-valued trace on J. Note that $\tau^J_{\alpha,\infty}$ is *not* $\prec$-positive. We have now the following analog of Theorem 3.4.

Theorem 5.2. *For any sequence $\alpha \in (0, \infty)^{\mathbb{Z}_+}$ and any ideal $J \subsetneq \mathscr{B}(H)$ contained in $\mathscr{L}_1$,*

$$\boldsymbol{A}_{\alpha,\infty}(J) = \bigcap_{\lambda \in \Sigma(J)} Z\left(\frac{\lambda}{\alpha\omega}\right). \tag{35}$$

Note that the right-hand side of (35) is *exactly* like in (26).

Regarding the proof, it is essential to observe that

$$\sigma(S) + \sigma(T) - \sigma(S+T) = -(\sigma_\infty(S) + \sigma_\infty(T) - \sigma_\infty(S+T))$$

and

$$(\ell^\bullet - 1^\bullet)\sigma(\lambda) = -(\ell^\bullet - 1^\bullet)\sigma_\infty(\lambda) \qquad (\lambda \in \ell_1^{\star}).$$

Now, the proof of Theorem 3.4 carries over word for word by replacing everywhere σ by σ_∞.

5.3. Renormalization with a convex sequence α. The sequence of remainders $\sigma_\infty(T)$ is convex. In accordance with this, we shall now analyze the case when the normalizing sequence α is convex and converges to 0. In such a case, $\alpha = \sigma_\infty(\pi)$ for a unique $\pi \in \ell_1^{\star}$.

The following theorem is an analog of Theorem 3.18.

Theorem 5.4. *For any nonzero sequence $\pi \in \ell_1^{\star}$, the additivity set $\boldsymbol{A}_{\sigma_\infty(\pi),\infty}((\pi))$ of the map*

$$T \mapsto \lim \frac{\sigma_\infty(T)}{\sigma_\infty(\pi)} \qquad (T \in (\pi)_+) \tag{36}$$

coincides with the set of slow variation $\mathbf{sv}(\sigma_\infty(\pi))$. □

This results from combining Theorem 5.2 with the second assertion of Theorem 1.19. If π satisfies the $\Delta_{\frac{1}{2}}$-condition (9), we obtain a more precise statement.

Corollary 5.5. *For any nonzero sequence $\pi \in \ell_1^{\star}$ satisfying the $\Delta_{\frac{1}{2}}$-condition and any pair of distinct positive real numbers s and t,*

$$\boldsymbol{A}_{\sigma_\infty(\pi),\infty}((\pi)) = Z\left(\frac{\pi}{\pi_{a,\infty}}\right) = Z_1\left(\frac{t^\bullet\sigma_\infty(\pi)}{s^\bullet\sigma_\infty(\pi)}\right) = \mathbf{sv}(\sigma_\infty(\pi)).$$

Proof. The inclusion $Z_1\big(\frac{t^{\bullet}\sigma_\infty(\pi)}{\sigma_\infty(\pi)}\big) \subseteq Z\big(\frac{\pi}{\pi_{a,\infty}}\big)$, $t > 1$, follows directly from inequality 1.11, and the inclusion $Z_1\big(\frac{s^{\bullet}\sigma_\infty(\pi)}{\sigma_\infty(\pi)}\big) \subseteq Z\big(\frac{\pi}{\pi_{a,\infty}}\big)$, $0 < s < 1$, from inequality 1.12. □

Finally, we establish an analog of Theorem 3.20.

Corollary 5.6. *For any nonzero sequence $\pi \in \ell_1^{\star}$, the following conditions are equivalent*:

(a) $\boldsymbol{A}_{\sigma_\infty(\pi),\infty}((\pi)) = \varnothing$;
(b) *any trace τ on (π) is a multiple of the ordinary trace* Tr;
(c) $\pi_{a,\infty} = O(D_m\pi)$ *for some $m \in \mathbb{Z}_+$.*

Proof. The equivalence of (a) and (c) follows from Theorem 5.4. The equivalence of (b) and (c) is an immediate consequence of Theorem 7.3 below. An alternative proof was given in [18, Theorem 5.11(iii)]. □

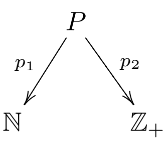
6. The renormalization of the sequence of interval sums

Guided by the constructions of traces in Sections 3 and 5, we shall now undertake a single construction of a positive vector-valued trace which encompasses the infinite hierarchy (5) of increasingly finer classes of scalar traces.

6.1. Consider the set of pairs of natural numbers

$$P := \{(m, n) \in \mathbb{N} \times \mathbb{Z}_+ : m < n\}.$$

The natural projections



induce maps between the sequence spaces $p_1^*\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^P$ and $p_2^*\colon \mathbb{C}^{\mathbb{Z}_+} \to \mathbb{C}^P$, and the compact space $\beta P \setminus P$ decomposes into the disjoint union of fibers of the induced map $\bar{p}_1\colon \beta P \to \mathbb{N} \cup \infty$:

$$\beta P \setminus P = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \dots \cup \mathcal{P}_\infty,$$

where $\mathcal{P}_m := \{q \in \beta P \setminus P\colon \bar{p}_1(q) = m\}$ and $\bar{p}_1$ denotes the map induced by p_1.

The function

$$r\colon P \to [1, \infty], \qquad (m, n) \mapsto \frac{n}{m+1},$$

induces the continuous map $\beta r\colon \beta P \to [1, \infty]$. The fiber of βr at ∞ will be denoted $\boldsymbol{P}$. It is a compact subspace of $\beta P \setminus P$ and, in fact, $\boldsymbol{P}$ is the disjoint union

$$\boldsymbol{P} = \bigcup_{m=0}^{\infty} \mathcal{P}_m \cup \boldsymbol{P}_\infty,$$

where $\boldsymbol{P}_\infty := \mathcal{P}_\infty \cap \boldsymbol{P}$.

Later, we shall encounter the space $\boldsymbol{P}_{0\infty} = \mathcal{P}_0 \cup \boldsymbol{P}_\infty$. As mentioned in Section 1.3, it will be convenient to extend any $\mathbb{Z}_+$-indexed sequence α to $\mathbb{N}$ by putting $\alpha_0 = 0$.

6.2. For any sequence $\alpha \in \mathbb{C}^{\mathbb{Z}_+}$, let $d\alpha \in \mathbb{C}^P$ be the double sequence $d\alpha = (p_2^* - p_1^*)\alpha$. Its terms are given by the formula

$$d\alpha_{mn} = \begin{cases} \alpha_n & \text{if } m = 0, \\ \alpha_n - \alpha_m & \text{if } m > 0. \end{cases}$$

6.3. Fix a nonzero sequence $\pi \in c_0^{\star}$ and let $\boldsymbol{P}(\pi)$ denote the additivity set of the map $(\pi)_+ \to [0, \infty)$ given by

$$T \mapsto \left(\lim \frac{d\sigma(T)}{d\sigma(\pi)} \right) \bigg|_{\boldsymbol{P}},$$

i. e., let

$$\boldsymbol{P}(\pi) := \bigcap_{S,T \in (\pi)_+} \left\{ p \in \boldsymbol{P} \colon \lim_p \frac{d\sigma(S) + d\sigma(T) - d\sigma(S+T)}{d\sigma(\pi)} = 0 \right\}.$$

The set $\boldsymbol{P}(\pi)$ is compact and, if it is nonempty, the correspondence

$$\operatorname{itr}_\pi \colon (\pi)_+ \to C(\boldsymbol{P}(\pi), [0, \infty)), \qquad T \mapsto \left(\lim \frac{d\sigma(T)}{d\sigma(\pi)} \right) \bigg|_{\boldsymbol{P}}$$

defines a trace on the principal ideal (π). We will refer to it as the *interval trace.*

The sequences π such that π/ω is slowly varying constitute a particularly important class of sequences for which $(\pi) \neq [\mathscr{B}(H), (\pi)]$. We shall now prove that the additivity set $\boldsymbol{P}(\pi)$ for such sequences is as large as possible.

Theorem 6.4. *If π/ω is slowly varying, then*

$$\boldsymbol{P}(\pi) = \begin{cases} \boldsymbol{P}_{0\infty} & \text{for } \pi \in \ell_1, \\ \boldsymbol{P} & \text{for } \pi \notin \ell_1. \end{cases}$$

Proof. For operators $S, T \in (\pi)_+$, inequality 2.21 gives the following double inequality on P:

$$-p_1^*(\lambda(S+T)/\omega) \leq d\sigma(S) + d\sigma(T) - d\sigma(S+T) \leq p_2^*(\lambda(S+T)/\omega), \tag{37}$$

or, in "point coordinates",

$$-m\lambda_m(S+T) \leq d\sigma_{mn}(S) + d\sigma_{mn}(T) - d\sigma_{mn}(S+T) \leq n\lambda_n(S+T).$$

The key to the proof of the theorem is provided by the following assertion.

Theorem 6.5. *Let $\pi \in c_0^{\star}$, and suppose that π/ω is slowly varying. Then, for every point $q \in \beta P \setminus P$,*

$$\lim_q \frac{p_2^*(\pi/\omega)}{d\sigma(\pi)} = \frac{1}{\log \beta r(q)} \tag{38}$$

and

$$\lim_q \frac{p_1^*(\pi/\omega)}{d\sigma(\pi)} = \begin{cases} \dfrac{m\pi_m}{\sum_{i=m+1}^{\infty} \pi_i} & \text{if } \pi \in \ell_1 \text{ and } \bar{p}_1(q) = m < \infty, \\[2ex] \dfrac{1}{\log \beta r(q)} & \text{if } \pi \notin \ell_1 \text{ or } \bar{p}_1(q) = \infty. \end{cases} \tag{39}$$

Note that when $\beta_r(q)$ is equal to 1 or ∞, $1/\log\beta r(q)$ obviously has the meaning of 0 or ∞.

Corollary 6.6. *For $\pi \in c_0^{\star}$ and slowly varying π/ω, the zero set of $\lim \dfrac{p_2^*(\pi/\omega)}{d\sigma(\pi)}$ on $\beta P\setminus P$ coincides with $\boldsymbol{P}$:*

$$Z\left(\frac{p_2^*(\pi/\omega)}{d\sigma(\pi)}\right) = \boldsymbol{P},$$

whereas

$$Z\left(\frac{p_1^*(\pi/\omega)}{d\sigma(\pi)}\right) = \begin{cases} \boldsymbol{P}_{0\infty} & \text{if } \pi \in \ell_1 \text{ and } \bar{p}_1(q) = m < \infty, \\ \boldsymbol{P} & \text{if } \pi \notin \ell_1. \end{cases}$$

Proof of Theorem 6.4 (continued). In view of inequality (37) and Corollary 6.6, it remains only to show that, in the case when π is summable, $\mathcal{P}_m \cap \boldsymbol{P}(\pi) = \varnothing$ for $0 < m < \infty$.

Take a $q \in \mathcal{P}_m$. Its image under the projection p_1 is the principal ultrafilter containing the singleton set $\{m\} \subset \mathbb{Z}_+$. In particular, the family

$$\mathcal{B}_q := \{E \in q \colon p_1(E) = \{m\}\} \tag{40}$$

is a base of q. Since $q \in \beta P\setminus P$, we have $\bigcap_{E\in\mathcal{B}_q} E = \varnothing$ and, consequently, for any operator $S \in (\pi)_+$, the point

$$\frac{\sum_{i=m+1}^{\infty} \lambda_i(S)}{\sum_{i=m+1}^{\infty} \pi_i} \in [0,\, \infty] \tag{41}$$

is the only common cluster point of the images under the map

$$\frac{d\sigma(S)}{d\sigma(\pi)} \colon P \to [0,\, \infty)$$

of *all* the sets $E \in \mathcal{B}_q$. It follows that $\lim\limits_q \dfrac{d\sigma(S)}{d\sigma(\pi)}$ is equal to the expression in (41). For $m = 0$, this is $\operatorname{Tr} S$ up to a multiplicative factor, while for $m > 1$, (41) fails to be a trace even on finite rank operators. □

For the proof of Theorem 6.5, we need the following lemma.

Lemma 6.7. *Suppose that π/ω is slowly varying, and let $\varepsilon > 0$.*

(a) *For any $1 < \rho_1 < \rho_2 < \infty$, there exists such an integer N that*

$$\frac{1-\varepsilon}{\log(n/m)} < \frac{n\pi_n}{d\sigma_{mn}(\pi)} < \frac{1+\varepsilon}{\log(n/m)} \tag{42}$$

if $n \geq N$ and $n/m \in [\rho_1,\, \rho_2]$.

(b) *For any $\rho > 1$, there exists such an integer N that*

$$\frac{n\pi_n}{d\sigma_{mn}(\pi)} > \frac{1-\varepsilon}{\log\rho} \tag{43}$$

if $n \geq N$ and $n/m \in (1,\, \rho]$.

(c) *For any* $\rho > 1$, *there exists such an integer* N *that*

$$\frac{n\pi_n}{d\sigma_{mn}(\pi)} < \frac{1+\varepsilon}{\log\rho}$$

if $n \geq N$ *and* $n/m \in \rho$.

Proof. (a) Since π/ω is slowly varying, there exists an integer N_0 such that

$$\frac{1}{\sqrt{1+\varepsilon}} < \frac{(\pi/\omega)_{\lceil tn\rceil}}{(\pi/\omega)_n} < \frac{1}{\sqrt{1-\varepsilon}} \tag{44}$$

if $n \geq N_0$ and $t \in [1/\rho_1, 1]$ (see Theorem 1.18(f) above). By applying (44) to the sequence of

$$t \in \left\{\frac{m+1}{n}, \frac{m+2}{n}, \dots, \frac{n-1}{n}, 1\right\},$$

we obtain the estimates

$$\frac{1}{k\sqrt{1+\varepsilon}} < \frac{\pi_k}{n\pi_n} < \frac{1}{k\sqrt{1-\varepsilon}},$$

which holds for $k \in \{m+1, \dots, n\}$ *provided* $n/m \leq \rho_2$ and $n \geq N_0$. Hence the inequalities

$$\frac{\sqrt{1-\varepsilon}}{d\sigma_{mn}(\omega)} < \frac{n\pi_n}{d\sigma_{mn}(\pi)} < \frac{\sqrt{1-\varepsilon}}{d\sigma_{mn}(\omega)}$$

are valid in the same range of (m, n). Since

$$\frac{\log(n/m)}{d\sigma_{mn}(\omega)} \to 1 \quad \text{as} \quad n \to \infty$$

uniformly on the set $\{(m, n) \in P\colon n/m \geq \rho_1 > 1\}$, there exists an $N \geq N_0$ for which the estimates in (42) hold.

To prove part (b), we apply the inequality

$$\frac{n\pi_n}{d\sigma_{mn}(\pi)} \geq \frac{n\pi_n}{d\sigma_{\lceil n/\rho\rceil,n}(\pi)}$$

valid when $n/m \in (1, \rho]$ and to prove part (c), the inequality

$$\frac{n\pi_n}{d\sigma_{mn}(\pi)} \leq \frac{n\pi_n}{d\sigma_{\lceil n/\rho\rceil,n}(\pi)},$$

which holds for $n/m \geq \rho$; then we proceed like in part (a). □

6.8. Remark. All three assertions of Lemma 6.7 remain valid for $n\pi_n$ replaced by $m\pi_m$ and $n \geq N$ by $m \geq N$ in (42)–(43). Only small changes to the proof are needed (in particular, the interval $[1/\rho_1, 1]$ is replaced by the interval $[1, \rho_2]$).

Proof of Theorem 6.5. Let $q \in \beta P \setminus P$ and $\beta r(q) = \rho \in (1, \infty)$. Then $q \in \boldsymbol{P}_\infty$ and, for every $\varepsilon \in (0, \rho)$, the ultrafilter q possesses a base $\mathcal{B}$ whose members $E \in \mathcal{B}$ are subsets of $\{(m, n) \in P\colon |n/m - \rho| < \varepsilon\}$. Let an integer N be chosen so that (42) holds for $\rho_1 = \rho - \varepsilon$ and $\rho_2 = \rho + \varepsilon$. The set

$$\mathcal{B}' := \{E \in \mathcal{B}\colon n \geq N\}$$

is cofinal in $\mathcal{B}$, hence it is a base of q itself. The image of each $E \in \mathcal{B}'$ under the function

$$\frac{p_2^*(\pi/\omega)}{d\sigma(\pi)}\colon P \to [0,\, \infty]$$

is contained, in view of part (a) of Lemma 6.7, in the closed interval

$$\left[\frac{1-\varepsilon}{\log(\rho+\varepsilon)},\, \frac{1+\varepsilon}{\log(\rho-\varepsilon)}\right]. \tag{45}$$

It follows that the number

$$\lim_q \frac{p_2^*(\pi/\omega)}{d\sigma(\pi)} \tag{46}$$

belongs to the interval (45). In view of the arbitrariness of $\varepsilon \in (0,\, \rho)$, this demonstrates formula (38) for $0 < \rho < \infty$. For $\beta r(q) = 1$, a similar argument with passing to a subbase of the base

$$\mathcal{B} := \{E \in q\colon n/m < 1+\varepsilon\}$$

shows, with the help of part (b) of Lemma 6.7 (for $\rho = 1+\varepsilon$), that the limit (46) belongs to the set

$$\bigcap_{0<\varepsilon<1} \left[\frac{1-\varepsilon}{\log(1+\varepsilon)},\, \infty\right] = \{\infty\},$$

i. e., equals ∞. Finally, in the same spirit, if $\rho = \infty$, we show with the help of part (c) of Lemma 6.7 (for $\rho = 1/\varepsilon$) that the limit (46) belongs to the set

$$\bigcap_{\varepsilon>0} \left[0,\, \frac{1+\varepsilon}{\log(1/\varepsilon)}\right] = \{0\},$$

i. e., equals 0.

In view of Remark 6.8, the same reasoning proves formula (39) for $q \in \mathcal{P}_\infty$.

When $q \in \mathcal{P}_m$, family (40) is a base of q. By passing to the subbases $\mathcal{B}_{q,N} := \{E \in \mathcal{B}_q\colon p_2(E) \subseteq \mathbb{Z}_{\geq N}\}$, we see that

$$\lim_q \frac{p_1^*(\pi/\omega)}{d\sigma(\pi)} = \lim_{n\to\infty} \frac{m\pi_m}{\sum_{i=m+1}^{\infty} \pi_i},$$

which gives the requested values, since $\beta r(q) = \infty$ if $\bar{p}_1(q) = m < \infty$. □

The interval trace itr generates a hierarchy of $C(\mathbb{N}_\infty)$-valued traces: just choose any pair of sequences $\ell,\, u \in \mathbb{Z}_+^{\mathbb{Z}_+}$ subject to the conditions

$$\ell \leq u \quad \text{and} \quad \ell = o(u) \tag{47}$$

and consider the sequence of *interval*-sums

$$\sigma_n(\alpha;\, \ell,\, u) := \sum_{i=\ell(n)}^{u(n)} \alpha_i \qquad (\alpha \in \mathbb{C}^{\mathbb{Z}_+}).$$

Corollary 6.9. *Let $\pi \in c_0^{\star}$ be a nonzero sequence such that π/ω is slowly varying, and let ℓ and u be a pair of integer-valued sequences satisfying conditions (47). Then the correspondence*

$$S \mapsto \lim \frac{\sigma(\lambda(S);\, \ell,\, u)}{\sigma(\pi;\, \ell,\, u)} \qquad (S \in (\pi)_+) \tag{48}$$

defines a positive $C(\mathbb{N}_\infty)$-valued trace on the principal ideal (π) if π is not *summable. The same is true for summable π provided $\lim \ell = \infty$.* □

We have

$$\frac{\sigma(\lambda(S);\, \ell,\, u)}{\sigma(\pi;\, \ell,\, u)} = f^* \left(\frac{d\sigma(\lambda)}{d\sigma(\pi)} \right),$$

where $f\colon \mathbb{Z}_+ \to P$ is the function $f(n) = (\ell(n) - 1,\, u(n))$, so tha trace (48) is the composition of the interval trace itr and the linear map $(\beta f)^*\colon C(\beta P \setminus P) \to C(\beta \mathbb{N} \setminus \mathbb{N})$.

7. Universal trace

7.1. The inclusion of the category of groups $\mathscr{G}r$ into the category of monoids $\mathscr{M}on$ has the group completion functor $K\colon \mathscr{M}on \to \mathscr{G}r$ as its left and the group-of-invertible-elements functor $G\colon \mathscr{M}on \to \mathscr{G}r$ as its right adjoint functor. Recall that, for a monoid M, KM can be realized as the quotient of the free group FM spanned by the set M modulo the normal subgroup generated by the relations between elements of the monoid M. For Abelian monoids, there is a simpler construction: $KM = M \times M/\sim$, where $(m,\, m') \sim (n,\, n')$ if $m + n' = m' + n$ in M. The equivalence class of $(m,\, m')$ will be denoted $m - m'$.

If C is a semimodule over the semifield $[0,\, \infty)$, then KC is automatically a real vector space with the obvious action of the multiplicative group $\mathbb{R}^*$:

$$a(v - w) := \begin{cases} av - aw & \text{if } a > 0, \\ 0 - 0 & \text{if } a = 0, \\ (-a)w - (-a)v & \text{if } a < 0. \end{cases}$$

7.2. For any semimodule $C \subseteq [0,\, \infty)^{\mathbb{Z}_+}$, the relation of equivalence on $\mathbb{C}^{\mathbb{Z}_+}$

$$\alpha \approx_C \beta \quad \text{if} \quad |\alpha_a - \beta_a| = O(\xi) \text{ for some } \xi \in C$$

is a *congruence*, i. e., it is compatible with the $[0,\, \infty)$-semimodule structure on C. We have encountered a similar relation in part (c) of Theorem 4.2. The quotient of any semimodule by a congruence is a semimodule again.

Theorem 7.3. *For any ideal $J \subsetneq \mathscr{B}(H)$, the correspondence*

$$\mathrm{utr}\colon S \mapsto \textit{the class of } \lambda(S) \textit{ in } K(\Sigma(J)/{\approx_{\Sigma(J)}}) \qquad (S \in J_+) \tag{49}$$

defines a trace on J. This trace is universal, *i. e., for any vector-valued trace $\tau\colon J \to V$, there exists a unique $\mathbb{C}$-linear map*

$$t\colon K(\Sigma(J)/{\approx_{\Sigma(J)}}) \to V$$

such that $\tau = t \circ \mathrm{utr}$.

Proof. Inequality 2.21 shows that, for any $S, T \in J_+$,

$$(\lambda(S) + \lambda(T)) \approx_{\Sigma(J)} \lambda(S + T).$$

It follows that $J_+ \to \Sigma(J)/{\approx_{\Sigma(J)}}$ is a homomorphism of $[0, \infty)$-semimodules and, therefore, the composite map

$$J_+ \to \Sigma(J)/{\approx_{\Sigma(J)}} \to K(\Sigma(J)/{\approx_{\Sigma(J)}}) \otimes_{\mathbb{R}} \mathbb{C}$$

induces a trace map

$$\mathrm{utr}\colon J \to K(\Sigma(J)/{\approx_{\Sigma(J)}}) \otimes_{\mathbb{R}} \mathbb{C}, \tag{50}$$

(see Lemma 2.11 above). The map (50) is surjective by construction. Every element in the kernel of utr has the form $S + iT$ for unique Hermitian operators $S, T \in \mathrm{Ker}(\mathrm{utr})$. Finally, every hermitian operator S in the kernel of (50) is equal modulo $[\mathscr{B}(H), J]$ to the operator

$$U \begin{pmatrix} \lambda_1 & & & & \\ & -\mu_1 & & & \\ & & \lambda_2 & & \\ & & & -\mu_2 & \\ & & & & \ddots \end{pmatrix} U^*, \tag{51}$$

where U is a suitable unitary operator and $\lambda \approx_{\Sigma(J)} \mu$. We know from Theorem 5.6 (the implication (f) ⇒ (b)) of [18] that any operator of the form (51) is a sum of at most three commutators from $[\mathscr{B}(H), J]$. Thus, $\mathrm{Ker}(\mathrm{utr}) = [\mathscr{B}(H), J]$, and (50) induces an isomorphism of vector spaces

$$J/[\mathscr{B}(H), J] \xrightarrow{\sim} K(\Sigma(J)/{\approx_{\Sigma(J)}}) \otimes_{\mathbb{R}} \mathbb{C}$$

proving that the trace utr is indeed universal. □

Corollary 7.4. *For any ideal $J \subsetneq \mathscr{B}(H)$ and positive vector space (V, V_+), there are natural identifications*

$$\left\{ \begin{array}{c} V\text{-valued} \\ \text{traces on } J \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} [0, \infty)\text{-semimodule maps} \\ \tau\colon \Sigma(J) \to V \text{ such that} \\ \tau(\lambda) = \tau(\mu) \text{ if } \lambda \approx_{\Sigma(J)} \mu \end{array} \right\}.$$

In this picture, real *V-valued traces correspond to maps $\tau\colon \Sigma(J) \to V_{\mathbb{R}}$ and positive V-valued traces to maps $\tau\colon \Sigma(J) \to V_+$.*[2]

Corollary 7.5 (cf. [19])**.** *For any ideal $J \subsetneq \mathscr{B}(H)$, the following conditions are equivalent*:

(a) $(\omega) \nsubseteq J$;
(b) *there exists a trace $\tau\colon J \to \mathbb{C}$ which extends the ordinary trace* Tr *from $J \cap \mathscr{L}_1$.*

[2]Note that the monotonicity of such semimodule maps is automatic in view of the additivity of the composite map $\tau \circ \lambda\colon J_+ \to V$.

Proof. Suppose that no trace extending Tr from $J \cap \mathscr{L}_1$ to J exists. This happens precisely when $\operatorname{Tr} T = 1$ for some $T \in J \cap \mathscr{L}_1 \cap [\mathscr{B}(H),\, J]$. Since $T^* \in J \cap \mathscr{L}_1 \cap [\mathscr{B}(H),\, J]$ and $\operatorname{Tr} T^* = 1$, we may assume that $T = T^*$.

We have

$$|\lambda(T_+)_a - \lambda(T_-)_a| \sim |\operatorname{Tr} T|\,\omega = \omega,$$

where $T = T_+ - T_-$ is the representation of T as the difference of its positive and negative parts. Thus, $\omega \in \Sigma(J)$, in view of Theorem 7.3.

On the other hand, any rank one projection is an element of $[\mathscr{B}(H),\, (\omega)]$. □

7.6. Remark. Our demonstration that the correspondence (49) defines a trace on J supplies a new proof of the nontrivial implication (a) ⇒ (f) of Theorem 5.6 of [18], in the case when one of the ideals equals $\mathscr{B}(H)$.

Acknowledgments. Except for the recently proved Theorem 6.4, the results of the present article, like the discovery that (5) defines a hierarchy of increasingly subtler positive traces on the ideal (ω), as well as the results of Sections 3–5 and 7 which preceded it, date back to my stay in the fall of 1998 at Institut Mittag-Leffler in Djursholm, Sweden. I am grateful to Jouko Mickelsson (in part, for the invitation to the Institute), Teoman Turgut and, especially, Tadeusz Figiel for the interest in my work and very stimulating discussions.

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Department of Mathematics, University of California, Berkeley, California 94720–3840

E-mail address: `wodzicki@math.berkeley.edu`