Report on the cyclic homology of symbols (of both unbounded and L^2_{loc}-bounded ψDOs), Dec. 86/Jan. 87, I.A.S.

1. Let $CS(X)$ denote the algebra of symbols on a C^∞ manifold X (for simplicity assumed to have at most countably many connected components). By this I mean the quotient of the algebra $CL_{prop}(X)$ of proper scalar ψDOs of integer order (and of 'classical type') by the ideal $L^{-\infty}_{prop}(X)$ of proper smoothing operators. As a topological vector space $CS(X)$ is a strict inductive limit of spaces $CS^m(X)$ ($m \in \mathbb{Z}$, $m \longrightarrow \infty$) of symbols of order m. Each $CS^m(X)$ can be identified, though un-canonically, with the infinite product $\prod_{l=-\infty}^{m} C^\infty(S^*X)$.

When speaking of homology of an arbitrary topological algebra $\mathcal{A}$ arises the question of specifying a kind of completion of the corresponding complex of <u>algebraic</u> chains. In the context of Hochschild and cyclic homology this is equivalent to questioning what completion of the algebraic tensor powers $T_k(\mathcal{A}) = \mathcal{A} \otimes \ldots \otimes \mathcal{A}$ is suitable. The standard way is to replace all "$\otimes$" by the corresponding locally convex tensor product $\tilde{\otimes}$ (in the sense of Grothendieck). For this purpose it is necessary that $\mathcal{A}$ be an algebra 'relative to $\tilde{\otimes}$' (in the terminology of J. L. Taylor, cf. Adv. Math. 9, 137-182 (1972)), i.e. a Hausdorff l.c. space with multiplication giving rise to a continuous linear map $\mathcal{A} \tilde{\otimes} \mathcal{A} \longrightarrow \mathcal{A}$.

For example, in order to make use of projective tensor product $\hat{\otimes}_\pi$ the algebra $\mathcal{A}$ has to have multiplication continuous in both arguments. Our algebra $CS(X)$ definitely doesn't satisfy this condition. This is the reason why $\hat{\otimes}_\pi$ will be absent from these notes. Multiplication in $CS(X)$, however, is separately continuous (and even hypocontinuous), therefore homological algebra based on $\bar{\otimes}_i$-tensor product is applicable to $CS(X)$, and yet inductive tensor product is not well designed for spaces with a mixture of inductive and projective topology like $CS(X)$. For instance, as a l.c. space $\mathcal{A} = CS(X)$ has a $\mathbb{Z}$-filtration by closed subspaces

$$\dots \subset F_p \subset F_{p+1} \subset \dots \quad \mathcal{A}$$

($F_p \equiv CS^p(X)$) such that

$$\mathcal{A} \xleftarrow{\simeq} \varinjlim_p F_p \qquad \text{and} \qquad \mathcal{A} \xrightarrow{\simeq} \varprojlim_m \mathcal{A}/F_m \quad . \quad (1)$$

Similarly, there are induced filtrations on the inductive tensor powers $\overline{T}_k(\mathcal{A})$:

$$\dots \subset F_{pk} \subset F_{p+1,k} \subset \dots \subset \overline{T}_k(\mathcal{A})$$

with this difference that, for $k \geq 2$, the second condition in (1) is no more valid ($\overline{T}_k(\mathcal{A}) \longrightarrow \varprojlim_m \overline{T}_k(\mathcal{A})/F_{mk}$ is injective with dense image).

A natural remedy is to replace $\overline{T}_k(\mathcal{A})$ by its formal completion at '$-\infty$', i.e. by $\overline{T}_k(\mathcal{A})^\wedge = \varprojlim_m \overline{T}_k / F_{mk}$, for which both conditions analogous to (1) hold.

Since, clearly, the boundary maps $b : \overline{T}_{k+1}(\mathcal{A}) \to \overline{T}_k(\mathcal{A})$ and $B : \overline{T}_{k+1}(\mathcal{A}) \longrightarrow \overline{T}_{k+2}(\mathcal{A})$ have a unique extension to formal completions, one can form the corresponding Hochschild complex $(C_*(\mathcal{A},\mathcal{A}), b)$ where $C_q(\mathcal{A},\mathcal{A})$ will mean in what follows $\overline{T}_{q+1}(\mathcal{A})^\wedge$ ($\mathcal{A} = CS(X)$). Its homology will be referred to as 'Hochschild homology of $\mathcal{A}$'. Similarly, one has a double complex $(\mathcal{B}_{**}(\mathcal{A}), b, B)$ where $\mathcal{B}_{kl}(\mathcal{A}) = C_{l-k}(\mathcal{A},\mathcal{A})$ in the above sense ($k, l \geq 0$). Homology of the associated total complex will be referred to as 'cyclic homology of $\mathcal{A}$'.

It is worthy noting that our complexes when restricted to the subalgebra of symbols of order zero $CS^0(X) \subset CS(X)$ turn out to be standard '$\hat{\otimes}_\pi$'-complexes. Similarly, when restricted to the subalgebra $\mathcal{D}(X) \subset CS(X)$ they turn out to be ordinary '$\bar{\otimes}_i$'-complexes (if X is in addition compact the latter coincide with '$\hat{\otimes}_\pi$'-complexes).

It should be also clear that Connes long exact sequence holds as usual.

2. By Y we shall usually denote $T^*X \setminus X$ equipped with the ordinary $\mathbb{R}_+^\times$-action and by Y^c its complexification $Y \times_{\mathbb{R}_+^\times} \mathbb{C}^\times$. As a $\mathbb{C}^\times$-bundle over S^*X Y^c is made trivial by any choice of a nowhere vanishing function on Y^c of homogeneity one. Associated with Y^c are : the $\mathbb{Z}$-graded algebra of complex valued C^∞ functions on Y^c algebraic along fibres of the projection $\tau^c : Y^c \longrightarrow S^*X$ and, the related de Rham complex $\Omega^*_{\mathcal{O}} \equiv \Omega^*_{\mathcal{O}/\mathbb{C}}$ whose cohomology will be denoted $H^*_{DR}(Y^c)$.

3. THEOREM. There are canonical isomorphisms

$$H_q(CS(X), CS(X)) \simeq H^{2n-q}_{DR}(Y^c) \quad , \quad (q \in \mathbb{N},\ n = \dim X). \tag{2}$$

A precise statement in the case of cyclic homology requires some notation to be introduced. Kernels of iterated S-mappings induce on each $HC_q(CS(X))$ the growing filtration

$$\{0\} \subset \mathcal{S}_{q0} \subset \mathcal{S}_{q1} \subset \ldots \subset \mathcal{S}_{qt} = HC_q(CS(X))$$

where $t = [\frac{q}{2}]$ and $\mathcal{S}_{qr} := \operatorname{Ker} S_*^{1+r} \cap HC_q(CS(X))$. Let $HC_{qr} = Gr^{\mathcal{S}}_r HC_q(CS(X)) \equiv \mathcal{S}_{qr} / \mathcal{S}_{q,r-1}$ be the associated subquotients.

4. THEOREM. *The natural mapping* $I : H_*(CS(X), CS(X)) \longrightarrow HC_*(CS(X))$ *is injective. In particular, there are canonical isomorphisms*

$$HC_{qr} \simeq H_{DR}^{2n-q+2r}(Y^c) \qquad (r=0,1,2,\ldots) \qquad (3)$$

for every $q \in \mathbb{N}$.

5. REMARKS. a) The isomorphisms in (2) and (3) are functorial with respect to embeddings of underlying manifolds $X' \hookrightarrow X$ of codimension zero.
b) $CS(X)$ contains the subalgebra $CS(X)_0$ of $\mathbb{R}^\times$ - homogeneous symbols (this is, of course, the closure of the localisation of the algebra of differential operators $\mathcal{D}(X)$). Theorems 3 and 4 remain valid if one replaces $CS(X)$ by $CS(X)_0$ and Y^c by $Y_0^c = Y \times_{\mathbb{R}^\times} \mathbb{C}^\times$ (the latter is a 'trivialisable' $\mathbb{C}^\times$-bundle over $\mathbb{P}(T_X^*)$). The inclusion $CS(X)_0 \subset CS(X)$ corresponds under (2) and (3) to the map in de Rham cohomology induced by the double covering $Y^c \longrightarrow Y_0^c$.

c) Both theorems are valid also for more general algebras of classical symbols CS_Y, namely those associated with general conical submanifolds $Y \subset T^*X \setminus X$

(Y^c has the previous meaning). The isomorphisms (2) and (3) in that situation are functorial with respect to inclusions $Y' \subset Y$ as well as isomorphisms $\tilde{F} : CS_Y \to CS_{Y'}$ which are defined by integral Fourier operators F (in the latter case action on homology corresponds to the action of the homogeneous canonical transformation $C_F : Y' \to Y$ associated with F on de Rham cohomology. More general algebras of symbols will be used in derivation of the structure of order filtration on $HC_*(CS(X))$ (cf. Remark 14.c) below).

d) Inclusion $\mathcal{D}(X) \subset CS(X)$ induces in homology homomorphisms corresponding to maps in de Rham cohomology induced by the composite projection $Y^c \to S^*X \to X$ (cf. the remark at the end of subsection 1 and invoke the results of [CHDO] ; proofs presented below will make clear that the results of [CHDO] hold for $\bar{\otimes}_i$ - continuous homology of $\mathcal{D}(X)$ without requiring compactness of X ; if I'm not mistaken the same results hold for $\hat{\otimes}_\pi$ - continuous homology if to replace $\mathcal{D}(X)$ by the algebra of differential operators of locally finite order).

HDO] = Cycl. ...om. of Diff. Ops., ...uke J. 54

The statement of Theorem 4 differs in its form from the similar theorem for differential operators (cf. [CHDO, Theorem 3]). The reason is that in general I am able to obtain a canonical description in terms of de Rham cohomology only of the graded space associated to the order filtration on cyclic homology (at least in C^∞ case - my method uses skeleton filtrations in several places), cf. Remark 14.c) below. A <u>canonical</u> description of $HC_*(\mathcal{D}(X))$ then turns out to be a consequence of 'purity' of cyclic homology of $\mathcal{D}(X)$. For the algebra of symbols, however the situation is different. Weights of the order filtration on $HC_q(CS(X))$ for q in the 'unstable' range, i.e. for $q \le 2n-2$, are usually non-vanishing in a certain interval (cf. (7) below).

Closely related is the question of splitting of the S-filtration. In the case of $\mathcal{D}(X)$ a canonical splitting is an immediate consequence of the fact that the natural map $HC_*(\mathcal{O}(X)) \longrightarrow HC_*(\mathcal{D}(X))$ which is an epimorphism admits a canonical splitting. Indeed, the abovementioned question concerning splitting of S-filtration on $HC_q(\mathcal{D}(X))$ arises only for $q > n+1$ whereas the map $S_q : HC_q(\mathcal{O}(X)) \longrightarrow HC_{q-2}(\mathcal{O}(X))$ is an isomorphism just for $q > n+1$.

In the case of symbols an analogous statement reads as follows: there is a canonical splitting of the 'S-filtration' on the space of weight zero (with respect to the order filtration on $HC_*(CS(X))$) which is canonically a quotient of $HC_*(CS(X))$ (notice that S-maps are compatible with the order filtration; the action of S on other 'weights' is trivial). All that follows from the description of spectral sequences $E^{(m),r}_{**}$ of subsection 10 which is sketched in Remark 14.c) below.

I should add in conclusion that all this 'complexities' occur only in the unstable range $q \leq 2n-2$. For $q \geq 2n-1$ cyclic homology of symbols is simple: $HC_q(CS(X))$ is 'pure' of weight zero and can be canonically identified with $H^{ev}_{DR}(Y^c)$ if q is even and $H^{odd}_{DR}(Y^c)$ if q is odd.

e) For a discussion of the inclusion $CS^0(X) \subset CS(X)$ cf. Remark 14.b) below.

6. Let us proceed to proofs. Recall, first of all, that $(C_*(CS(X), CS(X)), b)$ has a $\mathbb{Z}$-filtration $\ldots \subset F_p C_* \subset F_{p+1} \subset \ldots$ $\ldots \subset C_*$ induced by the order filtration on $CS(X)$. Since each $F_p C_j$ is closed in C_j, $j \in \mathbb{N}$, one can form the quotient-complexes $C^{(m)}_* := C_* / F_{m-1} C_*$ $(m \in \mathbb{Z})$ which inherit filtration from C_*: $\{0\} = F^{(m)}_{m-1,*} \subset F^{(m)}_{m,*} \subset \ldots$ (4) (where $F^{(m)}_{pj} \equiv F_p C_j / F_{m-1} C_j$). It is clear that

$$C_j^{(m)} = \varinjlim_p F_{pj}^{(m)} \qquad (p \to \infty;\ m \in \mathbb{Z},\ j \in \mathbb{N}) \qquad (5)$$

and, in view of definition of $C_j \equiv \overline{T}_{j+1}(CS(X))^{\wedge}$ (cf. subsection 1 above), also

$$C_j = \varprojlim_m C_j^{(m)} \qquad (m \to -\infty;\ j \in \mathbb{N}) \qquad (6)$$

Finally, let $H_*^{(m)}$ denote the homology of $C_*^{(m)}$ and H_* the homology of C_*. Our first objective will be to find $H_*^{(m)}$.

In view of (5), there is a spectral sequence (s.s.) ${}'E_{pq}^{(m),r}$ converging to $H_*^{(m)}$ which is associated with (4). Its complete description is provided in the following proposition (we restrict ourselves to the only important case $m \ll 0$; here it means $m \leqslant -n$).

7. PROPOSITION. Assume that $m \leq -n$. Then:

(a) the second term of ${}'E^{(m),r}_{**}$ is given by

$$ {}'E^{(m),2}_{pq} \simeq \begin{cases} H^{n-p}_{DR}(Y^c) & , q=n, \\ \Omega^{2n-m-q}(n-q)/d\Omega^{2n-1-m-q}(n-q) & , p=m, \\ 0 & \text{otherwise} \end{cases} $$

and the isomorphisms are functorial ;

(b) ${}'E^{(m),r}_{**}$ degenerates at E^2 ;

(c) the identifications in (a) are compatible with morphisms of s.s. induced by the canonical projections $C^{(l)}_* \longrightarrow C^{(m)}_*$ $(l \leq m)$.

8. COROLLARY. The projective system of homology $\{ H^{(m)}_j \, ; \, m \in \mathbb{Z}_{<-n} \}$ satisfies M.L. condition (j is an arbitrary natural number).

More precisely, if $\mathcal{H}^{(l,m)}_j$ denotes, for $l<m$, the image of $H^{(l)}_j$ in $H^{(m)}_j$ then

$$ \mathcal{H}^{(l_1,m)}_j = \mathcal{H}^{(l_2,m)}_j \qquad \text{for every } l_1, l_2 < m $$

(and $m < -n$).

Proof of the Corollary. Assume $l_2 < l_1$. For $m < -n$, we obtain from Proposition 7 the commutative diagrams whose rows are exact

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{\mathcal{O}}^{2n-j}(n+m-j)/d\Omega_{\mathcal{O}}^{2n-1-j}(n+m-j) \arrow[r] & H_j^{(m)} \arrow[r] & H_{DR}^{2n-j}(Y^c) \arrow[r] & 0 \\
0 \arrow[r] & \Omega_{\mathcal{O}}^{2n-j}(n+l_i-j)/d\Omega_{\mathcal{O}}^{2n-1-j}(n+l_i-j) \arrow[r] \arrow[u, "0"] & H_j^{(l_i)} \arrow[r] \arrow[u] & H_{DR}^{2n-j}(Y^c) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

$(i = 1,2)$. In particular, the projection $H_j^{(m)} \longrightarrow H_{DR}^{2n-j}(Y^c)$ maps $\mathcal{H}_j^{(l_i,m)}$ bijectively onto $H_{DR}^{2n-j}(Y^c)$. Since $\mathcal{H}_j^{(l_2,m)} \subset \mathcal{H}_j^{(l_1,m)}$ one obtains that $\mathcal{H}_j^{(l_2,m)} = \mathcal{H}_j^{(l_1,m)}$. ∎

9. In view of (6), $H_* = H_*(CS(X), CS(X))$ is the homology of the projective limit $\varprojlim_m C_*^{(m)}$. The projective system $C_*^{(m)}$ satisfies, of course, M.-L. condition. The same holds, in view of Corollary 8, for the projective systems of homology groups $\{H_*^{(m)};\, m \in \mathbb{Z}_{<-n}\}$. Hence, the standard argument gives us

$$H_j = \varprojlim_m H_j^{(m)} \simeq H_{DR}^{2n-j}(Y^c).$$

In order to prove Theorem 3 it remains to demonstrate Proposition 7.

Sketch of the proof of Proposition 7. Fix $m \in \mathbb{Z}$.

By definition, ${}'E^{(m),1}_{pq} = H_{p+q}(\mathcal{O},\mathcal{O})(p)$ is the component of weight p of the Hochschild homology of $\mathcal{O}$, if $p \geq m$, and otherwise vanishes.

Similarly as we did in [CHDO, proof of Prop. 9] we define natural surjective maps $\tau^p_j : C_j(\mathcal{O},\mathcal{O})(p) \longrightarrow \Omega^{2n-j}_{\mathcal{O}}(p-j+n)$ by

$$f_0 \otimes \ldots \otimes f_j \longmapsto \frac{(-1)^j}{j!} f_0 \, i_{f_1} \ldots i_{f_j} \, \omega^n .$$

Notice that up to a sign this is the composition of the projection $C_j(\mathcal{O},\mathcal{O})(p) \longrightarrow \mathcal{C}^{(k)}_j(\mathcal{P};\mathrm{ad})$, $k = p-j$, and of graded Poisson trace

$$\mathrm{ptr} : \mathcal{C}^{(k)}_*(\mathcal{P};\mathrm{ad}) \longrightarrow \Omega_{*,k+n}$$

[NC·I] = Nonc. Res., Chapt. I, LNM 1289

of [NC·I, 1.25]. Here $\mathcal{C}_*(\mathcal{P};\mathrm{ad}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{C}^{(k)}_*(\mathcal{P};\mathrm{ad})$ is the Koszul standard complex of the graded Poisson algebra $\mathcal{P} = (\mathcal{O}, \{\,,\,\})$.

Then one has to show that $\ker \tau^p_j$ consist entirely of boundaries what, in fact, reduces to determination of $H_*(\mathcal{A},\mathcal{A})$, for $\mathcal{A} = C^\infty(S^*X) \,\overline{\otimes}\, \mathbb{C}[\rho,\rho^{-1}] = C^\infty(S^*X) \otimes_{alg} \mathbb{C}[\rho,\rho^{-1}]$, if one chooses an everywhere positive function ρ on Y, of homogeneity one, and $H_*(\mathcal{A},\mathcal{A})$ denotes the topological Hochschild homology based on $\overline{\otimes}$. And this, by the ordinary Künneth formula reduces the problem to calculation of $H_*(C^\infty(Z), C^\infty(Z))$, for a general Z. Although this seems to be lacking in the existing literature (except your computation in the dual setting, for Z compact) I cannot see an obstacle for having an identification $H_*(C^\infty(Z), C^\infty(Z)) \xrightarrow{\sim} \Omega^*(Z)$ [1].

Having identified ${}'E^{(m),1}_{pq}$ with $\Omega^{2n-p-q}_{\mathcal{O}}(n-q)$, for $p \geq m$, we obtain a description of d^1 in almost exactly the same way as we did in [CHDO, ibid.] by using the relevant argument from [NC I, 1.25]. In particular, d^1 identifies with d_{DR}. Part (c) of the assertion is also clear.

To demonstrate (b) one has to show that the only possibly non-trivial differentials $d^{p-m}_{p,n}$: ${}'E^{p-m}_{p,n} \longrightarrow {}'E^{p-m}_{m,n-m-1+p}$ all vanish (to simplify notation I omitted the superscript (m)). And this is an immediate consequence of the commutativity of the diagram

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1) Notice that since $C^\infty(Z)$ is Fréchet there is no question here about distinction between $\bar{\otimes}$- and $\hat{\otimes}$-homology. However, the cohomology $H^*(C^\infty(Z), C^\infty(Z)')$ is <u>formally</u> defined using only $\bar{\otimes}$-tensor product — $C^\infty(Z)'$ is not an $\hat{\otimes}$-module over $C^\infty(Z)$!

$$
\begin{tikzcd}
{}'E^{(m),\,p-m}_{p,n} \arrow[r, "d^{p-m}"] & {}'E^{(m),\,p-m}_{m,\,n-m+p-1} \\
{}'E^{(l),\,p-m}_{p,n} \arrow[r, "d^{p-m}"] \arrow[u, "\wr"] & {}'E^{(l),\,p-m}_{m,\,n-m+p-1} = 0 \arrow[u]
\end{tikzcd}
$$

$(l < m)$. ■

10. Let us proceed to the proof of Theorem 4.

As was, in fact, already noticed in subsection 1 B-boundary map is compatible with the filtration by order. In particular $(\mathcal{B}_{**}(CS(X)), b, B)$ is a $\mathbb{Z}$-filtered double complex.

Put $\mathcal{B}^{(m)}_{**} = \mathcal{B}_{**}/F_{m-1}\mathcal{B}_{**}$ where $F_p\mathcal{B}_{kl} \equiv F_p C_{l-k}$. Much as we did before we consider the projective system of quotient complexes $\mathrm{Tot}\,\mathcal{B}^{(m)}_{**} = \mathrm{Tot}\,\mathcal{B}_{**}/F_{m-1}\mathcal{B}_{**}$ $(m \to -\infty)$. The analogs of (5) and (6) read as

$$\mathcal{B}^{(m)}_{kl} = \varinjlim_{p} F^{(m)}_{pkl} \qquad (p \to \infty;\ m \in \mathbb{Z};\ k,l \geq 0) \qquad (5')$$

and

$$\mathcal{B}_{kl} = \varprojlim_{m} \mathcal{B}^{(m)}_{kl} \qquad (m \to -\infty;\ k,l \geq 0) \qquad (6')$$

where $F^{(m)}_{pkl} \equiv F_p\mathcal{B}_{kl}/F_{m-1}\mathcal{B}_{kl}$.

Let, finally, $HC_*^{(m)}$ denote the homology of $\mathrm{Tot}\,\mathcal{B}_{**}^{(m)}$, and HC_* the homology of $\mathrm{Tot}\,\mathcal{B}_{**}$.

11. PROPOSITION. *Assume that $m \le 0$ and $q \ge 2n+1$. Then there exist isomorphisms*

$$HC_q^{(m)} \simeq \begin{cases} H_{DR}^{ev}(Y^c), & q = \text{even}, \\ H_{DR}^{odd}(Y^c), & q = \text{odd}, \end{cases} \tag{7}$$

compatible with the canonical maps $HC_q^{(m')} \longrightarrow HC_q^{(m)}$ $(m' \le m)$.

In particular, the systems $\{HC_q^{(m)};\ m \in \mathbb{Z}_{\le 0}\}$ satisfy, for $q \ge 2n+1$, M.-L. condition. This gives us, in view of (6'), the following corollary.

12. COROLLARY. *There are, for $q \ge 2n+1$, natural isomorphisms*

$$HC_q \simeq \varprojlim_m HC_q^{(m)} \simeq \begin{cases} H_{DR}^{ev}(Y^c), & q = \text{even}, \\ H_{DR}^{odd}(Y^c), & q = \text{odd}. \end{cases} \blacksquare$$

This corollary plus Theorem 3 plus Lemma 6 of [CHDO] imply immediately the assertion of Theorem 4 <u>if</u> <u>$\dim H_{DR}^*(Y^c) < \infty$</u>.

13. In the general case one can do the following. Represent X as $\cup X_j$ $(j \in \mathbb{N})$ where each X_j is compact (with smooth, or empty, boundary) and $X_j \subset \operatorname{Int} X_{j+1}$ [2] (it suffices to take a <u>proper</u> smooth function $f: X \longrightarrow [0, \infty)$ and $X_j = f^{-1}([0, a_j])$ for a sequence $\dots < a_{j-1} < a_j \longrightarrow \infty$ of its regular values). Then the restriction maps $CS(X) \longrightarrow CS(X_j)$ induce homomorphisms

$$\theta : H_*(CS(X), CS(X)) \longrightarrow \hat{H}_* \equiv \varprojlim_j H_*(CS(X_j), CS(X_j)) \qquad (8)$$

and

$$\vartheta : HC_*(CS(X)) \longrightarrow \widehat{HC}_* \equiv \varprojlim_j HC_*(CS(X_j)) .$$

In view of the naturality of isomorphisms (2) we have the commutative diagrams, for each q,

$$\begin{tikzcd} H_q(CS(X), CS(X)) \arrow[r, "\theta_q"] \arrow[d, "\wr"] & \hat{H}_q \arrow[d, "\wr"] \\ H^{2n-q}_{DR}(Y^c) \arrow[r] & \varprojlim_j H^{2n-q}_{DR}(Y^c_j) \end{tikzcd} .$$

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2) $\dim H^*_{DR}(Y^c) = \infty$ can happen only when X is not compact.

Notice that also the lower arrow is an isomorphism, since $\Omega^*_{\mathcal{O}} = \varprojlim \Omega^*_{\mathcal{O}_j}$ ($\mathcal{O}_j$ denotes the corresponding graded algebra of functions on Y^c_j, cf. subsection 2) and the projective systems $\{\Omega^*_{\mathcal{O}_j}\}$ and $\{H^*_{DR}(Y^c_j)\}$ both satisfy M.-L. condition. Thus (8) is an isomorphism.

The naturality of your long exact sequence gives us the commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{0} & \widehat{H}_q & \xrightarrow{\hat{I}} & \widehat{HC}_q & \xrightarrow{\hat{S}} & \widehat{HC}_{q-2} & \xrightarrow{0} & \widehat{H}_{q-1} & \longrightarrow & \cdots \\
 & & \wr\uparrow \theta_q & & \uparrow \vartheta_q & & \uparrow \vartheta_{q-2} & & \wr\uparrow \theta_{q-1} & & \\
\cdots & \xrightarrow{B} & H_q & \xrightarrow{I} & HC_q & \xrightarrow{S} & HC_{q-2} & \xrightarrow{B} & H_{q-1} & \longrightarrow & \cdots
\end{array}$$

with a priori only the lower sequence being exact. The exactness of the upper sequence, however, follows from $\lim^{(1)} H_q(CS(X_j), CS(X_j)) = 0$, $\forall q \in \mathbb{N}$, the latter being the consequence of the finite-dimensionality of the groups $H_q(CS(X_j), CS(X_j)) \simeq H^{2n-q}_{DR}(Y^c_j)$. Thus the « five lemma » and an easy inductive argument prove that ϑ is an isomorphism and $B=0$.

The proof of Theorem 4 will be therefore complete if we demonstrate Proposition 11.

Proof of 11. The filtration $\{ F^{(m)}_{p**} \; ; \; p = m, m+1, \ldots \}$ on $\mathcal{B}^{(m)}_{**}$ induces a filtration on $\mathrm{Tot}\, \mathcal{B}^{(m)}_{**}$. Denote by $E^{(m),r}_{pq}$ the associated s.s. Condition (5′) guarantees that it converges to $HC^{(m)}_{*}$.

This s.s. is a priori located in the region ($p \geq m$ and $p+q \geq 0$). We shall see that $E^{(m),r}_{pq}$ ($r \geq 1$) vanishes, in fact, outside the region shown below

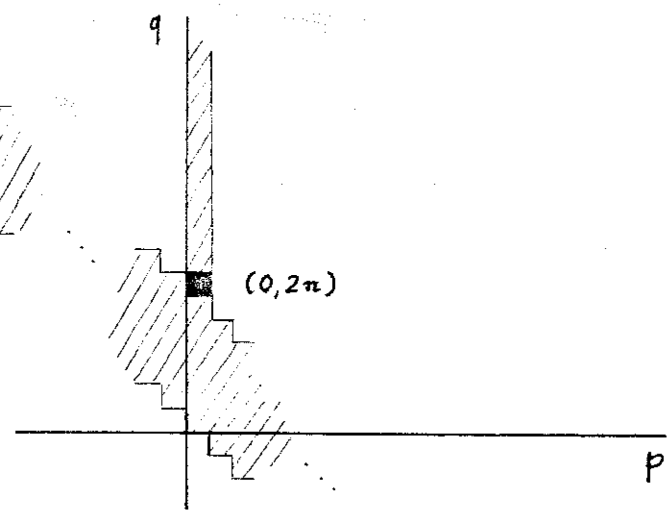


Fig. 1

i.e. $E^{(m),r}_{pq} = 0$ also if $p+q \geq 2n$ and $p \neq 0$.

Indeed, $E^{(m),1}_{pq}$ is equal, for $p \geq m$, to $H_{p+q}(\mathrm{Tot}\, \mathcal{B}_{**}(\mathcal{O})(p)) = HC_{p+q}(\mathcal{O})(p)$. Actually, the first spectral sequence of the double complex $\mathcal{B}_{**}(\mathcal{O})(p)$ degenerates at E^2 yielding thus that

$$E^{(m),1}_{pq} \simeq \Omega^{p+q}_{\mathcal{O}}(p)/d\Omega^{p+q-1}_{\mathcal{O}}(p) \qquad (p \geq m \text{ and } \neq 0)$$

and

$$E^{(m),1}_{0q} \simeq H^{(\tilde{q})}_{DR}(Y^c) \qquad (q \geq 2n)$$

where $\tilde{q}$ = parity of q and $H^*_{DR}(Y^c) \equiv H^{(0)}_{DR}(Y^c) \oplus H^{(1)}_{DR}(Y^c)$.

This implies the required location of non-vanishing $E^{(m),r}_{pq}$'s and as a corollary gives

$$HC^{(m)}_q \simeq E^{(m),1}_{0q} \simeq H^{(\tilde{q})}_{DR}(Y^c) ,$$

for $q \geq 2n+1$. Compatibility of these isomorphisms with the canonical mappings $HC^{(m')}_q \longrightarrow HC^{(m)}_q$ is clear as well. ■

14. FINAL REMARKS. a) It should be clear that the result of Theorem 4 has also the following meaning: the first s.s. of the double complex $\mathcal{B}_{**}(CS(X))$ degenerates at E^1.

b) For the subalgebra $CS^0(X) \subset CS(X)$ of symbols of order ≤ 0 (which correspond to symbols of operators bounded in $L^2(X)$, or $L^2_{loc}(X)$, if X is not compact) a very similar method yields at least the following:

(i) there exist canonical isomorphisms

$$HC_q(CS^0(X)) \simeq H^{(\tilde{q})}_{DR}(S^*X) \qquad (q \geq 2n+1) ;$$

(ii) the homomorphism of principal symbol $\sigma : CS^0(X) \longrightarrow C^\infty(S^*X)$ induces isomorphisms in cyclic homology in dimensions $q \geq 2n+1$;

(iii) in the same range of q's the map in cyclic homology induced by the embedding $CS^0(X) \subset CS(X)$ corresponds, via (i), to the map in de Rham cohomology induced by the projection $Y^c \longrightarrow S^*X$. (This means, in particular, that you couldn't detect noncommutative residue while working with $CS^0(X)$ and with cyclic homology in « stable » range. It must be added, however, that noncommutative residue makes its appearance in $HC_q(CS^0(X))$ for small q's.)

I determined, furthermore, Hochschild homology of $CS^0(X)$ but I have no room to dwell on it here. I hope to write on the case of $CS^0(X)$ separately.*)

c) Determining cyclic homology of $CS(X)$ didn't require the complete knowledge of differentials of the s.s. $E^{(m),r}_{**}$. The latter, however, provides an extra information, e.g. about the filtration on $HC_*(CS(X))$ induced by the order filtration on $CS(X)$.

I'll sketch below very briefly how to obtain this missing information. This leads ultimately to the following stronger version of Theorem 4 (recall that the order filtration was denoted by F).

---

*) see Appendix.

$$Gr_p^F \, HC_k \, (CS(X)) \equiv 0$$

unless $\min\left(\left[\frac{k-2n+1}{2}\right], 0\right) \leq p \leq \min(k-n, 0)$.

In the latter case there exist canonical isomorphisms

$$Gr_p^F \, HC_k \simeq \begin{cases} H_{DR}^{k-2p}(Y^c) & \text{if } p \leq \min(k-n,-1), \\ H_{DR}^{2n-k}(Y^c) \oplus H_{DR}^{2n-k+2}(Y^c) \oplus \dots \oplus H_{DR}^{k}(Y^c) & \end{cases} \qquad (7)$$

if $p=0$ and $k \geqslant n$.

This may be supplemented by an information about $Gr_*^F S_*$ (recall that $S_*: HC_*(CS(X)) \longrightarrow HC_*(CS(X))[2]$ preserves the F- filtration) :

$Gr_p^F S_* \equiv 0$ unless $p=0$. In the latter case it corresponds to the trivial 'forgetting' projection:

$$\begin{array}{c} H_{DR}^{2n-k} \oplus H_{DR}^{2n-k+2} \oplus \dots \oplus H_{DR}^{k-2} \oplus H_{DR}^{k} \\ \big\downarrow \\ H_{DR}^{2n-k+2} \oplus \dots \oplus H_{DR}^{k-2} \end{array}$$

Step 1. How to find $E^{(m),2}_{**}$.

I'll use the fact that both $B: HC_*(CS(X))[1] \longrightarrow H_*(CS(X), CS(X))$ and $I: H_*(CS(X), CS(X)) \longrightarrow HC_*(CS(X))$ come from morphisms of complexes compatible with the order filtration and, that we already know the term E^2 of the corresponding 'Hochschild' s.s. $'E^{(m),*}_{**}$ (see the proof of Proposition 7). The former means that we have maps compatible with differentials d^1:

$$E^{(m),1}_{*k} \xleftarrow{I_k} {'E^{(m),1}_{*k}} \xleftarrow{B_k} E^{(m),1}_{*,k-1}, \qquad (8)_k$$

for every $k \in \mathbb{Z}$.

Let us interprete $(8)_k$ as a sequence of chain complexes and notice that $(8)_k$ is exact at the middle term (= exactness of $HC_k \xleftarrow{I} H_k \xleftarrow{B} HC_{k-1}$ for the graded algebra Θ). Besides, the complexes $\operatorname{Ker} B_k$ and $\operatorname{Coker} I_k$ are concentrated in dimension 0:

$$(\operatorname{Ker} B_k)_j = \begin{cases} H^{k-1}_{DR} \oplus H^{k-3}_{DR} \oplus \dots, & j=0, \\ 0, & j \neq 0 \end{cases}$$

and

$$(\operatorname{Coker} I_k)_j = \begin{cases} H^{k-2}_{DR} \oplus H^{k-4}_{DR} \oplus \dots, & j=0, \\ 0, & j \neq 0 \end{cases}$$

(here and in what follows H^*_{DR} stands for $H^*_{DR}(Y^c)$).

Make an additional observation : $S_*: HC_*(CS(X)) \longrightarrow HC_*(CS(X))[2]$ is also compatible with the order filtration, hence it induces the morphisms of complexes

$$E^{(m),1}_{*,k-2} \xleftarrow{S_k} E^{(m),1}_{*k} . \qquad (9)_k$$

Denote, for brevity, $\Omega^j_{\mathcal{O}}(p)/d\,\Omega^{j-1}_{\mathcal{O}}(p)$ by $\bar{\Omega}^j(p)$. Then $(9)_k$ reads as

$$\begin{tikzcd}
\cdots & \bar{\Omega}^{k-1}(-1) \arrow[l, "d^1"'] \arrow[d, "0"] & \bar{\Omega}^k(0) \oplus H^{k-2}_{DR} \oplus H^{k-4}_{DR} \oplus \cdots \arrow[l, "d^1"'] \arrow[d, "S_{0k}"] & \bar{\Omega}^{k+1}(1) \arrow[l, "d^1"'] \arrow[d, "0"] & \cdots \arrow[l] \\
\cdots & \bar{\Omega}^{k-3}(-1) \arrow[l, "d^1"'] & \bar{\Omega}^{k-2}(0) \oplus H^{k-4}_{DR} \oplus \cdots \arrow[l, "d^1"'] & \bar{\Omega}^{k-1}(1) \arrow[l, "d^1"'] & \cdots \arrow[l]
\end{tikzcd}$$

where all the vertical arrows except S_{0k} vanish and S_{0k} is the composition of the standard projection

$$\bar{\Omega}^k(0) \oplus H^{k-2}_{DR} \oplus H^{k-4}_{DR} \oplus \cdots \longrightarrow H^{k-2}_{DR} \oplus H^{k-4}_{DR} \oplus \cdots$$

with the direct sum of the canonical injection $H^{k-2}_{DR} \hookrightarrow \bar{\Omega}^{k-2}(0)$ and of the identity map on $H^{k-4}_{DR} \oplus H^{k-6}_{DR} \oplus \cdots$.

Hence, we infer, by looking on $(9)_k$ and $(9)_{k+2}$ simultaneously, that $(E^{(m),1}_{*k}, d^1)$ splits into the direct sum

$$E^{(m),1}_{*k} = \bar{E}^{(m),1}_{*k} \oplus \left(H^{k-2}_{DR} \oplus H^{k-4}_{DR} \oplus \dots \right)_0 \qquad (10)$$

where $\bar{E}^{(m),1}_{pk} \equiv \bar{\Omega}^{p+k}(p)$ and the second summand is concentrated purely in dimension 0. If to denote the homology of $(\bar{E}^{(m),1}_{*k}, d^1)$ by $\bar{E}^{(m),2}_{*k}$ then we have

$$E^{(m),2}_{*k} = \bar{E}^{(m),2}_{*k} \oplus \left(H^{k-2}_{DR} \oplus H^{k-4}_{DR} \oplus \dots \right)_0 .$$

All this having said, $(8)_k$ simplifies to the exact sequence of chain complexes

$$0 \longleftarrow \bar{E}^{(m),1}_{*k} \xleftarrow{I_k} {}'E^{(m),1}_{*k} \xleftarrow{B_k} \bar{E}^{(m),1}_{*,k-1} \longleftarrow \left(H^{k-1}_{DR}\right)_0 \longleftarrow 0 \ . \qquad (11)_k$$

View $(11)_k$ as an acyclic double complex[3)] and let ${}_k\mathcal{E}^r_{**}$ be the associated second s.s. We know that

$${}_k\mathcal{E}^1_{p1} = \begin{cases} H^{2n-p}_{DR} & , \ k = n \\ \bar{\Omega}^{2n-p-m}(n-m) & , \ p = m \\ 0 & \text{otherwise} \end{cases} \qquad (\text{cf. Proposition } 7),$$

$${}_k\mathcal{E}^1_{p3} = \begin{cases} H^{k-1}_{DR} & , \ p = 0 \\ 0 & , \ p \neq 0 \end{cases}$$

and, we have

$${}_k\mathcal{E}^1_{*0} = \bar{E}^{(m),2}_{*k} \qquad , \qquad {}_k\mathcal{E}^1_{*2} = \bar{E}^{(m),2}_{*,k-1} .$$

3) with just four rows.

On the other hand, ${}_k\mathcal{E}^{\infty}_{**} = {}_k\mathcal{E}^{3}_{**}$ vanishes.

<u>NB.</u> All further considerations will be carried in assumption that $m \ll 0$ (probably $m < -n$ is enough).

We start playing with these 'toy' s.s. ${}_k\mathcal{E}^{r}_{**}$ <u>taking into account</u> that each $\bar{E}^{(m),2}_{*k}$ appears twice :

$$\text{as} \quad {}_k\mathcal{E}^{1}_{*0} \qquad \text{and as} \quad {}_{(k+1)}\mathcal{E}^{1}_{*2}$$

and, the fact that $\bar{E}^{(m),1}_{*k}$ (and, hence, $\bar{E}^{(m),2}_{*k}$) vanish outside the interval $-k \le * \le 2n-k$. The rule of the game is : ${}_k\mathcal{E}^{\infty}_{**} \equiv 0$ for all k.

In dependence on whether k is less, equal or greater than n we obtain essentially 3 different pictures :

${}_k\mathcal{E}^{1}_{**}$ (<u>$k < n$</u>)

$$\begin{array}{lccccccc}
 & & & H^{k-1}_{DR} & & & & \\
 & \dots & \bar{E}^{(m),2}_{-1,k-1} & \bar{E}^{(m),2}_{0,k-1} & \bar{E}^{(m),2}_{1,k-1} & \bar{E}^{(m),2}_{2,k-1} & \bar{E}^{(m),2}_{3,k-1} & \dots \\
 & \dots & 0 & 0 & 0 & 0 & 0 & \\
 & \dots & \bar{E}^{(m),2}_{-1,k} & \bar{E}^{(m),2}_{0k} & \bar{E}^{(m),2}_{1k} & \bar{E}^{(m),2}_{2k} & \bar{E}^{(m),2}_{3k} & \dots \\
q = & \dots & -1 & 0 & 1 & 2 & 3 & \dots
\end{array}$$

Fig. 2

By induction on $k < n$ we get easily that

$$\overline{E}^{(m),2}_{0k} \simeq \overline{E}^{(m),2}_{-1,k-1} \simeq \overline{E}^{(m),2}_{-2,k-2} \simeq \dots \simeq \overline{E}^{(m),2}_{-l,k-l} = 0$$

where $l = [\frac{k}{2}]+1$.

This implies as well that

$$\overline{E}^{(m),2}_{1k} = 0 \quad , \quad \text{for} \quad k \leq n .$$

Hence, all non-trivial differentials in ${}_k\mathcal{E}^{+}_{**}$, for $k<n$, must act as it is desplayed on Fig. 2; ${}_k\mathcal{E}^{1}_{pq}$'s not connected by any arrows simply vanish.

For $k > n$, the picture is the following:

${}_k\mathcal{E}^{1}_{**}$ ($\underline{k>n}$)

$$\begin{tikzcd}[column sep=small]
 & & & & H^{k-1}_{DR} \arrow[d] & & & \\
\cdots \arrow[ddr] & \overline{E}^{(m),2}_{-3,k-1} \arrow[ddr] & \overline{E}^{(m),2}_{-2,k-1} \arrow[ddr] & \overline{E}^{(m),2}_{-1,k-1} \arrow[ddr] & \overline{E}^{(m),2}_{0,k-1} & \overline{E}^{(m),2}_{1,k-1} & \overline{E}^{(m),2}_{2,k-1} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & \overline{E}^{(m),2}_{-3,k} & \overline{E}^{(m),2}_{-2,k} & \overline{E}^{(m),2}_{-1,k} & \overline{E}^{(m),2}_{0k} & \overline{E}^{(m),2}_{1k} & \overline{E}^{(m),2}_{2k} & \cdots
\end{tikzcd}$$

Fig. 3

By induction on $j \equiv k-1 \geqslant n$ we get easily that

$$\bar{\bar{E}}^{(m),2}_{2j} \simeq \bar{E}^{(m),2}_{3,j+1} \simeq E^{(m),2}_{4,j+1} \simeq \dots \simeq \bar{E}^{(m),2}_{2+\ell,\, j+\ell} = 0$$

where $\ell = n - [\frac{j+1}{2}]$.

This implies as well that

$$\bar{E}^{(m),2}_{1k} = 0 \qquad \text{also for } k > n .$$

Hence, all non-trivial differentials in ${}_k\mathcal{E}^{r}_{**}$, for $k > n$, must act as it is displayed on Fig. 3; as above, ${}_k\mathcal{E}^{1}_{pq}$'s not connected by arrows vanish.

We obtain, therefore, by gathering up an available information that ${}_n\mathcal{E}^{1}_{**}$ has the following form:

$$
{}_n\mathcal{E}^{1}_{**} \qquad
\begin{tikzcd}[column sep=small]
 & & & H^{n-1}_{DR} \arrow[ddr] & & & & \\
\dots & 0 & 0 & 0 & 0 & \bar{E}^{(m),2}_{2,n-1} \arrow[d] & \bar{E}^{(m),2}_{3,n-1} \arrow[d] & \dots \\
\dots & H^{n+2}_{DR} \arrow[d] & H^{n+1}_{DR} \arrow[d] & H^{n}_{DR} \arrow[d] & H^{n-1}_{DR} & H^{n-2}_{DR} & H^{n-3}_{DR} & \dots \\
\dots & \bar{\bar{E}}^{(m),2}_{-2,n} & \bar{\bar{E}}^{(m),2}_{-1,n} & \bar{E}^{(m),2}_{0n} & 0 & 0 & 0 & \dots
\end{tikzcd}
$$

Fig. 4

and that differentials must act as it is shown at Fig. 4.

This, in turn, provides the missing information about ${}_k\mathcal{E}^1_{**}$ ($k \neq n$; cf. Figures 2 & 3) and the description of $\bar{E}^{(m),2}_{*k}$ is complete. Taking into account decomposition (10) we obtain the following answer

$$E^{(m),2}_{pq} \simeq \begin{cases} H^{q-p}_{DR} \,, & p \leq -1 \,,\ q \geq n \,, \\ H^{q-p+1}_{DR} \,, & p \geq 2 \,,\ q \leq n-1 \end{cases}$$

and

$$E^{(m),2}_{0q} \simeq \begin{cases} H^{q}_{DR} \oplus H^{q-2}_{DR} \oplus \ldots \,, & q \geq n \,, \\ H^{q-2}_{DR} \oplus H^{q-4}_{DR} \oplus \ldots \,, & q < n \,. \end{cases}$$

All remaining $E^{(m),2}_{pq}$'s (except the extreme left column $E^{(m),2}_{m*}$ which I don't care of since it has little importance) vanish. I tried (seemingly with little success) to represent the answer on a separate sheet (Fig. 6). A better idea of the shape of $E^{(m),2}_{**}$ can be obtained from the following figure:

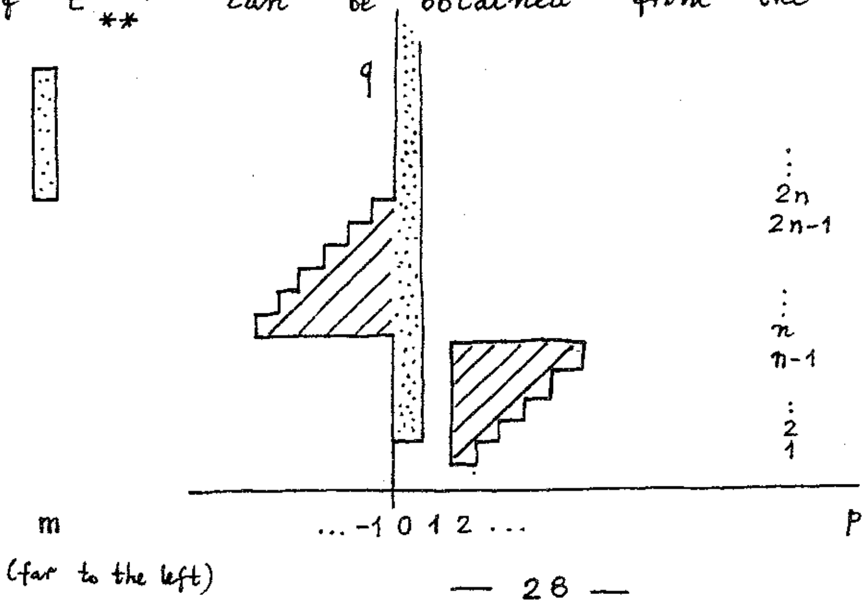
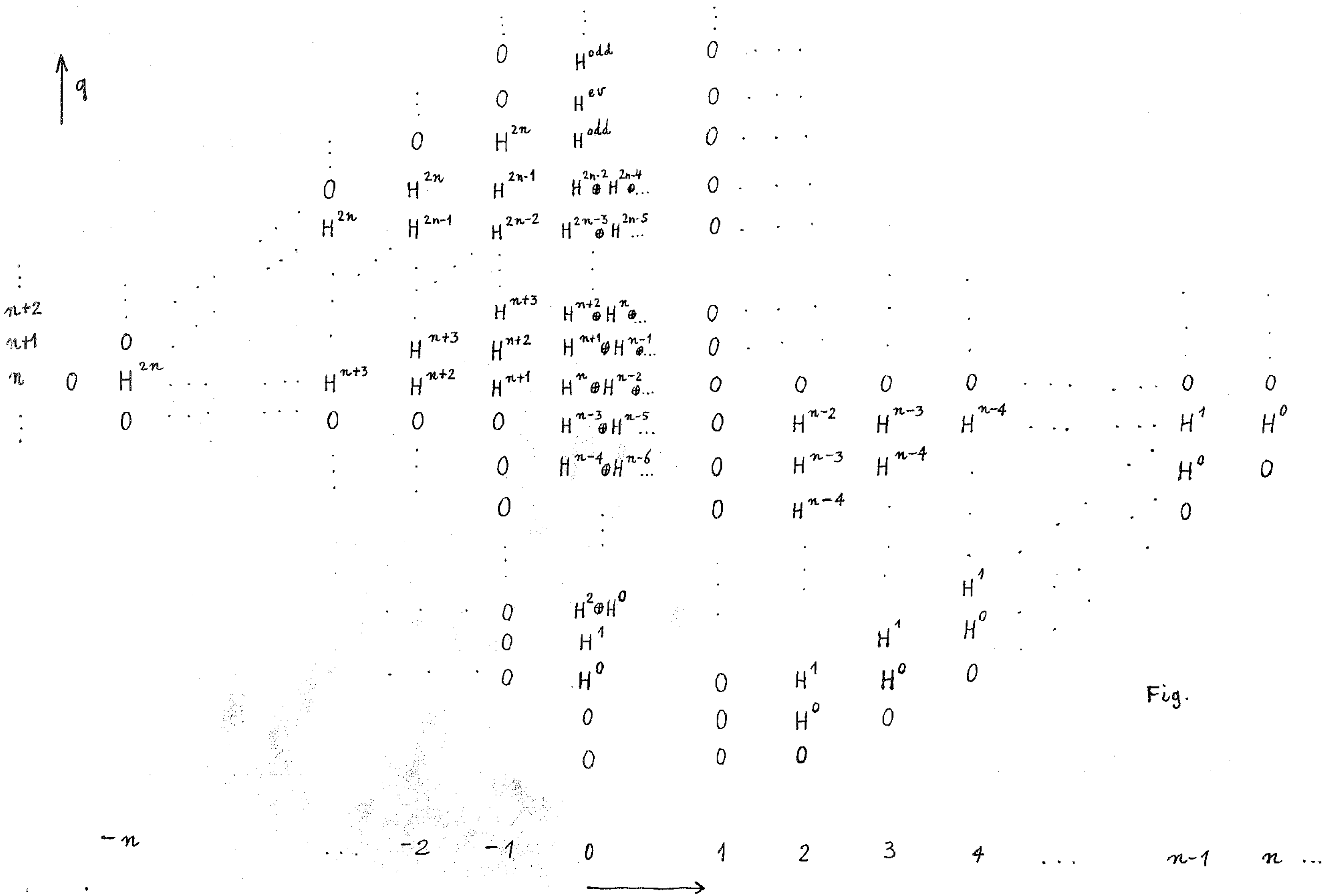


Fig. 5



q
0 H^odd 0
0 H^ev 0
0 H^2n H^odd 0
0 H^2n H^2n-1 H^2n-2 ⊕ H^2n-4 ⊕ … 0
H^2n H^2n-1 H^2n-2 H^2n-3 ⊕ H^2n-5 … 0
n+2 H^n+3 H^n+2 ⊕ H^n ⊕ … 0
n+1 0 H^n+3 H^n+2 H^n+1 ⊕ H^n-1 ⊕ … 0
n 0 H^2n H^n+3 H^n+2 H^n+1 H^n ⊕ H^n-2 ⊕ … 0 0 0 0 0 0
0 0 0 0 H^n-3 ⊕ H^n-5 … 0 H^n-2 H^n-3 H^n-4 H^1 H^0
0 H^n-4 ⊕ H^n-6 … 0 H^n-3 H^n-4 H^0 0
0 0 H^n-4 0
H^1
0 H^2 ⊕ H^0 H^0
0 H^1 H^1
0 H^0 0 H^1 H^0 0
0 0 H^0 0
0 0 0
-n … -2 -1 0 1 2 3 4 … n-1 n …

Fig.

Step 2. Higher differentials.

The only non-vanishing differentials are

$$d^{p}_{pq} : E^{(m),p}_{pq} \longrightarrow E^{(m),p}_{0,p+q-1} \qquad (12)$$

which inject $E^{(m),p}_{pq} = E^{(m),2}_{pq} \simeq H^{q-p+1}_{DR}(X)$ into $E^{(m),p}_{0,p+q-1}$.

This will be demonstrated by making use of algebras of symbols CS_Y associated with more general conical manifolds $Y \subset T^*X \setminus X$. As I mentioned at the beginning Theorems 3 and 4 extend to this situation with Y^c having the previous meaning (this indeed is clear from the presented proofs) and it will be convenient to establish (12) for a general Y.

Assume, first, that $H^k_{DR}(Y^c) = 0$, for $k > n$. Then the assertion follows from the simple 'dimensions counting' argument similar to that of Lemma 6 in [CHDO] ($\sum_{j=0}^{2n-2} \dim E^{(m),2}_{0j} - \sum_{p>0;q} \dim E^{(m),2}_{pq} = \sum_{j=0}^{2n-2} \dim HC_j(CS_Y)$; of course, Y is additionally assumed to satisfy the finite-dimensionality condition $\dim H^*_{DR}(Y) < \infty$).

Now, let Y be a general conic submanifold of $T^*X \setminus X$ (of codimension 0) with compact base $Z \subset S^*X$.

Let us consider the skeleton filtration $Z^0 \subset Z^1 \subset \ldots \subset Z^{2n-1} = Z$ associated with any finite cell decomposition and let $\tilde{Z}^0 \subset \tilde{Z}^1 \subset \ldots$ be its 'thickening', i.e. each $\tilde{Z}^k$ is of codimension zero in Z and $Z^k \hookrightarrow \tilde{Z}^k$ is a homotopy equivalence. Finally, if τ denotes the canonical projection $Y \to Z$ then let $Y^k = \tau^{-1}(\tilde{Z}^k)$. Clearly, the inclusion $Y^k \subset Y$ induces the maps $H^j_{DR}(Y^c) \longrightarrow H^j_{DR}((Y^k)^c)$ which are isomorphisms for $j < k$, a monomorphism for $j = k$ and zero, for $j > k+1$.

Take $k = n-1$ and consider the following assertions, for $r \geq 2$,

$(A)_r$ the natural maps $E^{(m),r}_{pq} \longrightarrow E^{(m),r}_{pq} \langle Y^{n-1} \rangle$ induced by the homomorphism $CS_Y \longrightarrow CS_{Y^{n-1}}$ are isomorphisms, for $p > 0$ (r fixed);

$(B)_r$ the differentials $d^r_{rq} : E^{(m),r}_{rq} \longrightarrow E^{(m),r}_{0,r+q-1}$ are injective;

$(C)_r$ the differentials $d^r_{pq} : E^{(m),r}_{pq} \longrightarrow E^{(m),r}_{p-r,r+q-1}$ are zero, for $p > r$.

From the functoriality of $E_{**}^{(m),r}$ and the fact that the differentials of $E_{**}^{(m),r}\langle Y^{n-1}\rangle$ have been already deciphered we obtain the following chain of implications

$$\begin{tikzcd}[column sep=small, row sep=small] & (B)_2 \arrow[dr, Rightarrow] & & (B)_3 \arrow[dr, Rightarrow] & \\ (A)_2 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (A)_3 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \cdots \\ & (C)_2 \arrow[ur, Rightarrow] & & (C)_3 \arrow[ur, Rightarrow] & \end{tikzcd}$$

and $(A)_2$ follows from the remark on the maps $H_{DR}^{j}(Y^c) \longrightarrow H_{DR}^{j}((Y^k)^c)$ and the description of terms $E_{pq}^{(m),2}$ $(p>0)$ obtained on the previous step.

This gives us the answer concerning differentials with sources $E_{pq}^{(m),r}$ with $p>0$ (still in assumption of compactness of Z). Having reached this point we conclude that all differentials with sources with $p\leq 0$ must vanish by using another variant of the 'dimensions counting' argument (on this occasion based on the equality

$$\sum_{m<p<0;q} \dim E_{pq}^{(m),2} + \sum_{j=0}^{2n-2} \dim \tilde{E}_{0q}^{(m),2} = \sum_{j=0}^{2n-2} \dim HC_j(CS_Y)$$

where $\tilde{E}_{0*}^{(m),2}$ is defined as the cokernel of all differentials coming into $E_{0*}^{(m),r}$).

Thus (12) is proved for Y with compact bases. For a general Y consider the exhaustion of Y by a sequence $Y_0 \subset Y_1 \subset \ldots$ with compact bases.

Let us consider, for every $r \geq 1$, the following assertions:

$(D)_r$ $\qquad d^r_{pq} = \varprojlim_j d^r_{pq, Y_j}$,

$(E)_r$ $\qquad E^{(m),r}_{pq} = \varprojlim_j E^{(m),r}_{pq} \langle Y_j \rangle \qquad (p > m)$.

It is clear that $(D)_r$ follows from $(E)_r$ and the functoriality of differentials. On the other hand $(E)_{r+1}$ follows from $(D)_r + (E)_r$ if the projective systems $\{E^{(m),r}_{pq} \langle Y_j \rangle ; j \in \mathbb{N}\}$ and $\{E^{(m),r+1}_{pq} \langle Y_j \rangle ; j \in \mathbb{N}\}$ satisfy M.L. condition. This is certainly true if $r \geq 2$ because in the category of finite-dimensional vector spaces every projective system has this property. For $r = 1$ the only thing that should be verified[4] is that $\{ \bar{\Omega}^{p+q}_{\mathcal{O}_j}(p) ; j \in \mathbb{N}\}$ satisfies M.L. condition (recall that $\bar{\Omega}^{p+q}(p) \equiv \Omega^{p+q}(p) / d\,\Omega^{p+q-1}(p)$; $\mathcal{O}_j$ denotes the corresponding graded algebra of functions on Y_j).

This follows easily from the commutativity of the diagram

---

4) we are speaking here only about M.L. condition

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^q_{DR}(Y^c_{k_2}) & \longrightarrow & \bar{\Omega}^q_{\mathcal{O}_{k_2}}(0) & \longrightarrow & d\Omega^q_{\mathcal{O}_{k_2}}(0) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & H^q_{DR}(Y^c_{k_1}) & \longrightarrow & \bar{\Omega}^q_{\mathcal{O}_{k_1}}(0) & \longrightarrow & d\Omega^q_{\mathcal{O}_{k_1}}(0) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & H^q_{DR}(Y^c_{j}) & \longrightarrow & \bar{\Omega}^q_{\mathcal{O}_{j}}(0) & \longrightarrow & d\Omega^q_{\mathcal{O}_{j}}(0) & \longrightarrow 0
\end{array}$$

if $p = 0$ (indeed , let $V_k = \mathrm{Im}\,\{\bar{\Omega}^q_{\mathcal{O}_k}(0) \to \bar{\Omega}^q_{\mathcal{O}_j}(0)\}$, then the sequence $H^q_{DR}(Y^c_j) \cap V_k$ stabilizes for a certain $k' \geqslant j$. Since $H^q_{DR}(Y^c_j) + V_k = \bar{\Omega}^q_{\mathcal{O}_j}(0)$, for every $k \geqslant j$, one obtains easily that $\dots \supset V_k \supset V_{k+1} \supset \dots$ stabilizes for the same k').

For $p \neq 0$ one has $\bar{\Omega}^{p+q}(p) \xrightarrow{\sim} d\Omega^{p+q}(p)$ and in consequence the restriction maps $\bar{\Omega}^{p+q}_{\mathcal{O}_k}(p) \longrightarrow \bar{\Omega}^{p+q}_{\mathcal{O}_j}(p)$ $(k \geqslant j)$ are surjective.

Finally $(E)_1$ reduces to verification that $\bar{\Omega}^{p+q}_{\mathcal{O}}(p) = \varprojlim_j \bar{\Omega}^{p+q}_{\mathcal{O}_j}(p)$. Notice that $\bar{\Omega}^{p+q}_{\mathcal{O}}(p)$ is the first cohomology of the two-term complex

$$0 \longrightarrow \bar{\Omega}^{p+q-1}_{\mathcal{O}}(p) \xrightarrow{\ d\ } \Omega^{p+q}_{\mathcal{O}}(p) \longrightarrow 0 \qquad (13)$$

Assume , by induction on $p+q$, that we already have shown that $\bar{\Omega}^{p+q-1}_{\mathcal{O}}(p) = \varprojlim_j \bar{\Omega}^{p+q-1}_{\mathcal{O}_j}(p)$. Then the complex (13) is a limit of the system

$$\{\, 0 \longrightarrow \bar{\Omega}^{p+q-1}_{\mathcal{O}_j}(p) \xrightarrow{d} \Omega^{p+q}_{\mathcal{O}_j}(p) \longrightarrow 0 \ ; \ j \in \mathbb{N} \,\}$$

satisfying M.L. condition and, since the projective system of 0-th cohomology certainly satisfies M.L. condition the 'standard lemma' on cohomology of projective limits applies yielding the required assertion.

Since $\varprojlim_j 0 = 0$ and $\varprojlim_j$ preserves monomorphisms (12) follows for general Y.

Step 3. More precise structure of $d^p_{p,q-p+1} : E^{(m),p}_{p,q-p+1} \longrightarrow E^{(m),p}_{0,q}$.

What we obtained so far concerning $E^{(m),\infty}_{**}$ can be stated as follows:

$E^{(m),\infty}_{**} = E^{(m),n+1}_{**}$ and it vanishes except the following cases:

(i) $E^{(m),\infty}_{pq} \simeq H^{q-p}_{DR}(Y^c)$ $\quad (p \leq -1\ ,\ q \geq n)$

(isomorphism canonical);

(ii) $E^{(m),\infty}_{0q}$, for $q \geq n$;

(iii) $E^{(m),\infty}_{mq}$.

(The latter case is, of course, of little importance.)

$E^{(m),\infty}_{**}$

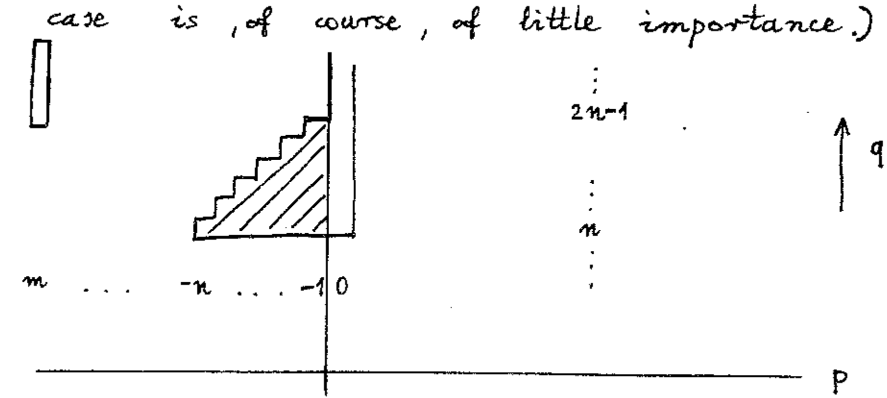


Fig. 7

I shall prove below that the inclusion

$$H^q_{DR} \oplus H^{q-2}_{DR} \oplus \dots \oplus H^{2n-q}_{DR} \subset E^{(m),2}_{0q} \qquad (q \geq n)$$

composed with the canonical projection onto $E^{(m),\infty}_{0q}$ induces a canonical isomorphism

$$E^{(m),\infty}_{0q} \simeq H^q_{DR} \oplus H^{q-2}_{DR} \oplus \dots \oplus H^{2n-q}_{DR} \qquad (14)$$

($q \geq n$; here and further on H^k_{DR} denotes $H^k_{DR}(Y^c)$).

For $q \geq 2n-1$ this has been already proved since $E^{(m),\infty}_{0q} \equiv E^{(m),2}_{0q}$ in this case, so that we can restrict to the case $q \leq 2n-2$. To establish (14) we need a better understanding of differentials. In order to make a precise statement let us introduce certain subspaces in $E^{(m),r}_{0q}$, for $q \leq 2n-2$. First of all recall that

$$E^{(m),2}_{0q} \equiv E^{(m),p'}_{0q} \simeq \begin{cases} H^q_{DR} \oplus H^{q-2}_{DR} \oplus \dots & , \; q \geq n, \\ H^{q-2}_{DR} \oplus H^{q-4}_{DR} \oplus \dots & , \; q < n, \end{cases}$$

where $p' = 2 + \max(0, q-n)$. For a given $p \geq p'$ let $W^p_+ \subset E^{(m),p}_{0q}$ denote the image of the right-hand side of (14) under the projection $E^{(m),2}_{0q} \to E^{(m),p}_{0q}$, if $q \geq n$, for $q < n$ we put simply $W_+^p = 0$. Similarly, let $W_0^p \subset E_{0q}^{(m),p}$ denote the image of H_{DR}^{q-2p+2} and, finally, $W_-^p \subset E_{0q}^{(m),p}$ denote the image of the space $H_{DR}^{q-2p} \oplus H_{DR}^{q-2p-2} \oplus \dots$.

Introduce the following assertions, for every $p \geq p'$,

$(F)_p \qquad E_{0q}^{(m),p} = W_-^p \oplus W_0^p \oplus W_+^p \qquad$ (q fixed)

and, in assumption of validity of $(F)_p$, let d_-^p, d_0^p and d_+^p be the corresponding components of $d^p : E_{p,q-p+1}^{(m),p} \longrightarrow E_{0q}^{(m),p}$. The two remaining assertions are

$(G)_p \qquad d_-^p \equiv 0$,

$(H)_p \qquad d_0^p : E_{p,q-p+1}^{(m),p} \xrightarrow{\sim} W_0^p$.

<u>Proof</u>. The implication $(F)_{p-1}$ & $(G)_{p-1}$ & $(H)_{p-1} \Rightarrow (F)_p$ is clear.

Next we prove the implication $(F)_r$ & $(G)_{r-1}$ $(r \leq p) \Rightarrow (G)_p$. For this purpose consider the inclusion $Y^{q-2p} \subset Y$ where $Y^{q-2p} = \tau^{-1}(\tilde{Z}^{q-2p})$ and $\tilde{Z}^{q-2p}$ is the corresponding thickened $(q-2p)$-skeleton of the base $Z \subset S^*X$

of Y (cf. Step 2; at this stage finiteness of the cell decomposition of Z is irrelevant). Denote by $W_i^p\langle Y^{q-2p}\rangle$ the corresponding subspaces of $E_{0q}^{(m),p}\langle Y^{q-2p}\rangle$ (for notation cf. assertion $(A)_r$ above; $i = -, 0$ or $+$).

We have the commutative diagram

$$\begin{tikzcd}
E_{0q}^{(m),p}\langle Y^{q-2p}\rangle = W_-^p\langle Y^{q-2p}\rangle & E_{p,q-p+1}^{(m),p}\langle Y^{q-2p}\rangle = 0 \arrow[l, "d^p"'] \\
E_{0q}^{(m),p} = W_-^p \oplus W_0^p \oplus W_+^p \arrow[u] & E_{p,q-p+1}^{(m),p} \arrow[l, "d^p"'] \arrow[u]
\end{tikzcd}$$

in which the left vertical arrow in view of apparent functoriality of the W-decomposition and in view of the inductive assumption injects W_-^p into $W_-^p\langle Y^{q-2p}\rangle$ and sends W_0^p and W_+^p to zero (notice that $W_0^p\langle Y^{q-2p}\rangle = W_+^p\langle Y^{q-2p}\rangle = 0$).

Then $(G)_p$ follows easily from the fact that $E_{p,q-p+1}^{(m),p}\langle Y^{q-2p}\rangle \simeq H_{DR}^{q-2p+2}((Y^{q-2p})^c) = 0$.

Finally, we shall prove the implication $(F)_r \,\&\, (G)_{r-1}$ $(r \leq p)$ $\Rightarrow (H)_p$. Injectivity of d_0^p can be demonstrated similarly to the previous case by using inclusion $Y^{q-2p+2} \subset Y$. This is sufficient if $\dim H_{DR}^{q-2p+2}(Y^c) < \infty$ for then d_0^p has to be an isomorphism. In the general case we consider, as usual, a 'compactly based' exhaustion $Y_0 \subset Y_1 \subset \dots \subset Y$ and notice that assertion $(E)_r$, for $2 \leq r \leq p$, proved in Step 2 and the inductive assumption imply that $W_0^p \xrightarrow{\sim} \varprojlim_j W_0^p \langle Y_j \rangle$. If d^p_{0,Y_j} denotes the 'd_0^p' corresponding to Y_j then $(D)_p$, also demonstrated in Step 2, shows that $d_0^p = \varprojlim_j d^p_{0,Y_j}$ and $(H)_p$ follows.

Since the initial step of induction is trivial ($(F)_{p'}$ is valid by definition and $(G)_{p'-1}$ and $(H)_{p'-1}$ are meaningless) we conclude the proof of $(F)_p - (H)_p$ and, in result, of (7).

Assertions $(F)_p - (H)_p$ above can be supplemented by an information about d_+^p if to take into account in addition the induced action of S-operator on our picture. I omit the details.

d) Relation to noncommutative residue.

In [NC·I] I mentioned that there exists a canonical map $HC_*(CS(X)) \longrightarrow H(\mathcal{A}_*(X))$ where $\mathcal{A}_*(X)$ denotes the complex of integral forms on X, which arises in the context of noncommutative residue. Up to normalizing constants this can be obtained by consecutive composition of the following maps:

$$
\begin{tikzcd}
HC_q(CS(X)) \arrow[r] \arrow[d, dashed, "\text{"res}_q\text{"}"'] & H_q(\mathrm{Tot}\,\mathcal{B}^{(m)}_{**}) \arrow[r, "\text{edge homomorphism}", "\text{of } E^{(m),r}_{**} \quad (m<-n)"'] & E^{(m),\infty}_{q-n,n} \arrow[d, equal] \\
H_q(\mathcal{A}_*(X)) & H^{2n-1-q}_{DR}(S^*X) \arrow[l, "\int_{S^*X/X}"'] & H^{2n-q}_{DR}(Y^c) \simeq E^{(m),2}_{q-n,n} \arrow[l, "\text{Cauchy}"', "\text{residue}"]
\end{tikzcd}
$$

(My original construction was quite different.)

Appendix

15. Complementing our previous results we shall determine below Hochschild and cyclic homology of the algebra $CS^0(X)$ of symbols of order ≤ 0. This will elucidate, as was already pointed out in Remark 14.6), your approach to cyclic homology of $C^\infty(S^*X)$ and to homological invariants of ψDOs via 'trace relations'.

Homology considered here means continuous homology as defined by $\hat{\otimes}_\pi$ -completion of the corresponding algebraic chain complexes. In the first section we noticed that this agreed with the completion chosen for the 'big' algebra of symbols $CS(X)$ (cf. p. 3).

Let us consider first Hochschild homology. Filtration by order induces on it a $\mathbb{Z}_-$-filtration:

$$\ldots \subset F_p \subset F_{p+1} \subset \ldots \subset F_0 = H_*(CS^0(X), CS^0(X)).$$

The associated graded space is described by the following theorem.

16. THEOREM. There are canonical identifications

$$Gr_p^F H_k \simeq \begin{cases} {}'G_{0k} & , p = 0 , \\ {}'G_{-1,1+k} & , p = -1 , \\ H_{DR}^{2n-k}(Y^c) & , p = k-n \leq -2 , \\ 0 & \text{otherwise} \end{cases}$$

where $H_k \equiv H_k\,(cS^0(X),\, cS^0(X))$ and

$${}'G_{0k} := \alpha \wedge \Omega^{2n-1-k}(n-1-k) \cap \operatorname{Ker}\left\{d_{DR} : \Omega^{2n-k}(n-k) \longrightarrow \Omega^{2n-k+1}(n-k)\right\} \tag{15}$$

and

$${}'G_{-1,1+k} := \frac{\operatorname{Ker}\{d_{DR} : \Omega^{2n-k}(n-1-k) \longrightarrow \Omega^{2n-k+1}(n-1-k)\}}{d\,(\alpha \wedge \Omega^{2n-2-k}(n-2-k))} . \tag{16}$$

(Here Ω^* has the previous meaning $\Omega^*_{\mathcal{O}/\mathbb{C}}$ (cf. subsection 2) and α is the canonical 1-form $i_\Xi \omega$ where Ξ is the Euler field and ω the symplectic form on T^*X).

17. In order to demonstrate Theorem 16 we consider the projective system of quotient-complexes $C_*^{(m)} \equiv C_*(CS^0(X), CS^0(X))/F_{m-1}$ much as we did in the proof of Theorem 3. On each $C_*^{(m)}$ the filtration induced by the order filtration is finite and the associated with it s.s. (denoted $'G_{**}^{(m),r}$) is located in the second quadrant.

An analogue of Proposition 7 reads as

18. PROPOSITION. *Assume that* $m \le \min(-n,-2)$. *Then:*

(a) *the second term of* $'G_{**}^{(m),r}$ *is given by*

$$'G_{pq}^{(m),2} \simeq \begin{cases} 'G_{0q} & , p=0, \\ 'G_{-1,q} & , p=-1, \\ H_{DR}^{n-p}(Y^c) & , p\le -2, q=n, \\ \Omega^{2n-m-q}(n-q)/d\Omega^{2n-1-m-q}(n-q) & , p=m, \\ 0 & \text{otherwise} \end{cases}$$

where the isomorphisms are functorial and $'G_{0q}$ *and* $'G_{-1,q}$ *were defined in* (15) - (16);

(b) $'G^{(m),\infty}_{**} = {}'G^{(m),2}_{**}$;

(c) the identifications in (a) are compatible with morphisms of s.s. induced by the canonical projections $C^{(l)}_{*} \longrightarrow C^{(m)}_{*}$ $(l \leq m)$.

19. For $m < -n$ and $p \leq 0$ let $F_p H^{(m)}_{*}$ denote the p-th term of the filtration induced on $H^{(m)}_{*} \equiv H_{*}(CS^0(X), CS^0(X))$ by the order filtration.

20. COROLLARY. For every $p \leq 0$ the projective system $\{F_p H^{(m)}_{*} \; ; \; m \in \mathbb{Z}_{<-n}\}$ satisfies M.L. condition.

Proof. By Proposition 18 this is clear for $p < -n$. On the other hand we have the exact sequence

$$0 \longrightarrow F_p H^{(m)}_{*} \longrightarrow F_{p+1} H^{(m)}_{*} \longrightarrow {}'G^{(m),2}_{p+1,*} \longrightarrow 0 \qquad (17)$$

and since $\{'G^{(m),2}_{p+1,*} \; ; \; m \in \mathbb{Z}_{<-n}\}$ obviously satisfies M.L., $\{F_{p+1} H^{(m)}_{*} \; ; \; m \in \mathbb{Z}_{<-n}\}$ satisfies M.L. if $\{F_p H^{(m)}_{*} \; ; m \in \mathbb{Z}_{<-n}\}$ does. Applying an inductive argument completes the proof. ■

21. Since $F_0 H_*(CS^0(X), CS^0(X)) = H_*(CS^0(X), CS^0(X))$ and, similarly, $F_0 H_*^{(m)} = H_*^{(m)}$ we obtain from Corollary 20 that

$$H_*(CS^0(X), CS^0(X)) \xrightarrow{\sim} \varprojlim H_*^{(m)} . \qquad (18)$$

Moreover, since $\varprojlim$ is left-exact, the following sequence

$$0 \longrightarrow \varprojlim_m F_p H_*^{(m)} \longrightarrow \varprojlim_m H_*^{(m)} \longrightarrow H_*^{(p+1)}$$

must be also exact. By combining this with (18) one obtains the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim_m F_p H_*^{(m)} \arrow[r] & \varprojlim_m H_*^{(m)} \arrow[r] & H_*^{(p+1)} \\
0 \arrow[r] & F_p H_* \arrow[r] \arrow[u] & H_* \arrow[r] \arrow[u, "\wr"'] & H_*^{(p+1)} \arrow[u, equal]
\end{tikzcd}$$

(where $F_p H_* \equiv F_p H_*(CS^0(X), CS^0(X))$ and $H_* \equiv H_*(CS^0(X), CS^0(X))$), hence the left vertical arrow is an isomorphism.

Finally, we have another commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim_m F_{p-1} H_*^{(m)} \arrow[r] & \varprojlim_m F_p H_*^{(m)} \arrow[r] & \varprojlim_m {}'G_{p*}^{(m),2} \arrow[r] & 0 \\
0 \arrow[r] & F_{p-1} H_* \arrow[r] \arrow[u, "\wr"] & F_p H_* \arrow[r] \arrow[u, "\wr"] & Gr_p^F H_* \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with the lower row exact by definition. The exactness of the upper row is a clear consequence of the exactness of (17) above and the fact that $\varprojlim_m^{(1)} F_{p-1} H_*^{(m)} = 0$.

Hence the canonical map $Gr_p^F H_* \longrightarrow \varprojlim_m {}'G_{p*}^{(m),2}$ is an isomorphism for all p and Theorem 16 follows.

Thus the proof of Theorem 16 will be complete if we prove Proposition 18.

Proof of Proposition 18. Recall that we have a morphism of s.s. ${}'G_{**}^{(m),r} \longrightarrow {}'E_{**}^{(m),r}$ induced by inclusion $CS^\circ(X) \subset CS(X)$ (cf. the discussion at the end of subsection 1).

The required assertion is essentially an abstract consequence of the fact that ${}'G^{(m),1}_{p*} \to {}'E^{(m),1}_{p*}$ is an isomorphism, for $p<0$, and a monomorphism for $p=0$.[5] The latter is clear if one recalls that ${}'G^{(m),1}_{p*} \longrightarrow {}'E^{(m),1}_{p*}$ is simply the natural map

$$H_*(\mathcal{O}_-, \mathcal{O}_-)(p) \longrightarrow H_*(\mathcal{O}, \mathcal{O})(p) \qquad (p \geq m)$$

induced by inclusion $\mathcal{O}_- \subset \mathcal{O}$ where $\mathcal{O}_- := \bigoplus_{p\leq 0} \mathcal{O}(p)$ and $\mathcal{O} \equiv \bigoplus_{p\in\mathbb{Z}} \mathcal{O}(p)$ was defined in subsection 2.

We obtain therefore that ${}'G^{(m),2}_{p*} \to {}'E^{(m),2}_{p*}$ is an isomorphism for all $p \leq 0$ except $p=0$ and -1.
In particular, in the following commutative diagram

$$\begin{tikzcd}
{}'E^{(m),2}_{p-2,q+1} & {}'E^{(m),2}_{pq} \arrow[l, "d^2_{pq}"'] \\
{}'G^{(m),2}_{p-2,q+1} \arrow[u, "\wr"] & {}'G^{(m),2}_{pq} \arrow[l, "d^2_{pq}"'] \arrow[u]
\end{tikzcd}$$

The left vertical arrow is <u>always</u> an isomorphism and hence the vanishing of d^2_{pq} on ${}'E^{(m),2}_{pq}$ implies

[5] and, of course, of Proposition 7 describing the structure of ${}'E^{(m),r}_{**}$.

the vanishing of d^2_{pq} on ${}'G^{(m),2}_{pq}$.

Similarly for higher differentials.

What remains is to obtain a description of ${}'G^{(m),2}_{p*}$ for $p=0$ and -1.

First of all recall that we have the commutative diagram

$$\begin{tikzcd} {}'E^{(m),1}_{-1,q} \arrow[d, equal] & {}'E^{(m),1}_{0q} \arrow[l, "d^1_{0q}"'] \\ {}'G^{(m),1}_{-1,q} & {}'G_{0q} \arrow[l, "d^1_{0q}"'] \arrow[u] \end{tikzcd}$$

which implies that ${}'G^{(m),2}_{pq}$ for $p=-1$ and 0 is simply H_0 and respectively H_1 of the following short complex (cf. the proof of Proposition 7):

$$0 \longleftarrow \Omega^{2n-q+1}_{\mathcal{O}}(n-q) \xleftarrow{\ d_{DR}\ } \tau^0_q\{C_q(\mathcal{O}(0),\mathcal{O}(0))\} \longleftarrow 0 \tag{19}$$

($\mathcal{O}(0)$ is simply the algebra of functions on Y^c constant along fibres of the projection onto S^*X and the right term in (19) is the image of $C_q(\mathcal{O}(0),\mathcal{O}(0)) \equiv C_q(\mathcal{O}_-,\mathcal{O}_-)(0)$ in $\Omega^{2n-q}_{\mathcal{O}}(n-q)$ under the map τ^p_q $(p=0)$ defined in the proof of Proposition 7).

I leave as an excersice to verify that τ_q^0 maps $C_q(\mathcal{O}(0), \mathcal{O}(0))$ onto the subspace $\alpha \wedge \Omega_{\mathcal{O}}^{2n-1-q}(n-1-q)$ in $\Omega_{\mathcal{O}}^{2n-q}(n-q)$ of forms divisible by $\alpha \in \Omega_{\mathcal{O}}^1(1)$ (I recall that $\alpha = i_{\Xi}\omega$ is the canonical 1-form; the kernel of τ_q^0 on $C_q(\mathcal{O}(0), \mathcal{O}(0))$ consists of boundaries since that is true if to replace $\mathcal{O}(0)$ by $\mathcal{O}$ (cf. the proof of Proposition 7) and because

$$\partial C_{q+1}(\mathcal{O}(0), \mathcal{O}(0)) = \partial C_{q+1}(\mathcal{O}, \mathcal{O}) \cap C_q(\mathcal{O}(0), \mathcal{O}(0)).)$$

This completes the proof of Proposition 18. ■

22. REMARK. It is clear from the presented proof that the natural map

$$F_p H_*(CS^0(X), CS^0(X)) \longrightarrow F_p H_*(CS(X), CS(X))$$

is an isomorphism for $p \leq -2$ and that

$$Gr_{-1}^F H_*(CS^0(X), CS^0(X)) \longrightarrow Gr_{-1}^F H_*(CS(X), CS(X))$$

is an epimorphism.

In particular, inclusion $F_{-2} H_*(CS^0(X), CS^0(X)) \subset H_*(CS^0(X), CS^0(X))$ admits a canonical splitting.

Besides, the top non-vanishing group $H_k(CS^0(X), CS^0(X))$ is H_{2n-1}. In this case we have:

$${}'G_{0,2n-1} = \mathcal{O}(-n)\cdot\alpha \ \cap\ \mathrm{Ker}\{d_{DR} : \Omega^1(1-n) \longrightarrow \Omega^2(1-n)\}$$

which is zero unless $n=1$ (indeed, then ${}'G_{0,2n-1} = \mathcal{O}(-n)\cdot\alpha \cap d\,\mathcal{O}(1-n)$; assume that $f\alpha = dg$ for $f \in \mathcal{O}(-n)$, $g \in \mathcal{O}(1-n)$, then $0 = i_\Xi(f\alpha) = i_\Xi dg = \mathcal{L}_\Xi g = (1-n)g \Rightarrow g \equiv 0$) and in the latter case, i.e. $n=1$, $\mathcal{O}(-1)\cdot\alpha \subset \mathrm{Ker}\{d_{DR} : \Omega^1(0) \to \Omega^2(0)\}$ so that ${}'G_{0,1} = \mathcal{O}(-1)\cdot\alpha \equiv \Omega^1_{\mathcal{O}(0)} \equiv \Omega^1(S^*X)$.

On the other hand

$${}'G_{-1,2n} = \mathrm{Ker}\{d_{DR} : \Omega^1(-n) \longrightarrow \Omega^2(-n)\} = d\,\mathcal{O}(-n) \underset{(i_\Xi)}{\simeq} \mathcal{O}(-n)$$

and the latter space is canonically isomorphic to $\Omega^{vol}(S^*X)$ via the symplectic residue map Res_0 (cf. [NC·I, Section 1.17]).

Hence the top non-vanishing group $H_{2n-1}(CS^0(X), CS^0(X))$ is, for $n \neq 1$, of pure weight -1 and canonically isomorphic to $\Omega^{vol}(S^*X)$:

$$H_{2n-1}(CS^0(X), CS^0(X)) \simeq \Omega^{vol}(S^*X) \qquad (n \neq 1)$$

whereas for $n=1$ it is mixed of weights -1 and 0 and can be included into the extension

$$0 \longrightarrow \Omega^1(S^*X) \longrightarrow H_{2n-1}(CS^0(X), CS^0(X)) \longrightarrow \Omega^1(S^*X) \longrightarrow 0 \quad (n=1).$$

23. Let us proceed to cyclic homology.

The associated graded space $Gr^F_* HC_*(CS^0(X))$ of the order filtration is described by the following theorem.

24. THEOREM. (a) For $p \leq -1$ the natural map

$$Gr^F_p HC_*(CS^0(X)) \longrightarrow Gr^F_p HC_*(CS(X))$$

is an isomorphism in the range $p \leq -2$ and an epimorphism if $p = -1$.

In particular, we have canonical isomorphisms for $p \leq -2$:

$$Gr^F_p HC_k(CS^0(X), CS^0(X)) \simeq \begin{cases} H^{k-2p}_{DR}(Y^c), & p \leq k-n, \\ 0 & \text{otherwise}. \end{cases}$$

(b) For $k \geq 2n-1$ the space $HC_k(CS^0(X))$ is purely of weight 0 and the natural map

$$HC_k(CS^0(X)) \longrightarrow HC_k(\mathcal{O}(0))$$

induced by the homomorphism of principal symbol is an isomorphism.

In particular, for $k \geq 2n-1$ there are canonical isomorphisms

$$HC_k(CS^0(X)) \simeq \begin{cases} H_{DR}^{ev}(S^*X) & \text{if } k = \text{even}, \\ H_{DR}^{odd}(S^*X) & \text{if } k = \text{odd}. \end{cases}$$

(It will be convenient to provide the description of the terms $Gr_{-1}^F HC_k(CS^0(X))$ and $Gr_0^F HC_k(CS^0(X))$, for $k \leq 2n-2$, a bit later; cf. 26 and Remark-Corollary 30.)

25. The proof is formally very similar to that of Theorem 16. One considers s.s. $G_{**}^{(m),r}$ associated with the filtration on the quotient-complexes $\mathrm{Tot}\, \mathcal{B}_{**} / F_{m-1} \mathcal{B}_{**}$ and compares them to the corresponding s.s. $E_{**}^{(m),r}$ for the algebra $CS(X)$.

In order to prove that the natural map $G^{(m),1}_{p*} \longrightarrow E^{(m),1}_{p*}$ is, for $p \leq 0$, an isomorphism for all p except $p=0$ we consider the commutative diagram

$$\begin{tikzcd}
{} \arrow[r, "B"] & H_k(\mathcal{O},\mathcal{O})(p) \arrow[r, "I"] & HC_k(\mathcal{O})(p) \arrow[r, "S"] & HC_{k-2}(\mathcal{O})(p) \arrow[r, "B"] & H_{k-1}(\mathcal{O},\mathcal{O})(p \\
{} \arrow[r, "B"] & H_k(\mathcal{O}_-,\mathcal{O}_-)(p) \arrow[r, "I"] \arrow[u] & HC_k(\mathcal{O}_-)(p) \arrow[r, "S"] \arrow[u] & HC_{k-2}(\mathcal{O}_-)(p) \arrow[r, "B"] \arrow[u] & H_{k-1}(\mathcal{O}_-,\mathcal{O}_-)(\arrow[u]
\end{tikzcd}$$

whose horizontal rows are the p-components of Connes long exact sequence for $\mathcal{O}_-$ and $\mathcal{O}$ respectively and the vertical arrows are natural maps induced by the inclusion $\mathcal{O}_- \subset \mathcal{O}$. Since $H_*(\mathcal{O}_-,\mathcal{O}_-)(p) \longrightarrow H_*(\mathcal{O},\mathcal{O})(p)$ is an isomorphism, for $p \leq -1$, this implies that $HC_k(\mathcal{O}_-)(p) \xrightarrow{\sim} HC_k(\mathcal{O})(p)$, for $p \leq -1$, as well.

Then part (a) of the theorem follows in precisely the same manner as in the case of Hochschild homology. In particular, the s.s. $G^{(m),r}_{**}$ degenerate at 'E^2-term'.

The proof of part (b) is even easier.

It is clear that $G_{pq}^{(m),1} \equiv HC_{p+q}(\mathcal{O}_-)(p)$ vanishes in the region $p+q \geqslant 2n$ unless $p=0$. In the latter case one has

$$G_{0q}^{(m),1} = HC_q(\mathcal{O}(0)) \simeq H_{DR}^{(\tilde{q})}(S^*X) \qquad (q \geqslant 2n-1).$$

This automatically implies that $G_{0q}^{(m),\infty} = G_{0q}^{(m),1}$ for at least $q \geqslant 2n+1$ and that

$$HC_k(CS^0(X)) \xrightarrow{\sim} \varprojlim_m H_k(\operatorname{Tot}\mathcal{B}_{**}/F_{m-1}\mathcal{B}_{**}) \simeq G_{0q}^{(m),1} \qquad (m<0)$$

for at least $k \geqslant 2n+1$. This bound can be immediately lowered to $k \geqslant 2n-1$ by combining the relevant parts of Connes long exact sequences for $CS^0(X)$ and $\mathcal{O}(0)$ with the information on Hochschild homology of $CS^0(X)$:

$$\begin{tikzcd}
 & HC_{2n+2}(\mathcal{O}(0)) \arrow[r, "\sim"] & HC_{2n}(\mathcal{O}(0)) & 0 \\
0 \arrow[r] & HC_{2n+2}(CS^0(X)) \arrow[r, "\sim"] \arrow[u, "\wr"'] & HC_{2n}(CS^0(X)) \arrow[r] \arrow[u] & H_{2n+1}(CS^0(X), CS^0(X)) \arrow[u, equal]
\end{tikzcd}$$

$$\begin{tikzcd}
 & HC_{2n+1}(\mathcal{O}(0)) \arrow[r, "\sim"] & HC_{2n-1}(\mathcal{O}(0)) & \\
\arrow[r] & HC_{2n+1}(CS^0(X)) \arrow[r, "\sim"] \arrow[u, "\wr"'] & HC_{2n-1}(CS^0(X)) \arrow[r] \arrow[u] & H_{2n}(CS^0(X), CS^0(X)) = 0 \,.
\end{tikzcd}$$

26. Our proof of Theorem 24 shows that $Gr_p^F HC_k(CS^0(X))$ identifies with $G_{p,k-p}^{(m),2}$ if $m \ll 0$ (the latter doesn't depend on m already for $m < p$). Hence the task of description of $Gr_p^F HC_k(CS^0(X))$ for $p = -1$ or 0 reduces to determining $G_{p,k-p}^{(m),2}$ for $p = -1$ and 0. This will be preceded by the following important digression.

27. <u>'De Rham' and 'Spencer' pictures.</u>

With every C^∞ manifold M there are associated two canonically constructed complexes. One arises as a <u>subcomplex</u> of the Koszul complex of <u>cochains</u> of the Lie algebra $\mathcal{T}M$ of vector fields with coefficients in the $\mathcal{T}M$-module $C^\infty(M)$. The other arises as a <u>quotient</u> of the Koszul complex of <u>chains</u> of $\mathcal{T}M$ with coefficients in the (right) $\mathcal{T}M$-module $\Omega^{vol}(M)$.

The two constructions are fairly symmetric but, in fact, provide two different definitions of the same complex[6], once viewed as a cochain complex $(\Omega^*(M) = \Gamma(M, \Lambda^*\Omega^1), d)$, the other time as a chain complex $(S_*(M) = \Gamma(M, \Omega_M^{vol} \otimes \Lambda^* \mathcal{T}), \partial)$.

---

[6] namely de Rham complex; note, however, that these constructions are really different, e.g. if M is a supermanifold and Ω_M^{vol} is replaced by Ber_M which is a dualizing sheaf in that category, one obtains two totally different complexes.

The identifying isomorphism $\imath : (S_*(M), \partial) \xrightarrow{\sim}$ $(\Omega^{d-*}(M), d)$, $d = \dim M$, is given by

$$\imath \;:\; \nu \otimes \eta_1 \wedge \ldots \wedge \eta_q \longmapsto i_{\eta_1} \cdots i_{\eta_q} \nu .$$

(cf. similar considerations in [NC·I, 1.25]).

We can also take arbitrary identifications of the corresponding sheaves (i.e. of underlying vector bundles) $\mathcal{J} : \mathcal{T}_M \longrightarrow \Omega^1_M$ and $K : \Omega^{vol}_M \longrightarrow \mathcal{O}_M$ and use them to transport Spencer boundary ∂ on $\Omega^*(M)$.

More precisely, let $\mathcal{J}(\eta) \equiv \varphi(\cdot, \eta)$ and $K(\nu) \equiv \nu / V$ $(\eta \in \mathcal{T}M,\ \nu \in \Omega^{vol}(M))$ for a certain non-degenerate quadratic differential $\varphi \in \Gamma(M, (\Omega^1)^{\otimes 2})$ and a volume form V. Denote these data by $\phi = (\varphi, V)$ and the corresponding boundary operator on $\Omega^*(M)$ by ∂^ϕ. The latter clearly depends only on the class $\bar{\Phi} = (\varphi, [V])$ where $[V] \in \mathbb{P}(\Omega^{vol}(M))$ denotes the line containing V.

If (M, ω) is a symplectic manifold there is an obvious choice for ϕ, namely $\varphi = \omega$ and $V = \omega^{d/2}$.

One can verify directly that in this case ∂^ϕ anti-commutes with $d \equiv d_{DR}$. In fact, the converse is also true.

28. LEMMA. Assume that ∂^ϕ anti-commutes with d (it suffices to require this to hold on functions and 1-forms).

Then φ is a symplectic form (i.e. skew-symmetric, closed and non-degenerate) and $[v] = [\varphi^{d/2}]$.

Proof. Let $I_\varphi : \Omega^1_M \longrightarrow \mathcal{T}_M$ be $\mathcal{J}^{-1}$. We define using $\phi = (\varphi, [v])$ the corresponding notions of a 'Hamiltonian' vector field and of 'Poisson' bracket:

$$H_f := I_\varphi(df) \qquad\qquad (f \in \mathcal{O}(M))$$

and
$$\{f,g\} := \mathcal{L}_{H_f} g \equiv \varphi(H_f, H_g). \qquad (20)$$

We shall show that, in fact, bracket (20) defines a Lie algebra structure on $\mathcal{O}(M)$ (under the assumptions of the Lemma) and that $f \longmapsto H_f$ becomes a homomorphism of Lie algebras $(\mathcal{O}(M), \{\cdot,\cdot\}) \longrightarrow (\mathcal{T}M, [\cdot,\cdot])$.

First of all notice that

$$\partial^{\phi}(g\,df) = \frac{d(g\, i_{H_f} v)}{v} = \mathcal{L}_{H_f} g + g\, \mathrm{div}_v(H_f)$$

$$= \{f, g\} + g\, \mathrm{div}_v(H_f) \quad . \qquad (21)$$

The condition $0 = \partial^{\phi} d + d\partial^{\phi}$ on functions translates as :

for every Hamiltonian field H_f $\quad \mathrm{div}_v H_f \equiv 0$. $\qquad$ (22)

This combined with (21) gives

$$\partial^{\phi}(g\,df) = \{f, g\} \qquad (23)$$

and hence

$$0 = \partial^{\phi} d(fg) = \{f, g\} + \{g, f\} \;,$$

i.e. (20) , and therefore φ , are skew-symmetric.

The condition $0 = \partial^{\phi} d + d\partial^{\phi}$ on 1-forms combined with (22) and (23) translates as

$$H_{\{f,g\}} = [H_f, H_g]$$

i.e. $f \longmapsto H_f$ is a homomorphism of the algebraic structure $(\mathcal{O}(M), \{\ ,\ \})$ into the Lie algebra of vector fields $\mathcal{T}M$.

In particular, for every $f, g \in \mathcal{O}(M)$ we have $0 = i_{H_f} \mathcal{L}_{H_g} \varphi$ (ideed, the latter equals $d\{f,g\} + i_{[H_f, H_g]}\varphi$). Since 'Hamiltonian' fields generate $\mathcal{T}M$ as $\mathcal{O}(M)$-module this in turn means that $\mathcal{L}_{H_g}\varphi = 0$, for every $g \in \mathcal{O}(M)$. Finally,

$$0 = \mathcal{L}_{H_g}\varphi = i_{H_g} d\varphi + d\, i_{H_g}\varphi = i_{H_g} d\varphi - d^2 g = i_{H_g} d\varphi$$

and this holds for every $g \in \mathcal{O}(M)$, hence $d\varphi = 0$ and $(\mathcal{O}(M), \{\ ,\ \})$ is the associated (with φ) Poisson algebra.

Finally, let $h = v / \varphi^{d/2}$. Since $\mathrm{div}_v H_f = 0$ for every Hamiltonian vector field, the function h belongs to the center of the Poisson algebra and therefore is constant. ■

Because in what follows we deal exclusively with the symplectic case we shall denote the corresponding boundary map simply by ∂. It can be also defined by the 'symplectic' $*$ which is the composite $\Omega^*_M \xrightarrow{K^{-1}\otimes\Lambda^* I} S_{*,M} \xrightarrow{i} \Omega^{d-*}_M$. The $*$-automorphism interchanges (up to signs) ∂ and d. The prospects of playing with these anti-commuting boundary and coboundary maps were already noted in a different context by J.-L. Koszul and especially by J.-L. Brylinski (they both used different notation).

After this brief digression let us return to our main subject.

29. LEMMA. There are canonical isomorphisms

$$G^{(m),2}_{0q} \simeq \frac{\{\psi \in \Omega^{q}_{\mathcal{O}(0)} \mid \partial\psi \in d\Omega^{q-2}_{\mathcal{O}}(-1)\}}{d\Omega^{q-1}_{\mathcal{O}(0)}} \oplus H^{q-2}_{DR}(S^*X) \oplus H^{q-4}_{DR}(S^*X) \oplus \dots \qquad (24)$$

$(m<0)$

and

$$G^{(m),2}_{-1,q} \simeq \frac{\{\chi \in \Omega^{q-1}_{\mathcal{O}}(-1) \mid \partial\chi \in d\Omega^{q-3}_{\mathcal{O}}(-2)\}}{d\Omega^{q-2}_{\mathcal{O}}(-1) + \partial\Omega^{q}_{\mathcal{O}(0)}} \qquad (25)$$

$(m<0)$.

($\Omega^{*}_{\mathcal{O}(0)}$ is viewed here as sitting inside $\Omega^{*}_{\mathcal{O}}(0)$; notice also that $\partial : \Omega^{p+q}_{\mathcal{O}}(p) \longrightarrow \Omega^{p+q-1}_{\mathcal{O}}(p-1)$.)

30. REMARK - COROLLARY. By combining remarks in subsection 26 with Lemma 29 and using somewhat abbreviated notation we obtain the canonical isomorphisms

$$Gr^{F}_{0} HC_{q} \simeq \frac{\Omega^{q}_{\mathcal{O}(0)} \cap \partial^{-1} d\Omega^{q-2}_{\mathcal{O}}(-1)}{d\Omega^{q-1}_{\mathcal{O}(0)}} \oplus H^{q-2}_{DR}(S^*X) \oplus H^{q-4}_{DR}(S^*X) \oplus \dots$$

and

$$Gr^{F}_{1} HC_{q} \simeq \frac{\partial^{-1} d\Omega^{q-2}_{\mathcal{O}}(-1)}{d\Omega^{q-1}_{\mathcal{O}}(-1) + \partial\Omega^{q+1}_{\mathcal{O}(0)}}$$

($HC_q \equiv HC_q(CS^0(X))$). ■

Proof. Let us recall, first of all, the initial piece of the determination of $E^{(m),2}_{**}$ (cf. pp. 22-24 above).

We noticed that as a chain complex each $(E^{(m),1}_{*q}, d^1)$ decomposes into the direct sum

$$\overline{E}^{(m),1}_{*q} \oplus \left(H^{q-2}_{DR}(Y^c) \oplus H^{q-4}_{DR}(Y^c) \oplus \dots \right)_0$$

where the second summand is concentrated in dimension zero (cf. (10) above) and $\overline{E}^{(m),1}_{pq} = \Omega^{p+q}_{\mathcal{O}}(p) / d\Omega^{p+q-1}_{\mathcal{O}}(p)$.

Next, we noticed that we have the commutative diagrams

$$\begin{array}{ccc} {}'E^{(m),1}_{p-1,q} & \xleftarrow{d^1} & {}'E^{(m),1}_{pq} \\ \downarrow & & \downarrow \\ \overline{E}^{(m),1}_{p-1,q} & \xleftarrow{d^1} & \overline{E}^{(m),1}_{pq} \end{array} \qquad (26)$$

(cf. $(11)_k$ above). Recall that the upper arrow reads simply as

$$\Omega^{2n-p-q+1}_{\mathcal{O}}(n-q) \xleftarrow{d_{DR}} \Omega^{2n-p-q}_{\mathcal{O}}(n-q) \qquad (27)$$

(cf. the proof of Proposition 7). We can use also a symplectically dual picture (i.e. to apply to (27) the symplectic '$*$'). Then (26) reads simply as

$$
\begin{tikzcd}
\Omega_{\mathcal{O}}^{p+q-1}(p-1) \arrow[d] & \Omega_{\mathcal{O}}^{p+q}(p) \arrow[l, "d^1"'] \arrow[d] \\
\Omega_{\mathcal{O}}^{p+q-1}(p-1)/d\Omega_{\mathcal{O}}^{p+q-2}(p-1) & \Omega_{\mathcal{O}}^{p+q}(p)/d\Omega_{\mathcal{O}}^{p+q-1}(p) \arrow[l, "d^1"']
\end{tikzcd}
$$

where the upper d^1 now is (up to a sign) equal to ∂ (cf. also the definition of τ-map in the proof of 7) and the vertical arrows are canonical projections. In particular the bottom d^1 is simply the upper d^1 passed to quotients. It is clear then that

$$\bar{E}_{pq}^{(m),2} \simeq \frac{\{\psi \in \Omega_{\mathcal{O}}^{p+q}(p) \mid \partial\psi \in d\Omega_{\mathcal{O}}^{p+q-2}(p-1)\}}{d\Omega_{\mathcal{O}}^{p+q-1}(p) + \partial\Omega_{\mathcal{O}}^{p+q+1}(p+1)} . \tag{28}$$

Precisely the same argument applied to $CS^0(X)$ expresses $G_{pq}^{(m),2}$ as the direct sum

$$\bar{G}_{pq}^{(m),2} \oplus (H_{DR}^{q-2}(S^*X) \oplus H_{DR}^{q-4}(S^*X) \oplus \dots)_0$$

where $\bar{G}_{pq}^{(m),2}$ is given by the same expression (28) except that $\mathcal{O}$ is replaced by $\mathcal{O}_-$.

Then the formulae (24) and (25) follow immediately if one recalls that

$$\Omega_{\mathcal{O}_-}^{p+q}(p) = \begin{cases} \Omega_{\mathcal{O}}^{p+q}(p) & , \ p<0 \\ \Omega_{\mathcal{O}(0)}^{q} & , \ p=0 \\ 0 & , \ p>0 \end{cases} \quad . \blacksquare$$

31. REMARK. By combining (28) with the description of $\bar{E}_{**}^{(m),2}$ obtained before (see pp. 24 – 28, esp. p. 28) we obtain rather nontrivially looking isomorphisms:

$$\frac{\{\psi \in \Omega_{\mathcal{O}}^{p+q}(p) \mid \partial\psi \in d\Omega_{\mathcal{O}}^{p+q-2}(p-1)\}}{d\Omega_{\mathcal{O}}^{p+q-1}(p) + \partial\Omega_{\mathcal{O}}^{p+q+1}(p+1)} \simeq \begin{cases} H_{DR}^{q-p}(Y^c) & , \ p\leq 0 \, , \, q \geq n, \\ H^{q-p+1}(Y^c) & , \ p\geq 2 \, , \, q \leq n-1, \\ 0 & \text{otherwise} \, . \end{cases} \tag{2$$

This in turn allows to simplify (25) by expressing $G_{-1,q}^{(m),2}$ as an extension of $E_{-1,q}^{(m),2}$ (given by (29) above) with the kernel

$$\frac{d\Omega_{\mathcal{O}}^{q-2}(-1) + \partial\Omega_{\mathcal{O}}^{q}(0)}{d\Omega_{\mathcal{O}}^{q-2}(-1) + \partial\Omega_{\mathcal{O}(0)}^{q}} \simeq \frac{\partial\Omega_{\mathcal{O}}^{q}(0)}{[\,d\Omega_{\mathcal{O}}^{q-2}(-1) \cap \partial\Omega_{\mathcal{O}}^{q}(0)\,] + \partial\Omega_{\mathcal{O}(0)}^{q}} \quad . \tag{30}$$

It may be sometimes convenient to write (30) also in the symplectically dual form

$$\frac{d\Omega_{\mathcal{O}}^{2n-q}(n-q)}{[\partial\Omega_{\mathcal{O}}^{2n+2-q}(n+1-q)\cap d\Omega_{\mathcal{O}}^{2n-q}(n-q)] + d[\alpha\wedge\Omega_{\mathcal{O}}^{2n-1-q}(n-1-q)]} \tag{31}$$

For example, (31) combined with (29) gives at once that $G_{-1,2n}^{(m),2}$ vanishes (as do all $G_{-1,q}^{(m),2}$ with $q>2n$) and that the top non-vanishing term is $G_{-1,2n-1}^{(m),2}$; the latter is included in the following extension:

$$\longrightarrow \frac{d\Omega_{\mathcal{O}}^{1}(1-n)}{[\partial\Omega_{\mathcal{O}}^{3}(2-n)\cap d\Omega_{\mathcal{O}}^{1}(1-n)] + d(\mathcal{O}(-n)\cdot\alpha)} \longrightarrow G_{-1,2n-1}^{(m),2} \longrightarrow H_{DR}^{2n}(Y^{c}) \longrightarrow \tag{32}$$

It is also clear that the game involving these two pictures ('de Rham' and 'Spencer') being switched by the symplectic '$*$' is 'responsible' for the symmetries in terms $E_{**}^{(m),2}$ and $G_{**}^{(m),2}$.

32. REMARK. As in the case of symbols of unbounded order our determination of Hochschild and cyclic homology is valid for the algebra CS^0_Y of symbols associated with a general conical submanifold Y of $T^*X \setminus X$. (In the corr. statements Y^c has an obvious meaning while S^*X should be replaced by the base $Z \subset S^*X$ of the cone Y).

33. EXAMPLE. $\dim X = 1$.

Assume that the orientation is chosen (this is equivalent to distinguishing between d/dx and $-d/dx$) and consider the subalgebra $CS^0_+(X) \subset CS^0(X)$ consisting of symbols of the form [7]

$$a_0 + a_{-1}\partial_x^{-1} + a_{-2}\partial_x^{-2} + \ldots$$

where $a_{-j} \in C^\infty(X) \; (\equiv \mathcal{O}(0))$ and $\partial_x = d/dx$.

Let $H_* \equiv H_*(CS^0_+(X), CS^0_+(X))$. Formulae (15)-(16) turn into:

$${}'G_{00} = \alpha \wedge \Omega^1_{\mathcal{O}}(0) = \Omega^{vol}_{\mathcal{O}}(1) \underset{(*)}{\simeq} \mathcal{O}(0) = C^\infty(X),$$

[7] an alternative approach is to view $CS^0_+(X) \subset CS^0(X)$ as the algebra of $\mathbb{R}^\times$-homogeneous symbols, then, in what follows, X will arise as $\mathbb{P}(T^*)$ and not as the 'positive' component of S^*X.

$$'G_{01} = \mathcal{O}(-1)\cdot\alpha \cap \operatorname{Ker}\{d_{DR} : \Omega^1_{\mathcal{O}}(0) \longrightarrow \Omega^2_{\mathcal{O}}(0)\}$$

$$= \Omega^1_{\mathcal{O}(0)} \cap \operatorname{Ker}\{d_{DR} : \Omega^1_{\mathcal{O}}(0) \longrightarrow \Omega^2_{\mathcal{O}}(0)\}$$

$$= \Omega^1_{\mathcal{O}(0)} = \Omega^1(X) ,$$

$$'G_{-1,1} = \Omega^2_{\mathcal{O}}(0) / d(\mathcal{O}(-1)\cdot\alpha) = \Omega^2_{\mathcal{O}}(0)/d\,\Omega^1_{\mathcal{O}(0)}$$

$$= \Omega^2_{\mathcal{O}}(0) \underset{(*)}{\simeq} \mathcal{O}(-1) \underset{Res_0}{\simeq} \Omega^1(X) ,$$

$$'G_{-1,2} = \operatorname{Ker}\{d_{DR} : \Omega^1_{\mathcal{O}}(-1) \longrightarrow \Omega^2_{\mathcal{O}}(-1)\}$$

$$= d\,\mathcal{O}(-1) \underset{-i_{\Xi}}{\simeq} \mathcal{O}(-1) \underset{Res_0}{\simeq} \Omega^1(X)$$

and other $'G_{pq}$'s ($p = 0$ or -1) vanish. Hence Theorem 16 gives us the following answer:

The groups $H_q \equiv H_q\,(CS^0_+(X), CS^0_+(X))$, for $q = 0$ and 1, are included into the extensions:

$$0 \longrightarrow \Omega^1(X) \longrightarrow H_0 \xrightarrow{\sigma} C^\infty(X) \longrightarrow 0 \qquad (33)$$

and

$$0 \longrightarrow \Omega^1(X) \longrightarrow H_1 \xrightarrow{\sigma} \Omega^1(X) \longrightarrow 0 \qquad (34)$$

where σ denotes the map induced by principal symbol. For $q \geq 2$ $H_q(CS^0_+(X), CS^0_+(X))$ vanish.

The situation concerning cyclic homology of $CS^0_+(X)$ is even simpler for Theorem 24(b) says that principal symbol induces the isomorphism

$$\sigma : HC_q(CS^0_+(X)) \xrightarrow{\sim} HC_q(C^\infty(X)) \simeq \begin{cases} H^0_{DR}(X), & q = \text{even}, \\ H^1_{DR}(X), & q = \text{odd} \end{cases}$$

if $q \geq 1$. On the other hand $HC_0(CS^0_+(X))$ which, of course, coincides with $H_0(CS^0_+(X), CS^0_+(X))$ includes into the extension (33).

34. REMARK. It follows from our results that the ideal $CS^{-1-n}(X)$ of symbols of ψDOs belonging to the trace class gives no direct contribution to homology of $CS^0(X)$. Indeed, both $H_*(CS^{-1-n}(X), CS^{-1-n}(X)) \longrightarrow H_*(CS^0(X), CS^0(X))$ and $HC_*(CS^{-1-n}(X)) \longrightarrow HC_*(CS^0(X))$ are simply zero maps. In particular, the natural map $CL^0_{tr.class} \longrightarrow CL^0/L^{-\infty} \equiv CS^0$ always induces zero in homology. This contrasts with the

well known fact that several non-trivial classes over $CS^0(X)$ (or its quotient-algebras) arise in connection with 'transgressing' certain classes from $CL_{tr.\,class}(X)$.

35. REMARK. As we already know Lie algebra homology of $gl_\infty(CS^0(X))$ it would be very interesting to compare that with homology of the Lie algebra $\mathfrak{f}_\infty$ of 'stable infinitesimal Fourier integral operators'. The latter is defined as $\mathfrak{f}_\infty = \varinjlim_r \mathfrak{f}_r$ where $\mathfrak{f}_r = CS^1_{scal}(X, \theta^r) \subset Mat_r(CS^1(X))$ consists of 1-symbols whose principal symbols are scalar (in fact, $\mathfrak{f}_\infty$ is rather the 'complexification' of infinitesimal FIOs, but this doesn't matter).

Unfortunately, we do not know much about homology of $\mathfrak{f}_\infty$ (except dimension 1 where the answer can be obtained in a closed nice way (e.g. $\mathfrak{f}_1$ has 13 (sic!) nontrivial central extensions when $X = S^1$)).

Since $\mathfrak{f}_\infty$ is an extension of the algebra $\mathfrak{p}^1$ of (complexified) infinitesimal homogeneous canonical automorphisms of $T^*X \setminus X$ with kernel $\mathfrak{gl}_\infty(CS^0(X))$:

$$0 \longrightarrow \mathfrak{gl}_\infty(CS^0(X)) \longrightarrow \mathfrak{f}_\infty \longrightarrow \mathfrak{p}^1 \longrightarrow 0 \qquad (35)$$

we have an obvious Hochschild - Serre[8] s.s. converging to $H_*(\mathfrak{f}_\infty)$ with $E^2_{pq} = H_p(\mathfrak{p}^1; H_q\, \mathfrak{gl}_\infty(CS^0(X)))$. Unfortunately it seems that the situation with homology of $\mathfrak{p}^1$ is not promising either. It is clear at least that the structure of $H_*\mathfrak{gl}_\infty(CS^0(X))$ as a $\mathfrak{p}^1$-module shouldn't be difficult to determine (e.g. $\mathfrak{p}^1$ acts trivially on a 'big' piece of $H_*\mathfrak{gl}_\infty(CS^0(X))$ generated by a homotopy invariant part of $HC_*(CS^0(X))$ (that includes all stable cyclic homology classes and all classes of filtration ≤ -2)).

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8) (35) splits in the category of l.c. spaces.

Princeton, 14 February 1987

Dear Professor Connes,

Prompted by an interest aroused by the publication of Chapter I of 'Noncommutative Residue' ([NC I]) I decided to write a preliminary report on my work on cyclic homology of symbols and to send it to you. These notes contain also rather detailed explanations of points touched in complementary remarks in the preprint 'Cyclic Homology of Differential Operators' ([CHDO]) sent to you some time ago.

The notes consist roughly speaking of two parts. In the first one I prove that the Hochschild homology $H_q(CS(X), CS(X))$ can be canonically identified with $H_{DR}^{2\dim X - q}(Y^c)$ where Y^c denotes a 'complexification' of the $\mathbb{R}_+^\times$-action on $Y = T^*X \setminus X$ (see Theorem 3 of the notes), and that the first spectral sequence of your double complex $(\mathcal{B}_{**}, b, B)$ which converges to $HC_*(CS(X))$ degenerates at E^1 (in the notes an equivalent assertion is proved, see Theorem 4). In the second part, bearing somewhat misleading heading 'Final Remarks' I show, in particular, how to determine a precise structure of the graded vector space associated with the filtration on $HC_*(CS(X))$ induced by order of symbol (see Remark 14.c), ibid.).

Homology considered in the notes is, of course, a kind of continuous homology. The character of the topology on the algebra of symbols of un-bounded order makes it, however, impossible to use $\hat{\otimes}_\pi$-completions (the composition law in $CS(X)$ is not continuous in both arguments). The completion which I choosed is intermediate between $\bar{\otimes}_i$- and $\hat{\otimes}_\pi$-completions and seems to be best suited to spaces with such a mixture of inductive and projective topology as that of $CS(X)$ (in fact, I used this completion, in a special case, already in [NC·I]). When restricted to the subalgebra of symbols of order zero $CS^0(X)$ it reduces to $\hat{\otimes}_\pi$-completion; restricted to differential operators coincides with $\bar{\otimes}_i$-completion (if X is compact the latter doesn't differ from $\hat{\otimes}_\pi$-completion).

The obtained answers seem to clarify several issues, e.g. how the mechanism of noncommutative residue matches a more common picture of homological invariants of symbols. An overall inference from them is that cyclic homology of symbols and closely related algebras drastically simplifies in the stable range (the latter always means: in dimensions $\geqslant 2\dim X - 1$), it becomes strictly periodic, 'pure' of weight zero (in the sense of order filtrations) and identifies with cyclic homology of the <u>commutative</u> algebra of functions on the corresponding space (Y^c in the case of $CS(X)$, S^*X in the case of $CS^0(X)$ and T^*X in the case of $\mathcal{D}(X)$).

In particular, your constructions of cyclic cocycles on S^*X <u>in the stable range</u> are reflected in the fact that the principal symbol map $CS^0(X) \longrightarrow C^\infty(S^*X)$ induces an isomorphism in cyclic homology precisely in the same range.

Most of the second part of notes deals with 'complexity' of the unstable range. What it reveals, however, is an ordered structure, much richer and looking even more interesting than that on stable homology.

I found it convenient to enclose another copy of [CHDO] with a few misprints corrected and complementary Remarks made more precise in order to make them better correspond to the material contained in the handwritten notes.

Sincerely yours,

Mariusz Wodzicki