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CYCLIC HOMOLOGY OF DIFFERENTIAL OPERATORS

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To Dear Yuri Ivanovich on His Fiftieth Birthday

1. Let $\mathcal{D}(X)$ denote the k-algebra of differential operators on a smooth manifold X in one of the following categories: algebraic, holomorphic or C^∞. In the first case X has to be an affine variety over the ground field k of characteristic zero, in the second case a Stein manifold ($k = \mathbb{C}$), assumed, for simplicity, to possess finitely many connected components, and in the last case a compact C^∞-manifold (possibly with boundary or nonorientable; $k = \mathbb{R}$ or $\mathbb{C}$). The purpose of this article is to determine Hochschild and cyclic homology of $\mathcal{D}(X)$ denoted, respectively, $H_*(\mathcal{D}(X), \mathcal{D}(X))$ and $HC_*(\mathcal{D}(X))$. In the holomorphic and C^∞ settings, $\mathcal{D}(X)$ is naturally a locally convex algebra with respect to $\hat{\otimes}_\pi$-tensor product, and the groups above mean the corresponding *topological* homology groups. For basic definitions and properties of cyclic homology see [5] and for basics on locally convex homological algebra consult [4] and [7].

2. THEOREM.

$$H_q(\mathcal{D}(X), \mathcal{D}(X)) \simeq H_{\mathrm{DR}}^{2n-q}(X) \qquad (q \in \mathbb{N};\ n = \dim X). \tag{1}$$

3. THEOREM.

$$HC_q(\mathcal{D}(X)) \simeq H_{\mathrm{DR}}^{2n-q}(X) \oplus H_{\mathrm{DR}}^{2n-q+2}(X) \oplus H_{\mathrm{DR}}^{2n-q+4}(X) \oplus \cdots \qquad (q \in \mathbb{N}). \tag{2}$$

4. Remark. In proof of the holomorphic case of Theorem 3 we shall assume, for simplicity, that $H_{\mathrm{DR}}^*(X)$ is finite-dimensional; the similar condition automatically holds in the two remaining cases.

The isomorphisms in (1) are canonical and functorial with respect to embeddings of codimension zero. The proof of Theorem 3 which is presented below will provide similarly functorial isomorphisms in (2), for $q \geqslant 2n - 1$. The existence of *canonical* isomorphisms in the "unstable" range $q < 2n - 1$ can be proved as well, at least in C^∞ case, but requires stronger means (cf. Remarks 8 and 13.1 below).

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5. Recall that for a general algebra $\mathscr{A}$ the groups $H_*(\mathscr{A}, \mathscr{A})$ and $HC_*(\mathscr{A})$ are related to each other by Connes long exact sequence (cf., e.g., [5, Thm. 1.6])

$$\cdots \to H_q(\mathscr{A}, \mathscr{A}) \xrightarrow{I} HC_q(\mathscr{A}) \xrightarrow{S} HC_{q-2}(\mathscr{A}) \xrightarrow{B} H_{q-1}(\mathscr{A}, \mathscr{A}) \to \cdots. \tag{3}$$

Assume that $\dim_{\Bbbk} H_*(\mathscr{A}, \mathscr{A}) < \infty$. Then the following simple lemma holds.

6. Lemma. *One has*

$$\dim HC_q(\mathscr{A}) \leqslant \sum_{i=0}^{[q/2]} \dim H_{q-2i}(\mathscr{A}, \mathscr{A}) \tag{4}$$

for every q. If there is equality in (4) *for $q \gg 0$ (suffices for just one q) sequence* (3) *splits into the short exact sequences*

$$0 \to H_k(\mathscr{A}, \mathscr{A}) \xrightarrow{I} HC_k(\mathscr{A}) \xrightarrow{S} HC_{k-2}(\mathscr{A}) \to 0 \qquad (k \in \mathbb{N}),$$

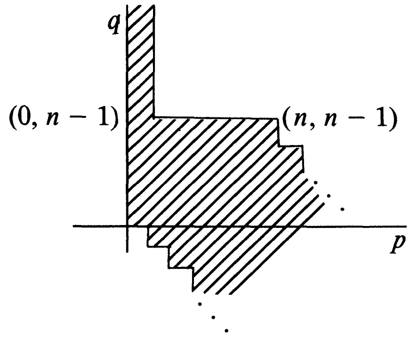
and equality holds in (4) *for all q.*

7. Recall that $HC_*(\mathscr{A})$ can be obtained as homology of $\operatorname{Tot} \mathscr{B}_{**}(\mathscr{A})$ where $(\mathscr{B}_{**}(\mathscr{A}), b, B)$ is Connes double complex ([5, (1.8)]). One has $\mathscr{B}_{kl}(\mathscr{A}) = \mathscr{C}_{l-k}(\mathscr{A}, \mathscr{A})$ $(k, l \geqslant 0)$ where $\mathscr{C}_j(\mathscr{A}, \mathscr{A}) = \mathscr{A} \hat{\otimes} \cdots \hat{\otimes} \mathscr{A}$ ($j + 1$ times). Here $\hat{\otimes}$ denotes $\hat{\otimes}_\pi$ if $\mathscr{A}$ is a locally convex $\hat{\otimes}_\pi$-algebra, and the ordinary (inductive) tensor product if $\mathscr{A}$ has no topology.

Assume that $\mathscr{A}$ is filtered: $\{0\} \subset \mathscr{A}^0 \subset \cdots \subset \mathscr{A}^p \subset \cdots$ $(p \in \mathbb{N})$, and $\mathscr{A}^i \mathscr{A}^j \subset \mathscr{A}^{i+j}$. This induces the filtrations $\{0\} \subset \mathscr{C}_j^0 \subset \cdots \subset \mathscr{C}_j^p \subset \cdots$ on spaces $\mathscr{C}_j(\mathscr{A}, \mathscr{A})$ which are defined by images in $\mathscr{C}_j(\mathscr{A}, \mathscr{A})$ of the spaces $\bigoplus (\mathscr{A}^{i_0} \hat{\otimes} \cdots \hat{\otimes} \mathscr{A}^{i_j})$ (summation over $i_0 + \cdots + i_j = p$). It is clear from the corresponding definitions that both Hochschild and Connes boundary maps (denoted above b and B respectively) preserve these filtrations.

If $\mathscr{A}$ is topologized we shall assume in addition that the canonical maps $\varinjlim_p \mathscr{C}_j^p \to \mathscr{C}_j(\mathscr{A}, \mathscr{A})$ are isomorphisms for all j. Then the spectral sequence E_{pq}^r associated with the considered filtration on $\operatorname{Tot} \mathscr{B}_{**}(\mathscr{A})$ *converges* to $HC_{p+q}(\mathscr{A})$. Similarly, the spectral sequence $'E_{pq}^r$ associated with the filtration on $(\mathscr{C}_*(\mathscr{A}, \mathscr{A}), b)$ *converges* to $H_{p+q}(\mathscr{A}, \mathscr{A})$.

Reduction of Theorem 3 *to Theorem* 2. The above remarks apply to $\mathscr{A} = \mathscr{D}(X)$ filtered by order of operator in any of the three cases quoted at the beginning. The spectral sequence $E_{pq}^r \Rightarrow HC_{p+q}(\mathscr{D}(X))$ is a priori located in the region $(p \geqslant 0$ and $p + q \geqslant 0)$. In fact, we shall see that E_{pq}^r $(r \geqslant 1)$ vanish outside the region shown below



i.e., $E^r_{pq} = 0$ if either $p \geqslant 1$ and $q \geqslant n$ or $p \geqslant 1$ and $p + q \geqslant 2n$.

Indeed, $\operatorname{gr} \mathscr{D}(X)$ is the graded algebra $\mathscr{O} = \oplus_{p=0}^{\infty} \mathscr{O}(p)$ of functions on $\mathscr{T}^*X$ polynomial along fibres of $\mathscr{T}^*X \to X$ and algebraic, holomorphic, or C^∞ in the X-direction (depending on the case). Thus $E^1_{pq} = HC_{p+q}(\mathscr{O})(p) = H_{p+q}(\operatorname{Tot} \mathscr{B}_{**}(\mathscr{O})(p))$. One can show that, for every $p \in \mathbb{N}$, the first spectral sequence (corresponding to filtering by columns) of the double complex $\mathscr{B}_{**}(\mathscr{O})(p)$ degenerates at E^2-term (for an algebraic X this is Theorem 2.9 of [5] plus the considerations which precede it). This yields, in particular, that

$$HC_{p+q}(\mathscr{O})(p) \simeq \Omega_{\mathscr{O}}^{p+q}(p)/d\Omega_{\mathscr{O}}^{p+q-1}(p) \qquad (p \geqslant 1) \tag{5}$$

(for brevity, $\Omega^*_{\mathscr{O}}$ will denote $\Omega^*_{\mathscr{O}/k}$). Since the right-hand side of (5) vanishes for $q \geqslant n$, as well as for $p + q \geqslant 2n$, we obtain the required location region for nonvanishing terms of E^1_{pq}.

The same spectral sequence for $\mathscr{B}_{**}(\mathscr{O})(p)$ yields, if $p = 0$, that

$$E^1_{0q} = HC_q(\mathscr{O})(0) = HC_q(\mathscr{O}(0)) = H^{(\tilde{q})}_{\mathrm{DR}}(X) \qquad (q \geqslant n)$$

where $\tilde{q} =$ parity of q and $H^{(\varepsilon)}(X)$ denotes the standard $\mathbb{Z}/2$-grading of $H^*_{\mathrm{DR}}(X)$. In view of the shape of $E^1{}_{**}$, the terms E^1_{0q} $(q \geqslant 2n - 1)$ survive to infinity and we have $HC_q(\mathscr{D}(X)) \simeq H^{(\tilde{q})}_{\mathrm{DR}}(X)$ $(q \geqslant 2n - 1)$. By comparing this with the statement of Theorem 2, we obtain equality in (4) for $q \geqslant 2n - 1$. An application of Lemma 6 then finishes the proof of Theorem 3.

8. Remark. As a corollary of the proof, we obtain that the natural embedding of the algebra $\mathscr{O}(X)$ of functions on X viewed as differential operators of order zero induces an isomorphism in cyclic homology $HC_q(\mathscr{O}(X)) \xrightarrow{\sim} HC_q(\mathscr{D}(X))$ for $q \geqslant 2n - 1$. This is an "additive analog" of the theorem of D. Quillen saying that $\mathscr{O}(X) \hookrightarrow \mathscr{D}(X)$ induces an isomorphism in algebraic K-theory (for X smooth affine). In the unstable range $q < 2n - 1$ the maps $HC_q(\mathscr{O}(X)) \to HC_q(\mathscr{D}(X))$

are always surjective, and usually with nontrivial kernel, as shows an easy inductive argument (q going from the stable range to zero) which involves comparing Connes exact sequences for $\mathcal{O}(X)$ and $\mathcal{D}(X)$.

Note that our proof does not necessitate information about the behavior of the spectral sequence *inside* the marked region. This additional information, however, can be useful, e.g., in proving that the surjections $HC_q(\mathcal{O}(X)) \to HC_q(\mathcal{D}(X))$ split or, equivalently, that the short exact sequences

$$0 \to H_q(\mathcal{D}(X), \mathcal{D}(X)) \to HC_q(\mathcal{D}(X)) \to HC_{q-2}(\mathcal{D}(X)) \to 0$$

admit a canonical splitting (cf. Remark 13.i below), for all q, therefore giving *canonical* identifications in (2).

Theorem 2 is a corollary of the proposition describing the term E^1 of the spectral sequence $'E^r_{pq} \Rightarrow H_{p+q}(\mathcal{D}(X), \mathcal{D}(X))$ of subsection 7 above.

9. PROPOSITION. *There is a natural identification* $'E^1_{pq} \simeq \Omega^{2n-p-q}_{\mathcal{O}}(n-q)$ ($p \geqslant 0$ *and* $p+q \geqslant 0$). *Under this identification* d^1_{pq} *corresponds to de Rham differential* $d_{\mathrm{DR}}\colon \Omega^{2n-p-q}_{\mathcal{O}}(n-q) \to \Omega^{2n-p-q+1}_{\mathcal{O}}(n-q)$.

10. COROLLARY. $'E^2_{pq} \simeq H^{2n-p-q}_{\mathrm{DR}}(X)$, *if* $q = n$, *otherwise* $'E^2_{pq} = 0$. *In particular,* $'E^r_{pq}$ *degenerates at* E^2*-term, and* $H_k(\mathcal{D}(X), \mathcal{D}(X)) \simeq {}'E^2_{k-n,n} \simeq H^{2n-k}_{\mathrm{DR}}(X)$.

(Notice that $\Omega^{2n-p-q}_{\mathcal{O}}(n-q) = 0$ for $p < 0$.)

We shall need one general construction. For a given algebra $\mathcal{A}$ and an $\mathcal{A}$-bimodule $\mathcal{M}$ let $\mathfrak{a}$ denote $\mathcal{A}$ regarded as a Lie algebra, and $\mathfrak{m}$ denote $\mathcal{M}$ as a *right* $\mathfrak{a}$-module (i.e. $m \cdot a \equiv ma - am$). *Let* $(C_*(\mathfrak{a}; \mathfrak{m}), \partial)$ *be the standard Koszul chain complex*: $C_q(\mathfrak{a}; \mathfrak{m}) = \mathfrak{m} \hat{\otimes} \Lambda^q \mathfrak{a}$, *and*

$$\begin{aligned}\partial(m \otimes a_1 \wedge \cdots \wedge a_q) &= \sum_{1 \leqslant i < j \leqslant q} (-1)^{i+j} m \otimes [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge \widehat{a_j} \wedge \cdots \wedge a_q \\ &\quad + \sum_{1 \leqslant i \leqslant q} (-1)^{i-1} m \cdot a_i \otimes a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_q. \qquad (6)\end{aligned}$$

Define a set-theoretic embedding $\eta\colon C_*(\mathfrak{a}; \mathfrak{m}) \to \mathcal{C}_*(\mathcal{A}, \mathcal{M})$ by

$$m \otimes a_1 \wedge \cdots \wedge a_q \mapsto \sum_{\sigma \in \mathcal{S}_q} (\operatorname{sign} \sigma) m \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(q)}, \qquad (7)$$

and let $\mathcal{L}_*(\mathcal{A}, \mathcal{M}) = \eta C_*(\mathfrak{a}; \mathfrak{m})$.

11. LEMMA. *One has* $b\eta = \eta\partial$, *i.e., Hochschild boundary induces Lie algebra boundary* (6) *on* $\mathcal{L}_*(\mathcal{A}, \mathcal{M})$.

12. Remark. The induced map η: $H_*(\mathfrak{a}; \mathrm{ad}) \to H_*(\mathcal{A}, \mathcal{A})$ coincides with the edge homomorphism of a certain spectral sequence $''E^2_{pq} = H_p(\mathcal{A}, H_q(\mathfrak{a}; \mathcal{A} \hat{\otimes} \mathcal{A})) \Rightarrow H_{p+q}(\mathfrak{a}; \mathrm{ad})$ where $\mathcal{A} \hat{\otimes} \mathcal{A}$ is an $\mathcal{A}$-bimodule via the action: $a'(a_1 \otimes a_2)a'' = a'a_1 \otimes a_2a''$, and a right $\mathfrak{a}$-module via: $(a_1 \otimes a_2) \cdot a = a_1a \otimes a_2 - a_1 \otimes aa_2$. There is a similar interpretation of η: $H_*(\mathfrak{a}; \mathfrak{m}) \to H_*(\mathcal{A}, \mathcal{M})$ for a general $\mathcal{A}$-bimodule $\mathcal{M}$.

Proof of Proposition 9. We have $'E^1_{pq} = H_{p+q}(\mathcal{O}, \mathcal{O})(p)$—the component of weight p of the Hochschild homology of the graded algebra $\mathcal{O}$. Let $\omega \in \Omega^2_{\mathcal{O}}(1)$ be the canonical symplectic form. For $\mathscr{f} \in \mathcal{O}$ we denote by $i_{\mathscr{f}}$ interior product with the Hamiltonian vector field corresponding to $\mathscr{f}$. Then the correspondence

$$\mathscr{f}_0 \otimes \mathscr{f}_1 \otimes \cdots \otimes \mathscr{f}_j \mapsto \frac{(-1)^j}{j!} \mathscr{f}_0 i_{\mathscr{f}_1} \dots i_{\mathscr{f}_j} \omega^n \qquad (j \in \mathbb{N})$$

defines surjections τ^p_j: $\mathcal{C}_j(\mathcal{O}, \mathcal{O})(p) \to \Omega^{2n-j}_{\mathcal{O}}(p - j + n)$. One verifies directly that $\tau b = 0$. One can also show that $\operatorname{Ker} \tau^p_j$ consists entirely of boundaries (in the algebraic case this is essentially the dual formulation of the Hochschild-Kostant-Rosenberg theorem [3, Thm. 3.1]; the other two cases are more subtle but not difficult). Thus τ^p_{p+q} induce identifications $'E^1_{pq} \simeq \Omega^{2n-p-q}_{\mathcal{O}}(p)$. In these terms $\mathscr{d}^1_{pq}$ are determined as follows. Notice that already the restriction of τ^p_{p+q} to $\mathcal{L}_{p+q}(\mathcal{O}, \mathcal{O})(p)$ is a surjection onto $\Omega^{2n-p-q}_{\mathcal{O}}(n - q)$. Denote this by $\bar{\tau}^p_{p+q}$. One has $\mathcal{L}_*(\mathcal{O}, \mathcal{O}) = \operatorname{gr} \mathcal{L}_*(\mathcal{D}(X), \mathcal{D}(X))$, and by Lemma 11 we know that Hochschild boundary induces on $\mathcal{L}_*(\mathcal{D}(X), \mathcal{D}(X))$ the Lie algebra boundary (6). Recall that commutator of two operators of orders, say, l and m modulo operators of order $l + m - 2$ induces on $\mathcal{O}$ the structure of the graded Poisson algebra (denote it by $\mathcal{P}$). Now, it should be clear that $\mathscr{d}^1_{pq}$ on $\Omega^{2n-p-q}_{\mathcal{O}}(n - q)$ is the corresponding homogeneous component of the projection of the boundary homomorphism in the Koszul complex $C_*(\mathcal{P}; \mathrm{ad})$, under the composite map $C_*(\mathcal{P}; \mathrm{ad}) \xrightarrow{\eta} \mathcal{L}_*(\mathcal{O}, \mathcal{O}) \xrightarrow{\bar{\tau}} \Omega_{\mathcal{O}, *}$ where $\Omega_{\mathcal{O}, *}$ denotes de Rham complex with the dual grading $\Omega_{\mathcal{O}, k} \equiv \Omega^{2n-k}_{\mathcal{O}}$ and η is the identification (7). This composite map is, in fact, a morphism of complexes (we proved this, e.g., in a slightly different context in [8, 1.25 and 1.19]; mind there the opposite sign convention for the boundary in Koszul complex). In particular, $\mathscr{d}^1_{pq}$ identifies with de Rham differential $\mathscr{d}_{\mathrm{DR}}$: $\Omega^{2n-p-q}_{\mathcal{O}}(n - q) \to \Omega^{2n-p-q+1}_{\mathcal{O}}(n - q)$. ∎

13. Final remarks. (i) It follows from the proof of Theorem 2 that the homomorphism $H_q(\mathcal{O}(X), \mathcal{O}(X)) \to H_q(\mathcal{D}(X), \mathcal{D}(X))$ induced by the natural embedding $\mathcal{O}(X) \hookrightarrow \mathcal{D}(X)$ is zero except $q = n$ where it corresponds to the projection $\Omega^n_{\mathcal{O}} \to H^n_{\mathrm{DR}}(X)$.

It is worth noting that Proposition 9 plus a rather substantial amount of diagram chasing plus an argument involving "counting dimensions" similar to that of Lemma 6 yield together an almost complete description of the spectral sequence $E^r_{pq} \Rightarrow HC_{p+q}(\mathscr{D}(X), \mathscr{D}(X))$ used previously to prove Theorem 3.

This can be summarized as follows. The term E^2 is given by

$$E^2_{pq} \simeq \begin{cases} H^q_{\mathrm{DR}} \oplus H^{q-2}_{\mathrm{DR}} \oplus \cdots, & p = 0, q \geqslant n, \\ H^{q-2}_{\mathrm{DR}} \oplus H^{q-4}_{\mathrm{DR}} \oplus \cdots, & p = 0, q < n, \\ H^{q-p+1}_{\mathrm{DR}}, & 2 \leqslant p \leqslant n, q < n, \\ 0, & \text{otherwise} \end{cases}$$

($H^*_{\mathrm{DR}} \equiv H^*_{\mathrm{DR}}(X)$). The only nontrivial differentials $\mathscr{d}^r_{pq}$ ($r \geqslant 2$) are $\mathscr{d}^p_{pq}$: $E^p_{pq} \to E^p_{0,p+q-1}$ which inject $E^p_{pq} = E^2_{pq} = H^{q-p+1}_{\mathrm{DR}}(X)$ into $E^p_{0,p+q-1}$. In particular, E^r_{**} never stops earlier than at E^{n+1} (the "last" differential being $\mathscr{d}^n_{n,n-1}$: $H^0_{\mathrm{DR}}(X) = E^2_{n,n-1} = E^n_{n,n-1} \rightarrowtail E^n_{0,2n-2}$), and the term $E^\infty_{**} = E^{n+1}_{**}$ vanishes except $E^\infty_{0,*}$. A slightly more precise information about differentials[1] is necessary to conclude that the composition of the inclusion

$$H^{2n-q}_{\mathrm{DR}} \oplus H^{2n-q+2}_{\mathrm{DR}} \oplus \cdots \hookrightarrow H^{(\tilde{q})}_{\mathrm{DR}} \simeq E^2_{0q} \qquad (q \geqslant n)$$

with the canonical projection $E^2_{0q} \to E^\infty_{0q}$ is, in fact, an isomorphism. Since $E^2_{0q} \equiv HC_q(\mathscr{O}(X))$, this gives, in equivalent terms, canonical splitting of the surjections $HC_q(\mathscr{O}(X)) \to HC_q(\mathscr{D}(X))$ and hence, canonical isomorphisms in (2) (cf. the discussion in Remarks 8 and 4 above).

For a fuller treatment of the presented results, their extension to noncompact C^∞ manifolds and yet another proof of Theorem 2 in the C^∞ case (actually chronologically prior; based on a certain kind of "noncommutative" Poincaré lemma) the reader will be referred to [9]. One may add that the point-of-view adopted here has its source in the author's work on noncommutative residue.

(ii) Homology of $\mathscr{D}(X)$ for an affine space $X = \mathbb{A}^n$ reduces to the single nonvanishing group $H_{2n}(\mathscr{D}(\mathbb{A}^n), \mathscr{D}(\mathbb{A}^n)) \simeq \mathscr{k}$ (this was found by B. B. Feigin and B. B. Tsygan, and also by T. Masuda (cf. [2], [6])).

Our approach gives immediately what the corresponding generator is. Denote by ∂_j, the partial derivative $\partial/\partial x_j$, and let $\mathscr{e}_j = 1 \otimes 1 \otimes 1 - 1 \otimes X_j \otimes \partial_j + 1 \otimes \partial_j \otimes X_j$. Then $\mathscr{e} = \mathscr{e}_1 \times \cdots \times \mathscr{e}_n$ is a nontrivial $2n$-cycle (the symbol $\times$ denotes the shuffle product that identifies $H_{2n}(\mathscr{D}(\mathbb{A}^n), \mathscr{D}(\mathbb{A}^n))$ with the n-th tensor power of $H_2(\mathscr{D}(\mathbb{A}^1), \mathscr{D}(\mathbb{A}^1))$). For a general smooth affine variety (in characteristic 0), Hochschild homology of $\mathscr{D}(X)$ was determined also by Ch. Kassel and C. Mitschi (as is reported by J.-L. Brylinski in [1]; we do not know, however, what

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[1] Which is known to the author at present in C^∞ case and seems very likely in the two remaining cases.

their method is[2]). The evidence towards the existence of isomorphisms like (1) and (2) was given also in the preprint of Brylinski mentioned above (the author thanks Ezra Getzler for drawing his attention to it).

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[2] Cf., however, the recent preprint by J.-L. Brylinski, *Some examples of Hochschild and cyclic homology*, Fall 1986 (note added in proof).