Annals of Mathematics, 129 (1989), 591–639

Excision in cyclic homology and in rational algebraic K-theory

By MARIUSZ WODZICKI*

We prove in this article that the following implication holds:

(1) *a ring $\mathscr{A}$ has the excision property in rational algebraic K-theory* $\Rightarrow$ *the* $\mathbf{Q}$*-algebra* $\mathscr{A} \otimes \mathbf{Q}$ *has the excision property in cyclic homology.*

Recall that a ring $\mathscr{A}$ (without unit) is said to possess the excision property in algebraic K-theory if for every ring extension $\mathscr{A} \overset{i}{\rightarrowtail} \mathscr{R} \overset{f}{\twoheadrightarrow} \mathscr{S}$ the K_*-groups of $\mathscr{A}$, $\mathscr{R}$ and $\mathscr{S}$ are related to each other by the natural long exact sequence

$$\cdots \longrightarrow K_{q+1}(\mathscr{S}) \longrightarrow K_q(\mathscr{A}) \xrightarrow{i} K_q(\mathscr{R}) \xrightarrow{f} K_q(\mathscr{S}) \longrightarrow \cdots$$

(for a precise definition, see §1 below). By replacing everywhere $K_*(\ )$ by $K_*(\ ) \otimes \mathbf{Q}$, one obtains the corresponding notion in rational algebraic K-theory.

The above definition has an obvious counterpart for cyclic homology and algebras over a fixed field. In the case of a general commutative ground ring k some restrictions on the class of allowable extensions seem inevitable (due to the well-known limitations of cyclic homology considered as a homology functor for algebras not flat over a ground ring). An extension of k-algebras will be called *pure* if it is pure as an extension of k-modules (in the sense of P. M. Cohn [6]; cf. also Appendix A.3 below). The class of pure extensions contains, e.g.,

(i) extensions which admit a k-module splitting,

(ii) extensions $A \rightarrowtail R \twoheadrightarrow S$ with S flat over k.

In fact, one among the several possible characterizations of purity says that an extension is pure if and only if the underlying extension of k-modules is an inductive limit of split extensions (see Theorem A.4 of Appendix A below). Everywhere in this paper the word "excision" used in the context of cyclic homology will mean "excision with respect to the class of pure extensions."

The second purpose of the present paper is to give a complete characterization of the class of algebras possessing the excision property in cyclic homology. Before stating the corresponding result we need the following definition. Let us

*Supported by NSF Grant No. DMS-8610730(2).

consider the chain complex $(B_*(A), b')$ given by

(2) $$B_q(A) = A^{\otimes q}\ (q \geq 1),\quad b'(a_1 \otimes \cdots \otimes a_q) = \sum_{i=1}^{q-1} (-1)^{i-1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q.$$

(A denotes an arbitrary non-unital algebra.) If A has a unit, $B_*(A)$ is up to a shift in dimension, its standard Bar resolution. The existence of a unit in A equips (2) and, more generally, $B_*(A) \otimes V$, where V is an arbitrary k-module, with an explicit contracting homotopy. If A lacks a unit the complexes $B_*(A) \otimes V$ may have a non-zero homology. In view of this, we propose to call a non-unital algebra A *homologically unital* (shortly, *H-unital*) if, for every k-module V, the complex $B_*(A) \otimes V$ is acyclic. It is easy to see that, if A is flat over k, the latter condition is equivalent to the acyclicity of (2) alone (in the flat case the homology of $B_*(A) \otimes V$ is equal to $\mathrm{Tor}_*^{\tilde{A}}(k, V)$, $* > 0$, where $\tilde{A}$ denotes the algebra with unit obtained by adjoining the unit to A, and k and V are viewed as $\tilde{A}$-modules via the augmentation map $\tilde{A} \to k$).

Having this definition we are ready to state the following result:

(3) *a k-algebra A has the excision property in cyclic homology* $\Leftrightarrow$ *A is H-unital.*

For algebras over a field of characteristic zero, (3) has been proved in [54]. The case of a general ground ring of coefficients requires, however, a completely different proof.

It seems to be rather self-evident that (3) admits extensions to the categories of differential, simplicial or super-algebras. We leave the corresponding modifications of the arguments used to prove (3) (see §3 below) to the interested reader.

By combining (1) and (3) we obtain the following necessary condition for excision in rational algebraic K-theory

(4) *a ring $\mathscr{A}$ has the excision property in $K_*(\ )\otimes \mathbf{Q}$* $\Rightarrow$ *the $\mathbf{Q}$-algebra $\mathscr{A} \otimes \mathbf{Q}$ is H-unital.*

What is known so far about the excision in algebraic K-theory (even this, mostly concerning K_1 and K_2) gives support to the suspicion that the necessary condition in (4) is also a sufficient one (cf. Remarks 1.3(1)–(3) below).

The above results are contained in Sections 1–3 of the present paper. The H-unitality of a number of interesting algebras is then established, using various techniques, in the subsequent Sections 4 to 8. An impression arises that the H-unitality (and, hence, the excision in cyclic homology) tends to occur where it is needed most, *viz.* for algebras which are, roughly speaking, the subject of

Alain Connes' "noncommutative differential geometry," [7]. We prove thus that to the class of H-unital algebras belong: *every* C^*-algebra and, more generally, every Banach algebra with bounded approximate unit (e.g. $L^1(G)$ of an arbitrary locally compact group G), the algebra $C^\infty(X, Z)$ of C^∞-functions on a smooth manifold X which are "flat" along a closed subset $Z \subset X$ (in particular, L. Schwartz' algebras $\mathscr{S}$ and $\mathscr{D}$), the corresponding algebra $\mathscr{D}(X, Z)$ of C^∞ differential operators "flat" along Z, L. Schwartz' algebra $\mathring{\mathscr{B}}$ of C^∞-functions on $\mathbf{R}^d$ which vanish at infinity with all derivatives, every left (or right) ideal in the algebra $\mathscr{D}^{\text{alg}}(V)$ of algebraic differential operators on a regular affine variety (independently of characteristic) et cetera. A noteworthy feature of these examples is that all, except the last one, are H-unital both as abstract and as topological algebras (with respect to their natural locally convex topologies and related completions of the corresponding chain complexes). We decided to consider the excision in continuous cyclic homology separately in [53], this will include an extension of (3) to that situation. In the present paper we are concerned exclusively with the excision in "discrete" homology.

The algebras which one encounters in "noncommutative differential geometry" have a tendency to enter into interesting extensions (many of them being fairly profound, e.g. the extension of pseudodifferential operators, cf. [52]). We exploit this, using (3) and the results of Sections 4–8, to derive a number of new long exact sequences in cyclic homology. This allowed us to determine the (continuous) cyclic homology of algebras like the algebra of Toeplitz operators with continuous symbols [54] and the algebra of pseudodifferential operators [52] which previously seemed to be rather intractable.

The remaining sections possess a more general character.

In Section 9 we introduce a useful class of modules over H-unital algebras which we propose to call the *homologically unitary* (*H-unitary*) modules. We prove then that an algebra B which contains a "pure" H-unital subalgebra A such that the quotient B/A is H-unitary as a left or right A-module is necessarily H-unital (Theorem 9.5). This has several applications. The most noteworthy seems to be the following one (Corollary 9.6).

Let C be an algebra with unit whose structural map $k \to k \cdot 1 \subset C$ is a "pure" monomorphism. Then, for every H-unital algebra A, the tensor product $A \otimes C$ is H-unital.

As a corollary we obtain the Morita invariance of the cyclic homology of H-unital algebras. The Morita invariance is evidently false for general algebras without unit.

In Section 10 we prove, relying on the material of previous sections, that the cone and the suspension functors known from algebraic K-theory (cf. e.g. [22],

[48]) retain their respective properties in cyclic homology. One immediate gain is the ability to replace the relative cyclic homology groups associated with an arbitrary algebra homomorphism $A \to B$ by the absolute ones (cf. Remark 10.2.4).

The last section is devoted to the algebras of triangular matrices. With no restrictions on the ground ring k, we prove that the natural projection "onto the diagonal matrices" induces an isomorphism in cyclic homology (Theorem 11.1). The same was previously proved in algebraic K-theory (D. G. Quillen, R. K. Dennis–S.C. Geller [9], A. J. Berrick–M. E. Keating [2]).

The original version of this article, written in December 1987, consisted of Sections 1–8. The last three sections were added in March 1988.

Throughout the paper we hold to the following convention: $\mathbf{Z}_+ = \{1, 2, \dots\}$ and $\mathbf{N} = \{0, 1, 2, \dots\}$. For a chain complex (K_*, ∂) (in particular, for a $\mathbf{Z}$-graded module), $K_*[j]$ is defined as $(K_*[j])_q \equiv K_{q-j}$ and $(\partial[j])_q = (-1)^j \partial_{q-j}$. Similarly, for a double complex $(L_{**}, \partial', \partial'')$ we define $L_{**}[i, j]$ as $(L_{**}[i, j])_{pq} = L_{p-i, q-j}$ and $(\partial'[i, j])_{pq} = (-1)^i \partial'_{p-i, q-j}$ and $(\partial''[i, j])_{pq} = (-1)^j \partial''_{p-i, q-j}$.

I would like to thank Yuri Ivanovich Manin for his distant yet much felt support.

1. A necessary condition for excision in rational algebraic K-theory

Let $\mathscr{A}$ be a ring without unit. For every ring extension $\mathscr{A} \overset{i}{\rightarrowtail} \mathscr{R} \overset{f}{\twoheadrightarrow} \mathscr{S}$ with kernel $\mathscr{A}$, one defines in algebraic K-theory the relative K-groups $K_*(\mathscr{R}, \mathscr{A})$ so that they fit into the natural long exact sequence

$$(5) \quad \cdots \longrightarrow K_{q+1}(\mathscr{S}) \xrightarrow{\partial} K_q(\mathscr{R}, \mathscr{A}) \xrightarrow{i_*} K_q(\mathscr{R}) \xrightarrow{f_*} K_q(\mathscr{S}) \xrightarrow{\partial} \cdots .$$

The group $K_*(\tilde{\mathscr{A}}, \mathscr{A})$, where $\tilde{\mathscr{A}} = \mathbf{Z} \ltimes \mathscr{A}$ is the ring obtained from $\mathscr{A}$ by adjoining the unit, is usually denoted $K_*(\mathscr{A})$. If $\mathscr{A}$ has a unit itself this agrees with the usual definition of $K_*(\mathscr{A})$.

In view of the naturality of (5), there is a canonical map $K_*(\mathscr{A}) \to K_*(\mathscr{R}, \mathscr{A})$, for every ring $\mathscr{R}$ containing $\mathscr{A}$ as a two-sided ideal. The ring $\mathscr{A}$ is said to possess the *excision property* (or to be an *excision ring*) if this map is an isomorphism for every $\mathscr{R}$. One can define a similar excision property in rational algebraic K-theory by requiring, instead, that $K_*(\mathscr{A}) \otimes \mathbf{Q} \to K_*(\mathscr{R}, \mathscr{A}) \otimes \mathbf{Q}$ be an isomorphism.

1.1. THEOREM. *Assume that a ring $\mathscr{A}$ has the excision property in rational algebraic K-theory. Then the supplemented $\mathbf{Q}$-algebra $\tilde{\mathscr{A}} \otimes \mathbf{Q}$ is acyclic, i.e.* $\operatorname{Tor}_n^{\tilde{\mathscr{A}} \otimes \mathbf{Q}}(\mathbf{Q}, \mathbf{Q}) = 0$ *for $n > 0$.*

Proof. Set $\mathscr{R} = \widetilde{\mathscr{A} \times \mathscr{I}}$ where $\mathscr{I}$ is a nilpotent ring without unit; i.e. $\mathscr{I}^k = 0$ for some $k > 0$. We are interested in the extension $\mathscr{A} \rightarrowtail \mathscr{R} \twoheadrightarrow \mathscr{S}$ where

$\mathscr{S} \xrightarrow{\varepsilon} \mathbf{Z}$ is the supplemented ring $\tilde{\mathscr{I}}$ having $\mathscr{I}$ as its augmentation ideal. The following commutative diagram of extensions

$$\begin{array}{ccccc}
 & & \mathscr{I} & = & \mathscr{I} \\
 & & \downarrow & & \downarrow \\
\mathscr{A} & \rightarrowtail & \mathscr{R} & \twoheadrightarrow & \mathscr{S} \\
\| & & \downarrow & & \downarrow \\
\mathscr{A} & \rightarrowtail & \tilde{\mathscr{A}} & \twoheadrightarrow & \mathbf{Z}
\end{array}$$

induces the related commutative diagram with exact rows and columns in algebraic K-theory:

$$(6)\qquad \begin{array}{ccccc}
 & & K_*(\mathscr{R}, \mathscr{I}) & \xrightarrow{\varphi_*^K} & K_*(\mathscr{I}) \\
 & & \downarrow & & \downarrow \\
K_*(\mathscr{R}, \mathscr{A}) & \rightarrowtail & K_*(\mathscr{R}) & \twoheadrightarrow & K_*(\mathscr{S}) \\
\psi_*^K \downarrow & & \downarrow & & \downarrow \\
K_*(\mathscr{A}) & \rightarrowtail & K_*(\tilde{\mathscr{A}}) & \twoheadrightarrow & K_*(\mathbf{Z})
\end{array}$$

and the similar diagram for the corresponding $\mathbf{Q}$-algebras, in cyclic homology

$$(7)\qquad \begin{array}{ccccc}
 & & HC_*(\mathscr{R} \otimes \mathbf{Q}, \mathscr{I} \otimes \mathbf{Q}) & \xrightarrow{\varphi_*^{HC}} & HC_*(\mathscr{I} \otimes \mathbf{Q}) \\
 & & \downarrow & & \downarrow \\
HC_*(\mathscr{R} \otimes \mathbf{Q}, \mathscr{A} \otimes \mathbf{Q}) & \rightarrowtail & HC_*(\mathscr{R} \otimes \mathbf{Q}) & \twoheadrightarrow & HC_*(\mathscr{S} \otimes \mathbf{Q}) \\
\psi_*^{HC} \downarrow & & \downarrow & & \downarrow \\
HC_*(\mathscr{A} \otimes \mathbf{Q}) & \rightarrowtail & HC_*(\tilde{\mathscr{A}} \otimes \mathbf{Q}) & \twoheadrightarrow & HC_*(\mathbf{Q}).
\end{array}$$

If $\mathscr{A}$ is an excision ring for rational K-theory, the map $\psi_*^K \otimes \mathbf{Q}$ is an isomorphism. On the other hand, from (6) we get $\mathrm{Ker}(\psi_*^K \otimes \mathbf{Q}) \simeq \mathrm{Ker}(\varphi_*^K \otimes \mathbf{Q})$; by a theorem of T. G. Goodwillie [16, Main Theorem, p. 348] we have $\mathrm{Ker}(\varphi_*^K \otimes \mathbf{Q}) \simeq \mathrm{Ker}(\varphi_{*-1}^{HC})$; and from (7) it follows that $\mathrm{Ker}(\varphi_*^{HC}) \simeq \mathrm{Ker}(\psi_*^{HC})$. Hence, $\mathrm{Ker}\,(\psi_*^K \otimes \mathbf{Q}) \simeq \mathrm{Ker}(\psi_{*-1}^{HC})$.

In the case $\mathscr{I}^2 = 0$ the kernel of the corresponding map in cyclic homology has been completely described in [54] (cf. also §3 of the present paper). In particular, one has, for every $n > 0$, the following inclusion (cf. [54, Proof of Cor. 2] or (22) below):

$$(8)\qquad \mathrm{Ker}\,\psi_n^{HC} \supset V_{n,1} \oplus V_{n-1,2} \oplus \cdots \oplus V_{1,n}$$

where $V_{i,j} = \mathrm{Tor}_i^{\tilde{\mathscr{A}} \otimes \mathbf{Q}}(\mathbf{Q}, (\mathscr{I} \otimes \mathbf{Q})^{\otimes j}) = \mathrm{Tor}_i^{\tilde{\mathscr{A}} \otimes \mathbf{Q}}(\mathbf{Q}, \mathbf{Q}) \otimes (\mathscr{I} \otimes \mathbf{Q})^{\otimes j}$.

If ψ_*^{HC} is an isomorphism, it follows from (8) that all the groups $\mathrm{Tor}_i^{\tilde{\mathscr{A}}\otimes\mathbf{Q}}(\mathbf{Q},\mathbf{Q})$, $i > 0$, must vanish. □

We proved in [54] (cf. also §3 of the present paper) that the condition $\mathrm{Tor}_*^{\tilde{A}}(\mathbf{Q},\mathbf{Q}) = 0$, $* > 0$, characterizes $\mathbf{Q}$-algebras A satisfying excision in cyclic homology. We obtain, therefore, the following:

1.2. COROLLARY (cf. [54, Remark 2]). *If a ring without unit $\mathscr{A}$ satisfies excision in rational algebraic K-theory, the $\mathbf{Q}$-algebra $\mathscr{A}\otimes\mathbf{Q}$ satisfies excision in cyclic homology.* □

1.3. *Remarks.* (1) The vanishing of $\mathrm{Tor}_n^{\tilde{\mathscr{A}}\otimes\mathbf{Q}}(\mathbf{Q},\mathbf{Q})$, for $n > 0$, very well may be a necessary and *sufficient* condition for excision in $K_*(\ )\otimes\mathbf{Q}$. It *is* a necessary and sufficient condition on the class of split extensions $\mathscr{A}\rightarrowtail\mathscr{R}\twoheadrightarrow\mathscr{S}$ of the form $\mathscr{R} = \widetilde{\mathscr{A}\times\mathscr{I}}$ where $\mathscr{I}$ is a nilpotent ideal. When we consider excision in cyclic homology, extensions of this type are proved to be a kind of "touchstone"—if the excision holds for them, it holds for all the other extensions (cf. [54, Proof of Cor. 2] and the proof of Theorem 3.1 below). If anything like this can also be established in algebraic K-theory, this will immediately solve the excision problem for $K_*(\ )\otimes\mathbf{Q}$.

(2) When dealing with particular problems one is often confronted with the question of whether the excision holds in a specific dimension. In this respect Theorem 1.1 can be stated more precisely as

$$\begin{matrix}\mathscr{A}\ \textit{is an excision ring}\\ \textit{for}\ K_n(\ )\otimes\mathbf{Q}\end{matrix}\ \Rightarrow\ \mathrm{Tor}_q^{\tilde{\mathscr{A}}\otimes\mathbf{Q}}(\mathbf{Q},\mathbf{Q}) = 0,\ \textit{for}\ 1\le q\le n-1.$$

(cf. Theorem 3.5 below). If $K_n(\ )\otimes\mathbf{Q}$ is replaced by $HC_{n-1}(\ )$ this implication is, actually, an equivalence if the condition on the right-hand side is slightly sharpened by adding an extra condition involving $\mathrm{Tor}_n^{\tilde{\mathscr{A}}\otimes\mathbf{Q}}(\mathbf{Q},\mathbf{Q})$ (Theorem 3.5 below; vanishing of either $\mathrm{Tor}_n^{\tilde{\mathscr{A}}\otimes\mathbf{Q}}(\mathbf{Q},\mathbf{Q})$ or $HC_{n-1}(\mathscr{A}\otimes\mathbf{Q})$ would suffice anyway). Considering cyclic homology we also have the following implications ($n\ge 2$):

$$(\mathrm{E}_n)\qquad \begin{matrix}A\ \textit{is an excision}\\ \textit{algebra for}\ HC_{n-1}(\ )\end{matrix}\ \Rightarrow\ \begin{matrix}A\ \textit{is an excision}\\ \textit{algebra for}\ HC_{n-2}(\ ).\end{matrix}$$

In rational algebraic K-theory the corresponding implications hold for K_2 and K_1 (cf. Remark 3 below) and nothing seems to be known about the situation in higher dimensions.

(3) The necessary condition stated in Theorem 1.1 is also a sufficient one at least in dimensions 1 and 2 (on the level of K_0, excision holds universally). More precisely, from the results of L. N. Vaserstein [45] and W. van der Kallen [21] it

follows that

$$\text{(9)} \qquad \operatorname{Tor}_1^{\tilde{\mathscr{A}} \otimes \mathbf{Q}}(\mathbf{Q}, \mathbf{Q}) = 0 \;\Rightarrow\; \begin{array}{c}\mathscr{A} \textit{ is an excision} \\ \textit{ring for } K_1(\ ) \otimes \mathbf{Q}\end{array}$$

and

$$\text{(10)} \qquad \begin{array}{c}\operatorname{Tor}_q^{\tilde{\mathscr{A}} \otimes \mathbf{Q}}(\mathbf{Q}, \mathbf{Q}) = 0 \\ (q = 1,2)\end{array} \;\Rightarrow\; \begin{array}{c}\mathscr{A} \textit{ is an excision} \\ \textit{ring for } K_2(\ ) \otimes \mathbf{Q}\end{array}$$

(compare this with the previous remark; $\operatorname{Tor}_1^{\tilde{\mathscr{A}} \otimes \mathbf{Q}}(\mathbf{Q}, \mathbf{Q}) = \mathscr{A}/\mathscr{A}^2 \otimes \mathbf{Q}$ and $\operatorname{Tor}_2^{\tilde{\mathscr{A}} \otimes \mathbf{Q}}(\mathbf{Q}, \mathbf{Q})$ identifies with the kernel of the multiplication map $\mathscr{A} \otimes_{\mathscr{A}} \mathscr{A} \to \mathscr{A}$ tensored with $\mathbf{Q}$). It is noteworthy that both (9) and (10) hold not just for rational K-groups. Indeed, (9) and (10) continue to be true if one replaces $K_n(\ ) \otimes \mathbf{Q}$ by $K_n(\ )$ and $\operatorname{Tor}_q^{\tilde{\mathscr{A}} \otimes \mathbf{Q}}(\mathbf{Q}, \mathbf{Q})$, $q = 1$ or 2, by the "Bar homology groups" $HB_q(\mathscr{A})$ (to be introduced in the next section; if $\mathscr{A}$ as an abelian group has no torsion, $HB_q(\mathscr{A}) \equiv \operatorname{Tor}_q^{\tilde{\mathscr{A}}}(\mathbf{Z}, \mathbf{Z})$, $q > 0$). This and a theorem of L. N. Vaserstein (that excision holds in $K_*(\ )$ for rings having left or right "local" units, [45, Thm. 17.1]) provided the primary motivation to extend the theorem on excision in cyclic homology over a field of characteristic zero [54, Thm. 3] to the case of a general ground ring. This will be done in the next few sections.

2. The standard chain complexes

Let A be an algebra (without unit) over a commutative ring k. One can associate with A the following two chain complexes: $(C_*(A, A), b)$, where $C_q(A, A) = A^{\otimes(1+q)}$,

$$b(a_0 \otimes \cdots \otimes a_q) = \sum_{i=0}^{q-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q + (-1)^q a_q a_0 \otimes a_1 \otimes \cdots \otimes a_{q-1},$$

and $(B_*(A), b')$ where

$$B_q(A) = A^{\otimes q} \quad (q \geq 1),$$

$$b'(a_1 \otimes \cdots \otimes a_q) = \sum_{i=1}^{q-1} (-1)^{i-1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q.$$

The homology of $C_*(A, A)$ will be denoted $\mathscr{H}_*(A, A)$ and the homology of $B_*(A)$ (called hereafter the Bar homology of A) by $HB_*(A)$. If V is a k-module we shall denote by $HB_*(A; V)$ the homology of the complex $B_*(A) \otimes V$. For algebras with unit, the complex $B_*(A)$ is, up to a shift in dimension, the standard (augmented) Bar resolution of A, and it comes equipped with the contracting homotopy $a_1 \otimes \cdots \otimes a_q \mapsto 1 \otimes a_1 \otimes \cdots \otimes a_q$ which is a (k, A)-bimodule map. In particular, $HB_*(A; V) \equiv 0$ if A has a unit.

In terms of the complexes $C_*(A, A)$ and $B_*(A)$ the Hochschild homology $H_*(A, A)$ and the cyclic homology $HC_*(A)$ are defined as the homology groups of the double complexes $\mathscr{K}(A) = (C_*(A, A) \xleftarrow{1-t} B_*(A)[-1])$ and, respectively,

$$\mathscr{C}(A) = (C_*(A, A) \xleftarrow{1-t} B_*(A)[-1] \xleftarrow{N} C_*(A, A) \xleftarrow{1-t} \cdots)$$

$(t_q(a_1 \otimes \cdots \otimes a_{q+1}) = (-1)^q(a_{q+1} \otimes a_1 \otimes \cdots \otimes a_q)$ and $N_q = 1 + t_q + \cdots + t_q^q$; cf. [27, §4]).

For A flat over k, one has $HB_*(A; V) = \overline{\mathrm{Tor}}_*^{\tilde{A}}(k, V)$ and $H_*(A, A) = \overline{\mathrm{Tor}}_*^{\tilde{A} \otimes \tilde{A}}(\tilde{A}, \tilde{A})$ where $\tilde{A} = k \ltimes A$ denotes the k-algebra obtained by adjoining a unit to A and $\overline{\mathrm{Tor}}_*$ denotes the corresponding reduced Tor-groups:

$$\mathrm{Tor}_*^{\tilde{A}}(k, V) \equiv \mathrm{Tor}_*^{k}(k, V) \oplus \overline{\mathrm{Tor}}_*^{\tilde{A}}(k, V)$$

and

$$\mathrm{Tor}_*^{\tilde{A} \otimes \tilde{A}}(\tilde{A}, \tilde{A}) \equiv \mathrm{Tor}_*^{k \otimes k}(k, k) \oplus \overline{\mathrm{Tor}}_*^{\tilde{A} \otimes \tilde{A}}(\tilde{A}, \tilde{A}).$$

With no restrictions on A, one can still identify $HB_*(A; V)$ with the corresponding reduced *relative* Tor-group:

$$HB_*(A; V) = \overline{\mathrm{Tor}}_*^{(\tilde{A}, k)}(k, V).$$

Recall that the latter is defined using resolutions consisting of $\tilde{A}$-modules which are projective *relative* to the class of k-split epimorphisms (for more details cf. e.g. [18], [28, IX.8] or, within the general framework of relative homological algebra, [5] and [11]).

3. Excision in cyclic homology

An extension of (not necessarily unital) k-algebras $A \overset{i}{\rightarrowtail} R \overset{f}{\twoheadrightarrow} S$ will be called *pure* if the underlying short exact sequence of k-modules is pure (see Appendix A). Of particular importance are the following two classes of pure extensions:

(I) extensions admitting a k-module splitting, and

(II) extensions $A \rightarrowtail R \twoheadrightarrow S$ with S flat over k.

For a given extension the relative cyclic groups $HC_*(R, A)$ (denoted also $HC_*(R \to S)$) are defined as the homology groups of the double complex $\mathscr{C}(R \to S) := \mathrm{Ker}(\mathscr{C}(R) \xrightarrow{f} \mathscr{C}(S))$. Precisely in the same manner one can define the relative Hochschild, Bar and $\mathscr{H}$-groups.

We will say that an algebra A *satisfies excision in cyclic homology* if for every pure extension the natural inclusion i: $\mathscr{C}(A) \hookrightarrow \mathscr{C}(R \to S)$ is a quasi-isomorphism (i.e. induces an isomorphism in homology; notice that for extensions other than pure, the map $i\colon \mathscr{C}(A) \to \mathscr{C}(R \to S)$ need not be injective). Similarly defined are the corresponding excision properties for Hochschild, Bar and $\mathscr{H}$-homology.

It follows directly from the definition that any pure extension $A \overset{i}{\rightarrowtail} R \overset{f}{\twoheadrightarrow} S$ whose kernel satisfies excision in one of the above homology theories $F_* = HC_*$, H_*, $\mathscr{H}_*$ or HB_* (the latter will be called shortly the F_*-excision property) induces the natural long exact sequence

$$(11)\qquad \cdots \xrightarrow{f} F_{q+1}(S) \xrightarrow{\partial} F_q(A) \xrightarrow{i} F_q(R) \xrightarrow{f} F_q(S) \longrightarrow \cdots .$$

Almost as obvious is that every Cartesian square of (not necessarily unital) k-algebras

$$(12)\qquad \begin{array}{ccc} P & \xrightarrow{\psi} & R \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Q & \xrightarrow{\varphi} & S \end{array}$$

where $f\colon R \to S$ is a *pure* epimorphism (see Appendix A) and the ideal $\operatorname{Ker} f$ possesses the F_*-excision property, induces the corresponding Mayer-Vietoris long exact sequence

$$(13)\qquad \cdots \longrightarrow F_{q+1}(S) \xrightarrow{\partial} f_q(P) \longrightarrow F_q(Q) \oplus F_q(R) \longrightarrow F_q(S) \longrightarrow \cdots .$$

Indeed, if we denote $\operatorname{Ker} f$ by A and $\operatorname{Ker} g$ by A', then (12) is equivalent to the following morphism of extensions

$$\begin{array}{ccccc} A & \xrightarrow{i} & R & \xrightarrow{f} & S \\ \uparrow{\scriptstyle \wr} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \varphi} \\ A' & \xrightarrow{j} & P & \xrightarrow{g} & Q \end{array}$$

and under the above assumptions on A we have, in view of (11), the following commutative diagram with exact rows

$$(14)\qquad \begin{array}{ccccccccc} \cdots \longrightarrow & F_{q+1}(S) & \xrightarrow{\partial} & F_q(A) & \xrightarrow{i} & F_q(R) & \xrightarrow{f} & F_q(S) & \longrightarrow \cdots \\ & \uparrow{\scriptstyle \varphi} & & {\scriptstyle \wr}\uparrow{\scriptstyle \psi_{|A'}} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \varphi} & \\ \cdots \longrightarrow & F_{q+1}(Q) & \xrightarrow{\partial} & F_q(A') & \xrightarrow{j} & F_q(P) & \xrightarrow{g} & F_q(Q) & \longrightarrow \cdots \end{array}$$

where the natural maps $\psi_{|A'}\colon F_q(A') \to F_q(A)$ are isomorphisms. Treating (14) as a double complex we may simplify it in the standard way without changing its homology (which is here, of course, zero) by removing columns containing isomorphisms $\psi_{|A'}$. We arrive at the following acyclic double complex

$$\begin{array}{ccccccc}
 & & & & F_q(R) & \xrightarrow{f} & F_q(S) \longrightarrow \cdots \\
 & & & & \uparrow\psi & & \uparrow\varphi \\
F_{q+1}(R) & \xrightarrow{f} & F_{q+1}(S) & \xrightarrow{j\circ(\psi_{|A'})^{-1}\circ\partial} & F_q(P) & \xrightarrow{g} & F_q(Q) \\
\uparrow\psi & & \uparrow\varphi & & & & \\
\cdots \longrightarrow F_{q+1}(P) & \xrightarrow{g} & F_{q+1}(Q) & & & &
\end{array}$$

whose total complex is the Mayer-Vietoris long exact sequence (13).

Another formal feature of pure extensions with kernels satisfying excision in cyclic and in Hochschild homology is the existence of the following lattice with exact rows and columns

(15)

$$\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longleftarrow & \mathbf{HC}_{q+1}(\mathbf{R}) & \xleftarrow{I} & H_{q+1}(R,R) & \xleftarrow{B} & HC_q(R) & \xleftarrow{S} & HC_{q+2}(R) & \longleftarrow \cdots \\
 & \downarrow f & & \downarrow f & & \downarrow f & & \downarrow f & \\
\cdots \longleftarrow & HC_{q+1}(S) & \xleftarrow{I} & H_{q+1}(S,S) & \xleftarrow{B} & HC_q(S) & \xleftarrow{S} & HC_{q+2}(S) & \longleftarrow \cdots \\
 & & & & \circlearrowright & & & & \\
 & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & \\
\cdots \longleftarrow & HC_q(A) & \xleftarrow{I} & H_q(A,A) & \xleftarrow{B} & HC_{q-1}(A) & \xleftarrow{S} & HC_{q+1}(A) & \longleftarrow \cdots \\
 & \downarrow i & & \downarrow i & & \downarrow i & & \downarrow i & \\
\cdots \longleftarrow & HC_q(R) & \xleftarrow{I} & H_q(R,R) & \xleftarrow{B} & HC_{q-1}(R) & \xleftarrow{S} & \mathbf{HC}_{q+1}(\mathbf{R}) & \longleftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

in which all squares commute except those containing B and ∂ which anti-commute. The columns in (15) are the long exact sequences (11) while the rows are the corresponding Connes' long exact sequences (cf. [27, Thm. 1.6]).

H-unitality. A k-algebra A is said to be *homologically unital* (*H-unital*) if its Bar homology with coefficients in an arbitrary k-module V vanishes, $HB_*(A; V) \equiv 0$. For A flat over k, the spectral sequence of "universal coefficients" $E^2_{pq} = \mathrm{Tor}^k_p(HB_q(A), V) \Rightarrow HB_{p+q}(A; V)$ implies that in that case H-unitality is equivalent to requiring that $HB_*(A) \equiv \overline{\mathrm{Tor}}^{\tilde{A}}_*(\Bbbk, k) = 0$ (this was precisely the definition of H-unitality given in [54] in the case of the ground ring being a field of characteristic zero).

In the theorem stated below the word *excision* is shorthand for "excision with respect to pure extensions $A \rightarrowtail R \twoheadrightarrow S$ of non-unital algebras."

Restricting oneself only to extensions of unital algebras does not make any difference if one considers the excision problem in Hochschild or in cyclic homology. In H_* or HC_* the excision holds for a given extension $A \rightarrowtail R \twoheadrightarrow S$ precisely when it holds for the extension $A \rightarrowtail \tilde{R} \twoheadrightarrow \tilde{S}$; this follows from the identities $H_*(\tilde{R}, \tilde{R}) = H_*(R, R) \oplus H_*(k, k)$ and $HC_*(\tilde{R}) = HC_*(R) \oplus HC_*(k)$; cf. [27, §4].

3.1. THEOREM. *For any algebra A over an arbitrary commutative ground ring k the following conditions are equivalent*:

(a) *A satisfies excision in cyclic homology*,
(b) *A satisfies excision in Hochschild homology*,
(c) *A satisfies excision in $\mathscr{H}$-homology*,
(d) *A satisfies excision in Bar homology*,
(e) *A is H-unital.*

The proof of Theorem 3.1 is divided into several parts.

(d) $\Leftrightarrow$ (e). Let A be an arbitrary k-algebra, V an arbitrary k-module and $A \rightarrowtail R \twoheadrightarrow S$ a pure extension with kernel A. Then the complex $L_* = \operatorname{Ker}(B_*(R; V) \to B_*(S; V))$ comes equipped with the natural filtration

(16) $F_p L_{p+q} = \operatorname{lin\,span}\{ r_1 \otimes \cdots \otimes r_{p+q} \otimes v \,|\, \text{at least } q \ r_j\text{'s belong to } A \}$

(*lin span* is an abbreviation for "linear span").

The associated spectral sequence (s.s.) $E^k_{pq} \Rightarrow H_{p+q}(L_*)$ is located in the region $(p \geq 0,\ q \geq 1)$. In order to describe its E^0-term, we need the following construction.

Let l be a positive integer and $\mathscr{M} = (n_1, \ldots, n_{l+1}) \in \mathbf{N} \times \mathbf{Z}_+ \times \cdots \times \mathbf{Z}_+ \times \mathbf{N}$ (i.e. $n_1, n_{l+1} \geq 0$ and $n_i > 0$, for $1 < i < l+1$). We introduce the notation $|\mathscr{M}| := n_1 + \cdots + n_{l+1}$ and $l(\mathscr{M}) := l$. For a given $\mathscr{M}$, let $Y_*(\mathscr{M}; V)$ denote the total complex of the following $(l+1)$-tuple complex

$$B_*(A)[n_1] \otimes \cdots \otimes B_*(A)[n_l] \otimes S^{\otimes |\mathscr{M}|} \otimes V[-|\mathscr{M}| + n_{l+1}]$$

where $S^{\otimes |\mathscr{M}|} \otimes V[-|\mathscr{M}| + n_{l+1}]$ is treated as a trivial complex concentrated in dimension $-|\mathscr{M}| + n_{l+1}$.

3.2. LEMMA. *For every $p \geq 0$, there is a canonical isomorphism*

$$(17) \qquad \left(E^0_{p*}, d^0\right) \simeq \bigoplus_{\substack{\mathscr{M}:\ |\mathscr{M}| = p \\ l(\mathscr{M}) \geq 1}} Y_*(\mathscr{M}; V).$$

Proof. Let us introduce the abbreviated notation

$$s_{(n_i)} \equiv s_{n_1+\cdots+n_{i-1}+1} \otimes \cdots \otimes s_{n_1+\cdots+n_i} \qquad (s_\nu \in S)$$

and

$$a_{(k_i)} \equiv a_{k_1+\cdots+k_{i-1}+1} \otimes \cdots \otimes a_{k_1+\cdots+k_i} \qquad (a_\kappa \in A).$$

Then the identification in (17) is given by

$$\begin{aligned} &s_{(n_1)} \otimes a_{(k_1)} \otimes \cdots \otimes s_{(n_l)} \otimes a_{(k_l)} \otimes s_{(n_{l+1})} \\ &\quad \mapsto a_{(k_1)} \otimes \cdots \otimes a_{(k_l)} \otimes s_{(n_1)} \otimes \cdots \otimes s_{(n_{l+1})}. \end{aligned}$$ □

If A is H-unital, all the complexes $Y_*(\mathcal{M}; V)$ are acyclic and, in view of Lemma 3.2, $HB_*(A; V) \xrightarrow{\sim} H(L_*) = 0$. In particular, A satisfies excision in Bar homology.

In order to prove the opposite implication, we take $S = V$, $V^2 = 0$, and $R = A \oplus V$ (with V acting trivially on A). In this situation the s.s. $E^k_{**} \Rightarrow HB_*(R \to S)$ obviously stops at E^1. Since $E^1_{0*} = HB_*(A)$, the excision property of A forces all E^1_{p*}, $p > 0$, to vanish. It remains to notice that, in view of Lemma 3.2, $E^1_{p*} \supset HB_*(A; V^{\otimes p})$.

(c) ⇔ (e). Let A be again an arbitrary algebra and $A \rightarrowtail R \twoheadrightarrow S$ a pure extension. The complex $M_* = \operatorname{Ker}(C_*(R, R) \to C_*(S, S))$ carries the natural filtration

$$(18) \quad F_p M_{p+q} = \operatorname{lin\,span}\{ r_0 \otimes \cdots \otimes r_{p+q} | \text{at least } q+1 \quad r_j\text{'s belong to } A \}.$$

The associated s.s. $D^k_{pq} \Rightarrow H_{p+q}(M_*)$ is a s.s. of the first quadrant. It is clear that (D^0_{0*}, d^0) coincides with $(C_*(A, A), b)$ (cf. Lemma A.6 of the Appendix). For $p > 0$, the complex (D^0_{p*}, d^0) decomposes, as is easily seen, into the direct sum of two subcomplexes

$$D^0_{p*} \simeq S^{\otimes p} \otimes B_*(A)[-1] \oplus \overline{D}^0_{p*} \qquad (p > 0)$$

where, after D^0_{pq} has been identified with $(A \oplus S)^{\otimes(1+p+q)}$, $\overline{D}^0_{pq}$ is given by

$$\overline{D}^0_{pq} = \operatorname{lin\,span}\{ \rho_0 \otimes \cdots \otimes \rho_{p+q} \in D^0_{pq} | \exists\, 0 \le j < i \le p+q \text{ such that } \rho_i \in S \text{ and } \rho_j \in A \}.$$

Let $\overline{D}^1_{p*}$ denote the homology of $(\overline{D}^0_{p*}, d^0)$, $p > 0$; then

$$(19) \qquad D^1_{pq} \simeq HB_{q+1}(A; S^{\otimes p}) \oplus \overline{D}^1_{pq} \qquad (q \ge 0).$$

In order to compute $\overline{D}^1_{p*}$, we introduce on $(\overline{D}^0_{p*}, d^0)$ yet another filtration

$$'F_u\overline{D}^0_{p,u+v} = \operatorname{lin\,span}\{\rho_0 \otimes \cdots \otimes \rho_{p+u+v} | \exists\, p+v \le i \le p+u+v \text{ such that } \rho_i \in S\}.$$

The associated s.s. $'D^k_{uv}(p) \Rightarrow \overline{D}^1_{p,u+v}$ is a s.s. of the first quadrant. Its E^0-term is described by the following:

3.3. LEMMA. *For every $p > 0$ and $u \ge 0$, there is a canonical isomorphism*

$$('D^0_{u*}(p), 'd^0) \simeq \bigoplus_{\substack{\mathscr{M}: |\mathscr{M}|=p \\ l(\mathscr{M}) \ge 1,\, n_{l+1} \ge 1}} Y_*(\mathscr{M}) \otimes A^{\otimes u}[-1] \tag{20}$$

where $A^{\otimes u}[-1]$ is treated as a trivial complex concentrated in dimension -1 and $Y_*(\mathscr{M}) \equiv Y_*(\mathscr{M}; k)$.

Proof. The identification in (20) is given by

$$\begin{aligned} &s_{(n_1)} \otimes a_{(k_1)} \otimes \cdots \otimes s_{(n_l)} \otimes a_{(k_l)} \otimes s_{(n_{l+1})} \otimes a_{(u)} \\ &\quad \mapsto a_{(k_1)} \otimes \cdots \otimes a_{(k_l)} \otimes s_{(n_1)} \otimes \cdots \otimes s_{(n_{l+1})} \otimes a_{(u)} \end{aligned}$$

where $a_{(u)} \equiv a_{k_1+\cdots+k_l+1} \otimes \cdots \otimes a_{k_1+\cdots+k_l+u}$. □

If A is H-unital all the complexes on the right-hand side of (20) are acyclic. Hence $'D^1_{u*}(p) = 0$, for $u \ge 0$ and $p > 0$, and $D^1_{p*} = HB_*(A; S^{\otimes p})[-1] \oplus \overline{D}^1_{p*} = 0$, for $p > 0$. Therefore $\mathscr{H}_*(A, A) = D^1_{0*} \to H(M_*)$ is an isomorphism for an arbitrary pure extension.

To prove the opposite implication, take again $S = V$, $V^2 = 0$, and $R = A \oplus V$. The corresponding s.s. D^k_{**} stops at the "E^1-term" and the excision property means in this case that D^1_{p*} has to vanish for $p > 0$. The comparison with (19) shows that A is H-unital.

(a) ⇒ (e). As in the case of Bar and $\mathscr{H}$-homology it will be sufficient to examine the excision property for $R = A \oplus V$, $V^2 = 0$ (with V acting trivially on A). The double complex $\mathscr{C}(A \oplus V)$ decomposes into the direct sum of its subcomplexes

$$\mathscr{C}(A \oplus V) = \mathscr{C}(A) \oplus \mathscr{C}(V) \oplus \bigoplus_{\mathscr{N}} \mathscr{C}\langle \mathscr{N} \rangle \tag{21}$$

where the summands $\mathscr{C}\langle \mathscr{N} \rangle$ are labelled by orbits of $\mathbf{Z}/l\mathbf{Z}$, $l = 1, 2, 3, \ldots$, acting by cyclic permutations on $\mathbf{Z}_+ \times \cdots \times \mathbf{Z}_+$ (l times; in [54] such orbits were called *patterns of length l*). The subcomplexes $\mathscr{C}\langle \mathscr{N} \rangle$ are defined as follows. Let $(n_1, \ldots, n_l)$ be any representative of $\mathscr{N}$; for every $m \ge l + |\mathscr{N}|$, where $|\mathscr{N}| = n_1 + \cdots + n_l$, let $W(n_1, \ldots, n_l; m) \subset (A \oplus V)^{\otimes m}$ denote the submodule

$$\bigoplus_{\substack{k_1, \ldots, k_l \ge 1 \\ k_1 + \cdots + k_l = m - |\mathscr{N}|}} V^{n_1} \otimes A^{k_1} \otimes \cdots \otimes V^{n_l} \otimes A^{k_l}.$$

On $(A \oplus V)^{\otimes m}$, the group $\mathbf{Z}/m\mathbf{Z}$ acts by cyclic permutations. Denote by $\overline{W}(\mathscr{N}; m)$ the smallest $\mathbf{Z}/m\mathbf{Z}$-invariant submodule of $(A \oplus V)^{\otimes m}$ which contains $W(n_1, \ldots, n_l; m)$ (it clearly does not depend on the choice of a representative $(n_1, \ldots, n_l) \in \mathscr{N}$). Then the subcomplex $\mathscr{C}\langle \mathscr{N} \rangle$ is defined by putting

$$\mathscr{C}\langle \mathscr{N} \rangle_{pq} = \overline{W}(\mathscr{N}; q+1) \qquad (p, q \geq 0).$$

If $l(\mathscr{N}) = 1$, i.e. $\mathscr{N} = (n)$ for some $n \in \mathbf{Z}_+$, the $\mathbf{Z}/m\mathbf{Z}$-module $\overline{W}(\mathscr{N}; m)$ is free

$$\overline{W}((n); m) \simeq k[\mathbf{Z}/m\mathbf{Z}] \bigotimes_k (A^{\otimes(m-n)} \otimes V^{\otimes n}) \qquad (m \geq n+1).$$

In particular, the natural augmentation from the extreme left column of $\mathscr{C}\langle(n)\rangle$ to $B_*(A)[n-1] \otimes V^{\otimes n}$ induces a quasi-isomorphism $B_*(A)[n-1] \otimes V^{\otimes n} \leftarrow \operatorname{Tot} \mathscr{C}\langle(n)\rangle$. Therefore, by taking into account in (21) only patterns of length one, we arrive at

$$\begin{aligned}(22) \qquad HC_q(A \oplus V) \supset\ & HC_q(A) \oplus HC_q(V) \oplus HB_q(A; V) \\ & \oplus HB_{q-1}(A; V^{\otimes 2}) \oplus \cdots \oplus HB_1(A; V^{\otimes q})\end{aligned}$$

$(q \geq 1)$. Since A is assumed to satisfy excision in cyclic homology, all the summands on the right-hand side of (22), except $HC_q(A)$ and $HC_q(V)$, must vanish. Hence A is H-unital.

The remaining implications (c) & (d) $\Rightarrow$ (b) $\Rightarrow$ (a) follow easily and purely formally from the corresponding relative versions of the obvious long exact sequences linking cyclic with Hochschild and Hochschild with Bar and $\mathscr{H}$-homology groups

$$\begin{array}{ccc} & H_* & \\ \nearrow & & \searrow \\ \mathscr{H}_* & \xleftarrow[-1]{} & HB_* \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & H_* & \\ {\scriptstyle +1}\nearrow & & \searrow \\ HC_* & \xleftarrow[-2]{} & HC_* . \end{array}$$

The proof of Theorem 3.1 is complete.

Remark. The assumption of purity of the corresponding extensions was used repeatedly in the above proof in conjunction with Lemma A.6 of Appendix A to ensure "proper" E^0-terms of the spectral sequences associated with filtrations (16) and (18).

3.4. Corollary. *If A is H-unital and $A \rightarrowtail R \twoheadrightarrow S$ is a pure extension, then*

$$R \text{ is } H\text{-unital} \Leftrightarrow S \text{ is } H\text{-unital}.$$

In other words, the category of H-unital algebras is closed under pure extensions and under passing to pure quotients. □

This is a corollary of the implication (e) ⇒ (c) of Theorem 3.1.

From our proof of Theorem 3.1 it is not difficult to extract also the necessary and sufficient conditions under which excision holds in a specific dimension. For brevity, we shall state the corresponding result only for cyclic homology and under the additional assumption that the algebra A is flat over k.

3.5. THEOREM. *Fix $n \geq 0$, and let A be a flat k-algebra.*

(I) *The following two conditions are equivalent:*

(a) *For every pure extension $A \rightarrowtail R \twoheadrightarrow S$, the natural map*

$$HC_n(A) \xrightarrow{\psi_n} HC_n(R; A),$$

where $HC_n(R; A) \equiv HC_n(R \to S)$ is the corresponding relative cyclic group, is surjective;

(b) *the groups $HB_q(A)$ vanish for $1 \leq q \leq n$.*

(II) *Assume that A satisfies any one of the equivalent conditions above. Then there exists a functorial exact sequence*

$$HB_{n+1}(A; S) \longrightarrow HC_n(A) \xrightarrow{\psi_n} HC_n(R; A) \longrightarrow 0. \tag{23}$$

In particular, if $HB_q(A)$ vanishes for $1 \leq q \leq n+1$, the excision map ψ_n: $HC_n(A) \to HC_n(R; A)$ is an isomorphism for an arbitrary pure extension with kernel A. □

Description of the map $HB_{n+1}(A; S) \to HC_n(A)$. Let E^k_{**} be the s.s. associated with the following filtration on the double complex $\mathscr{D} = \operatorname{Ker}(\mathscr{C}(R) \to \mathscr{C}(S))$:

$$F_p\mathscr{D}_{kl} = \text{lin span}\{ r_0 \otimes \cdots \otimes r_l | \text{at least } l-p+1 \quad r_j\text{'s belong to } A \}.$$

One has $(E^0_{p*}, d^0) = \operatorname{Tot}\operatorname{Gr}^F_p \mathscr{D}$; in particular, $(E^0_{0*}, d^*) = \operatorname{Tot}\mathscr{C}(A)$. The double complex $\operatorname{Gr}^F_1 \mathscr{D}$ coincides with the complex $\mathscr{C}\langle \mathscr{N} \rangle[0, -1]$, for $\mathscr{N} = (1)$ and $V = S$, from the proof of the implication (a) ⇒ (e) in Theorem 3.1. It has therefore an augmentation to the (single) complex $B_*(A) \otimes S[-1]$ which induces a quasi-isomorphism $(E^0_{1*}, d^0) = \operatorname{Tot}\operatorname{Gr}^F_1 \mathscr{D} \to B_*(A) \otimes S[-1]$. The mapping $HB_{n+1}(A; S) \to HC_n(A)$ of (23) is then defined as the composite map $HB_{n+1}(A; S) \xleftarrow{\sim} E^1_{1,n} \xrightarrow{d^1} E^1_{0,n} \equiv HC_n(A)$.

Knopfmacher formulae. In the case when the ground ring k is a field, the Bar homology groups $HB_*(A)$, which by Theorem 3.1 are the only obstructions to the excision in Hochschild and in cyclic homology, admit an interesting description due to J. Knopfmacher [23]. Assume that A is represented as a quotient of a free associative algebra without unit $\mathfrak{A}$ by an ideal of relations I. Let us consider the following four decreasing filtrations by ideals

$\mathfrak{A} = F_\nu^0 \supset F_\nu^1 \supset \cdots, (\nu = 1, \ldots, 4)$:

$$F_1^p = I^p, \qquad F_2^p = \mathfrak{A} I^{p-1} \mathfrak{A}, \qquad F_3^p = \mathfrak{A} I^p \quad \text{and} \quad F_4^p = I^p \mathfrak{A}$$

which are linked by the obvious inclusions

$$\begin{array}{ccccc} & & F_1^p & & \\ & \cup & & \cup & \\ F_3^p & & & & F_4^p \\ & \cup & & \cup & \\ & & F_2^p & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & F_1^{p+1} & & \\ & \cap & & \cap & \\ F_3^p & & & & F_4^p \\ & \cup & & \cup & \\ & & F_2^{p+1} & & \end{array}$$

(we take $F_2^1 = \mathfrak{A}^2$).

3.6. Theorem (J. Knopfmacher [23]). *There are canonical isomorphisms*

$$HB_{2p}(A) \simeq \frac{F_1^p \cap F_2^p}{F_3^p + F_4^p} \quad \text{and} \quad HB_{2p+1}(A) = \frac{F_3^p \cap F_4^p}{F_1^{p+1} + F_2^{p+1}}$$

$(p \geq 0)$. □

These beautiful formulae follow easily from the existence of the following $\tilde{A}$-free resolution of k (*Gruenberg resolution*):

$$0 \leftarrow k \leftarrow \tilde{\mathfrak{A}}/F_1^1 \leftarrow F_4^0/F_4^1 \leftarrow F_1^1/F_1^2 \leftarrow F_4^1/F_4^2 \leftarrow F_1^2/F_1^3 \leftarrow F_4^2/F_4^3 \leftarrow \cdots.$$

We obtain therefore the following:

3.7. Corollary. *An algebra A over a field is H-unital if and only if for some (and therefore for any) free presentation $I \rightarrowtail \mathfrak{A} \twoheadrightarrow A$, the ideal of relations I satisfies the identities*

$$I^p \cap \mathfrak{A} I^{p-1} \mathfrak{A} = \mathfrak{A} I^p + I^p \mathfrak{A} \quad \text{and} \quad I^p + \mathfrak{A} I^{p-1} \mathfrak{A} = \mathfrak{A} I^{p-1} \cap I^{p-1} \mathfrak{A}$$

$(p \geq 1)$. □

If A is commutative, $HB_*(A)$ admits yet another description in terms of André-Quillen deformation homology groups. We will formulate this under assumptions slightly more restrictive than necessary, supposing that k is a field of characteristic zero. In this case $HB_*(A)$ has a natural structure of a commutative graded Hopf algebra (cf. [1]) and its Lie coalgebra of indecomposables $Q(HB_*(A))$ identifies with the André-Quillen homology group $D_*(k/\tilde{A})$ (cf. [34, Thm. 7.3]). Since the canonical map $HB_*(A) \to D_*(k/\tilde{A})$ extends to an isomorphism of k-algebras $HB_*(A) \xrightarrow{\sim} S_k(D_*(k/\tilde{A}))$, where S_k denotes the functor of symmetric k-algebra (in the graded sense), we arrive at the following:

3.8. Observation. *A commutative algebra over a field k of characteristic zero is H-unital if and only if $D_*(k/\tilde{A}) = 0$ (note that $D_0(k/\tilde{A}) \equiv 0$).* □

For the generalization of 3.8 to the case of an arbitrary ground ring of characteristic zero, see [34, Thm. 7.8].

We will end this section with one more observation concerning commutative H-unital algebras. Let $\mathscr{I}$ be a finitely generated ideal in a commutative unital ring $\mathscr{R}$ which satisfies $\mathscr{I} = \mathscr{I}^2$. Then, in view of one of the variants of Nakayama's Lemma (cf. e.g. [12, Ex. 3.30(b).1 or Lemma 12.1]) there exists an idempotent $e \in \mathscr{R}$ such that $\mathscr{I} = \mathscr{R}e$. In particular, $\mathscr{I}$ possesses a unit. This leads to the following:

3.9. Observation. *A commutative H-unital algebra A is either infinitely generated (as a module over itself) or there exists a unit in A.* □

Notice that there are examples of *noncommutative* finite k-algebras which are H-unital but do not possess even a one-sided unit. Probably the simplest example of this kind is provided by the algebra $A = k^{\oplus 4}$ with the multiplication given by

$$(a, b, c, d) \cdot (a', b', c', d') = (aa', ab', cc', dc'),$$

which is realized as the direct sum of the subalgebras of row- and, respectively, column-vectors in $M_2(k)$:

$$A = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}.$$

Its H-unitality follows from Corollaries 4.5 and 3.4. More general H-unital matrix algebras will be considered in Section 11.

4. H-unitality criterion

The following proposition may serve as a useful tool in establishing H-unitality of several algebras.

4.1. Proposition. *Assume that a k-algebra A can be represented in the form $A = \bigcup_{i \in \mathscr{I}} A_i$ where:*

(1) *All A_i's are right (respectively, left) A-submodules of A;*

(2) *Each $A_i \subset A$ is pure as a k-submodule;*

(3) *For every $i \in \mathscr{I}$, there exists an A-linear map ϕ_i: $A_i \to A \otimes A$ making the following diagram*

$$\begin{array}{ccc} A_i & \xrightarrow{\phi_i} & A \otimes A \\ & \searrow & \downarrow \text{multiplication} \\ & & A \end{array}$$

commutative;

(4) *Every finite subset $\mathscr{P} \subset A$ is contained in some A_i.*
Then A is H-unital.

Proof. First a word of explanation about the A-linearity condition for ϕ_i. The module $A \otimes A$ has the right A-action via $(a' \otimes a'') \cdot a = a' \otimes a''a$ and the left A-action via $a \cdot (a' \otimes a'') = aa' \otimes a''$. Then ϕ_i's are assumed to be maps of either right A-modules, if all A_i's are right A-modules, or of left A-modules, if A_i's are left A-modules.

Let V be a k-module and $\alpha \in B_q(A; V)$ be a Bar q-chain with coefficients in V. Assume that A is a union of right A-submodules satisfying conditions (2), (3) and (4). Then $\alpha \in A_i \otimes A^{\otimes(q-1)} \otimes V$, for some $i \in \mathscr{I}$, and we have the following diagram

$$\begin{array}{ccccc} B_{q+1}(A;V) & \xrightarrow{b'} & B_q(A;V) & & \\ & \nwarrow^{\phi_{i,q}} & \cup & \nwarrow^{\phi_{i,q-1}} & \\ & & A_i \otimes A^{\otimes(q-1)} \otimes V & \xrightarrow{b'} & A_i \otimes A^{\otimes(q-2)} \otimes V \end{array}$$

where $\phi_{i,q} = \phi_i \otimes \mathrm{id}_{A^{\otimes(q-1)} \otimes V}$ and $\phi_{i,q-1} = \phi_i \otimes \mathrm{id}_{A^{\otimes(q-2)} \otimes V}$. One easily verifies, using conditions (1) and (3), that $\alpha = b' \circ \phi_{i,q}(\alpha) + \phi_{i,q-1} \circ b'(\alpha)$.

Assume now that α is a cycle, i.e. $b'(\alpha) = 0$ in $B_{q-1}(A; V)$. Then, in view of condition (2), $b'(\alpha) = 0$ also in $A_i \otimes A^{\otimes(q-2)} \otimes V$; hence $\alpha = b' \circ \phi_{i,q}(\alpha)$, i.e. α is a boundary and the complex $B_*(A; V)$ is acyclic.

If A, instead, is a union of left A-submodules satisfying conditions (2)–(4), we set

$$\phi_{i,q} = (-1)^{q+1} \mathrm{id}_{A^{\otimes(q-1)}} \otimes \phi_i \otimes \mathrm{id}_V \quad \text{and} \quad \phi_{i,q-1} = (-1)^{q} \mathrm{id}_{A^{\otimes(q-2)}} \otimes \phi_i \otimes \mathrm{id}_V.$$

□

4.2. *Remark.* Condition (2) of Proposition 4.1 is in fact too strong. All we need is that the mapping $\phi_{i,q-1}$: $A_i \otimes A^{\otimes(q-2)} \otimes V \to B_q(A; V)$ vanish on

$$\mathrm{Ker}\left\{ j_i \otimes \mathrm{id}_{A^{\otimes(q-2)} \otimes V} \colon A_i \otimes A^{\otimes(q-2)} \otimes V \to B_{q-1}(A; V) \right\},$$

where j_i denotes the natural inclusion $A_i \hookrightarrow A$, or, in other words, that $\phi_{i,q-1}$ pass to the image of $A_i \otimes A^{\otimes(q-2)} \otimes V$ in $B_{q-1}(A; V)$.

This is so, for example, if each ϕ_i: $A_i \to A \otimes A$ admits an extension to a k-module map $A \to A \otimes A$ (we do not require that this extension continue to satisfy the conditions imposed on ϕ_i).

4.3. Corollary. *Let A be a left or right ideal in a unital algebra B which is contained in no proper two-sided ideal. Then A is H-unital.*

Proof. Let A be, say, a left ideal. The condition on A means that $A \cdot B = B$, i.e. that $1 = \sum_{\nu=1}^{n} a^{\nu} b^{\nu}$ for certain $n \in \mathbf{N}$, $a^{\nu} \in A$ and $b^{\nu} \in B$; $\nu = 1, \ldots, n$. Define the *right* A-module map $\phi \colon A \to A \otimes A$ by setting $a \mapsto \sum_{\nu} a^{\nu} \otimes b^{\nu} a$. It remains to apply Proposition 4.1. □

4.4. COROLLARY. *Every left and every right ideal in a simple algebra with unit is H-unital.*

This follows immediately from the previous corollary.

4.5. COROLLARY. *Assume that in an algebra A every finite subset has a common left (respectively, right) unit. Then A is H-unital.*

Proof. Let $\mathscr{I}$ be the set of finite subsets of A, and for each $i = \{a^1, \ldots, a^n\} \in \mathscr{I}$, let $A_i = a^1 A + \cdots + a^n A$ be the corresponding right A-submodule. If e_i is the corresponding left unit, we set $\phi_i = \tilde{\phi}_{i|A_i}$ where $\tilde{\phi}_i \colon A \to A \otimes A$ is defined as $a \mapsto e_i \otimes a$. In general, $A_i \subset A$ need not be pure. The conclusion of Proposition 4.1 still applies, however (cf. Remark 4.2).

4.6. COROLLARY. *The* **Z**-*algebra* $C^{\infty}_{\mathrm{comp}}(X)$ *of* C^{∞}-*functions with compact support on a* C^{∞}-*manifold* X *is H-unital.* □

The algebra $C^{\infty}_{\mathrm{comp}}(X)$ satisfies the hypothesis of the previous corollary.

4.7. *Examples.* (1) Let $\mathscr{D}^{\mathrm{alg}}(X)$ denote the algebra of differential operators on a regular affine algebraic variety X over a field K. It is a simple ring (the fact well-known in characteristic zero and proved in positive characteristic by S. P. Smith, cf. [42, Prop. 3.4]; we take an algebraic variety to be irreducible by definition). Therefore, according to Corollary 4.4, every one-sided ideal in $\mathscr{D}^{\mathrm{alg}}(X)$, treated as a k-algebra over an arbitrary subring $k \subset K$, is H-unital.

(2) Let S be a semigroup satisfying the following two conditions:

(a) For every $s \in S$ the left (respectively, right) multiplication by s defines an injective mapping $S \to S$;

(b) Every finite subset $\Sigma \subset S$ is contained in a certain cyclic right ideal sS (respectively, cyclic left ideal Ss).

Then the semigroup ring $A = \mathbf{Z}[S]$ satisfies the assumption of Proposition 4.1 (with $k = \mathbf{Z}$); it is free over $\mathbf{Z}$, the $\phi_s \colon s\sigma \mapsto s \otimes \sigma$ define the required mappings $\phi_s \colon sA \to A \otimes A$ and every finite subset $\mathscr{P} \subset A$ is contained in some sA.

An interesting example of a semigroup satisfying conditions (a) and (b) above appeared recently in connection with conformal field theory. Graeme Segal introduced in [39] the so called "semigroup of annuli" $\mathscr{E}$ whose elements are triples (X, f_1, f_2) where X is a Riemann surface with oriented boundary

$\partial X = - Y_1 \cup Y_2$, which is homeomorphic to $S^1 \times [0,1]$, and f_i: $S^1 \to Y_i$, $i = 1,2$, are real-analytic parametrizations of the connected components of ∂X. Two triples (X, f_1, f_2) and (X', f_1', f_2') are identified if there exists a holomorphic homeomorphism φ: $X \to X'$ such that $f_i' = \varphi \circ f_i$, $i = 1,2$ (cf. [39]); the semigroup $\mathscr{E}$ is some sort of complexification of the group $\mathrm{Diff}_+(S^1)$ of orientation preserving diffeomorphisms of S^1. In particular, $\mathscr{E}$ is H-unital; i.e. its semigroup algebra $\mathbf{Z}[\mathscr{E}]$ is H-unital.

There is a variant of the above with parametrizations of the boundary components only of a certain class of smoothness. The resulting semigroup in general, does not satisfy condition (b). Its semigroup algebra, however, is a direct sum of (right) ideals satisfying the assumptions of Proposition 4 and, therefore, is H-unital too.

In Segal's theory $\mathscr{E}$ is the "identity" component of the semigroup of morphisms between "simple" strings. Also this bigger semigroup, as well as the "string" semi-category $\mathscr{C}$ containing it (cf. [39]), are H-unital (in the latter case this means, of course, that the corresponding algebra $\mathbf{Z}[\mathscr{C}]$ which generalizes the semigroup algebra, is H-unital).

(3) Here is another example related to semigroups. Let us fix an integer $l \geq 2$ and let I denote the augmentation ideal of the algebra

$$k\left[t, t^{l^{-1}}, t^{l^{-2}}, \dots\right] = \bigcup_{n=0}^{\infty} k\left[t^{l^{-n}}\right]$$

(k is an arbitrary commutative ring). In I one recognizes easily the semigroup algebra of the additive semigroup $\mathbf{Z}_+[l^{-1}]$. The latter clearly satisfies the conditions (a) and (b) mentioned in the previous example, hence I is H-unital.

Let $J \subset I$ be the principal ideal tI. The quotient algebra $A = I/J$ can be thought of as the "semigroup algebra of the semigroup with zero" $\mathbf{Z}_+[l^{-1}]/(\mathbf{Z}_+[l^{-1}] \cap \{z > 1\})$.

We will prove that A is H-unital, by refining the argument that was used to prove Proposition 4.1.

Let $\alpha \in B_q(A; V)$ be a Bar q-cycle over A with coefficients in a k-module V. It can be represented as a finite sum

$$\sum_{c;\, v} c t^{m_1/l^n} \otimes \cdots \otimes t^{m_q/l^n} \otimes v$$

of monomials with common $n \in \mathbf{N}$ where $c \in k$, $v \in V$ and $m_1, \dots, m_q \in \{1, 2, \dots, l^n\}$. Then also

$$\beta := \sum c t^{\frac{m_1}{l^n} - \frac{1}{l^{n+1}}} \otimes t^{m_2/l^n} \otimes \cdots \otimes t^{m_q/l^n} \otimes v$$

is a well-defined Bar q-chain and one clearly has

$$\alpha = b'\left(t^{1/l^{n+1}} \otimes \beta\right) + t^{1/l^{n+1}} \otimes b'(\beta).$$

The boundary of β belongs to the k-submodule $\mathscr{X} \subset B_{q-1}(A; V)$ spanned by monomials

$$t_1^{\frac{p_1}{l^n} - \frac{1}{l^{n+1}}} \otimes t_2^{p_2/l^n} \otimes \cdots \otimes t_{q-1}^{p_{q-1}/l^n} \otimes v$$

$(p_1, \dots, p_{q-1} \in \{1, 2, \dots, l^n\})$, and $(t^{1/l^{n+1}} \otimes 1 \otimes \cdots \otimes 1)b'(\beta) = b'(\alpha) = 0$. Since the multiplication by $t^{1/l^{n+1}} \otimes 1 \otimes \cdots \otimes 1$ defines an *injective* map $\mathscr{X} \to B_{q-1}(A; V)$, we obtain thus that $b'(\beta) = 0$ and α is a boundary

$$\alpha = b'\left(t^{1/l^{n+1}} \otimes \beta\right).$$

This proves that A is H-unital.

The above example demonstrates that the algebraic properties of H-unital algebras may be very different from those of unital ones. Every element in A is nilpotent (in particular, the only idempotent in A is 0) and $\mathrm{Ann}(A) = \{a \in A \mid aA = Aa = 0\}$ is not zero:[1]

$$\mathrm{Ann}(A) = k \cdot t.$$

In the case when k is a field, A is a maximal ideal in the local ring $\tilde{A}$ and the latter is a union of artinian rings.

(4) Let $\mathfrak{g}$ be a Lie k-algebra. The augmentation ideal $\mathscr{I}\mathfrak{g}$ of its enveloping algebra $U\mathfrak{g}$ is H-unital precisely in the case when $\mathfrak{g}$ is acyclic; i.e. $H_*(\mathfrak{g}; k) = 0$, for $* > 0$.

Example. $\mathfrak{g} = \mathfrak{gl}_\infty(A_\infty)$ where A_∞ is the Weyl algebra in infinitely many variables $k\langle p_1, p_2, \dots; q_1, q_2, \dots \rangle$, $[p_i, p_j] = [q_i, q_j] = 0$, $[p_i, q_j] = \delta_{ij}$, over a field of characteristic zero [14].

Similarly, the augmentation ideal I_G of the group algebra $k[G]$ is H-unital precisely in the case when the group G is k-acyclic; i.e. $H_*(G; k) = 0$, for $* > 0$.

5. Locally convex algebras

Let $\mathscr{A}$ be a locally convex k-algebra ($k = \mathbf{R}$ or $\mathbf{C}$), i.e. a locally convex vector space (not necessarily complete), such that the multiplication $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is separately continuous. Let us consider the following multiple factorization

[1] This answers a question by Christian Kassel.

property:

(F) *For every* $n = 1, 2, \ldots$ *and every* $(a^1, \ldots, a^n) \in \mathscr{A}^{\oplus n}$ *there exist such* $z \in \mathscr{A}$ *and* $(x^1, \ldots, x^n) \in \mathscr{A}^{\oplus n}$ *that*

$$z \cdot (x^1, \ldots, x^n) = (a^1, \ldots, a^n) \quad \text{and}$$
$$(x^1, \ldots, x^n) \in \overline{\mathscr{A} \cdot (a^1, \ldots, a^n)}.$$

(the horizontal bar denotes the topological closure in $\mathscr{A}^{\oplus n}$). The corresponding property of right factorization will be denoted (F°).

5.1. PROPOSITION. *Let* $\mathscr{A}$ *be a locally convex algebra and B be an arbitrary unital k-algebra* ($k = \mathbf{R}$ *or* $\mathbf{C}$). *Assume that* $\mathscr{A}$ *satisfies condition* (F) *or* (F°) *above. Then the algebra* $\mathscr{A} \otimes_k B$ *is H-unital (as an abstract k-algebra).*

Proof. Let us equip B with the strongest locally convex topology, i.e. the topology of the inductive limit $\varinjlim_{B_0 \subset B} B_0$ taken over all finite-dimensional linear subspaces $B_0 \subset B$. With respect to this inductive topology *every* linear map into a locally convex vector space is automatically continuous. In particular, B becomes a locally convex algebra.

We shall consider the algebraic tensor products $(\mathscr{A} \otimes B)^{\otimes q}$, $q = 1, 2, \ldots,$ with their strongest locally convex and "compatible-with-the-structure-of-tensor-product" (in the sense of Grothendieck [17, I.3.3]) topologies. With respect to them the boundary maps $b'\colon (\mathscr{A} \otimes B)^{\otimes q} \to (\mathscr{A} \otimes B)^{\otimes (q-1)}$ become continuous.

Let $\alpha = \sum_{\nu=1}^{n} (a_1^\nu \otimes b_1^\nu) \otimes \cdots \otimes (a_q^\nu \otimes b_q^\nu)$ be a q-chain. Assume that $\mathscr{A}$ has the property of left factorization (F). There exist, thus, such $z, x^1, \ldots, x^n \in \mathscr{A}$ that $a_1^\nu = z x^\nu$ ($\nu = 1, \ldots, n$) and

$$\beta := \sum_{\nu=1}^{n} (x^\nu \otimes b_1^\nu) \otimes (a_2^\nu \otimes b_2^\nu) \otimes \cdots \otimes (a_q^\nu \otimes b_q^\nu) \in \overline{\mathscr{A} \cdot \alpha} \text{ in } (\mathscr{A} \otimes B)^{\otimes q}.$$

In particular, by the continuity and $\mathscr{A}$-linearity of the boundary map, $b'(\beta) = 0$ if α is a cycle. Since in general, as one verifies easily, $\alpha = b'((z \otimes 1_B) \otimes \beta) + (z \otimes 1_B) \otimes b'(\beta)$, we obtain a representation of every q-cycle as an explicit boundary.

If $\mathscr{A}$, instead, satisfies (F°) we use a presentation of α as $b'(\gamma \otimes (y \otimes 1_B)) + b'(\gamma) \otimes (y \otimes 1_B)$ where

$$\gamma = (-1)^{q-1} \sum_{\nu=1}^{n} (a_1^\nu \otimes b_1^\nu) \otimes \cdots \otimes (a_{q-1}^\nu \otimes b_{q-1}^\nu) \otimes (w^\nu \otimes b_q^\nu)$$

and $a_q^\nu = w^\nu y$ ($\nu = 1, \ldots, n$). □

5.2. *Remark.* The gain in generality in Proposition 5.1 over the special case $B = k$ is only apparent. If $\mathscr{A}_1, \ldots, \mathscr{A}_m$ are locally convex algebras, their algebraic tensor product $\mathscr{A}_1 \otimes_\pi \cdots \otimes_\pi \mathscr{A}_m$ equipped with the projective tensor product topology is again locally convex with separately continuous multiplication. If all the $\mathscr{A}_i$'s possess the property (F) (respectively, property (F°)), then also $\mathscr{A}_1 \otimes_\pi \cdots \otimes_\pi \mathscr{A}_m$ possesses the property (F) (respectively, property (F°)).

6. Ideals of flat C^∞-functions

Given a closed subset Y in a smooth manifold X we shall denote by $C^\infty(X, Y)$ the algebra of C^∞-functions *flat* (i.e. vanishing with all derivatives) on Y. Elements of the quotient-algebra $C_Y^\infty(X) = C^\infty(X)/C^\infty(X, Y)$ will then be called *Whitney C^∞-functions on* Y. As the correspondence $U \mapsto C^\infty(U)/C^\infty(U, Y \cap U)$ defines a (soft) sheaf, the notion of "Whitney C^∞-function" is local. By using local coordinates and Whitney's extension theorem (cf. [29, Thm. I.3.2 and Prop. I.5.3]), we can view elements of $C_Y^\infty(X)$ locally as classical "C^∞-functions on Y in the sense of Whitney" (cf. [29, I.2.3 and I.5]).

Finally, if $Z \subset Y$ is a closed subset we shall introduce also the algebra $C_Y^\infty(X, Z) = \mathrm{Ker}(C_Y^\infty(X) \to C_Z^\infty(X))$ of "Whitney C^∞-functions on Y" which are flat on Z.

Examples. For $Y = \overline{\mathrm{Int}\, Y}$, $C_Y^\infty(X) = C^\infty(Y)$ where $C^\infty(Y)$ has the usual sense (C^∞-functions on Int Y admitting a C^∞-extension in a neighborhood of Y). For $Y = \{x_0\}$ a point, $C_Y^\infty(X) = \mathscr{J}_{x_0}^\infty$ (the algebra of ∞-jets at x_0).

6.1. Theorem. *The algebra $C_Y^\infty(X, Z)$ is H-unital and its "discrete" cyclic homology fits into the following two long exact sequences*

$$(24)\qquad \cdots \to HC_{q+1}(C_Z^\infty(X)) \to HC_q(C_Y^\infty(X, Z)) \to HC_q(C_Y^\infty(X)) \to HC_q(C_Z^\infty(X)) \to \cdots$$

and

$$(25)\qquad \cdots \to HC_{q+1}(C_Y^\infty(X, Z)) \to HC_q(C^\infty(X, Y)) \to HC_q(C^\infty(X, Z)) \to HC_q(C_Y^\infty(X, Z)) \to \cdots .$$

There are also similar exact sequences in Hochschild homology.

6.2. Corollary. *Let Z_1 and Z_2 be two closed subsets of X which are either disjoint or satisfy the so called Condition* (Λ) (*cf.* [29, *Thm.* I.5.5], *also called Łojasiewicz's Condition*). *Then there is the following Mayer-Vietoris long exact sequence in cyclic homology*

$$(26)\qquad \cdots \to HC_{q+1}\big(C_{Z_1 \cap Z_2}^\infty(X)\big) \to HC_q\big(C_{Z_1 \cup Z_2}^\infty(X)\big) \to HC_q(C_{Z_1}^\infty(X)) \oplus HC_q(C_{Z_2}^\infty(X)) \to \cdots$$

and a similar exact sequence in Hochschild homology.

Proof. Given a pair of closed subsets $Z_1, Z_2 \subset X$ with $Z_1 \cap Z_2 \neq \varnothing$, the square

$$\begin{array}{ccc} C^\infty_{Z_1 \cup Z_2}(X) & \longrightarrow & C^\infty_{Z_2}(X) \\ \downarrow & & \downarrow \\ C^\infty_{Z_1}(X) & \longrightarrow & C^\infty_{Z_1 \cap Z_2}(X) \end{array} \tag{27}$$

is Cartesian if and only if the subsets Z_1 and Z_2 satisfy the condition (Λ) (S. Łojasiewicz, cf. [29, Thm. I.5.5]). □

6.3. COROLLARY. *The algebra $\mathscr{S}(\mathbf{R}^d)$ of C^∞-functions of fast decay is H-unital and its "discrete" cyclic homology is included into the long exact sequence*

$$\begin{aligned} \cdots \to HC_{q+1}(k[[u^1, \ldots, u^d]]) \to HC_q(\mathscr{S}(\mathbf{R}^d)) \to HC_q(C^\infty(S^d)) \\ \to HC_q(k[[u^1, \ldots, u^d]]) \to \cdots . \end{aligned} \tag{28}$$

There is a similar exact sequence in Hochschild homology ($k = \mathbf{R}$ or $\mathbf{C}$; $u^1, \ldots, u^d$ denote local parameters at $\infty \in S^d = \mathbf{R}^d \cup \{\infty\}$). □

Proof of Theorem 6.1. Let us first consider the commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & C^\infty(X, Y) & \longrightarrow & C^\infty(X, Z) & \dashrightarrow & C^\infty_Y(X, Z) \dashrightarrow 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & C^\infty(X) & = & C^\infty(X) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^\infty_Y(X, Z) & \longrightarrow & C^\infty_Y(X) & \longrightarrow & C^\infty_Z(X) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{29}$$

The exactness of columns implies that $C^\infty_Y(X) \to C^\infty_Z(X)$ is surjective (hence the lowest row is exact too). As a result the composite homomorphism $C^\infty(X, Z) \to C^\infty_Y(X)$ identifies $C^\infty_Y(X, Z)$ with the quotient algebra $C^\infty(X, Z)/C^\infty(X, Y)$. In view of Corollary 3.4, the question reduces then to proving that $C^\infty(X, Z)$ is H-unital for an arbitrary closed subset $Z \subset X$.

Let $\mathscr{U} = \{\mathscr{U}_{i\alpha}\}$, $(i = 1, \ldots, m;\ \alpha \in \mathscr{I}_i)$, be a locally finite family of open pre-compact domains of local coordinates $\mathscr{U}_{i\alpha} \to \mathbf{R}^d$ such that $\bigcup_{i;\alpha} \mathscr{U}_{i\alpha} \supset \overline{X \setminus Z}$ and $\mathscr{U}_{i\alpha} \cap \mathscr{U}_{i\beta} = \varnothing$, for every i and $\alpha \neq \beta$. We may consider only $\mathscr{U}_{i\alpha}$'s with $\mathscr{U}_{i\alpha} \cap \overline{X \setminus Z} \neq \varnothing$.

For such a family, every compact set $K \subset X$ has a non-empty intersection with not more than finitely many $\mathscr{U}_{i\alpha}$'s. Finally, we choose an arbitrary partition of unity $\{\varphi_{i\alpha}\}_{i;\alpha}$ on $\bigcup_{i;\alpha} \mathscr{U}_{i\alpha}$ satisfying $W_{i\alpha} := \operatorname{supp} \varphi_{i\alpha} \subset \mathscr{U}_{i\alpha}$.

Let $\psi = \sum_{\nu=1}^{n} f_1^\nu \otimes \cdots \otimes f_q^\nu \in B_q(C^\infty(X, Z))$ be an arbitrary Bar q-chain over $C^\infty(X, Z)$. By using local coordinates we may view $W_{i\alpha} \cap \overline{X \setminus Z}$ as a compact subset of $\mathbf{R}^d$ and $\varphi_{i\alpha} f_1^\nu$ as an element of $C^\infty(\mathbf{R}^d, \mathscr{Z}_{i\alpha})$ where $\mathscr{Z}_{i\alpha} := \mathbf{R}^d \setminus [(\operatorname{Int} W_{i\alpha} \cap (X \setminus Z))]$.

If $\mathscr{Z} \subset \mathbf{R}^d$ is an arbitrary closed subset with a pre-compact complement, it follows from a theorem of J. Voigt [46, Thm. 3.4] that the algebra $C^\infty(\mathbf{R}^d, \mathscr{Z})$ has the factorization property (F). There exist, therefore, such $h_{i\alpha}$ and $g_{i\alpha}^\nu \in C^\infty(\mathbf{R}^d, \mathscr{Z}_{i\alpha})$ that $\varphi_{i\alpha} f_1^\nu = h_{i\alpha} g_{i\alpha}^\nu$ and $\operatorname{supp} g_{i\alpha}^\nu = \operatorname{supp}(\varphi_{i\alpha} f_1^\nu)$. Both $h_{i\alpha}$ and $g_{i\alpha}^\nu$ may be viewed as elements of $C^\infty(X, Z)$ with supports in $W_{i\alpha} \cap \overline{X \setminus Z}$. Notice that, in view of the properties of $\mathscr{U}$, each $\varphi_{(i)} = \sum_{\alpha \in \mathscr{I}_i} \varphi_{i\alpha}$ is a well-defined element of $C^\infty(\bigcup_{i;\alpha} \mathscr{U}_{i,\alpha})$, each $h_{(i)} = \sum_{\alpha \in \mathscr{I}_i} h_{i\alpha}$ and $g_{(i)}^\nu = \sum_{\alpha \in \mathscr{I}_i} g_{i\alpha}^\nu$ is a well-defined element of $C^\infty(X, Z)$ and one has

$$\varphi_{(i)} f_1^\nu = h_{(i)} g_{(i)}^\nu, \qquad (i = 1, \ldots, m;\ \nu = 1, \ldots, n).$$

If we put $\psi_{(i)} = \sum_{\nu=1}^{n} \varphi_{(i)} f_1^\nu \otimes f_2^\nu \otimes \cdots \otimes f_q^\nu$ and

$$\phi_{(i)} = \sum_{\nu=1}^{n} g_{(i)}^\nu \otimes f_2^\nu \otimes \cdots \otimes f_q^\nu,$$

we shall have $\psi = \psi_{(1)} + \cdots + \psi_{(m)}$ and $\psi_{(i)} = b'(h_{(i)} \otimes \phi_{(i)}) + h_{(i)} \otimes b'(\phi_{(i)})$. It remains to prove that $b'(\phi_{(i)}) = 0$, $i = 1, \ldots, m$, whenever ψ is a cycle.

Let us view each $h_{i\alpha} \otimes 1 \otimes \cdots \otimes 1$ and $b'(\sum_{\nu=1}^{n} g_{i\alpha}^\nu \otimes f_2^\nu \otimes \cdots \otimes f_q^\nu)$ as a C^∞-function on $X \times \cdots \times X$ ($q - 1$ times). Their product, being equal to $(\varphi_{i\alpha} \otimes 1 \otimes \cdots \otimes 1) \cdot b'(\psi)$, vanishes. On the other hand

$$\operatorname{supp}(h_{i\alpha} \otimes 1 \otimes \cdots \otimes 1) \supset \operatorname{supp} b'\left(\sum_{\nu=1}^{n} g_{i\alpha}^\nu \otimes f_2^\nu \otimes \cdots \otimes f_q^\nu \right);$$

hence $b'(\sum_{\nu=1}^{n} g_{i\alpha}^\nu \otimes f_2^\nu \otimes \cdots \otimes f_q^\nu) = 0$ and ψ is equal to the boundary $b'(\sum_{i=1}^{m} h_{(i)} \otimes \phi_{(i)})$. □

7. Ideals of flat differential operators

Let $\mathscr{D}(X)$ denote the algebra of C^∞ differential operators on X. For a closed set $Y \subset X$ we define "Whitney differential operators" on Y as $\mathscr{D}_Y(X) = C_Y^\infty(X) \otimes_{C^\infty(X)} \mathscr{D}(X)$; similarly defined are Whitney differential operators *flat* on a closed subset $Z \subset Y$, $\mathscr{D}_Y(X, Z) = C_Y^\infty(X, Z) \otimes_{C^\infty(X)} \mathscr{D}(X)$. It is clear that Whitney differential operators can be composed as usual.

Notice that tensoring by $\mathscr{D}(X)$ the corresponding rows of (29) ($\mathscr{D}(X)$ is projective as a left $C^\infty(X)$-module) yields the following two short exact sequences:

$$0 \to \mathscr{D}_Y(X, Z) \to \mathscr{D}_Y(X) \to \mathscr{D}_Z(X) \to 0 \quad \text{and} \tag{30}$$

$$0 \to \mathscr{D}(X, Y) \to \mathscr{D}(X, Z) \to \mathscr{D}_Y(X, Z) \to 0. \tag{31}$$

In particular, $\mathscr{D}_Y(X, Z)$ is an ideal in $\mathscr{D}_Y(X)$.

7.1. Theorem. *The algebra $\mathscr{D}_Y(X, Z)$ is H-unital and its "discrete" cyclic homology is included in the following two long exact sequences*

$$\cdots \to HC_{q+1}(\mathscr{D}_Z(X)) \to HC_q(\mathscr{D}_Y(X, Z)) \to HC_q(\mathscr{D}_Y(X)) \to HC_q(\mathscr{D}_Z(X)) \to \cdots \quad \text{and} \tag{32}$$

$$\cdots \to HC_{q+1}(\mathscr{D}_Y(X, Z)) \to HC_q(\mathscr{D}(X, Y)) \to HC_q(\mathscr{D}(X, Z)) \to HC_q(\mathscr{D}_Y(X, Z)) \to \cdots. \tag{33}$$

There are similar long exact sequences in Hochschild homology.

7.2. Corollary. *Let Z_1 and Z_2 be a pair of closed subsets of X which are either disjoint or satisfy the Condition (Λ) (cf. §6). Then there is the following Mayer-Vietoris exact sequence in "discrete" cyclic homology*

$$\cdots \to HC_{q+1}\big(\mathscr{D}_{Z_1 \cap Z_2}(X)\big) \to HC_q\big(\mathscr{D}_{Z_1 \cup Z_2}(X)\big) \to HC_q\big(\mathscr{D}_{Z_1}(X)\big) \oplus HC_q\big(\mathscr{D}_{Z_2}(X)\big) \to \cdots \tag{34}$$

and a similar long exact sequence in Hochschild homology.

This follows from Theorem 7.1 and the observation that the square

$$\begin{array}{ccc} \mathscr{D}_{Z_1 \cup Z_2}(X) & \longrightarrow & \mathscr{D}_{Z_2}(X) \\ \downarrow & & \downarrow \\ \mathscr{D}_{Z_1}(X) & \longrightarrow & \mathscr{D}_{Z_1 \cap Z_2}(X) \end{array}$$

is Cartesian precisely when (27) is Cartesian.

Before proving Theorem 7.1 we will need the following variant of Theorem 6.1. Let $\pi\colon V \to X$ be a vector bundle on X and $\mathscr{O}(V; Z) = \oplus_{p=0}^{\infty} \mathscr{O}(V; Z)(p)$ be the corresponding graded algebra of C^∞-functions on V which are polynomial of finite order along fibres of π and are flat on $\pi^{-1}(Z)$.

7.3. Proposition. *The algebra $\mathscr{O}(V; Z)$ is H-unital.*

Proof. Embed V as a direct summand into a trivial bundle $W = X \times \mathbf{R}^N$ (by using partitions of unity similar to those used in the proof of Theorem 6.1, this can be achieved with $N \leq m \cdot \mathrm{rk} V$). The existence of the natural mappings $V \hookrightarrow W$ and $W \to V$ implies then that $HB_*(\mathcal{O}(V; Z))$ is a direct summand in $HB_*(C^\infty(X, Z) \otimes k[\xi_1, \ldots, \xi_N])$. The proof that $C^\infty(X, Z) \otimes k[\xi_1, \ldots, \xi_N]$ is H-unital parallels the proof of Theorem 6.1. □

Proof of Theorem 7.1. In view of (31) and of Corollary 3.4, the question reduces to proving H-unitality of $\mathcal{D}(X, Z)$ for an arbitrary closed subset $Z \subset X$.

Let $E^k_{pq} \Rightarrow HB_{p+q}(\mathcal{D}(X, Z))$ be the spectral sequence associated with the filtration by the order of operator. Its E^1-term is equal to $HB_*(\mathcal{O}(T^*X; Z))$ which vanishes in view of Proposition 7.3.

Long exact sequences (32) and (33) are associated with extensions (30) and (31) respectively. □

7.4. *Remark*. Similarities between ideals of flat functions and of flat differential operators which are suggested by Theorems 6.1 and 7.1 respectively are, in fact, limited only to the case of the flatness of infinite order. Let $C^\infty_{(m)}(X, Z)$ denote the ideal of C^∞-functions vanishing up to order m ($1 \leq m < \infty$) on a "thin" closed subset $Z \subset X$ (a subset Z is called *thin* if, for every $z \in Z$, there are a neighborhood $V \ni z$ and a function $f \in C^\infty(V)$ vanishing on $Z \cap V$ whose ∞-jet at z is not zero). The algebra $C^\infty_{(m)}(X, Z)$ is then manifestly non-H-unital: $(C^\infty_{(m)}(X, Z))^2 \subset C^\infty_{(2m)}(X, Z) \subsetneq C^\infty_{(m)}(X, Z)$. On the other hand the least two-sided ideal in $\mathcal{D}(X)$ which contains the corresponding (left) ideal $\mathcal{D}_{(m)}(X, Z)$ of differential operators whose coefficients are m-flat on a (compact) thin $Z \subset X$ is $\mathcal{D}(X)$ itself. This implies, in view of Corollary 4.3, that $\mathcal{D}_{(m)}(X, Z)$ is H-unital.

8. Locally multiplicatively-convex algebras with approximate units

Recall that a locally convex algebra $\mathcal{A}$ is said to be *locally multiplicatively-convex* (locally m-convex or, simply, m-convex) if $\mathcal{A}$ has a basis of absolutely convex open neighborhoods of the origin $\{\mathcal{U}_\alpha\}$ with the property that $\mathcal{U}_\alpha \cdot \mathcal{U}_\alpha \subset \mathcal{U}_\alpha$; i.e. $\mathcal{A}$ is topologized by a system of semi-norms $\{p_\alpha\}$ satisfying

$$p_\alpha(a_1 a_2) \leq p_\alpha(a_1) p_\alpha(a_2),$$

for all $a_1, a_2 \in \mathcal{A}$ and all α (cf. [30]). It is almost immediate from this definition that a complete locally m-convex algebra is a projective limit of Banach algebras, and vice versa [30, Thm. 5.1]. A locally m-convex algebra whose topology is Fréchet will be called a *Fréchet m-convex algebra*.

A net $\{e_{\lambda \in \Lambda}\}$ in $\mathcal{A}$ (indexed by elements of some directed set Λ) is called a *uniformly bounded left approximate unit* (u.b.l.a.u.) if, for every $a \in \mathcal{A}$, $e_\lambda a \to a$ and $\sup_{\alpha, \lambda} p_\alpha(e_\lambda) < \infty$. The right approximate unit is similarly defined.

8.1. THEOREM. *Every Fréchet m-convex algebra with left or right* u.b.a.u. *is H-unital as an abstract k-algebra* ($k = \mathbf{R}$ *or* $\mathbf{C}$).

Proof. Every Fréchet m-convex algebra with left (respectively, right) u.b.a.u. has the property (F) (respectively, property (F°)). This follows immediately from the extension of the Cohen-Hewitt factorization theorem (established originally for Banach algebras and Banach modules) to Fréchet m-convex algebras; cf. [31], [8], [44, Thm. 2.1], [47, Cor. 5]. It remains then to apply Proposition 5.1. □

8.2. COROLLARY (cf. [54, Prop. 5]). *Every Banach algebra with left or right* b.a.u. *is H-unital as an abstract k-algebra* ($k = \mathbf{R}$ *or* $\mathbf{C}$). □

Since every C^*-algebra has a two-sided b.a.u. ([40, Lemma 1.1]) we obtain also:

8.3. COROLLARY (cf. [54, Cor. 5]). *Every C^*-algebra is H-unital as an abstract k-algebra* ($k = \mathbf{R}$ *or* $\mathbf{C}$). □

8.4. *Examples*. (1) Let G be a locally compact group. The group algebra $L^1(G)$, which is also a Banach $*$-algebra, has a unit only if G is a discrete group. However, it has always a two-sided bounded approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ where Λ is the directed set of compact neighborhoods of the group identity and e_λ denotes the characteristic function of $\lambda \in \Lambda$ normalized by dividing it by the volume of λ. Therefore $L^1(G)$ is, in view of Corollary 8.2, H-unital.

Let $H \subset G$ be a closed normal subgroup. Integration along fibres of the projection $G \to G/H$ is then a surjective homomorphism of Banach algebras T_H: $L^1(G) \to L^1(G/H)$. Its kernel is usually denoted $\mathscr{I}^1(G, H)$. According to [36, p. 883] the ideal $\mathscr{I}^1(G, H)$ has a right b.a.u. if H is *amenable* (i.e. there exists a left-invariant mean on $L^\infty(H)$). Therefore, as a corollary of Theorem 3.1 and Corollary 8.2, we arrive at the following conclusion:

For every locally compact group G and its closed normal amenable subgroup H integration along fibres of the projection $G \to G/H$ induces the long exact sequence in "discrete" cyclic homology

$$(35) \qquad \cdots \longrightarrow HC_{q+1}(L^1(H)) \longrightarrow HC_q(\mathscr{I}^1(G,H)) \longrightarrow HC_q(L^1(G)) \xrightarrow{T_H} HC_q(L^1(H)) \longrightarrow \cdots$$

and a similar long exact sequence in Hochschild homology.

For $H = G$, $\mathscr{I}^1(G, G)$ is the augmentation ideal corresponding to the augmentation $\int_G$: $L^1(G) \to \mathbf{C}$. It has a right b.a.u. if *and only if* G is amenable (cf. [36, p. 883]).

On an equal footing with $L^1(G)$ in harmonic analysis occur the group algebras with weights $L^1_w(G)$, alias *Beurling algebras* (introduced by A. Beurling [3, §2] in the special case $G = \mathbf{R}^n$). The Beurling algebra $L^1_w(G)$ consists of functions such that $fw \in L^1(G)$ where w is a *weight function* satisfying the following three conditions:

(I) w is a real-valued measurable and locally bounded function on G;

(II) For every $g_1, g_2 \in G$, one has $w(g_1g_2) \leq w(g_1)w(g_2)$;

(III) $w \geq 1$.

Equipped with the norm $\|f\|_{1,w} = \int |f(g)|w(g)\,dg$ the convolution algebra $L^1_w(G)$ is a Banach algebra with a b.a.u. (cf. [35, Ch. 3, §7.1–2]) and hence is H-unital.

If $w_1 < w_2 < \cdots$ is an increasing sequence of weight functions which is *uniformly bounded* on some neighborhood of the group identity, the limit $L^1(G; \{w_n\}) = \bigcap_{n=1}^{\infty} L^1_{w_n}(G)$ is then an example of a Fréchet m-convex algebra with a u.b.a.u. (cf. [31]). In particular, $L^1(G; \{w_n\})$ is then H-unital.

(2) Harmonic analysis on locally compact groups provides also interesting examples of non-H-unital algebras. There is an important class of Banach algebras, realized as dense (left) G-invariant ideals in $L^1(G)$, called *Segal algebras* (cf. [35, VI.2], [37], [4], [15], [49]). Here belong, e.g.

(a) Wiener algebra $W(G)$ defined, for a non-discrete locally compact abelian group G with a co-compact discrete subgroup Γ, as the convolution algebra of continuous functions on G satisfying

$$\|f\|_W = \sup_{g\in G} \sum_{\gamma\in\Gamma} \sup_{x\in K} |f(gx\gamma)| < \infty$$

($K \subset G$ is a compact subset such that $K\Gamma = G$; [35, VI.2.1]). In the special case of $G = \mathbf{R}$ and $\Gamma = \mathbf{Z}$ this algebra was introduced by N. Wiener in his work on Tauberian theorems (this is his class M_1 of [50, §3; pp. 21–22] and [51, §10, p. 73]);

(b) Algebra $L^1(G) \cap L^p(G)$, $1 < p < \infty$, introduced by K. Iwasawa in the early 1940's [19] in his work on representation theory.

Proper, i.e. distinct from $L^1(G)$, Segal algebras are generally believed to have the property $\mathscr{A} \neq \mathscr{A}^2$ (this is actually proved in a number of cases, including Wiener algebra and $L^1(G) \cap L^p(G)$, for G compact or locally compact abelian, cf. [49, 8.10] and [13]). In particular, they cannot be H-unital. An interesting circumstance is that all Segal algebras possess (left) approximate units, of course, unbounded (cf. [37, §8]).

Closely related examples of *convolution* Banach algebras with the property $\mathscr{A} \neq \mathscr{A}^2$ (and therefore not H-unital) include $C^k(S^1)$, $1 \leq k < \infty$, the algebra $BV(S^1)$ of continuous functions with bounded total variation, the algebra

$\mathrm{Lip}_\alpha(S^1)$, $0 < \alpha \leq 1$, of Lipschitz functions [49, 8.12] and the convolution Hardy algebra $H^1(S^1) = \{f \in L^1(S^1) | \hat{f}(n) = 0,\ n < 0\}$ (cf. [49, 8.16] and [13, 2.8]).

(3) Quite challenging seems to be the question of H-unitality of the algebra of compact operators $\mathscr{K}(E)$ on an arbitrary Banach space E. If E has the bounded approximation property (BAP, cf. [24, 43.8]), $\mathscr{K}(E)$ has, almost by definition, a left b.a.u. If the strong dual E' has a basis, $\mathscr{K}(E)$ has even a two-sided b.a.u. [20]. In particular, $\mathscr{K}(E)$ is H-unital for Banach spaces with BAP. Does $\mathscr{K}(E)$, for an arbitrary E, possess the factorization property (F) or (F°) above? Is $\mathscr{K}(E)$, for an arbitrary E, H-unital? We have to leave these questions open.

(4) Let $\mathring{\mathscr{B}}^k(\mathbf{R}^d)$, $0 \leq k \leq \infty$, denote the closure of $C^\infty_{\mathrm{comp}}(\mathbf{R}^d)$ in the space $\mathscr{B}^k(\mathbf{R}^d)$ of k-differentiable functions on $\mathbf{R}^d$ which are bounded with all derivatives. It is easy to see that

$$\mathring{\mathscr{B}}^k(\mathbf{R}) = \left\{f \in C^d(\mathbf{R}^n) \middle| \forall_{0 \leq |\alpha| \leq k} \partial^\alpha f \text{ vanishes at } \infty\right\}$$

and the system of norms $p_j(f) \equiv \sum_{0 \leq |\alpha| \leq j} 1/\alpha! \sup_x |\partial^\alpha f|$ makes $\mathring{\mathscr{B}}^k(\mathbf{R}^n)$ a Banach algebra, for $k < \infty$, and a Fréchet m-convex algebra, for $m = \infty$. The space $\mathring{\mathscr{B}}^k(\mathbf{R}^d)$ and its dual, denoted $\mathscr{D}'_{L^1}(\mathbf{R})$ and called the space of "integrable" distributions, were introduced by L. Schwartz (cf. [38, VI.8]).

The sequence $e_j(x) = e^{-|x|^2/j^2}$, $j = 1, 2, \ldots$, is easily seen to be a u.b.a.u. in each $\mathring{\mathscr{B}}^k(\mathbf{R}^d)$, $0 \leq k \leq \infty$ (cf. e.g. [46, Prop. 1.6]). In particular, Theorem 8.1 implies that all $\mathring{\mathscr{B}}^k(\mathbf{R}^d)$, $0 \leq k \leq \infty$, are H-unital.

8.5. *Remarks*. (1) It is worth noting that the algebras discussed in Sections 6–8 constitute two essentially disjoint classes of algebras to which Proposition 5.1 applies. All the algebras of Sections 6 and 7 are nuclear, and any quasi-complete (i.e. every *bounded* Cauchy net converges) nuclear space E is semi-reflexive (i.e. the canonical inclusion $E \hookrightarrow E''$ is bijective); cf. [32, Prop. 4.4.11]. However, a locally convex and semi-reflexive algebra $\mathscr{A}$ which has *no* (left) unit cannot have a bounded approximate one (not even mentioning uniformly bounded). By the standard criterion of semi-reflexivity, cf. [24, 23.3(1)], every bounded net $\{e_\lambda\}_{\lambda \in \Lambda}$ in a semi-reflexive $\mathscr{A}$ has a sub-net weakly converging to a certain $e \in \mathscr{A}$; if $\{e_\lambda\}_{\lambda \in \Lambda}$ is a (left) b.a.u., e must be then a genuine (left) unit. The above argument which I quote in a slightly stronger form after [46, p. 335] shows that the use of bounded approximate units is limited necessarily to non-semi-reflexive algebras.

(2) In [53] (cf. also [54]) we will establish the relevant "continuous" versions of Theorems 6.1, 7.1 and 8.1 and derive continuous analogs of the corresponding long exact sequences (see (24)–(26), (28), (32)–(34) and (35) above).

9. *H*-unitary modules

Let A be an algebra over an arbitrary commutative ring k. We will say that a left A-module M is *homologically unitary* (H-unitary) if

(a) A is H-unital;

(b) For every k-module V, the complex $(B'_*(A; M \otimes V), b')$ which is defined as follows

$$B'_0 = 0 \quad \text{and} \quad B'_q = A^{\otimes(q-1)} \otimes M \otimes V, \qquad q \geq 1,$$

and

$$\begin{aligned} &b'(a_1 \otimes \cdots \otimes a_{q-1} \otimes m \otimes v) \\ &\quad = \sum_{i=1}^{q-2} (-1)^{i-1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{q-1} \otimes m \otimes v \\ &\quad\quad + (-1)^{q-1} a_1 \otimes \cdots \otimes a_{q-2} \otimes a_{q-1} m \otimes v, \end{aligned}$$

is acyclic.

In general, the homology of $B'_*(A; M \otimes V)$ will be denoted $HB'_*(A; M \otimes V)$. If M is a unitary module over an algebra with unit, the complex $B'_*(A, M)$ is, up to a shift in dimension, the standard (augmented) Bar resolution of M and has a canonical contracting homotopy (cf. §2).

For A flat over k, one has $HB'_q(A; M \otimes V) = \mathrm{Tor}^{\tilde{A}}_{q-1}(k, M \otimes V)$, $q \geq 0$, while the equality $HB'_q(A; M \otimes V) = \mathrm{Tor}^{(\tilde{A},k)}_{q-1}(k, M \otimes V)$, $q \geq 0$, holds quite generally, with no restrictions on A (cf. §2, the last paragraph). If both A and M are k-flat, the condition (b) above reduces to the single requirement that $\mathrm{Tor}^{\tilde{A}}_*(k, M) = 0$.

Since $HB'_*(A; A \otimes V) \equiv HB_*(A; V)$, an H-unital algebra A is H-unitary as a left module over itself.

The definition of H-unitarity for right A-modules is similar. In that case condition (b) is replaced by the requirement of acyclicity of the suitably modified complexes (B'_*, b'):

$$B'_0 = 0 \quad \text{and} \quad B'_q = V \otimes M \otimes A^{\otimes(q-1)}, \qquad q \geq 1,$$

and

$$\begin{aligned} &b'(v \otimes m \otimes a_1 \otimes \cdots \otimes a_{q-1}) \\ &\quad = v \otimes m a_1 \otimes a_2 \otimes \cdots \otimes a_{q-1} \\ &\quad\quad - \sum_{i=1}^{q-2} (-1)^{i-1} v \otimes m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{q-1}, \end{aligned}$$

for all k-modules V.

9.1. LEMMA. *Let M be an A-bimodule which is either left or right H-unitary. Then the semi-direct product $A \ltimes M$ is H-unital.*

Proof. We will write elements of the Bar complex $B_*(A \ltimes M; V)$, where V denotes an arbitrary k-module, as the sums of monomials $r_1 \otimes \cdots \otimes r_q \otimes v$ with $r_j \in A$ or M ($j = 1, \dots, q$). Then the number of M-entries defines the decomposition of $B_*(A \ltimes M; V)$ into the direct sum of subcomplexes

$$B_*(A \ltimes M; V) = \bigoplus_{l=0}^{\infty} B_*(l)$$

where

$$B_q(l) = \operatorname{lin\,span}\{r_1 \otimes \cdots \otimes r_q \otimes v \mid \text{precisely } l \;\; r_j\text{'s belong to } M\}.$$

Assuming M to be H-unitary as a left A-module we will filter each subcomplex $B_*(l)$, $l > 0$, by

$$F_p B_{p+q}(l) = \operatorname{lin\,span}\Big\{r_1 \otimes \cdots \otimes r_{p+q} \otimes v \in B_{p+q}(l) \Big| r_j \in A \text{ for } j \le q - l\Big\}. \tag{36}$$

The associated s.s. $E^k_{pq} \Rightarrow H_{p+q}(B_*(l))$ is located in the region ($p \ge 0$, $q \ge l$). Its E^0-term is given by:

9.2. LEMMA. *For every $p \ge 0$ and $l > 0$, there is a canonical isomorphism*

$$\left(E^0_{p*}(l), d^0\right) \simeq B'_*(A; M) \otimes B_{p+l-1}(l-1)[l-1] \tag{37}$$

where the complex $B'_*(A; M)$ is formed with respect to the left A-module structure on M and $B_{p+l-1}(l-1)[l-1]$ is viewed as a trivial complex concentrated in dimension $l - 1$. □

The complex on the right-hand side of (37) is acyclic, since M is assumed to be left H-unitary. Hence all $B_*(l)$, $l > 0$, are acyclic. Finally, $B_*(0) \equiv B_*(A; V)$ is acyclic because A is H-unital.

If M happens to be H-unitary as a *right* A-module, one can use the filtration opposite to (36)

$$F_p^{\circ} B_{p+q}(l) = \operatorname{lin\,span}\Big\{v \otimes r_1 \otimes \cdots \otimes r_{p+q} \in B_{p+q}(l) \Big| r_j \in A \text{ for } j \ge p + l + 1\Big\}$$

or one can reduce that case formally to the previous one by treating M as an A°-bimodule (A° denotes the opposite algebra). □

9.3. PROPOSITION. *Let $I \rightarrowtail R \twoheadrightarrow A$ be a pure extension of an H-unital algebra A by an ideal I with $I^2 = 0$. If I is H-unitary with respect to either left or right A-module structures, the algebra R is H-unital.*

Proof. The number of I-entries defines the filtration on $B_*(R; V)$:

$$F_p B_{p+q}(R; V) = \text{lin span}\{ r_1 \otimes \cdots \otimes r_{p+q} \otimes v \,|\, \text{at least } q \quad r_j\text{'s belong to } I \}$$

(cf. (16)). The associated s.s. $E^k_{pq} \Rightarrow H_{p+q}(R; V)$ is a s.s. of the first quadrant and its E^1-term clearly equals $B_*(A \ltimes I; V)$. Hence, according to Lemma 9.1, $E^2_{**} = 0$ and the assertion follows. □

9.4. Assume that for a given k-algebra S

(a) there exists an H-unital subalgebra $A \subset S$,

(b) the associated extension of k-modules $0 \to A \to S \to S/A \to 0$ is pure (see Appendix A).

The latter condition is reminiscent of the situation encountered in Section 3.

9.5. THEOREM. *For A and S as above, S is H-unital provided S/A is H-unitary as a left or as a right A-module.*

Proof. For a given k-module V, we will filter $B_*(S; V)$ by

$$G_p B_{p+q}(S; V) = \text{lin span}\{ s_1 \otimes \cdots \otimes s_{p+q} \otimes v \,|\, \text{at least } q \quad s_j\text{'s belong to } A \}.$$

The associated graded complex naturally identifies (Lemma A.6 in Appendix A) with the Bar complex of the semi-direct product $A \ltimes M$,

$$\text{(38)} \qquad \text{Gr}^G_* B_*(S; V) = B_*(A \ltimes M; V),$$

where $M = S/A$ and $M^2 = 0$. In view of Lemma 9.1 and the hypothesis, the right-hand side of (38) is acyclic. □

9.6. COROLLARY. *Let B be a k-algebra with unit such that the structural homomorphism $k \to B$ is injective and $k \cdot 1 \subset B$ is a pure k-submodule. Then, for every H-unital k-algebra A, the algebra $A \otimes B$ is H-unital.*

Proof. Apply Theorem 9.5 to $S = A \otimes B$ and $A = A \otimes k \cdot 1$. □

9.7. COROLLARY. *In the case of the ground ring k being a field, tensor product defines a bifunctor*

$$\otimes : \{\textit{H-unital k-algebras}\} \times \{\textit{unital k-algebras}\} \to \{\textit{H-unital k-algebras}\}. \quad □$$

9.8. COROLLARY (*Morita invariance of Hochschild and cyclic homology on the category of H-unital algebras*). *Let A be an H-unital algebra. Then, for every $n = 1, 2, \ldots,$ the matrix algebra $M_n(A)$ is H-unital and the natural* *inclusion*

$$A \hookrightarrow M_n(A), \qquad a \mapsto \begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

induces isomorphisms

(39) $H_*(A, A) \tilde{\to} H_*(M_n(A), M_n(A))$ and $HC_*(A) \tilde{\to} HC_*(M_n(A))$.

Proof. The H-unitality of $M_n(A)$ follows from Corollary 9.6. In order to prove (39), let us consider the obvious morphism of split extensions

(40)
$$\begin{array}{ccccc} A & \rightarrowtail & \tilde{A} & \twoheadrightarrow & k \\ \downarrow & & \downarrow & & \downarrow \\ M_n(A) & \rightarrowtail & M_n(\tilde{A}) & \twoheadrightarrow & M_n(k) \end{array}$$

($\tilde{A}$ denotes, as usual, the result of adjoining the unit to A).

Since $M_n(A)$ is H-unital, (40) induces the following commutative diagram with exact rows (cf. Theorem 3.1):

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_*(A, A) & \longrightarrow & H_*(\tilde{A}, \tilde{A}) & \longrightarrow & H_*(k, k) & \longrightarrow 0 \\ & \downarrow & & \downarrow\wr & & \downarrow\wr & \\ 0 \longrightarrow & H_*(M_n(A), M_n(A)) & \longrightarrow & H_*(M_n(\tilde{A}), M_n(\tilde{A})) & \longrightarrow & H_*(M_n(k), M_n(k)) & \longrightarrow 0 \end{array}$$

whose middle and right vertical arrows are isomorphisms by the classical Morita invariance of the Hochschild homology for algebras with unit (cf. [10, Part A, Thm. 3.4]). □

Neither Hochschild nor cyclic homology remain Morita invariant for general algebras without unit; e.g. $H_0(A, A) \neq H_0(M_n(A), M_n(A))$, $n \geq 2$, if $A \neq A^2$.

9.9. Assume that an algebra A is embedded into an algebra with unit R. Let

$$\mathscr{E}(R, A) = \{(r_1, r_2) \in R \times R \mid Ar_1 \subset A,\ r_2A \subset A \quad \text{and} \quad r_1r_2 = 1\}.$$

The set $\mathscr{E}(R, A)$ has an obvious monoid structure

$$(r_1, r_2) \cdot (s_1, s_2) = (r_1s_1, s_2r_2)$$

with $(1, 1)$ as its neutral element and it acts naturally on the algebra A via

(41) $$\phi^{(r_1, r_2)}: a \mapsto r_2ar_1 \qquad ((r_1, r_2) \in \mathscr{E}(R, A)).$$

9.10. Corollary (cf. [7, Prop. II.5.1]). *Let A be an H-unital algebra and R an arbitrary algebra with unit containing A. The induced actions of the monoid*

$\mathscr{E}(R, A)$ on $H_*(A, A)$ and on $HC_*(A)$ are trivial; i.e. each $\phi_*^{(r_1, r_2)}$ acts as the identity map.

Proof. The two natural embeddings of $\mathscr{E}(R, A)$ into $\mathscr{E}(M_2(R), M_2(A))$

$$(r_1, r_2) \mapsto \left(\begin{pmatrix} r_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} r_2 & 0 \\ 0 & 1 \end{pmatrix}\right) \quad \text{and} \quad (r_1, r_2) \mapsto \left(\begin{pmatrix} 1 & 0 \\ 0 & r_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & r_2 \end{pmatrix}\right)$$

are conjugated. Under the first embedding, $\mathscr{E}(R, A)$ acts on $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) \subset M_2(A)$ as in (41), under the second one as id_A. It follows from Corollary 9.8 that the induced actions on $H_*(A, A)$ (respectively, on $HC_*(A)$) are conjugated. □

10. The cone and the suspension functors

The following construction is a slight modification of the original construction due to J.B. Wagoner [48].

For an algebra A (with or without unit), let ΓA denote the algebra of matrices (a_{ij}), $1 \leq i, j < \infty$, with entries from A, such that

(I) the set $\{a_{ij}; 1 \leq i, j < \infty\} \subset A$ is finite,

(II) the number of non-zero entries in each row and each column is finite.

The algebra ΓA will be referred to as the *cone* of A. It contains $M_\infty(A) = \varinjlim M_n(A)$ as a two-sided ideal; hence the functorial extension

$$M_\infty(A) \rightarrowtail \Gamma A \twoheadrightarrow \Sigma A \tag{42}$$

where the quotient algebra $\Sigma A \equiv \Gamma A / M_\infty(A)$ will be referred to as the *suspension* of A.

Since (42) is the inductive limit of k-split short exact sequences

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ M_n(A) & \longrightarrow & \Gamma A & \longrightarrow & \Gamma A/M_n(A) \\ \downarrow & & \| & & \downarrow \\ M_{n+1}(A) & \longrightarrow & \Gamma A & \longrightarrow & \Gamma A/M_{n+1}(A) \\ \vdots & & \vdots & & \vdots \end{array}$$

it is pure (in the sense of §3).

10.1. Theorem. *Let A be an H-unital algebra over an arbitrary commutative ring k. Then*

(a) *The algebras ΓA and ΣA are H-unital.*

(b) $H_*(\Gamma A, \Gamma A) = HC_*(\Gamma A) = 0$.

(c) *There are canonical isomorphisms*

$$H_*(\Sigma A, \Sigma A) \simeq H_*(A, A)\,[1] \quad \text{and} \quad HC_*(\Sigma A) \simeq HC_*(A)\,[1]. \tag{43}$$

None of the above remains valid for general algebras without unit; e.g. $H_0(\Gamma A, \Gamma A) \neq 0$ if $A \neq A^2$.

Proof. (a) We will view A as a subalgebra of ΓA via the diagonal embedding $a \mapsto \begin{pmatrix} a & & \\ & a & \\ & & \ddots \end{pmatrix}$. Then ΓA decomposes as an A-bimodule into the direct sum $\Gamma A = A \oplus \Gamma_0 A$ where $\Gamma_0 A = \{(a_{ij}) \in \Gamma A | a_{11} = 0\}$. It is clear that $\Gamma_0 A$ is an inductive limit of free A-bimodules of finite type. In particular, if A is H-unital, $\Gamma_0 A$ is H-unitary (in the sense of §9) both as a left and as a right A-module. Theorem 9.5 implies then that ΓA is H-unital. By combining this with Corollaries 9.8 and 3.4, we obtain the H-unitality of ΣA.

(b) Let us consider the following three maps $\Gamma A \to \Gamma^2 A \equiv \Gamma(\Gamma A)$,

$$\varphi : \alpha \mapsto \begin{pmatrix} 0 & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \ddots \end{pmatrix}, \quad \psi : \alpha \mapsto \begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \ddots \end{pmatrix}$$

and

$$\sigma : \alpha \mapsto \begin{pmatrix} \alpha & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix} \qquad (\alpha \in \Gamma A).$$

The second one, ψ, can be written as a composite map $\Gamma A \xrightarrow{(p_1, p_2)} \Gamma A \oplus \Gamma^2 A \overset{i}{\hookrightarrow} \Gamma^2 A$, such that $ip_1 = \sigma$ and $ip_2 = \varphi$ where i: $\Gamma A \oplus \Gamma^2 A \hookrightarrow \Gamma^2 A$ denotes the bloc-diagonal embedding

$$\alpha \oplus (\beta_{ij}) \mapsto \left(\begin{array}{c|c} \alpha & \\ \hline & (\beta_{i-1, j-1}) \end{array}\right), \qquad (\alpha; \beta_{ij} \in \Gamma A).$$

Since A is H-unital, it follows from Corollary 3.4 and Part (a) of the theorem that $HC_*(\Gamma A \oplus \Gamma^2 A) = HC_*(\Gamma A) \oplus HC_*(\Gamma^2 A)$ and hence

$$\psi_* = \sigma_* + \varphi_*. \tag{44}$$

Let us embed $\Gamma^2 A$ into the algebra $\Gamma(\widetilde{\Gamma A})$ where $\widetilde{\Gamma A}$ denotes the result of adjoining the unit to ΓA. The monoid $\mathscr{E}(\Gamma(\widetilde{\Gamma A}), \Gamma^2 A)$ acts on $\Gamma^2 A$ via (41) and one clearly has $\varphi = \phi^{(r_1, r_2)}\psi$ where $(r_1, r_2) \in \mathscr{E}(\Gamma(\widetilde{\Gamma A}), \Gamma^2 A)$ is given by

$$r_1 = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad r_2 = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & \ddots \end{pmatrix}.$$

Corollary 9.10 implies then that $\varphi_* = \psi_*$. By combining this with (44) we obtain:

$$\sigma_*: HC_*(\Gamma A) \to HC_*(\Gamma^2 A) \text{ is a zero map.} \tag{45}$$

After a bijection $\mathbf{Z}_+ \times \mathbf{Z}_+ \leftrightarrow \mathbf{Z}_+$ has been fixed one can view $\Gamma^2 A$ as being embedded in $M_2(\Gamma A)$ in such a way that

$$\begin{pmatrix} \alpha & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix} \in \Gamma^2 A \quad \text{goes to} \quad \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\Gamma A),$$

i.e. that the composition with $\sigma: \Gamma A \to \Gamma^2 A$ is the canonical stabilization map $\Gamma A \to M_2(\Gamma A)$. According to Corollary 9.8, the latter induces an isomorphism $HC_*(\Gamma A) \xrightarrow{\sim} HC_*(M_2(\Gamma A))$. However, in view of (45) the same map is a zero map. We conclude that $HC_*(\Gamma A) = 0$.

(c) Extension (42) is pure and $M_\infty(A)$ is H-unital (cf. Corollary 9.8). Thus (42) induces, in view of Theorem 3.1, the long exact sequence

$$\cdots \to HC_q(\Gamma A) \to HC_q(\Sigma A) \xrightarrow{\partial} HC_{q-1}(M_\infty(A)) \to HC_{q-1}(\Gamma A) \to \cdots \tag{46}$$

and a similar sequence in Hochschild homology. By combining (46) with the already proven Part (b) of the theorem and with Corollary 9.8, we obtain the canonical isomorphisms $HC_q(\Sigma A) \xrightarrow[\sim]{\partial} HC_{q-1}(M_\infty(A)) \xrightarrow[\sim]{\mathrm{tr}_*} HC_{q-1}(A)$, $q \in \mathbf{Z}$, cf. (43), and the similar isomorphisms in Hochschild homology. □

10.2. *Remarks.* (1) In Wagoner's original definition of the cone functor [48] the matrices (a_{ij}) were not required to have entries belonging to a finite subset of A (see Condition (I) above). Denote this "bigger" algebra by $\Gamma^W A$ and let $\Sigma^W A := \Gamma^W A / M_\infty(A)$. The extension

$$M_\infty(A) \rightarrowtail \Gamma^W A \twoheadrightarrow \Sigma^W A \tag{47}$$

is still pure and Theorem 10.1 is valid for (47) (with the same proof) if A has a unit. However, for general H-unital algebras, there is a potential trouble with the H-unitality of $\Gamma^W A$. This difficulty can be removed by considering instead the pair of functors $\Gamma^L A := A \otimes \Gamma^W k$ and $\Sigma^L A := A \otimes \Sigma^W k \simeq \Gamma^L A / M_\infty(A)$ (this variant of Wagoner's construction was considered e.g. by J.-L. Loday [26, 1.4.4]). The H-unitality of $\Gamma^L A$ follows then from Corollary 9.6; the proof of the remaining assertions of Theorem 10.1 is unchanged.

Another possibility is to reduce the H-unital case to the unital one by considering $(\Gamma^W A)' := \Gamma^W \tilde{A}$ and $(\Sigma^W A)' := \Gamma^W \tilde{A} / M_\infty(A)$.

(2) An alternative construction of the cone and the suspension functors was proposed earlier by M. Karoubi (cf. e.g. [22, p. 269]). His cone algebra $\Gamma^K A$ consists of matrices satisfying Condition (I) above and the following stronger version of Condition (II):

(II′) The number of non-zero entries in every row and every column is bounded.

This definition of $\Gamma^K A$, quoted by A. Connes in [7, p. 103], is equivalent to the original definition of Karoubi.

Theorem 10.1 holds for $\Gamma^K A$ and $\Sigma^K A := \Gamma^K A / M_\infty(A)$ with the same proof (in the case: $k = \mathbf{C}$ and A has a unit, the acyclicity of $\Gamma^K A$ was proved by A. Connes [7, Cor. II.6]).

(3) Let $\mathscr{F}(H)$ denote the algebra of finite rank operators on an infinite-dimensional Hilbert space H and $\mathscr{L}(H)$ be the corresponding algebra of all bounded linear operators. The latter contains $\mathscr{F}(H)$ as a two-sided ideal. For an arbitrary H-unital $\mathbf{C}$-algebra A,

(a) the algebra $A \otimes \mathscr{F}(H)$ is H-unital;

(b) the embedding $A \hookrightarrow A \otimes \mathscr{F}(H)$, $a \mapsto a \otimes p_0$, where p_0 is the orthogonal projector onto the first basis vector $e_0 \in H$, induces isomorphisms in Hochschild and in cyclic homology;

(c) $A \otimes \mathscr{L}(H)$ is H-unital and acyclic.

Assertions (a) and (b) follow from the representation of $\mathscr{F}(H)$ as the inductive limit $\varinjlim_{V \subset H} \operatorname{End} V$, with V running over all finite-dimensional linear subspaces of H, and from Corollary 9.8. The H-unitality of $A \otimes \mathscr{L}(H)$ follows from Corollary 9.6. In order to prove that $A \otimes \mathscr{L}(H)$ is acyclic, one may consider the short exact sequence

$$
\begin{aligned}
(48)\quad 0 \to H_*(A \otimes \mathscr{L}(H), A \otimes \mathscr{L}(H)) &\to H_*(\tilde{A} \otimes \mathscr{L}(H), \tilde{A} \otimes \mathscr{L}(H)) \\
&\to H_*(\mathscr{L}(H), \mathscr{L}(H)) \to 0
\end{aligned}
$$

associated with the split extension $A \otimes \mathscr{L}(H) \rightarrowtail \tilde{A} \otimes \mathscr{L}(H) \twoheadrightarrow \mathscr{L}(H)$. The middle and the right terms in (48) vanish in view of Proposition 5 of [55] and Künneth's formula in Hochschild homology.

Letting $\Gamma^H A := A \otimes \mathscr{L}(H)$ and $\Sigma^H A := A \otimes \mathscr{L}(H)/\mathscr{F}(H)$, one thus obtains an alternative construction of the cone and the suspension functors, for algebras over $k = \mathbf{C}$, which possesses all the desired properties.

(4) Any of the above constructions of the cone and suspension functors Γ and, respectively, Σ allows us to replace the relative Hochschild and cyclic homology groups associated with an arbitrary algebra homomorphism f: $A \to B$ by the corresponding *absolute* homology groups of the algebra with unit $R'f$

defined by the Cartesian square

$$\begin{array}{ccc} R'f & \longrightarrow & \Gamma\tilde{B} \\ \downarrow & & \downarrow \\ \Sigma\tilde{A} & \xrightarrow{\Sigma\tilde{f}} & \Sigma\tilde{B} \end{array} .$$

(here $\tilde{f}(a) = f(a)$ and $\tilde{f}(1) = 1$). If both A and B are H-unital, one can use instead the H-unital algebra Rf defined by the Cartesian square

$$\begin{array}{ccc} Rf & \longrightarrow & \Gamma B \\ \downarrow & & \downarrow \\ \Sigma A & \xrightarrow{\Sigma f} & \Sigma B \end{array} .$$

11. Triangular matrix algebras

With an arbitrary $(A - B)$-bimodule M one can associate the corresponding triangular matrix algebra

$$T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| a \in A,\ m \in M \text{ and } b \in B \right\} \tag{49}$$

with obvious addition and multiplication.

11.1. THEOREM. (a) *Let us assume that M is H-unitary as a left A-module or that it is H-unitary as a right B-module. Then the canonical (split) epimorphism $T \twoheadrightarrow A \oplus B$ induces isomorphisms*

$$H_*(T, T) \tilde{\to} H_*(A, A) \oplus H_*(B, B) \quad \text{and} \quad HC_*(T) \tilde{\to} HC_*(A) \oplus HC_*(B).$$

(b) *If, moreover, both A and B are H-unital, T is H-unital.*

Proof. Without loss of generality, we can assume that M is H-unitary as a left A-module. Let us consider the (split) projection $T \to B$, $\left(\begin{smallmatrix} a & m \\ 0 & b \end{smallmatrix}\right) \mapsto b$. Its kernel is the ideal of row vectors $R \doteq \left(\begin{smallmatrix} A & M \\ 0 & 0 \end{smallmatrix}\right)$. As a k-algebra, R can be identified with $A \ltimes M$, where M is viewed as an A-bimodule with the identically zero right A-module structure (so that M is not a unitary A-bimodule even if A has a unit). By Lemma 9.1 the ideal of row vectors is H-unital. Thus, in view of Theorem 3.1, we have the (split) short exact sequence

$$0 \to H_*(R, R) \to H_*(T, T) \to H_*(B, B) \to 0.$$

In order to prove Part (a) of the assertion, it suffices to show that the map $H_*(A, A) \to H_*(R, R)$ induced by the natural inclusion $A \hookrightarrow R$ is an isomorphism.

The standard Hochschild complex $C_*(R, R)$ splits into the direct sum

$$C_*(R, R) = C_*(A, A) \oplus \bigoplus_{l=1}^{\infty} C_*(l) \tag{50}$$

where

$$C_q(l) = \text{lin span}\{ r_0 \otimes \cdots \otimes r_q | \text{precisely } l \quad r_j\text{'s belong to } M \}.$$

We will equip each $C_*(l)$ with a filtration resembling (36)

$$F_p C_{p+q}(l) = \text{lin span}\left\{ r_0 \otimes \cdots \otimes r_{p+q} \in C_{p+q}(l) \middle| r_j \in A,\ j \le q - l \right\}.$$

The associated s.s. $E^k_{pq}(l) \Rightarrow H_{p+q}(C_*(l))$ is located in the region $(p \ge 0,$ $q \ge l - 1)$ and its E^0-term is given by the following:

11.2. Lemma. *For every $p \ge 0$ and $l \ge 1$, there is a canonical isomorphism*

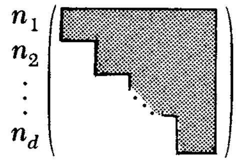
$$\left(E^0_{p*}(l), d^0\right) \simeq B'_*(A; M) \otimes T(p, l-1)[l-2] \tag{51}$$

where $T(i, j) \subset (A \oplus M)^{\otimes(i+j)}$ denotes the homogeneous component of bidegree (i, j) (i is the number of A-entries, j the number of M-entries). □

Since M is supposed to be left H-unitary, the right-hand side of (51) has no homology and, accordingly, all the complexes $C_*(l)$, $l \ge 1$, are acyclic. In view of this, the embedding $C_*(A, A) \hookrightarrow C_*(R, R)$, cf. (50), is a quasi-isomorphism. Recall that both A and R are H-unital, and for H-unital algebras the Hochschild homology is the homology of the standard complex C_* (cf. §2). Thus $H_*(A, A) \to H_*(R, R)$ is an isomorphism.

Part (b) of Theorem 10.1 follows from Corollary 3.4, applied to the extension $R \rightarrowtail T \twoheadrightarrow B$, and from Lemma 9.1. □

For a given set $\nu = (n_1, \ldots, n_d)$ of d positive integers, let us consider the corresponding bloc-triangular matrix algebra $T_\nu(A)$ consisting of $n \times n$ matrices $(n \equiv n_1 + \cdots + n_d)$ of the format:



with entries in A (the unshaded region contains zeros).

11.3. Corollary. *For every $\nu = (n_1, \ldots, n_d)$, the triangular algebra $T_\nu(A)$ over an H-unital algebra A is H-unital itself and its homology is given by*

$$H_*(T_\nu(A), T_\nu(A)) \simeq H_*(A, A)^{\oplus d} \quad \text{and} \quad HC_*(T_\nu(A)) \simeq HC_*(A)^{\oplus d}. \tag{52}$$

Proof. The assertion follows immediately from Theorem 11.1 and Corollary 9.8 provided one demonstrates that, for every $m > 0$ and $n > 0$, the set of $(m \times n)$-matrices $M_{m,n}(A)$ is H-unitary both as a left $M_m(A)$- and a right $M_n(A)$-module.

Fix an arbitrary k-module V. One clearly has

$$B'_*\big(M_m(A); M_{m,n}(A) \otimes V\big) = B'_*\big(M_m(A); M_{m,1}(A) \otimes V\big)^{\oplus n}$$

and $B'_*(M_m(A); M_{m,1}(A) \otimes V)$ is a direct summand in $B_*(M_m(A); V)$, hence it is acyclic, according to Corollary 9.8. The case of the right $M_n(A)$-module structure can be treated similarly. The maps in (52) are induced by the canonical projection

$$T_\nu(A) \twoheadrightarrow M_{n_1}(A) \oplus \cdots \oplus M_{n_d}(A). \qquad \square$$

11.4. *Remarks.* (1) Theorem 11.1 and Corollary 11.3 constitute precise additive analogues of the results known to hold in algebraic K-theory for rings with unit. By a theorem of D. G. Quillen [33, Thm. 2′] the canonical map $K_*(T_2(\mathscr{A})) \to K_*(\mathscr{A}) \oplus K_*(\mathscr{A})$ is an isomorphism for an arbitrary ring with unit $\mathscr{A}$. This result was extended to general triangular matrix rings like (49) by R. K. Dennis and S. C. Geller (for K_i, $i \leq 2$, [9]) and by A. J. Berrick and M. E. Keating (for all K_i's, [2]).

(2) It is noteworthy that even if one is primarily interested in the case when algebras have units and bimodules are unitary, the proof of Theorem 11.1 bears on the ideas related to H-unitality. If, however, in addition to that one has $k \supset \mathbf{Q}$, Theorem 11.1 becomes almost trivial. Here is the relevant argument.

The Hochschild complex $C_*(T, T)$ decomposes into a direct sum of its subcomplexes

$$C_*(T,T) = \bigoplus_{l=0}^{\infty} C_*(T,T)(l)$$

such that the adjoint action of the unit $1 \in A$ on $C_*(T,T)(l)$ is just the multiplication by l. Here

$$C_q(T,T)(l) = \operatorname{lin\,span}\left\{ t_0 \otimes \cdots \otimes t_q \;\middle|\; \begin{array}{c} \text{each } t_j \text{ belongs to } A,\ M \text{ or } B \\ \text{and precisely } l\ \ t_j\text{'s belong to } M \end{array} \right\}.$$

A suitable analog of the Cartan identity which holds in the Hochschild complex (see Appendix B) implies then that the multiplication by l on $C_*(T,T)(l)$ is null-homotopic. As $k \supset \mathbf{Q}$, all $C_*(T,T)(l)$ with $l > 0$ are contractible. It remains to notice that $C_*(T,T)(0) = C_*(A \oplus B,\ A \oplus B)$ and that the latter complex is quasi-isomorphic to $C_*(A, A) \oplus C_*(B, B)$.

Appendices

A. The hyper-cube lemma

For the better part of this appendix, k is an arbitrary (not necessarily commutative) ring. Assume that there is given a collection of (left) k-modules $T(i_1, \ldots, i_l)$ ($i_\nu = 0$, 1 or 2; $\nu = 1, \ldots, l$; l is a certain fixed positive integer) together with short exact sequences

$$T(i_1, \ldots, i_{\mu-1}, 0, i_{\mu+1}, \ldots, i_l) \overset{\varepsilon}{\rightarrowtail} T(i_1, \ldots, i_{\mu-1}, 1, i_{\mu+1}, \ldots, i_l) \overset{\phi}{\twoheadrightarrow} T(i_1, \ldots, i_{\mu-1}, 2, i_{\mu+1}, \ldots, i_l), \tag{53}$$

one for every $(i_1, \ldots, i_{\mu-1}, i_{\mu+1}, \ldots, i_l)$ and $\mu \in \{1, \ldots, l\}$.

We will assume that the hyper-cube with 2^l edges, formed by the short exact sequences (53), is commutative. This is equivalent to the commutativity of all the squares:

$$\begin{array}{ccccc}
T(i_1, \ldots, \overset{\mu}{0}, \ldots, \overset{\nu}{0}, \ldots, i_l) & \rightarrowtail & T(i_1, \ldots, \overset{\mu}{0}, \ldots, \overset{\nu}{1}, \ldots, i_l) & \twoheadrightarrow & T(i_1, \ldots, \overset{\mu}{0}, \ldots, \overset{\nu}{2}, \ldots, i_l) \\
\downarrow & & \downarrow & & \downarrow \\
T(i_1, \ldots, 1, \ldots, 0, \ldots, i_l) & \rightarrowtail & T(i_1, \ldots, 1, \ldots, 1, \ldots, i_l) & \twoheadrightarrow & T(i_1, \ldots, 1, \ldots, 2, \ldots, i_l) \\
\downarrow & & \downarrow & & \downarrow \\
T(i_1, \ldots, 2, \ldots, 0, \ldots, i_l) & \rightarrowtail & T(i_1, \ldots, 2, \ldots, 1, \ldots, i_l) & \twoheadrightarrow & T(i_1, \ldots, 2, \ldots, 2, \ldots, i_l)
\end{array}$$

($\mu, \nu = 1, \ldots, l$).

For $\alpha = (i_1, \ldots, i_l)$, $\beta = (j_1, \ldots, j_l)$ and $i \in \{0, 1, 2\}$ we set

$$|\alpha|_i = \#\{\nu | i_\nu = i\}$$

and

$$\alpha \prec \beta \Leftrightarrow i_\nu \le j_\nu \qquad \text{for all } \nu = 1, \ldots, l.$$

The iterations of the monic arrows occurring in the sequences (53) define the canonical monomorphisms

$$T(\alpha') \rightarrowtail T(\alpha'') \tag{54}$$

where $\alpha' \prec \alpha''$ and $|\alpha'|_2 = |\alpha''|_2$ (we shall often identify $T(\alpha')$ with its image in $T(\alpha'')$). Similarly, the iterations of the epic arrows occurring in the sequences (53) define the canonical epimorphisms

$$T(\beta') \twoheadrightarrow T(\beta'') \tag{55}$$

where $\beta' \prec \beta''$ and $|\beta'|_0 = |\beta''|_0$.

There is a filtration on $T(1, \ldots, 1)$

$$\{0\} = F_{-1} \subset F_0 \subset \cdots \subset F_l = T(1, \ldots, 1)$$

which is defined by

$$F_p = \sum_{\substack{|\alpha|_1 = p \\ |\alpha|_2 = 0}} T(\alpha).$$

The following lemma describes $\mathrm{Gr}_*^F T(1, \ldots, 1)$.

A.1. LEMMA ("Hyper-cube lemma"). *For every $p \geq 0$, there is a canonical isomorphism*

$$\Phi_p \colon F_p / F_{p-1} \xrightarrow{\sim} \bigoplus_{\substack{|\gamma|_1 = 0 \\ |\gamma|_2 = p}} T(\gamma). \tag{56}$$

Proof. The map Φ_p in (56) is uniquely defined by the requirement of commutativity of the following diagram

$$\begin{array}{ccc} T(1, \ldots, 1) & \xrightarrow{\Phi_p'} & \bigoplus_{\substack{|\beta|_0 = 0 \\ |\beta|_2 = p}} T(\beta) \\ \uparrow & & \uparrow \\ F_p & \xrightarrow{\Phi_p} & \bigoplus_{\substack{|\gamma|_1 = 0 \\ |\gamma|_2 = p}} T(\gamma) \end{array} \tag{57}$$

where Φ_p' is induced by the epimorphisms (55) with $\beta' = (1, \ldots, 1)$ and $|\beta''|_0 = 0$ and $|\beta''|_2 = p$, and the right vertical arrow is the direct sum of the canonical monomorphisms $T(\gamma) \rightarrowtail T(\beta)$, cf. (54), where

$$\gamma \prec \beta \quad \text{and} \quad |\beta|_0 = |\gamma|_1 = 0 \quad \text{and} \quad |\beta|_2 = |\gamma|_2 = p \tag{58}$$

(notice that the relation $\gamma \sim \beta$ defined by (58) is one-to-one).

It is clear from (53) and the definition of the filtration F that the maps Φ_p are surjective and that $F_{p-1} \subset \operatorname{Ker} \Phi_p$. We will prove by induction with respect to l that $F_{p-1} = \operatorname{Ker} \Phi_p$.

Let us set

$$F_p\{i\} = \sum T(\alpha) \subset T(\underbrace{1, \ldots, 1}_{l-1}, i) \tag{59}$$

where $i \in \{0, 1, 2\}$ and the sum in (59) extends over all $\alpha = (\alpha', i) \equiv (i_1, \ldots, i_{l-1}, i)$ satisfying $|\alpha'|_1 = p$, $|\alpha'|_2 = 0$. Similarly, let

$$G_p = \bigoplus_{\substack{|\gamma|_1 = 0 \\ |\gamma|_2 = p}} T(\gamma)$$

and

$$G_p\{i\} = \bigoplus T(\gamma) \tag{60}$$

where the direct sum in (60) extends over all $\gamma = (\gamma', i) \equiv (i_1, \dots, i_{l-1}, i)$ satisfying $|\gamma'|_1 = 0$ and $|\gamma'|_2 = p$. One has clearly

(61) $$F_p = F_p\{0\} + F_{p-1}\{1\}$$

and

(62) $$G_p = G_p\{0\} \oplus G_{p-1}\{2\}.$$

Finally, let $\Phi_p\{i\}\colon F_p\{i\} \to G_p\{i\}$ be the corresponding map Φ_p, cf. (57), where the last coordinate $i_l = i$ is "frozen". By the hypothesis of induction the following sequences are supposed to be exact:

(63) $$F_{p-1}\{i\} \rightarrowtail F_p\{i\} \xrightarrow{\Phi_p\{i\}} \twoheadrightarrow G_p\{i\}, \qquad (i = 0, 1).$$

By combining (61) and (62) with (63) we obtain then the commutative diagram

(64) $$\begin{array}{ccccc}
F_{p-1}\{0\} + F_{p-2}\{1\} & \longrightarrow & F_p\{0\} + F_{p-1}\{1\} & \xrightarrow{\quad \Phi_p \quad} & G_p\{0\} \oplus G_{p-1}\{2\} \\
\twoheaduparrow & & \twoheaduparrow & & \uparrow {\scriptstyle (\mathrm{id}, \phi)} \\
F_{p-1}\{0\} \oplus F_{p-2}\{1\} & \rightarrowtail & F_p\{0\} \oplus F_{p-1}\{1\} & \xrightarrow{\Phi_p\{0\} \oplus \Phi_{p-1}\{1\}} \!\!\!\twoheadrightarrow & G_p\{0\} \oplus G_{p-1}\{1\} \\
\uparrow & & \uparrow & & \uparrow {\scriptstyle (0, \varepsilon)} \\
F_{p-1}\{0\} \cap F_{p-2}\{1\} & \longrightarrow & F_p\{0\} \cap F_{p-1}\{1\} & \longrightarrow & G_{p-1}\{0\}
\end{array}$$

whose middle row and all columns are exact (ε and ϕ in the extreme right column denote the arrows (53) corresponding to $\mu = l$).

Note the obvious inclusions

$$F_{p-1}\{0\} \subset F_p\{0\} \cap F_{p-1}\{1\} \subset \operatorname{Ker} \Phi_p\{0\}.$$

Besides, according to the induction hypothesis, $F_{p-1}\{0\} = \operatorname{Ker} \Phi_p\{0\}$. Thus, $F_{p-1}\{0\} = F_p\{0\} \cap F_{p-1}\{1\}$ and, similarly, $F_{p-2}\{0\} = F_{p-1}\{0\} \cap F_{p-2}\{1\}$, and the bottom row of (64) identifies with

$$F_{p-2}\{0\} \to F_{p-1}\{0\} \xrightarrow{\Phi_{p-1}\{0\}} G_{p-1}\{0\} .$$

The latter is exact according to the same induction hypothesis. The exactness of the upper row follows. □

A.2. *Remark*. In the case when $T(\alpha)$'s are topological vector spaces ($k = \mathbf{R}$ or $\mathbf{C}$) and all the short sequences (53) are *topologically* exact (i.e. the corresponding maps ε and ϕ are *open*) the above proof yields that the subspaces $F_p \subset T(1, \dots, 1)$ are closed and $\Phi_p\colon F_p/F_{p-1} \xrightarrow{\sim} G_p$ are homeomorphisms.

A.3 *Pure exact sequences*. Recall that a short exact sequence $M_0 \rightarrowtail M_1 \twoheadrightarrow M_2$ of left modules over a ring k is said to be *pure* (in the sense of P. M. Cohn

[6, p. 383]) if for every right k-module N the sequence $N \otimes M_0 \rightarrowtail N \otimes M_1 \twoheadrightarrow N \otimes M_2$ is exact.

A monomorphism $i\colon M' \rightarrowtail M$ and an epimorphism $f\colon M \twoheadrightarrow M''$ are said to be pure if the corresponding short exact sequences $M' \rightarrowtail M \twoheadrightarrow \operatorname{Coker} i$ and $\operatorname{Ker} f \rightarrowtail M \twoheadrightarrow M''$ are pure.

Every split and every exact sequence with M_2 flat are pure. An inductive limit of pure sequences is again pure.

A.4. Theorem (cf. [6, Thm. 2.4], [43, Prop. 9.1], [25, Thm. 2.3]). *The following conditions are equivalent to the purity of an exact sequence* $M_0 \overset{i}{\rightarrowtail} M_1 \overset{f}{\twoheadrightarrow} M_2$:

(a) *For every commutative diagram*

$$\begin{array}{ccc} M_0 & \xrightarrow{\;i\;} & M_1 \\ {\scriptstyle \chi}\uparrow & \overset{\xi}{\nwarrow} & \uparrow \\ L_0 & \xrightarrow{\;j\;} & L_1 \end{array}$$

where L_0 *and* L_1 *are free k-modules of finite type, there exists a k-module map* $\xi\colon L_1 \to M_0$ *such that* $\chi = \xi \circ j$.

(b) *For every finitely presented k-module* P *the canonical mapping* $f^*\colon \operatorname{Hom}_k(P, M_1) \to \operatorname{Hom}_k(P, M_2)$ *is surjective.*

(c) *The sequence of right k-modules* $M_2^* \rightarrowtail M_1^* \twoheadrightarrow M_0^*$ *is split exact* $(M^* := \operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}))$.

(d) *The exact sequence* $M_0 \rightarrowtail M_1 \twoheadrightarrow M_2$ *is an inductive limit of split exact sequences*

$$M_0 \rightarrowtail M_0 \oplus P_\lambda \twoheadrightarrow P_\lambda$$

where $\{P_\lambda\}$ *is an inductive system of finitely presented k-modules.* □

Additional information on various aspects of "purity" can be found in [43] and [41].

A.5. Let us consider a pure exact sequence $M_0 \rightarrowtail M_1 \twoheadrightarrow M_2$ of modules over a *commutative* ring k. For an arbitrary k-module V one has the natural filtration on $M_1^{\otimes l} \otimes V$, $l \in \mathbf{N}$,

$$F_p\big(M_1^{\otimes l} \otimes V\big) = \operatorname{lin\,span}\{ m_1 \otimes \cdots \otimes m_l \otimes v \mid \text{at least } l - p \quad m_j\text{'s belong to } M_0 \}.$$

A.6. Lemma. *For every* $l \in \mathbf{N}$ *and* $p = 0, \dots, l$, *there is a canonical isomorphism of graded k-modules*

$$\operatorname{Gr}^F\big(M_1^{\otimes l} \otimes V\big) \xrightarrow{\sim} (M_0 \oplus M_2)^{\otimes l} \otimes V. \tag{65}$$

Proof. This is a particular case of Lemma A.1 with

$$T(i_1, \dots, i_l) = M_{i_1} \otimes \cdots \otimes M_{i_l} \otimes V. \qquad \square$$

The isomorphism (65) is functorial with respect to morphisms of pure exact sequences.

B. The Cartan identity in the Hochschild complex

Let M be a bimodule over a not necessarily unital k-algebra A. The Lie algebra $A_{\text{Lie}} = (A; [\ ,\ ])$ acts on the Hochschild complex $C_*(A, M)$ via the Lie derivative:

$$(66) \quad L_a(m \otimes a_1 \otimes \cdots \otimes a_q) = [a, m] \otimes a_1 \otimes \cdots \otimes a_q + \sum_{j=1}^{q} m \otimes a_1 \otimes \cdots \otimes \overset{j}{[a, a_j]} \otimes \cdots \otimes a_q$$

$(a \in A)$. For every $a \in A$, one can also define the suitable *exterior* product

$$e_a(m \otimes a_1 \otimes \cdots \otimes a_q) = - m \otimes a \otimes a_1 \otimes \cdots \otimes a_q + \sum_{i=1}^{q} (-1)^{i+1} m \otimes a_1 \otimes \cdots \otimes a_i \otimes \overset{i+1}{a} \otimes a_{i+1} \otimes \cdots \otimes a_q.$$

As in the case of the Chevalley complex for Lie algebras, one has the familiar looking analogue of the Cartan identity:

B.1. $L_a = e_a b + b e_a$ *where b denotes the Hochschild boundary map* (*cf.* §2 *where the case $M = A$ is considered*).

Proof. A straightforward computation shows that $e_a b(m \otimes a_1 \otimes \cdots \otimes a_q)$ is the sum of four terms:

$$\text{(I)} \qquad \sum_{1 \le i < j \le q-1} (-1)^{i+j} m \otimes \cdots \otimes \overset{i}{a} \otimes \cdots \otimes \overset{j+1}{a_j a_{j+1}} \otimes \cdots \otimes a_q,$$

$$\text{(II)} \qquad \sum_{1 \le i \le q} (-1)^{i+q} a_q m \otimes \cdots \otimes \overset{i}{a} \otimes \cdots \otimes a_{q-1},$$

$$\text{(III)} \qquad \sum_{1 \le j < i-1 \le q} (-1)^{(i-1)+j} m \otimes \cdots \otimes \overset{j}{a_j a_{j+1}} \otimes \cdots \otimes \overset{i-1}{a} \otimes \cdots \otimes a_q,$$

$$\text{(IV)} \qquad \sum_{2 \le i \le q+1} (-1)^{i-1} m a_1 \otimes \cdots \otimes \overset{i-1}{a} \otimes a_i \otimes \cdots \otimes a_q.$$

Similarly, $be_a(m \otimes a_1 \otimes \cdots \otimes a_q)$ is the sum of the following seven terms:

$$\text{(V)} \qquad \sum_{1 \le j < i-1 \le q} (-1)^{i+j} m \otimes \cdots \otimes \overset{j}{a_j a_{j+1}} \otimes \cdots \otimes \overset{i-1}{a} \otimes a_i \otimes \cdots \otimes a_q,$$

$$\text{(VI)} \qquad \sum_{1 \le i \le q+1} (-1)^{i+(i-1)} m \otimes \cdots \otimes \overset{i-1}{a_{i-1} a} \otimes a_i \otimes \cdots \otimes a_q,$$

$$\text{(VII)} \qquad \sum_{1 \le i \le q} (-1)^{i+(q+1)} a_q m \otimes \cdots \otimes \overset{i}{a} \otimes \cdots \otimes a_{q-1},$$

$$\text{(VIII)} \qquad \sum_{1 \le i < j \le q-1} (-1)^{i+(j+1)} m \otimes \cdots \otimes \overset{i}{a} \otimes \cdots \otimes \overset{j+1}{a_j a_{j+1}} \otimes \cdots \otimes a_q,$$

$$\text{(IX)} \qquad \sum_{1 \le i \le q} (-1)^{i+i} m \otimes \cdots \otimes \overset{i}{a a_i} \otimes \cdots \otimes a_q,$$

$$\text{(X)} \qquad am \otimes a_1 \otimes \cdots \otimes a_q,$$

$$\text{(XI)} \qquad \sum_{2 \le i \le q+1} (-1)^i m a_1 \otimes \cdots \otimes \overset{i-1}{a} \otimes a_i \otimes \cdots \otimes a_q.$$

Terms (VI), (IX) and (X) give the right-hand side of (66), while (I) cancels with (VIII), (II) with (VII), (III) with (V) and (IV) with (XI). □

Since the homology of $C_*(A, M)$ is equal to the Hochschild homology $H_*(A, M)$ in the case when A has a unit, and to $H_*(\tilde{A}, M)$ in the general case, we obtain:

B.2. Corollary. *Let A be a not necessarily unital algebra. Then the adjoint action of A_{Lie} on $H_*(\tilde{A}, M)$, where M is an arbitrary A-bimodule, is trivial.*

In particular, A_{Lie} acts trivially on $\mathscr{H}_*(A, A)$, $HB_*(A; V)$ and $H_*(A, A)$ (cf. §2 above). □

The Institute for Advanced Study, Princeton, NJ

References

[1] E. F. Assmus, Jr., On the homology of local rings, Illinois J. Math. 3 (1959), 187–199.

[2] A. J. Berrick and M. E. Keating, The K-theory of triangular matrix rings, Proc. Conf. on Applications of Alg. K-theory to Alg. Geometry and Number Theory (Boulder 1983), Part I, Contemp. Math., Vol. 55, pp. 69–74, Providence, 1986.

[3] A. Beurling, Sur les intégrales de Fourier absolument convergentes, IX^e Congrès Math. Scand., pp. 345–366, Helsinki, 1938.

[4] J. T. Burnham, Segal algebras and dense ideals in Banach algebras, in *Functional Analysis and its Applications* (Madras 1973), Lecture Notes in Mathematics, Vol. 399, pp. 33–58, Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[5] M. C. R. Butler and G. Horrocks, Classes of extensions and resolutions, Phil. Trans. Royal Soc. London, Ser. A, **254** (1961), 155–222.
[6] P. M. Cohn, On the free product of associative rings, Math. Z. 71 (1959), 380–398.
[7] A. Connes, Non-commutative differential geometry, Publ. Math. I.H.E.S. **62** (1986), 41–144.
[8] I. G. Craw, Factorisation in Fréchet algebras, J. London Math. Soc. 44 (1969), 607–611.
[9] R. K. Dennis and S. C. Geller, K_i of upper triangular matrix rings, Proc. A. M. S. **56** (1976), 73–78.
[10] R. K. Dennis and K. Igusa, Hochschild homology and the second obstruction for pseudo-isotopy, Proc. Conf. on Alg. K-theory (Oberwolfach 1980), Part I, Lecture Notes in Mathematics, Vol. 966, pp. 7–58, Springer-Verlag, 1982.
[11] S. Eilenberg and J. C. Moore, Foundations of relative homological algebra, Mem. A.M.S. **55**, Providence, 1965.
[12] C. Faith, *Algebra* I: *Rings, Modules, and Categories*, Die Grundlehren der mathematischen Wissenschaften, Vol. 190, Springer-Verlag, 1973, 1981 (corrected repr.).
[13] H. G. Feichtinger, C. C. Graham and E. H. Lakien, Nonfactorization in commutative, weakly self-adjoint Banach algebras, Pacific J. Math. **80** (1979), 117–125.
[14] B. L. Feigin and B. L. Tsygan, Cohomology of Lie algebras of generalized Jacobi matrices, Funct. Anal. and its Appl. **17:2** (1983), 86–87 (in Russian).
[15] R. R. Goldberg, Recent results on Segal algebras, in *Functional Analysis and its Applications* (Madras 1973), Lecture Notes in Mathematics, Vol. 399, pp. 220–229, Springer-Verlag, 1974.
[16] T. G. Goodwillie, Relative algebraic K-theory and cyclic homology, Ann. of Math. **124** (1986), 347–402.
[17] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Mem. A.M.S. **16**, Providence, 1955.
[18] G. Hochschild, Relative homological algebra, Trans. A.M.S. **82** (1956), 246–269.
[19] K. Iwasawa, On group rings of topological groups, Proc. Imp. Acad. Japan **20** (1944), 67–70.
[20] W. B. Johnson, H. P. Rosenthal and M. Zippin, On bases, finite dimensional decompositions and weaker structures in Banach spaces, Israel J. Math. 9 (1971), 488–506.
[21] W. van der Kallen, A note on excision for K_2, in *Algebraic K-theory, Number Theory, Geometry and Analysis* (Bielefeld 1982), Lecture Notes in Mathematics, Vol. 1046, pp. 173–177, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.
[22] M. Karoubi and O. Villamayor, K-théorie algébrique et K-théorie topologique I, Math. Scand. **28** (1971), 265–307.
[23] J. Knopfmacher, Some homological formulae, J. Algebra **9** (1968), 212–219.
[24] G. Köthe, Topological vector spaces I, Die Grundlehren der mathematischen Wissenschaften, Vol. 159; II. *idem.*, Vol. 237, Springer-Verlag, 1969 and 1979.
[25] D. Lazard, Autour de la platitude, Bull. Soc. Math. France **97** (1969), 81–128.
[26] J.-L. Loday, K-théorie algébrique et représentations de groupes, Ann. Scient. Éc. Norm. Sup., 4^e série, **9** (1976), 309–377.
[27] J.-L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, Comment. Math. Helv. **59** (1984), 565–591.
[28] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Vol. 114, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
[29] B. Malgrange, *Ideals of Differentiable Functions*, Oxford University Press, Oxford, 1966.
[30] E. A. Michael, Locally multiplicatively-convex topological algebras, Mem. A.M.S. 11, Providence, 1952.
[31] J.-L. Ovaert, Factorisation dans les algèbres et modules de convolution, C.R. Acad. Sci. Paris **265** (1967), 534–535.

[32] A. Pietsch, *Nuclear Locally Convex Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 66, Springer-Verlag, 1972.

[33] D. Quillen, Characteristic classes of representations, Proc. Conf. on Alg. K-theory (Evanston 1976), Lecture Notes in Mathematics, Vol. 551, pp. 189–216, Springer-Verlag, 1976.

[34] ______, On the (co-)homology of commutative rings, Proc. Symp. Pure Math. A.M.S. 17 (1970), 65–87.

[35] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Oxford University Press, Oxford, 1966.

[36] ______, Sur certains idéaux dans $L^1(G)$, C.R. Acad. Sci. Paris **267** (1968), 882–885.

[37] ______, *L^1-Algebras and Segal Algebras*, Lecture Notes in Mathematics, Vol. 231, Springer-Verlag, 1971.

[38] L. Schwartz, *Théorie des Distributions*, t. II, Hermann, Paris, 1951.

[39] G. Segal, The definition of conformal field theory (manuscript), Oxford, 1987.

[40] I. E. Segal, Irreducible representations of operator algebras, Bull. A.M.S. **53** (1947), 73–88.

[41] E. G. Sklyarenko, Relative homological algebra in the category of modules (in Russian), Uspekhi mat. nauk **33:3** (1978), 85–120; (English transl.) Russian Math. Surveys **33**, n°3, 97–137.

[42] S. P. Smith, Differential operators on commutative algebras, Proc. Conf. on Ring Theory (Antwerp 1985), Lecture Notes in Mathematics, Vol. 1197, pp. 165–177, Springer-Verlag, 1986.

[43] B. T. Stenström, Pure submodules, Arkiv för Matematik **7** (1967), 159–171.

[44] M. K. Summers, Factorization in Fréchet modules, J. London Math. Soc. (2), **5** (1972), 243–248.

[45] L. N. Vaserstein, Foundations of algebraic K-theory (in Russian), Uspekhi mat. nauk **31:4** (1976), 87–149; (English transl.) Russian Math. Surveys **31**, n°4, 89–156.

[46] J. Voigt, Factorization in some Fréchet algebras of differentiable functions, Stud. Math. **77** (1984), 333–348.

[47] J. Voigt, Factorization in Fréchet algebras, J. London Math. Soc. (2) **29** (1984), 147–152.

[48] J. B. Wagoner, Delooping classifying spaces in algebraic K-theory, Topology **11** (1972), 349–370.

[49] H.-c. Wang, *Homogeneous Banach Algebras*, Lecture Notes in Pure and Applied Math., Vol. 29, Marcel Dekker, New York-Basel, 1977.

[50] N. Wiener, Tauberian theorems, Ann. of Math. **33** (1932), 1–100.

[51] ______, *The Fourier Integral and Certain of its Applications*, Cambridge University Press, Cambridge, 1933.

[52] M. Wodzicki, Cyclic homology of pseudodifferential operators and noncommutative Euler class, C.R. Acad. Sci. Paris **306** (1988), 321–325.

[53] ______, Excision in cyclic homology, Continuous case (in preparation).

[54] ______, The long exact sequence in cyclic homology associated with an extension of algebras, C.R. Acad. Sci. Paris **306** (1988), 399–403.

[55] ______, Vanishing of cyclic homology of stable C^*-algebras, C.R. Acad. Sci. Paris **307** (1988), 329–334.

(Received April 20, 1988)