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Commutator structure of operator ideals

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0. Introduction

For any ideals I and J in the algebra of bounded operators, $\mathscr{B}(H)$, on a separable infinite dimensional Hilbert space H, we determine the commutator space $[I,J] = \bigcup_{r=1}^{\infty} [I,J]_r$ where

$$[I,J]_r := \left\{ T = \sum_{i=1}^{r} [A_i, B_i] \mid A_i \in I,\ B_i \in J \right\}. \tag{1}$$

Not much has been known about these spaces except for some special cases. It was known since 1953 that every operator $A \in \mathscr{B}(H)$ is the sum of two commutators of bounded linear operators [16,35] while the commutator spaces of Schatten ideals $\mathscr{L}_p$, $p > 0$, were the subject of several studies over the last 30 years, notably [3,4,18,38,49,66–68]. Aside from these efforts, the little that was known about the commutator spaces $[I,J]$ did not even suffice for deciding whether equality

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$[I, J] = [\mathscr{B}(H), IJ]$ held for general operator ideals. The study of this classical subject acquired a renewed sense of urgency since the introduction of cyclic cohomology by Alain Connes [20–22] in the early 1980s. The work of the last author exhibited in 1991 a direct link between the cyclic homology and algebraic K-theory of operator ideals and their commutator structure ([70,72] and Remark 5.14 below); in order to calculate the former one needs to tackle first the latter. Thus, the determination of the commutator structure of operator ideals became a prerequisite for the future theory of higher index invariants; cf. [70,72].

In order to state main results we need to recall some basic facts. The lattice of proper ideals in $\mathscr{B}(H)$ is naturally isomorphic to the lattice of symmetric proper ideals in the *commutative* algebra ℓ_∞ (cf. [19]) and both of these are isomorphic to the lattice of characteristic subsets of $c_0^{\star}$, the latter being the set of nonnegative sequences λ monotonically convergent to zero ([32] and Chapters 1 and 2). This triple correspondence is realized as follows: to any ideal $J \subsetneq \mathscr{B}(H)$ correspond the symmetric sequence space

$$S(J) := \left\{ \alpha \in c_0 \,\middle|\, \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \ddots \end{pmatrix} \in J \right\}$$

and the *characteristic set* $\Sigma(J) := S(J) \cap c_0^{\star}$. Definitions and constructions of ideals are often carried through most easily in terms of the corresponding characteristic sets. For example, the operation that associates with a solid subset Σ of $c_0^{\star}$ the solid set Σ_a which is generated by the arithmetic mean-sequences λ_a of sequences λ from Σ, allows us to associate with any ideal J in $\mathscr{B}(H)$, the *arithmetic mean ideal* J_a. Similarly, the naturally defined operation of tensor product of monotonic sequences leads to the operation of *internal tensor product* $\diamondsuit$ on the operator ideal lattice; see Section 4.3 below.

Characterization of the commutator space $[I, J]$ for an arbitrary pair of ideals I and J is an immediate consequence of the following result which is a focal point of the present article:

A normal operator $T \in IJ$ belongs to the commutator space $[I, J]$ if and only if

$$\left| \frac{\lambda_1(T) + \cdots + \lambda_n(T)}{n} \right| = O(\mu_n)$$

for some sequence $\mu \in \Sigma(IJ)$.

Here I and J are arbitrary ideals in $\mathscr{B}(H)$, at least one of them assumed to be proper, and $\lambda(T) = (\lambda_1(T), \lambda_2(T), \ldots)$ denotes the sequence of eigenvalues of T, the nonzero ones counted according to their multiplicities, but taken in any order subject only to the condition that $|\lambda_n(T)| = O(\nu_n)$ for some $\nu \in \Sigma(IJ)$. See Theorem 5.6 for the full statement.

The above result has numerous other applications. We mention here just a few:

1. $[I, J] = [\mathscr{B}(H), IJ]$ holds for any pair of ideals in $\mathscr{B}(H)$.
2. $I \subseteq [\mathscr{B}(H), J]$ *if and only if* $I_a \subseteq J$.
   Note that $I \subseteq [\mathscr{B}(H), J]$ precisely when every trace functional on J, i.e., a linear functional $\tau : J \to \mathbb{C}$ which annihilates the commutator space $[\mathscr{B}(H), J]$, identically vanishes on I.
3. The usual trace functional Tr $: \mathscr{L}_1 \to \mathbb{C}$ extends to a trace on an ideal $J \supsetneq \mathscr{L}_1$ *if and only if*

$$\omega := \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) \notin \Sigma(J).$$

   It is easy to see that no such extension can be positive or continuous. This result proves to be an essential ingredient in establishing the following "vanishing theorem" in Hochschild homology:

   The Hochschild homology groups $H_*(\mathscr{B}(H); J)$ vanish in all dimensions if and only if $H_0(\mathscr{B}(H); J) = 0$, i.e., $J = [\mathscr{B}(H), J]$, (2)

   which, in turn, is essential for the calculation of the cyclic homology and algebraic K-theory of operator ideals (cf. [70,72]).
4. If a positive operator T belongs to $[I, J]$ then the whole principal ideal (T) generated by T is contained in $[I, J]$ (Theorem 5.11(i)).

Theorem 5.6 also allows us to obtain the following result:

A finitely generated ideal J admits a complete norm $\|\ \|$ if and only if it admits no nonzero trace $\tau : J \to \mathbb{C}$. In this case, J coincides with the Marcinkiewicz ideal $\mathscr{M}(1/\pi_a)$ for some sequence $\pi \in c_0^{\star}$. (3)

(See Theorem 5.20.)

Comparison of (3) with (2) leads to a very interesting conclusion, namely that the existence of a complete norm on a finitely generated ideal J in $\mathscr{B}(H)$ is *equivalent* to the vanishing of the Hochschild homology groups of the algebra $\mathscr{B}(H)$ with coefficients in the bimodule J. In other words, the nontriviality of these purely algebraic homology groups is a faithful obstruction to the existence of a complete norm on the ideal!

The result cited in (3) shows also some of the limitations of the class of symmetrically normed ideals which, from the time of the publication of Schatten's book [54], has been occupying a privileged position in the study of operator ideals, largely due to the influence of [33]. Another well-known limitation of this class concerns the fact that the powers of Banach ideals are, generally, not Banach, *vide* the Schatten ideals $\mathscr{L}_p = (\mathscr{L}_1)^{1/p},\ 0 < p < 1$.

We overcome these limitations by introducing a new concept of an *e-complete* ideal (*e* denotes a positive real number, cf. Section 4.6 below). Now, every principal ideal (T) is *e*-complete as long as the sequence of reciprocals of singular numbers $s(T)$ satisfies the so-called Δ_2-condition (cf. (22) below). Unlike Banach ideals, the class of *e*-complete ideals is closed with respect to forming powers $J^s,\ s > 0$. Thus, any positive power of a Banach ideal, as well as numerous "classical" nonlocally convex ideals (like arbitrary Lorentz ideals $\mathscr{L}_p(\varphi)$; cf. Section 4.7), are *e*-complete.

The concept of a rearrangement invariant norm on the symmetric sequence space associated with a given ideal is replaced in the theory of *e*-complete ideals by the concept of a *gauge*, the latter being a homogeneous monotonic functional on the corresponding characteristic set (see Section 2.9 for details). Any gauge $\mathfrak{q}$ on a given characteristic set Σ induces the corresponding gauges $\mathfrak{q}_p$ for all powers $\Sigma^{1/p},\ p > 0$. We say that Σ is *e*-complete if Σ^e is complete with respect to $\mathfrak{q}_p$ for $p = 1/e$. When Σ is the characteristic set of an ideal J, we say in that case that ideal J is *e*-complete.

An exact relationship between the two classes of ideals is established by Theorem 3.6 below:

$$\textit{A certain power of any e-complete ideal is a symmetrically normed ideal with respect to an equivalent norm.} \tag{4}$$

It is worth emphasizing that this result does not detract from the usefulness of the concept of *e*-completeness. There are several reasons for this. The characteristic sets of not necessarily Banach powers of Banach ideals often admit natural and simple gauges as illustrated by principal ideals (cf. Sections 2.22–24). Even in the case of Banach ideals, a simple and natural gauge may be available on the corresponding characteristic set while an equivalent rearrangement invariant norm may be much harder to use: Lorentz ideals $\mathscr{L}_{pq}$ supply a classical example; cf. the last paragraph of Section 4.11.

Each *e*-complete ideal J has naturally attached to it a metric invariant $\alpha(J) \in [0, \infty)$ which we call the *Boyd α-index* of the ideal in question (cf. Section 4.6) following an example of Boyd, who defined his α- and β-indices only for rearrangement invariant Banach function spaces (note that for a Banach space both of these indices can take only values in the interval [0,1]; cf. [14]). We prove the following "index theorem" (Theorem 5.13):

$$\textit{The Boyd index of an e-complete ideal } J \textit{ does not exceed } p > 0 \textit{ if and only if ideal } J^{1/p} \textit{ admits a nonzero trace.} \tag{5}$$

A striking consequence of (5) is that:

$$J = [\mathscr{B}(H), J] \ \textit{if}\ J^{1+\varepsilon} \ \textit{admits a complete norm for some}\ \varepsilon > 0. \tag{6}$$

This follows from the equality $\alpha(J^{1+\varepsilon}) = (1+\varepsilon)\alpha(J)$ and the fact that the Boyd α-index of a Banach ideal does not exceed 1.

Several important classes of ideals like Lorentz ideals $\mathscr{L}_p(\varphi)$, Marcinkiewicz ideals $\mathscr{M}_p(\psi)$, Orlicz ideals $\mathscr{L}_M$ and $\mathscr{L}_M^{(0)}$ (cf. Section 4.7 below) are defined in terms of data consisting of a positive real parameter $p > 0$ and a certain auxiliary sequence or function (in the case of Orlicz ideals). We give complete characterizations of the condition $J = [\mathscr{B}(H), J]$ *purely in terms of the associated data* for each of the above-mentioned classes of ideals (see Theorems 5.21, 5.24 and 5.25). The proofs of these characterizations combine Theorem 5.6 with the analysis performed in Chapter 3.

In order to keep this introduction within a reasonable length, we shall only signal two additional topics worthy of being mentioned here: the $\otimes$-operation (the *internal* tensor product mentioned earlier) and an intriguing double inequality in which it appears (cf. inequalities (70) and (75)). The $\otimes$-operation plays an essential role in Section 7 where we devote our attention to the single commutator space $[\mathscr{B}(H), J]_1$ (see especially the proofs of Theorem 7.1 and Corollary 7.10). It is also of central importance for determining the cyclic homology and algebraic K-theory of operator ideals.

The article is organized as follows. The first two sections are preliminary and should be viewed as a helpful compendium of notations, definitions and constructions. We strived for a clear and thorough exposition in the hope that this material may become, in conjunction with Section 4, a "standard reference" for the subject of operator ideals. Section 3 contains a number of results important independently as well as in conjunction with the material of Section 5. The latter is devoted entirely to the statement and proof of the main result, Theorem 5.6, and many of its consequences and other results relying on it. Several of the latter combine Theorem 5.6 with results of Section 3. In Section 6 we establish upper bounds on the number of commutators required to represent an element of $[I, J]$ as a sum of commutators. In Section 7 we prove that at least some of these bounds are optimal, and we also give a sufficient condition for an operator to be represented as a single commutator. In these last two sections we were influenced by some results and techniques developed by Anderson et al. (cf. [3,5,18]).

The following remarks are designed to help the first-time reader in navigating to Theorem 5.6 in the shortest amount of time.

Begin by reading the first three sections of Section 4. All the necessary terminology and notation is explained in Sections 1–3, 6 and 8 of Chapter 1, and in Sections 1, 5 and 7 of Chapter 2. Then continue by reading the first five sections of Chapter 5 which lead to Theorem 5.6 and its proof.

The present article represents the final stage of a long development. Several of its main results have been obtained during the period 1994–1996, some as long ago as in the late 1980s. This earlier phase of the development was reflected in the Odense preprint [27]. The work reached its final form, except for some very minor details, in the Fall 2000. An alternative approach to some results of Section 5 is presented in [29,71].

0.1 Preliminaries. We will treat interval $[0, \infty]$ as an ordered monoid in which $0 \cdot \infty = \infty \cdot 0 = 0$. Note that $[0, \infty]$ coincides with $((\mathbb{R}^*_+)_\sim)_\sim$, where $S \mapsto S_\sim = S \amalg \{z\}$ denotes the functor that attaches a zero element to a semigroup S.[4] By using the *set* automorphism $t \mapsto t^{-1}$, we define division in $[0, \infty]$ as $a/b := ab^{-1}$.

The absolute value map $|\,|$ extends to the map $|\,| : \overline{\mathbb{C}} \to [0, \infty]$ where $\overline{\mathbb{C}} = \overline{\mathbb{R}} + \overline{\mathbb{R}}i$ and $\overline{\mathbb{R}} = [-\infty, \infty]$. It is also convenient to extend the usual "p-norms", $p > 0$, to functions $\alpha : \overline{\mathbb{C}}^\Gamma \to [0, \infty]$

$$||\alpha||_p := \sup_{F \subset \Gamma} \left(\sum_{\gamma \in F} |\alpha_\gamma|^p \right)^{1/p}$$

where the supremum is taken over all finite subsets F of a given set Γ, and to put

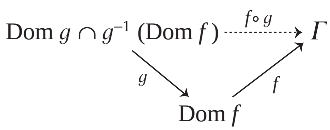
$$||\alpha||_\infty := \sup|\alpha| \equiv \sup_{\gamma \in \Gamma} |\alpha_\gamma|.$$

1. Symmetric vector subspaces of $\mathbb{C}^\Gamma$

1.1. The monoid $\mathrm{Emb}(\Gamma)$. For a given set Γ, the collection of all injective maps

$$\mathrm{Emb}(\Gamma) := \{ f : \Gamma' \rightarrowtail \Gamma \mid \Gamma' \subseteq \Gamma \}$$

forms naturally a monoid under the composition law



Note that id_Γ is the neutral element and the empty mapping $\varnothing : \varnothing \to \Gamma$ is the zero element.

The *antipode* map $f \mapsto f^\dagger$, where $f^\dagger : f(\Gamma) \to \Gamma$ is given by $f^\dagger(\gamma) = f^{-1}(\gamma)$, is an anti-involution of the monoid $\mathrm{Emb}(\Gamma)$.

There are four important submonoids in $\mathrm{Emb}(\Gamma)$:

$\mathscr{E}_\Gamma$ consists of all self-embeddings $\Gamma \to \Gamma$;

$\mathscr{E}^\dagger_\Gamma$ which is the image of $\mathscr{E}_\Gamma$ under the antipode † consists of all bijections $f : \Gamma' \xrightarrow{\sim} \Gamma$ where Γ' ranges over arbitrary subsets $\Gamma' \subseteq \Gamma$ of the same cardinality as Γ;

[4] Recall that an element $z \in S$ of a semigroup S is said to be *a zero* if $zs = sz = z$ for all $s \in S$. The category of semigroups with zero Semigr_0 is reflective in the category Semigr of all semigroups, $S \mapsto S_\sim$ being the left adjoint functor to the inclusion $\mathrm{Semigr}_0 \hookrightarrow \mathrm{Semigr}$.

$\mathscr{E}_\Gamma^* = \mathscr{E}_\Gamma \cap \mathscr{E}_\Gamma^\dagger$ is the group of all bijections $\Gamma \xrightarrow{\sim} \Gamma$ which is precisely the group of invertible elements of the monoid $\mathrm{Emb}(\Gamma)$;

the submonoid formed by all *inclusion* maps $\Gamma' \hookrightarrow \Gamma$ is canonically isomorphic to the monoid $(2^\Gamma, \cap)$; this allows us to view $(2^\Gamma, \cap)$ as a submonoid of $\mathrm{Emb}(\Gamma)$.

For an infinite set Γ, every element in $\mathrm{Emb}(\Gamma)$ admits a factorization $f^\dagger g$ for some $f, g \in \mathscr{E}_\Gamma$, i.e., $\mathrm{Emb}(\Gamma) = \mathscr{E}_\Gamma^\dagger \mathscr{E}_\Gamma := \{f^\dagger g \,|\, f, g \in \mathscr{E}_\Gamma\}$. On the other hand, $\mathscr{E}_\Gamma \mathscr{E}_\Gamma^\dagger \neq \mathrm{Emb}(\Gamma)$. In fact, $\mathscr{E}_\Gamma \mathscr{E}_\Gamma^\dagger$ is not even a submonoid.

1.2. The action on $\mathbb{C}^\Gamma$. The monoid $\mathrm{Emb}(\Gamma)$ acts naturally on the product vector space $\mathbb{C}^\Gamma$:

$$(f_*\alpha)_\gamma := \begin{cases} \alpha_\delta & \text{if } \gamma \in f(\Gamma') \text{ and } f(\delta) = \gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

(We shall write the argument of a function $\alpha : \Gamma \to \mathbb{C}$ as a subscript to emphasize that we think of α as being a "Γ-indexed sequence".)

This action is continuous in the product topology. The subalgebra of the algebra $\mathscr{L}(\mathbb{C}^\Gamma)$ of continuous linear operators on $\mathbb{C}^\Gamma$, which is generated by the group $\mathscr{E}_\Gamma^*$ and the monoid $2^\Gamma \subset \mathrm{Emb}(\Gamma)$, automatically contains also $\mathscr{E}_\Gamma$ and $\mathscr{E}_\Gamma^\dagger$, and therefore contains the whole monoid $\mathrm{Emb}(\Gamma)$. Even more: if $\Gamma' \subseteq \Gamma$ is a subset such that $|\Gamma'| = |\Gamma \backslash \Gamma'| = |\Gamma|$, then the aforementioned subalgebra is generated by $\mathscr{E}_\Gamma^*$ and $\Gamma' \in 2^\Gamma$ alone.

We conclude that any vector subspace $V \subseteq \mathbb{C}^\Gamma$ which is *symmetric* (i.e., $\mathscr{E}_\Gamma^*$-invariant; cf. [40, Section 14]), and *divisible* (i.e., invariant under the projection $\Gamma'_* : \alpha \mapsto \alpha_{|\Gamma'}$ for at least one subset $\Gamma' \subset \Gamma$ such $|\Gamma'| = |\Gamma \backslash \Gamma'| = |\Gamma|$) is automatically invariant under the action of the whole monoid $\mathrm{Emb}(\Gamma)$.

An example of a divisible vector subspace $V \subseteq \mathbb{C}^\Gamma$ is provided by a *solid* subspace, i.e., an $\ell_\infty(\Gamma)$-submodule of $\mathbb{C}^\Gamma$.

1.3. Rearrangements. For an infinite set Γ, the relation on $\mathbb{C}^\Gamma$ defined by membership in an $\mathscr{E}_\Gamma$-orbit:

$$\alpha' \in (\mathscr{E}_\Gamma)_*\alpha \tag{8}$$

is not an equivalence relation, since $\mathscr{E}_\Gamma$ is not a group. The smallest equivalence relation containing (8) is

$$\alpha \in (\mathscr{E}_\Gamma)_*\alpha'' \quad \text{and} \quad \alpha' \in (\mathscr{E}_\Gamma)_*\alpha'' \quad \text{for some } \alpha'' \in \mathbb{C}^\Gamma$$

or, equivalently,

$$(\mathscr{E}_\Gamma)_*\alpha \cap (\mathscr{E}_\Gamma)_*\alpha' \neq \emptyset.$$

In this case, we shall say that α' is a *rearrangement* of the function α. Thus, for an infinite set Γ, the rearrangements of α are obtained, loosely speaking, by possibly adding or removing some zero entries and reindexing the result with set Γ. The set of all rearrangements of α

$$[\![\alpha]\!] := \bigcup_{\substack{\beta \in \mathbb{C}^\Gamma \\ (\mathscr{E}_\Gamma)_* \alpha \,\cap\, (\mathscr{E}_\Gamma)_* \beta \neq \emptyset}} (\mathscr{E}_\Gamma)_* \beta \tag{9}$$

will be called the *quasiorbit* of α.

We shall say also that $V \subseteq \mathbb{C}^\Gamma$ is a *rearrangement invariant* (r.i.) *subspace* of $\mathbb{C}^\Gamma$ if $[\![\alpha]\!] \subset V$ whenever $\alpha \in V$.

For divisible vector subspaces the properties of being symmetric, $\mathscr{E}_\Gamma$-invariant, $(\mathscr{E}_\Gamma)^\dagger$-invariant, rearrangement invariant and $\mathrm{Emb}(\Gamma)$-invariant are all equivalent.

1.4. Rearrangement invariant Banach–Köthe (BK) spaces. A Banach space $V \subseteq \mathbb{C}^\Gamma$ such that the coordinate functionals $\alpha \mapsto \alpha_\gamma$ $(\gamma \in \Gamma)$ are bounded on its unit ball is called a BK-space (cf. [73, p. 29]). Every continuous linear map $A : \mathbb{C}^\Gamma \to \mathbb{C}^\Gamma$ which preserves V is automatically bounded in view of the Closed Graph Theorem. In particular, if a BK-space V is divisible and symmetric then the whole monoid $\mathrm{Emb}(\Gamma)$ acts on V in a bounded way. This action is actually uniformly bounded. We record this fact, which generalizes [42], without proof.

1.5. Proposition. *If a* BK-*space* $V \subseteq \mathbb{C}^\Gamma$ *is* $\mathrm{Emb}(\Gamma)$-*invariant then*

$$\sup\{\|f_*\| \mid f \in \mathrm{Emb}(\Gamma)\} < \infty \tag{10}$$

where $\|f_*\|$ *denotes the norm of the operator* $f_* : V \to V$.

Thus the assignment

$$\|\alpha\|' := \sup\{\|f_*\alpha\| \mid f \in \mathrm{Emb}(\Gamma)\}$$

defines an equivalent norm on V enjoying the following properties:

$$\text{for every } f \in \mathscr{E}_\Gamma, \quad f_* \text{ is an isometry,} \tag{11a}$$

$$\begin{aligned} &\text{for every } f \in \mathscr{E}_\Gamma,\ f^* := (f^\dagger)_* \text{ has norm 1 and is an isometry on the image} \\ &\text{of the projection } f(\Gamma)_* : \alpha \mapsto \alpha_{|f(\Gamma)} \ (f^* \text{ is 0 on the kernel of } f(\Gamma)_*), \end{aligned} \tag{11b}$$

$$\text{every projection } \Gamma'_*, \text{ where } \emptyset \neq \Gamma' \subseteq \Gamma, \text{ has norm 1.} \tag{11c}$$

Technically speaking, a *Köthe* norm on an r.i.-subspace $U \subseteq \mathbb{C}^\Gamma$ (i.e., a norm stronger than the product topology on $\mathbb{C}^\Gamma$) is *rearrangement invariant* if it satisfies just condition (11a). However, if the norm-completion $U^\sim$ is divisible the additional conditions (11b)–(11c) follow. An r.i. vector space $U \subseteq \mathbb{C}^{\mathbb{Z}_+}$, equipped with an r.i. norm will be called an *r.i. normed* space, and an *r.i.* BK-*space* if complete.

1.6. Symmetric solid subspaces $V \subseteq \mathbb{C}^{\Gamma}$. When a monoid N acts on a monoid M (an action of a monoid N on an object c of a category $\mathscr{C}$ is the same as a monoid homomorphism $\rho : N \to \mathrm{Hom}_{\mathscr{C}}(c, c)$), there is an associated semi-direct product $M \rtimes N$ which is itself a monoid: $(m, n)(m', n') = (m\rho_n(m'), nn')$.

The semidirect product $\ell_\infty(\Gamma) \rtimes \mathrm{Emb}(\Gamma)$ of $\mathrm{Emb}(\Gamma)$ and the multiplicative monoid of $\ell_\infty(\Gamma)$ acts naturally on $\mathbb{C}^{\Gamma}$:

$$(\beta, f)_* : \alpha \mapsto \beta f_* \alpha$$

and $\ell_\infty(\Gamma) \rtimes \mathrm{Emb}(\Gamma)$-invariant vector subspaces $V \subseteq \mathbb{C}^{\Gamma}$ are precisely symmetric solid subspaces.

1.7. Proposition. *If a* BK-*space* $V \subseteq \mathbb{C}^{\Gamma}$ *is symmetric and solid then*

$$\sup\{||(\beta, f)_*|| \mid \beta \in \ell_\infty(\Gamma),\ f \in \mathrm{Emb}(\Gamma)\} < \infty. \tag{12}$$

Indeed, one shows that

$$\sup\{||\beta_*|| \mid \beta \in \ell_\infty(\Gamma)\} < \infty \tag{13}$$

quite similarly to how one proves Proposition 1.5, and (13) combined with (10) gives (12). Since operators $(\beta, f)_*$ form a monoid, the $\ell_\infty(\Gamma) \rtimes \mathrm{Emb}(\Gamma)$-action on V is contractive with respect to the equivalent norm:

$$||\alpha||' := \sup\{||(\beta, f)_*\alpha|| \mid \beta \in \ell_\infty(\Gamma),\ f \in \mathrm{Emb}(\Gamma)\}.$$

An $\ell_\infty(\Gamma) \rtimes \mathrm{Emb}(\Gamma)$-invariant BK-space will be called a *contractive symmetric solid* (c.s.s.) BK-space if $||(\beta, f)_*|| \leqslant 1$ for all $\beta \in \ell_\infty(\Gamma)$ and $f \in \mathrm{Emb}(\Gamma)$.

Returning to general symmetric solid subspaces $V \subseteq \mathbb{C}^{\Gamma}$, if Γ is finite the only symmetric solid subspaces of $\mathbb{C}^{\Gamma}$ are 0 and $\mathbb{C}^{\Gamma}$. For Γ countable, it is well known (cf. e.g. [32]) and easy to see that $V = \mathbb{C}^{\Gamma}$ unless $V \subseteq \ell_\infty(\Gamma)$ and, in the latter case, $V = \ell_\infty(\Gamma)$ unless $V \subseteq c_0(\Gamma)$. For an uncountable set Γ, the situation is not so simple but is essentially governed by the structure of the cardinal number $|\Gamma|$. This will be of no interest to us in the present article. Subsequently, the focus will be on the countable case, in which, with the single exception of $\mathbb{C}^{\Gamma}$, symmetric solid subspaces of $\mathbb{C}^{\Gamma}$ are precisely *symmetric*, i.e., $\mathscr{E}^*_{\Gamma}$-invariant ideals in the ring of $\ell_\infty(\Gamma)$.

1.8. Two binary operations on $\mathbb{C}^{\Gamma}$. The disjoint union of sets induces the direct sum operation

$$\mathbb{C}^{\Gamma} \times \mathbb{C}^{\Gamma'} \to \mathbb{C}^{\Gamma \amalg \Gamma'}, \quad (\alpha, \alpha') \mapsto \alpha \oplus \alpha',$$

where

$$(\alpha \oplus \alpha')_\gamma := \begin{cases} \alpha_\gamma & \text{if } \gamma \in \Gamma, \\ \alpha'_\gamma & \text{if } \gamma \in \Gamma' \end{cases}$$

while the cartesian product of sets induces the tensor product operation

$$\mathbb{C}^{\Gamma} \times \mathbb{C}^{\Gamma'} \to \mathbb{C}^{\Gamma \times \Gamma'}, \quad (\alpha, \alpha') \mapsto \alpha \otimes \alpha',$$

where $(\alpha \otimes \alpha')_{(\gamma,\gamma')} := \alpha_\gamma \alpha'_{\gamma'}$.

We shall make use of these operations in Section 3.

2. Characteristic sets

2.1. We shall refer frequently to a number of operations on sequences. Thus, the *partial-sum sequence*:

$$\alpha \mapsto \sigma(\alpha), \quad \sigma_n(\alpha) := \alpha_1 + \cdots + \alpha_n, \tag{14}$$

the *difference sequence*:

$$\alpha \mapsto \Delta\alpha, \quad \Delta_n \alpha := \begin{cases} \alpha_1 & \text{for } n = 1, \\ \alpha_n - \alpha_{n-1} & \text{for } n > 1 \end{cases}$$

the *reverse difference sequence*:

$$\alpha \mapsto \Delta^- \alpha, \quad \Delta_n^- \alpha := \alpha_n - \alpha_{n+1},$$

the *arithmetic mean sequence*:

$$\alpha \mapsto \alpha_a, \quad (\alpha_a)_n := \sigma_n(\alpha)/n$$

and the *t-scansion*, $t \in (0, \infty)$,

$$\alpha \mapsto t^\bullet \alpha, \quad (t^\bullet \alpha)_n := \alpha_{\lceil tn \rceil}$$

define,[5] linear operators on $\mathbb{C}^{\mathbb{Z}_+}$. The operator D_m of *m-fold repetition* of each term coincides with $(1/m)^\bullet$.

In what follows, we shall encounter the sequence $\mathbb{1} = (1, 0, 0, \ldots)$ as well as the sequences $\mathbb{1}_m = D_m \mathbb{1} = (\underbrace{1, \ldots 1}_{m \text{ times}}, 0, 0, \ldots)$ and $\omega = \mathbb{1}_a = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$. Any sequence

[5] $\lceil x \rceil = -[-x]$ denotes the *upper entier* ("ceiling") function.

$\alpha \in c_0 = c_0(\mathbb{Z}_+)$ is represented by the series

$$\alpha = \sum_{m=1}^{\infty} (\Delta_m^- \alpha) \mathbb{1}_m \tag{15}$$

which converges in c_0.

If $\alpha \in \ell_1(\mathbb{Z}_+)$ then we have also the operation of the "*arithmetic mean at infinity*":

$$\alpha \mapsto \alpha_{a_\infty}, \quad (\alpha_{a_\infty})_n := \frac{1}{n} \sum_{i=n+1}^{\infty} \alpha_i. \tag{16}$$

2.2. If X is a nonempty *subset* of $\overline{\mathbb{R}}^{\mathbb{Z}_+}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$, then $\sup X$ and $\inf X$ are the corresponding supremum and infimum in the partially ordered set $\overline{\mathbb{R}}^{\mathbb{Z}_+}$; i.e., they are the sequences

$$(\sup X)_n = \sup\{\alpha_n \mid \alpha \in X\}$$

and

$$(\inf X)_n = \inf\{\alpha_n \mid \alpha \in X\}.$$

It is customary to write $\sup\{\alpha, \beta\} = \alpha \vee \beta$ and $\inf\{\alpha, \beta\} = \alpha \wedge \beta$.

On the other hand, for a *sequence* $\alpha \in \overline{\mathbb{R}}^{\mathbb{Z}_+}$, $\sup \alpha := \sup\{\alpha_n \mid n \in \mathbb{Z}_+\} \in \overline{\mathbb{R}}$, and $\inf \alpha$ is defined similarly.

2.3. Monotonic envelopes. For a sequence $\alpha \in \overline{\mathbb{R}}^{\mathbb{Z}_+}$, we shall occasionally refer to the upper or lower nondecreasing, or nonincreasing, envelopes of α. They are defined as follows:

$$\begin{aligned} \mathrm{und}_n(\alpha) &:= \sup_{i \leqslant n} \alpha_i, \\ \mathrm{lnd}_n(\alpha) &:= \inf_{i \geqslant n} \alpha_i \end{aligned} \tag{17}$$

and

$$\begin{aligned} \mathrm{uni}_n(\alpha) &:= \sup_{i \geqslant n} \alpha_i, \\ \mathrm{lni}_n(\alpha) &:= \inf_{i \leqslant n} \alpha_i. \end{aligned} \tag{18}$$

2.4. The Matuszewska indices of monotonic sequences. For any $\xi \in (0, \infty)^{\mathbb{Z}_+}$ the sequence

$$\bar{\xi}_m := \sup \frac{m^\bullet \xi}{\xi} \tag{19a}$$

is submultiplicative while

$$\underline{\xi}_m := \inf \frac{m^\bullet \xi}{\xi} \tag{19b}$$

is supermultiplicative. Note that $1/\bar{\xi} = (\underline{1/\xi})$ and $1/\underline{\xi} = (\overline{1/\xi})$.

By analogy with the Matuszewska indices of functions we shall consider the following invariants of a *monotonic* sequence ξ:

$$\alpha(\xi) := \lim_{m\to\infty} \frac{\log \bar{\xi}_m}{\log m} = \inf_{m\geqslant 2} \frac{\log \bar{\xi}_m}{\log m} \tag{20a}$$

and

$$\beta(\xi) := \lim_{m\to\infty} \frac{\log \underline{\xi}_m}{\log m} = \sup_{m\geqslant 2} \frac{\log \underline{\xi}_m}{\log m} \tag{20b}$$

(cf. [44,47] and the two part article [45]–[46], and Sections 1.52–56 of [46], especially). By definition, $-\infty \leqslant \beta(\xi) \leqslant \alpha(\xi) \leqslant \infty$, $\alpha(\xi\xi') \leqslant \alpha(\xi) + \alpha(\xi')$ and also

$$\alpha(1/\xi) = -\beta(\xi). \tag{21}$$

For a nondecreasing sequence the condition $\alpha(\xi) < \infty$ is equivalent to the so called *Δ_2-condition*

$$\bar{\xi}_2 < \infty. \tag{22}$$

2.5. Let $c_0^{\star}$ denote the set of all nonincreasing real sequences $\lambda \in c_0(\mathbb{Z}_+)$.

For every $\alpha \in c_0(\mathbb{Z}_+)$, the quasiorbit $[\![|\alpha|]\!]$ (cf. Section 1.3 above) contains a unique element of $c_0^{\star}$ which will be denoted $\alpha^{\star}$. This defines a nonadditive map $\star\colon c_0(\mathbb{Z}_+) \to c_0^{\star}$. If Γ is a countable set then a bijection of $\phi : \Gamma \xrightarrow{\sim} \mathbb{Z}_+$ induces an isomorphism of algebras $c_0(\Gamma) \xrightarrow{\sim} c_0(\mathbb{Z}_+)$ whose composition with $\star$ does not depend on the choice of ϕ, since $\star$ is constant on quasiorbits. The canonical map $c_0(\Gamma) \to c_0^{\star}$ thus obtained will be denoted $\alpha \mapsto \alpha^{\star}$ too.

Using the $\star$-operation, one can define the *internal direct sum*:

$$\diamond\!\!\!\!+\colon c_0^{\star} \times c_0^{\star} \to c_0^{\star}, \quad (\lambda, \mu) \mapsto \lambda \,\diamond\!\!\!\!+\, \mu := (\lambda \oplus \mu)^{\star},$$

and the *internal tensor product*

$$\diamond\!\!\!\!\times : c_0^{\star} \times c_0^{\star} \to c_0^{\star}, \quad (\lambda, \mu) \mapsto \lambda \,\diamond\!\!\!\!\times\, \mu := (\lambda \otimes \mu)^{\star}.$$

Both are associative and commutative operations on $c_0^{\star}$. The sequence $\mathbb{1} = (1, 0, 0, \ldots)$ serves as the neutral element for $\diamond\!\!\!\!\times$.

The following two simple double inequalities are sometimes useful:

2.6. Lemma. (a) *For any* $\lambda, \mu \in c_0^{\star}$,

$$\lambda \vee \mu \leqslant \lambda \oplus \mu \leqslant (\lambda \vee \mu)^{\oplus 2} = D_2(\lambda \vee \mu).$$

(b) *For any* $\lambda, \mu \in [0, \infty)^{\mathbb{Z}_+}$,

$$\lambda \vee \mu \leqslant \lambda + \mu = \lambda \vee \mu + \lambda \wedge \mu \leqslant 2(\lambda \vee \mu).$$

A subset $C \subseteq c_0^{\star}$ will be called:

(a) *solid* if $\lambda \leqslant \mu$ and $\mu \in C$ imply $\lambda \in C$,
(b) *radial* if $[0, \infty)C = C$,
(c) *additive* if $\lambda, \mu \in C$ implies $\lambda + \mu \in C$.

A radial additive subset $C \subseteq c_0^{\star}$ will be also called a *cone*.

2.7. Let Γ be countable. It is easy to see that the image under the $\star$-operation $\Sigma := S^{\star} \subseteq c_0^{\star}$ of any symmetric solid vector subspace $S \subseteq c_0(\Gamma)$, possesses the following property:

$$(\mathrm{Ch}\, S) \quad \text{If } \lambda \in c_0^{\star},\ \mu, \nu \in \Sigma \quad \text{and} \quad \lambda = O(\mu \oplus \nu) \text{ then } \lambda \in \Sigma.$$

Any subset Σ of $c_0^{\star}$ with this property will be called a *characteristic set*. In view of Lemma 2.6, a subset $\Sigma \subset c_0^{\star}$ is characteristic precisely when Σ is a D_2-invariant solid subcone of $c_0^{\star}$, i.e., $D_2\Sigma \subset \Sigma$.

2.8. There is a number of operations involving characteristic sets. The following two give rise to binary operations on the lattice of characteristic sets which will be denoted Λ; they are associative, commutative and respect the partial order of Λ:

(i) (Product): $\Sigma\Sigma' := \{\lambda \in c_0^{\star} \mid \lambda = O(\mu\nu) \text{ for some } \mu \in \Sigma \text{ and } \nu \in \Sigma'\}$,
(ii) (Internal tensor product):

$$\Sigma \otimes \Sigma' := \{\lambda \in c_0^{\star} \mid \lambda \leqslant \mu \otimes \mu' \text{ for some } \mu \in \Sigma \text{ and } \mu' \in \Sigma'\}.$$

The set $c_{\mathrm{f}}^{\star} := \{\lambda \in c_0^{\star} \mid \mathrm{card}(\mathrm{supp}\, \lambda) < \infty\}$ is the neutral element for $\otimes$.

The following operations involve an arbitrary nonempty subset $X \subset [0, \infty]^{\mathbb{Z}_+}$.

(iii) (Quotient): $\Sigma : X := \{\lambda \in c_0^{\star} \mid \lambda^{\oplus m} x \in \star^{-1}(\Sigma) \text{ for all } x \in X \text{ and } m \in \mathbb{Z}_+\}$.

If every element of X is dominated by one that satisfies the condition

$$x_{2n} + x_{2n+1} = O(x_n)$$

then $\Sigma{:}X = \{\lambda \in c_0^{\star} \mid \lambda x \in \star^{-1}(\Sigma) \text{ for all } x \in X\}$ (cf. Lemma 2.6). If, furthermore, X consists of nonincreasing sequences then $\Sigma{:}X = \{\lambda \in c_0^{\star} \mid \lambda X \subseteq \Sigma\}$.

(iv) (Köthe dual): $X^{\times} := \ell_1^{\star}{:}X$.

The next two operations require that X be a directed subset of $[0,\infty]^{\mathbb{Z}_+}$ in order for the following to be characteristic sets:

(v) (Pre-arithmetic mean): ${}_aX := \{\lambda \in c_0^{\star} \mid \lambda_a = O(x) \text{ for some } x \in X\}$,
(vi) (Arithmetic mean): $X_a := \{\lambda \in c_0^{\star} \mid \lambda = O(x_a) \text{ for some } x \in X\}$.

Finally we mention

(vii) (Real powers). $X^s := \{\lambda^s \mid \lambda \in X\}$, $s \in (0,\infty)$.

The fact that ${}_aX$ and X_a are characteristic sets follows from the inequalities $(D_2(x))_a \leqslant 2x_a$ and $D_2(x_a) \leqslant 2x_a$, which hold for any $x \in [0,\infty]^{\mathbb{Z}_+}$, and from Lemma 2.6. We shall call X_a the arithmetic mean (characteristic) set of X (briefly, *am*-set of X) and ${}_aX$ the pre-arithmetic mean set of X (briefly, pre-*am* set of X). The induced operations on Λ

$$\Sigma \mapsto \Sigma^{\circ} := ({}_a\Sigma)_a \quad \text{and} \quad \Sigma \mapsto \Sigma^{-} := {}_a(\Sigma_a)$$

are idempotent and are involved in the chain of inclusions:

$$\Sigma^{\circ} \subseteq \Sigma \subseteq \Sigma^{-}.$$

The set Σ° will be called the *am-interior* and Σ^{-} the *am-closure* of Σ. It is logical to call Σ *am-open* if $\Sigma = \Sigma^{\circ}$ and am-*closed* if $\Sigma = \Sigma^{-}$.

2.9. Gauged radial sets. A monotonic and homogeneous function $\mathcal{G} : C \to [0,\infty]$ on a radial set $C \subseteq c_0^{\star}$ will be called a *gauge*:

(a) (Monotonicity): $\lambda \leqslant \mu \Rightarrow \mathcal{G}(\lambda) \leqslant \mathcal{G}(\mu)$,
(b) (Homogeneity): $\mathcal{G}(t\lambda) = t\mathcal{G}(\lambda)$ for all $\lambda \in C$ and $t \in [0,\infty)$.

If C is a cone and $\mathcal{G}$ satisfies also the following conditions:

(c) (Triangle inequality). $\mathcal{G}(\lambda + \mu) \leqslant \mathcal{G}(\lambda) + \mathcal{G}(\mu)$ for all $\lambda, \mu \in C$,
(d) (Nondegeneracy). $\mathcal{G}(\lambda) > 0$ if $\lambda \neq 0$,

then $\mathcal{G}$ will be called a *cone norm* (or *c-norm* for short). Note that a finite gauge $\mathcal{G}$ on a cone C satisfies the triangle inequality if and only if the set

$$B(C,\mathcal{G}) := \{\lambda \in C \mid \mathcal{G}(\lambda) \leqslant 1\} \tag{23}$$

is convex.

Returning to the general case, a gauge $\mathcal{G}$ on a radial set C induces associated gauges on $C^{1/p} := \{\lambda^{1/p} \mid \lambda \in C\}$ for all $p > 0$:

$$\mathcal{G}_p(\mu) := (\mathcal{G}(\mu^p))^{1/p} \quad (\mu \in C^{1/p}). \tag{24}$$

2.10. Lemma. *If a finite gauge* $\mathfrak{q}$ *on a solid cone* C *satisfies the triangle inequality so do the gauges* $\mathfrak{q}_p$ *for all* $p \geqslant 1$.

Proof. It follows from Hölder's inequality that the powers $B^{1/p}$, for $p \geqslant 1$, of a given convex solid subset $B \subseteq c_0^{\star}$ are convex. By applying this to the set (23) we infer that the sets

$$(B(C,\mathfrak{q}))^{1/p} = B(C^{1/p},\mathfrak{q}_p) \quad (p \geqslant 1)$$

are convex.

Note that $C^{1/p}$ is a cone, for any $p \geqslant 1$, in view of the special case of Hölder's inequality $(\lambda + \mu)^p \leqslant 2^{p-1}(\lambda^p + \mu^p)$ and the hypothesis that C is a solid cone. $\square$

We shall say that a finite gauge $\mathfrak{q} : C \to [0, \infty)$ is *complete* if $\sum_{m=0}^{\infty} \lambda_m \in C$ whenever $\lambda_m \in C$ and $\sum_{m=0}^{\infty} \mathfrak{q}(\lambda_m) < \infty$. Given $e > 0$, we shall say that $\mathfrak{q}$ is e-complete if $\mathfrak{q}_p : C^{1/p} \to [0, \infty)$ is complete for $p = 1/e$. Note that an e-complete gauge is nondegenerate; i.e., it satisfies condition (d) above.

2.11. Lemma. *Let* $L : C \to C'$ *be a homogeneous order preserving map between radial sets equipped with gauges* $\mathfrak{q}$ *and* $\mathfrak{q}'$, *respectively. If* $\mathfrak{q}'$ *is finite and* $\mathfrak{q}$ *is e-complete for at least one* $e > 0$ *then* L *is bounded, i.e.*,

$$||L||_{\mathfrak{q},\mathfrak{q}'} := \sup_{\lambda \in C} \frac{\mathfrak{q}'(L(\lambda))}{\mathfrak{q}(\lambda)} < \infty.$$

Proof. Consider the map $L_p : C^{1/p} \to (C')^{1/p}$ given by

$$L_p(\mu) := L(\mu^p)^{1/p} \quad (\mu \in C^{1/p}).$$

Noting that

$$||L_p||_{\mathfrak{q}_p,\mathfrak{q}'_p} = (||L||_{\mathfrak{q},\mathfrak{q}'})^{1/p} \tag{25}$$

reduces the proof of the lemma to the special case $e = 1$.

Suppose that $||L|| = \infty$. Since the gauge $\mathfrak{q}$ is nondegenerate, there exist nonzero $\lambda_m \in C,\ m \geqslant 1$, such that $\mathfrak{q}'(L(\lambda_m)) \geqslant m^3 \mathfrak{q}(\lambda_m) > 0$. Set $\mu_m := \frac{1}{m^2 \mathfrak{q}(\lambda_m)} \lambda_m$. Then

$$\mathfrak{q}'(L(\mu_m)) \geqslant \frac{\mathfrak{q}'(L(\lambda_m))}{m^2 \mathfrak{q}(\lambda_m)} \geqslant m.$$

In view of the completeness of C, the sequence $\mu := \sum_{m=1}^{\infty} \mu_m$ belongs to C but $\mathfrak{q}'(L(\mu)) \geqslant \mathfrak{q}'(L(\mu_m)) \geqslant m$ for all $m \in \mathbb{Z}_+$. $\square$

2.12. Corollary. *If C admits an e-complete gauge φ, for some $e > 0$, then*

$$\sup_{\lambda \in C} \frac{\varphi'(\lambda)}{\varphi(\lambda)} < \infty$$

for any finite gauge φ'. In particular, if φ' is e'-complete, for some $e' > 0$, then φ' is equivalent to φ.

2.13. In view of the above, we shall say that a radial set $C \subseteq c_0^{\star}$ is e-complete if it *admits* an e-complete gauge. Note that such a set is automatically a cone. In fact, a complete *solid* cone C always admits an equivalent cone norm.

2.14. Proposition. *If φ is an e-complete gauge on a solid cone C, then C^e admits a complete c-norm equivalent to φ_p where $p = 1/e$.*

Proof. Since the powers of a complete solid cone are solid, it suffices to consider only the case $e = 1$. Define a function $\varphi' : C \to [0, \infty)$ by the formula

$$\varphi'(\lambda) := \inf\{\sum_{\ell=1}^{\infty} \varphi(\lambda_\ell) \mid \lambda_\ell \in C \text{ and } \lambda \leqslant \sum_{\ell=1}^{\infty} \lambda_l\}.$$

If $\lambda, \mu \in C$ then clearly $\varphi'(\lambda) \leqslant \varphi(\lambda)$, $\varphi'(\lambda + \mu) \leqslant \varphi'(\lambda) + \varphi'(\mu)$ and also $\varphi'(\lambda) \leqslant \varphi'(\mu)$ whenever $\lambda \leqslant \mu$. Thus φ' is a c-norm on C and is dominated by φ. Note that $\varphi'(\lambda) \geqslant a\|\lambda\|_\infty$, where $a = \varphi(\mathbb{1})$; hence φ' dominates the ℓ_∞ norm. The equivalence of φ and φ' will follow from Corollary 2.12 as soon as we prove that φ' is complete.

To this end, suppose that $\lambda_1, \lambda_2, \ldots \in C$ and $\sum_{\ell=1}^{\infty} \varphi'(\lambda_\ell) < \infty$.

Let us choose sequences $\lambda_{lm} \in C$ so that, for each $\ell \in \mathbb{Z}_+$, one has $\lambda_\ell \leqslant \sum_{m=1}^{\infty} \lambda_{\ell m}$ and $\sum_{m=1}^{\infty} \varphi(\lambda_{\ell m}) \leqslant 2\varphi'(\lambda_\ell)$. The sequence $\mu := \sum_{\ell,m=1}^{\infty} \lambda_{\ell m}$ belongs to C in view of the completeness of φ. Since $\lambda := \sum_{\ell=1}^{\infty} \lambda_\ell \leqslant \sum_{\ell,m=1}^{\infty} \lambda_{\ell m} = \mu$ and C is solid, the latter set contains λ. $\square$

The following result besides possessing an intrinsic interest will be used in the proof of Theorem 3.5 below.

2.15. Proposition. *Any e-complete solid cone $C \subseteq c_0^{\star}$ is e'-complete for any $e' \in (0, e]$.*

The proof will be based on the following.

2.16. Lemma. *Let a gauge $\varphi : C \to [0, \infty)$ satisfying the triangle inequality be given on a solid cone $C \subseteq c_0^{\star}$. Then, for any $\lambda, \mu \in C^{1/p}$ and $p \geqslant 1$, the sequence $(\lambda + \mu)^p - (\mu)^p$ belongs to C and*

$$\varphi((\lambda + \mu)^p - \mu^p) \leqslant p\varphi_p(\lambda)\varphi_p(\lambda + \mu)^{p-1}. \tag{26}$$

Proof. Since the sequence $(\lambda + \mu)^p - (\mu)^p$ is monotonic and C is solid, the former belongs to C.

Inequality (26) needs a proof only for $p > 1$. To do that we begin by invoking the numeric double inequality:

$$(b+c)^p - c^p \leqslant pb(b+c)^{p-1} \leqslant (ab)^p + (p-1)a^{\frac{p}{1-p}}(b+c)^p \quad (a,b,c \geqslant 0;\ p > 1). \tag{27}$$

The first estimate in (27) follows easily from the inequality $1-(1-x)^p \leqslant px$, while the other is a consequence of Young's inequality $xy \leqslant (1/p)x^p + (1/q)y^q$ where $q := p/(p-1)$.

By applying inequality (27) coordinatewise in combination with the triangle inequality for $\mathcal{G}$, we obtain the inequality

$$\mathcal{G}((\lambda+\mu)^p - \mu^p) \leqslant a^p \mathcal{G}_p(\lambda)^p + (p-1)a^{\frac{p}{1-p}}\mathcal{G}_p((\lambda+\mu))^p. \tag{28}$$

If $\mathcal{G}(\lambda) = 0$ then inequality (28) becomes

$$\mathcal{G}((\lambda+\mu)^p - \mu^p) \leqslant (p-1)a^{\frac{p}{1-p}}\mathcal{G}_p((\lambda+\mu))^p$$

and the arbitrariness of $a > 0$ shows that $\mathcal{G}((\lambda+\mu)^p - \mu^p) = 0$. If $\mathcal{G}(\lambda) > 0$ then letting $a = (\mathcal{G}_p(\lambda+\mu)/\mathcal{G}_p(\lambda))^{(p-1)/p}$ in (28) gives inequality (26). □

Proof of Proposition 2.15. By replacing C with the corresponding power of C, the assertion reduces to the case when C is complete with respect to some finite gauge $\mathcal{G}$. We shall prove that C is then also e-complete for $e \in (0,1]$. In view of Proposition 2.14, we can assume $\mathcal{G}$ to be a c-norm.

Suppose that $\lambda_1, \lambda_2, \ldots \in C^e$ and $M := \sum_{m=1}^{\infty} \mathcal{G}_p(\lambda_m) < \infty$ where $p = 1/e$. In view of Lemma 2.16, the sequences

$$\tau_m := (\lambda_1 + \cdots + \lambda_m)^p - (\lambda_1 + \cdots + \lambda_{m-1})^p \quad (m \geqslant 2),$$

belong to C and satisfy the inequality

$$\sum_{m=2}^{\infty} \mathcal{G}(\tau_m) \leqslant p \sum_{m=2}^{\infty} \mathcal{G}_p(\lambda_m)\mathcal{G}_p(\lambda_1 + \cdots + \lambda_m)^{p-1} \leqslant pM^{p-1} \sum_{m=2}^{\infty} \mathcal{G}_p(\lambda_m) = pM^p < \infty$$

(note that $\mathcal{G}_p(\lambda_1 + \cdots + \lambda_m) \leqslant \mathcal{G}_p(\lambda_1) + \cdots + \mathcal{G}_p(\lambda_m)$ in view of Lemma 2.10). Since $\mathcal{G}$ is complete, the sequence $\mu := \lambda_1^p + \sum_{m=2}^{\infty} \tau_m$ belongs to C. Accordingly, the sequence $\sum_{m=1}^{\infty} \lambda_m = \mu^e$ belongs to C^e, which shows that the c-norm $\mathcal{G}_p$ is complete. □

We shall now introduce certain numeric invariants of e-complete characteristic sets.

2.17. The Boyd α-index. If a characteristic set $\Sigma \subset c_0^{\star}$ is e-complete, for some $e > 0$, then the maps $D_m : \Sigma \to \Sigma$ are bounded in view of Lemma 2.11 and the nondecreasing sequence $m \mapsto ||D_m||$ is submultiplicative. In particular, its Matuszewska's α-index (20a) equals

$$\alpha(\Sigma) := \lim_{m \to \infty} \frac{\log ||D_m||}{\log m} = \inf_{m \geqslant 2} \frac{\log ||D_m||}{\log m}. \tag{29}$$

The number $\alpha(\Sigma) \in [0, \infty)$ depends only on Σ and not on the choice of an e-complete gauge on Σ as follows from Corollary 2.12. We shall call it the *Boyd* α*-index* of the e-complete characteristic set Σ.

Equality (25) when applied to $L = D_m$ and $p = 1/s$, yields the useful equality

$$||\Sigma^s \xrightarrow{D_m} \Sigma^s|| = ||\Sigma \xrightarrow{D_m} \Sigma||^s, \tag{30}$$

which implies that

$$\alpha(\Sigma^s) = s\alpha(\Sigma).$$

By evaluating a gauge $\mathcal{G}$ on finitely supported sequences $\mathbb{1}_n$, one obtains a nondecreasing sequence

$$n \mapsto \phi_n(\mathcal{G}) := \mathcal{G}(\mathbb{1}_n) \tag{31}$$

which is called the *fundamental sequence* of $\mathcal{G}$. Let $\bar{\phi}_m(\mathcal{G})$ denote the submultiplicative sequence (19a) associated with $\phi(\mathcal{G})$. The inequality

$$\bar{\phi}_m(\mathcal{G}) \leqslant ||\Sigma \xrightarrow{D_m} \Sigma|| < \infty \tag{32}$$

shows that the fundamental sequence (31) of an e-complete gauge satisfies the Δ_2-condition, (22). Vice versa, any nondecreasing sequence φ which satisfies the Δ_2-condition occurs as the fundamental sequence of a number of complete characteristic sets, e.g., the normed characteristic sets $\ell_\infty^{\star}(\varphi)$ and $\ell^{\star}(\varphi)$ discussed in Sections 2.20–32. In either of these special cases $\bar{\phi}_m(\mathcal{G}) = ||D_m||$.

The Matuszewska α-index $\alpha(\mathcal{G}) := \alpha(\phi(\mathcal{G}))$ of $\phi(\mathcal{G})$ will be called the *fundamental* α*-index* of $\mathcal{G}$. The fundamental index satisfies the inequality

$$\alpha(\mathcal{G}) \leqslant \alpha(\Sigma),$$

which follows from (32), and it depends only on the equivalence class of $\mathcal{G}$ restricted to $c_{\mathrm{f}}^{\star}$. In particular,

$$\alpha_{\mathrm{fun}}(\Sigma) := \alpha(\mathcal{G}) \tag{33}$$

is independent of the choice of gauge $\mathcal{G}$ on Σ, provided $\mathcal{G}$ is *e-complete* for some $e > 0$. We shall call (33) the *fundamental* α*-index* of a characteristic set Σ.

2.18. A symmetric (i.e., $\mathscr{E}^*_\Gamma$-invariant) norm $|| \, ||$ on a symmetric solid sequence space $S \subset c_0(\Gamma)$ induces, when Γ is countable, a c-norm on the characteristic set $S^{\star} \subset c_0^{\star}$. However, for a c-norm $\mathcal{G} : S^{\star} \to [0, \infty)$, the symmetric extension of $\mathcal{G}$ to S:

$$\xi \mapsto ||\xi||_{\mathcal{G}} := \mathcal{G}(\xi^{\star}) \quad (\xi \in S) \tag{34}$$

need not satisfy the triangle inequality on S. If it does, then the fundamental sequence $\phi = \phi(\mathcal{G})$ is *quasiconcave*; i.e.,

$$\phi \text{ is } \textit{nondecreasing} \text{ and } \phi\omega \text{ is } \textit{nonincreasing}. \tag{35}$$

This well-known fact has a simple proof. Let E be a set of cardinality $n+1$. By removing single elements one obtains $n+1$ subsets $E_1, \dots, E_{n+1} \subset E$, each of cardinality n. Since $\chi_{E_1} + \cdots + \chi_{E_{n+1}} = n\chi_E$, we obtain the inequality

$$n\phi_{n+1}(\mathcal{G}) = ||n\chi_E||_{\mathcal{G}} \leqslant ||\chi_{E_1}||_{\mathcal{G}} + \cdots + ||\chi_{E_{n+1}}||_{\mathcal{G}} = (n+1)\phi_n(\mathcal{G}).$$

Every quasiconcave sequence is equivalent to a concave sequence (the proof of Proposition 5.10 in Chapter 2 of [9] can be easily adapted to the sequence case).

We shall also mention without proof another fact which is better known in the case of functions on intervals $[0, \infty)$ or $[0, b]$:

$$\text{If } \alpha(\varphi) < 1 \text{ then } \varphi \asymp \psi \text{ for some concave sequence } \psi \in (0, \infty)^{\mathbb{Z}_+}. \tag{36}$$

Conversely, every concave sequence must satisfy $\psi_{\ell m} \leqslant m\psi_\ell$, for all $\ell, m \in \mathbb{Z}_+$, which implies that $\alpha(\psi) \leqslant 1$. If $\varphi \asymp \psi$ then $\alpha(\varphi) = \alpha(\psi) \leqslant 1$. The sequence $\varphi_m = 1 + \log m!$ is not equivalent to a concave sequence yet $\alpha(\varphi) = 1$.

2.19. Remark. David W. Boyd defined his α- and β-indices (also called the *upper* and the *lower Boyd* indices, respectively) for Banach spaces of functions on measure spaces [14,15]. In this article we are concerned with the α-index and in the *sequence* case only. However, we define it, as well as the fundamental α-index, in the context of e-complete characteristic sets and gauges which is significantly more general than the traditional context of Banach sequence spaces and the corresponding Banach norms. When the functional (34) satisfies the triangle inequality our definition agrees with Boyd's. His β-index can be likewise defined in the context of e-complete characteristic sets.

The subadditivity of the operation $c_0 \to c_0^{\star}$

$$\xi \mapsto (\xi^{\star})_a \tag{37}$$

(see [25], Example (4.1); cf. also [41]) implies that, for a c-norm $\mathcal{G}$ on any cone $C \subset c_0^{\star}$, the correspondence

$$\xi \mapsto \mathcal{G}((\xi^{\star})_a) \tag{38}$$

defines a symmetric norm on the symmetric sequence space $S \subseteq c_0(\Gamma)$ defined by the equality $S^{\star} = {}_aC$; cf. Section 2.8(v). Probably a majority of Banach symmetric sequence spaces present in the literature appear in this way.

The remainder of this section is devoted to a brief discussion of certain important types of characteristic sets. Practically all standard examples are included among or are easily derived from them.

2.20. For any subset

$$\Psi \subset \{\alpha \in [0, \infty]^{\mathbb{Z}_+} \mid \sup \alpha = \infty\}, \tag{39}$$

let $\ell_\infty(\Psi) := \{\xi \in c_0 \mid ||\xi||_{\infty,\psi} := ||\xi^{\star}\psi||_\infty < \infty$ for some $\psi \in \Psi\}$.

The c-norms $||\,||_{\infty,\psi}$ do not change if we replace ψ by the upper nondecreasing envelope $\phi := \mathrm{und}(\psi)$; cf. Section 2.3 above. When Ψ consists of a single element ψ then we shall write $\ell_\infty(\psi)$ instead of $\ell_\infty(\{\psi\})$.

2.21. Lemma. *For any nonempty and at most countable set Ψ (cf. (39)) the following conditions are equivalent*:

(a) $\ell_\infty^{\star}(\Psi)$ *is e-complete for every* $e > 0$,
(b) $\ell_\infty^{\star}(\Psi)$ *is e-complete for some* $e > 0$,
(c) *there exists* $\psi_0 \in \Psi$ *such that* $\ell_\infty^{\star}(\Psi) = \ell_\infty^{\star}(\psi_0)$, *i.e.*,

$$\psi_0 = O(\mathrm{und}(\psi)) \tag{40}$$

for any $\psi \in \Psi$.

Proof. For any unbounded sequence $\psi \in [0, \infty)^{\mathbb{Z}_+}$ the cone $\ell_\infty^{\star}(\psi) := \ell_\infty(\psi)^{\star}$ is obviously complete. Since $(\ell_\infty^{\star}(\psi))^e = \ell_\infty^{\star}(\psi^e)$, the previous remark applied to ψ^e shows that $\ell_\infty^{\star}(\psi)$ is e-complete for every $e > 0$.

Suppose that $\mathfrak{q}$ is an e-complete gauge on $\ell_\infty^{\star}(\Psi)$, for some $e > 0$. Let us consider the corresponding set of upper nondecreasing envelopes $\Phi := \{\mathrm{und}(\psi) \mid \psi \in \Psi\}$. In view of Lemma 2.11 applied to the inclusion $\ell_\infty^{\star}(\varphi) \hookrightarrow \ell_\infty^{\star}(\Psi)$, one has

$$b_\varphi := \sup_{\lambda \in \ell_\infty^{\star}(\varphi)} \frac{\mathfrak{q}(\lambda)}{||\lambda||_{\infty,\varphi}} < \infty \quad (\varphi \in \Phi).$$

Choose a function $a : \Phi \to (0, \infty)$ such that $\sum a_\varphi (b_\varphi)^e < \infty$.

In view of the e-completeness of $\ell_\infty^{\star}(\Psi) = \ell_\infty^{\star}(\Phi)$ and the fact that

$$\sum \mathfrak{q}_p(a_\varphi/\varphi^e) \leqslant \sum a_\varphi (b_\varphi)^e ||1/\varphi^e||_{\infty,\varphi^e} = \sum a_\varphi (b_\varphi)^e < \infty,$$

where $p = 1/e$, the sequence

$$\mu := \sum_{\varphi \in \Phi} a_\varphi (1/\varphi^e)$$

belongs to $(\ell_\infty^\star(\Phi))^e$. The membership in $(\ell_\infty^\star(\Phi))^e$ means that $\|\mu\|_{\infty,\varphi_0^e} < \infty$ for some $\varphi_0 \in \Phi$. This, in turn, means that

$$\|\varphi_0/\varphi\|_\infty \leqslant \left(\frac{\|\mu\|_{\infty,\varphi_0^e}}{a_\varphi}\right)^p < \infty \quad (\varphi \in \Phi),$$

which implies (40). $\square$

2.22. Characteristic sets $\mathcal{O}_X$. For any subset $X \subseteq c_0^\star$,

$$\mathcal{O}_X := \{\lambda \in c_0^\star \mid \lambda = O(\pi_1 \oplus \cdots \oplus \pi_\ell) \text{ for some } \pi_1, \ldots, \pi_\ell \in X\}$$

is the smallest characteristic set which contains X. We shall call it the characteristic set *generated* by $X \subseteq c_0^\star$. When $X = \{\pi\}$ we shall call $\mathcal{O}_\pi := \mathcal{O}_{\{\pi\}}$ the *principal* characteristic set associated with the sequence $\pi \in c_0^\star$. Finally, for $\alpha \in c_0$, we set $\mathcal{O}_\alpha := \mathcal{O}_{\alpha^\star}$. The set of monotonic sequences having finitely many nonzero terms, $c_{\mathrm{f}}^\star$, coincides with $\mathcal{O}_{\mathbb{1}}$.

Finitely generated characteristic set are principal. This follows from the equality

$$\mathcal{O}_{\{\pi_1,\ldots,\pi_\ell\}} = \mathcal{O}_{\pi_1 + \cdots + \pi_\ell}.$$

Any characteristic set Σ is the filtered union $\bigcup_{\lambda \in \Sigma} \ell_\infty^\star(1/\lambda)$, so for countably generated ones we have the following corollary of Lemma 2.21.

2.23. Corollary. *For any countably generated characteristic set $\Sigma \subseteq c_0^\star$, the following conditions are equivalent*:

(a) *Σ is e-complete for every $e > 0$,*
(b) *Σ is e-complete for some $e > 0$,*
(c) *Σ is the principal ideal $\mathcal{O}_\pi$ for some sequence $\pi \in c_0^\star$ whose reciprocal satisfies the Δ_2-condition* (22).

In this case the Boyd and the fundamental indices coincide. Note that $1/\pi$ satisfies the Δ_2-condition precisely when $\pi \asymp \pi^{\oplus 2}$.

2.24. Lemma. *If $\pi \asymp \pi^{\oplus 2}$ then the principal characteristic set $\mathcal{O}_\pi$ coincides with the gauged radial set $\ell_\infty^\star(1/\pi)$ and thus is e-complete for every $e > 0$, with $1/\pi$ serving as its* *fundamental sequence and* $||D_m||$ *given by*

$$||D_m|| = \sup \frac{\pi}{m^\bullet \pi} = 1/\underline{\pi}_m, \tag{41}$$

cf. (19a). *In particular, the Boyd and the fundamental indices coincide and are equal to the negative of the Matuszewska* β*-index of sequence* π:

$$\alpha(\mathcal{O}_\pi) = -\beta(\pi). \tag{42}$$

Proof. For $\lambda \in c_0^{\star}$, $||D_m \lambda/\pi||_\infty = ||\lambda^{\oplus m}/\pi||_\infty = ||\lambda/m^\bullet \pi||_\infty$ and therefore

$$||D_m|| = \sup_{\lambda \in \mathcal{O}_\pi} ||\lambda/m^\bullet \pi||_\infty / ||\lambda/\pi||_\infty \leqslant \sup_{\lambda \in \mathcal{O}_\pi} ||(\lambda/m^\bullet \pi)/(\lambda/\pi)||_\infty = ||\pi/m^\bullet \pi||_\infty = 1/\underline{\pi}_m$$

which equals $\bar{\phi}_m(\mathcal{O}_\pi)$. Equality (42) follows from (41); cf. (21). □

2.25. Lorentz sequence spaces. For a nondecreasing sequence $\varphi \in [0, \infty)^{\mathbb{Z}_+}$, let

$$\ell(\varphi) := \{\alpha \in c_0 \mid ||\alpha||_{\ell(\varphi)} := ||\alpha^{\star} \Delta\varphi||_1 < \infty\}.$$

The gauge $|| \, ||_{\ell(\varphi)}$ is a c-norm if $\varphi_1 > 0$. This condition can be enforced by modifying finitely many initial terms of φ if φ is not identically zero. The space $\ell(\varphi)$ does not change in the process. This allows us to focus on the case of $\varphi \in (0, \infty)^{\mathbb{Z}_+}$. Equipped with the c-norm $|| \, ||$, the *Lorentz cone* $\ell^{\star}(\varphi) := \ell(\varphi)^{\star}$ is e-complete for every $e \leqslant 1$. In fact, the cone $\ell_p^{\star}(\varphi) := \ell^{\star}(\varphi)^{1/p}$ is equipped, for $p \geqslant 1$, with the complete c-norm

$$||\lambda||_{\ell_p(\varphi)} := (||\lambda^p||_{\ell(\varphi)})^{1/p}$$

as follows, for example, from Lemma 2.10. The characteristic sets $\ell_p^{\star}$, $0 < p < 1$, demonstrate that $\ell^{\star}(\varphi)$ is not, generally, e-complete for $e > 1$. The fundamental sequence $\phi(|| \, ||_{\ell(\varphi)})$ coincides with φ. Many properties of the Lorentz cone follow from the following lemma.

2.26. Hardy's Lemma. *For any two nondecreasing sequences* $\varphi, \psi \in (0, \infty)^{\mathbb{Z}_+}$, *one has*

$$\sup_{\lambda \in c_0^{\star}} \frac{||\lambda||_{\ell(\varphi)}}{||\lambda||_{\ell(\psi)}} = ||\varphi/\psi||_\infty. \tag{43}$$

Moreover, $\ell(\psi) \subseteq \ell(\varphi)$ *if and only if* $\varphi = O(\psi)$.

Recall from the introduction (p. 6) that $\infty/\infty = \infty \cdot 0 = 0$ and, similarly, $0/0 = 0 \cdot \infty = 0$. Consequently, only $\lambda \in \ell^{\star}(\psi)$ contribute to the left-hand side of (43).

Proof. Denote the left-hand side of (43) by K. Since $\varphi_m = ||\mathbb{1}_m||_{\ell(\varphi)}$ and, likewise, $\psi_m = ||\mathbb{1}_m||_{\ell(\psi)}$, we have $K \geqslant ||\varphi/\psi||_\infty$. On the other hand, identity (15) combined with the fact that $\Delta^-\lambda$ is nonnegative for $\lambda \in c_0^{\star}$ yields the inequality

$$||\lambda||_{\ell(\varphi)} = ||(\Delta^-\lambda)\varphi||_1 \leqslant ||(\Delta^-\lambda)\psi||_1 ||\varphi/\psi||_\infty = ||\lambda||_{\ell(\psi)} ||\varphi/\psi||_\infty.$$

If $\ell(\psi) \subseteq \ell(\varphi)$ then the inclusion must be bounded by Lemma 2.11. Hence

$$||\varphi/\psi||_\infty = ||\ell(\psi) \hookrightarrow \ell(\varphi)|| < \infty. \qquad \square \tag{44}$$

2.27. Corollary. *The Lorentz cone $\ell^{\star}(\varphi)$ is D_2-invariant, i.e., is a characteristic set, if and only if φ satisfies Δ_2-condition* (22). *In this case, $\ell^{\star}(\varphi)$ is the Köthe dual $\{\Delta\varphi\}^\times$ of the singleton set $\{\Delta\varphi\}$ and*

$$||D_m|| = \overline{\varphi}_m \equiv ||m^\bullet \varphi/\varphi||_\infty. \tag{45}$$

In particular, the Boyd and the fundamental α-indices of $\ell^{\star}(\varphi)$ coincide.

Proof. Formula (45) follows from the identity

$$||D_m\lambda||_{\ell(\varphi)} = ||\lambda||_{\ell(m^\bullet\phi)}$$

combined with (44). $\square$

2.28. Corollary. *$\ell(\varphi) = \ell(\psi)$ if and only if $\varphi \asymp \psi$.*

2.29. Lemma. *$|| \, ||_{\ell(\varphi)}$ is a norm on $\ell(\varphi)$ if and only if φ is concave.*

Note that a concave sequence φ satisfies the Δ_2-condition (22).

Proof. If φ is concave then $\Delta\varphi \in c_0^{\star}$ and $\Delta^-(\Delta\varphi)$ is nonnegative. Since $\sigma((\alpha+\beta)^{\star}) \leqslant \sigma(\alpha^{\star}) + \sigma(\beta^{\star})$, cf. (37), this yields

$$\begin{aligned} ||\xi_1 + \xi_2||_{\ell(\varphi)} &= ||(\xi_1+\xi_2)^{\star}\Delta\varphi||_1 = ||\sigma((\xi_1+\xi_2)^{\star})\Delta^-(\Delta\varphi)||_1 \\ &\leqslant ||(\sigma(\xi_1^{\star}) + \sigma(\xi_2^{\star}))\Delta^-(\Delta\varphi)||_1 \\ &= ||(\xi_1^{\star} + \xi_2^{\star})\Delta\varphi||_1 = ||\xi_1||_{\ell(\varphi)} + ||\xi_2||_{\ell(\varphi)}. \end{aligned}$$

Vice versa, suppose that $|| \, ||_{\ell(\varphi)}$ satisfies the triangle inequality on $\ell(\varphi)$. Let

$$\xi_n = \begin{cases} 1 & \text{if } 1 \leqslant n \leqslant m-1 \text{ or } n = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(\mathbb{1}_m + \xi)_n = \begin{cases} 2 & \text{if } 1 \leqslant n \leqslant m-1, \\ 1 & \text{if } n = m \text{ or } m+1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\varphi_{m-1} + \varphi_{m+1} = ||\mathbb{1}_m + \xi||_{\ell(\varphi)} \leqslant ||\mathbb{1}_m||_{\ell(\varphi)} + ||\xi||_{\ell(\varphi)} = 2\varphi_m,$$

i.e., φ is concave. □

It follows that $\ell(\varphi)$ is a symmetric Banach sequence space with an equivalent norm precisely when $\phi \asymp \psi$ for some concave ψ. A necessary condition for this is that $\alpha(\varphi) \leqslant 1$; while $\alpha(\varphi) < 1$ is sufficient, cf. (36) and the sentence following it.

2.30. Lemma. $\ell^{\star}(\varphi)^{1/p}$ *is am-closed (cf. Section 2.8) if φ is equivalent to a concave sequence and $p \geqslant 1$.*

Proof. Function $x \mapsto x^p$, $x \in [0, \infty)$, is increasing and convex for $p \geqslant 1$. Hence, the inequality $\sigma(\lambda) \leqslant \sigma(\mu)$ implies that $\sigma(\lambda^p) \leqslant \sigma(\mu^p)$ for $p \geqslant 1$ (cf. [25], (4.1), and other references mentioned in the proof of Lemma 2.41 below). Since $\Delta^-(\Delta\varphi)$ is nonnegative, the inequality $||\lambda^p||_{\ell(\varphi)} \leqslant ||\mu^p||_{\ell(\varphi)}$ follows. □

2.31. Proposition. *For any characteristic set $\Sigma \subsetneq c_0^{\star}$, equipped with a complete gauge $\mathcal{g}$, one has the following double inclusion:*

$$\ell^{\star}(\phi(\mathcal{g})) \subseteq \Sigma \subseteq \ell^{\star}_{\infty}(\phi(\mathcal{g})) = \mathcal{O}_{1/\phi(\mathcal{g})}.$$

More precisely,

$$||\lambda\phi(\mathcal{g})||_{\infty} \leqslant \mathcal{g}(\lambda) \quad (\lambda \in \Sigma) \tag{46}$$

and there exists a constant $K > 0$ such that

$$\mathcal{g}(\lambda) \leqslant K||\lambda||_{\ell(\phi(\mathcal{g}))} \quad (\lambda \in \ell^{\star}(\phi(\mathcal{g}))). \tag{47}$$

Proof. On one hand, the inequality

$$||\lambda\phi(\mathcal{g})||_{\infty} = \sup_m \lambda_m\phi_m(\mathcal{g}) = \sup_m \mathcal{g}(\lambda_m\mathbb{1}_m) \leqslant \mathcal{g}(\lambda)$$

follows from the fact that $\lambda_m \mathbb{1}_m \leqslant \lambda$ for any $\lambda \in c_0^{\star}$ and $m \in \mathbb{Z}_+$. On the other hand, for any $\lambda \in c_0^{\star}$,

$$||\lambda||_{\ell(\phi(\varphi))} = \sum_{m=1}^{\infty} \Delta_m^- \lambda \varphi(\mathbb{1}_m) = \sum_{m=1}^{\infty} \varphi((\Delta_m^- \lambda)\mathbb{1}_m).$$

The completeness of Σ implies that $\lambda = \sum_{m=1}^{\infty} (\Delta_m^- \lambda)\mathbb{1}_m$ belongs to Σ if $||\lambda||_{\ell(\phi(\varphi))} < \infty$. Inequality (47) then follows from Lemma 2.11. Finally, if $\sup \phi(\varphi) < \infty$, then $\Delta\phi(\varphi) \in \ell_1$ and $\ell^{\star}(\phi(\varphi)) = c_0^{\star}$. This would force Σ to coincide with $c_0^{\star}$, contrary to the hypothesis that $\Sigma \subsetneq c_0^{\star}$. Thus $1/\phi(\varphi) \in c_0^{\star}$ and inequality (46) implies that $\Sigma \subseteq \ell_\infty^{\star}(\phi(\varphi)) = \mathcal{O}_{1/\phi(\varphi)}$. $\square$

2.32. Corollary. *Any e-complete characteristic set Σ properly contains the union $\ell_{p-}^{\star} := \bigcup_{s<p} \ell_s^{\star}$ where $p = 1/\alpha_{\mathrm{fun}}(\Sigma)$.*

Proof. If Σ is equipped with a complete gauge φ then $\Sigma \supseteq \ell^{\star}(\phi(\varphi))$, by the previous proposition. Since $\phi(\varphi) = O(\omega^{-r})$ for any $r > \alpha(\phi(\varphi))$, one has the inclusion

$$\ell^{\star}(\omega^{-r}) \subsetneq \ell^{\star}(\phi(\varphi)) \quad (r > \alpha(\phi(\varphi))).$$

By combining this with the inclusions

$$\ell_s^{\star} \subsetneq \mathcal{O}_{\omega^{1/s}} \subsetneq \ell^{\star}(\omega^{-r}) \quad (s < 1/r),$$

we deduce the inclusion $\ell_{p-}^{\star} \subset \Sigma$, where $p = 1/\alpha_{\mathrm{fun}}(\Sigma)$. This inclusion is proper, since $\ell_{p-}^{\star}$ is not complete for any $p > 0$ as is easy to see.

If φ is e-complete then $\ell_{r-}^{\star} \subsetneq \Sigma^e$ for $r = 1/\alpha_{\mathrm{fun}}(\Sigma^e) = 1/(e\alpha_{\mathrm{fun}}(\Sigma))$ by the already proven part of the corollary. $\square$

2.33. Marcinkiewicz characteristic sets. For a nonzero $\psi \in [0, \infty)^{\mathbb{Z}_+}$, the Marcinkiewicz sequence space

$$m(\psi) := \{\xi \in c_0 \mid ||\xi||_{m(\psi)} := ||(\xi^{\star})_a \psi||_\infty < \infty\}$$

is the pre-arithmetic mean space of $\ell_\infty(\psi)$, the latter was defined in Section 2.20. Since $|| \, ||_{m(\psi)}$ does not change if one replaces ψ by its upper nondecreasing envelope $\mathrm{und}(\psi)$, cf. (17) above, we may assume that ψ is nondecreasing in which case the cone $m^{\star}(\psi) := m(\psi)^{\star}$ is the pre-*am* set of the singleton set $\{1/\psi\}$:

$$m^{\star}(\psi) = {}_a\{1/\psi\}. \tag{48}$$

One has $m(\psi) = 0$, ℓ_1 or c_0 precisely when $\sup \psi\omega = \infty$, $0 < \overline{\lim}\, \psi\omega < \infty$ or $\sup \psi < \infty$, respectively. The case when $\sup \psi = \infty$ and $\lim \psi\omega = 0$ is therefore the most interesting one with $\ell_1 \subsetneq m(\psi) \subsetneq c_0$.

It follows from Proposition 2.15 or, more directly, from Minkowski's Inequality that $m^{\star}(\psi)$ is e-complete for any $e \leqslant 1$. The fundamental sequence $\phi = \phi(\|\,\|_{m(\psi)})$ is given by

$$\phi(\|\,\|_{m(\psi)}) = \operatorname{und}\left(\frac{\operatorname{uni}(\psi\omega)}{\omega}\right); \tag{49}$$

cf. (17)–(18) and $\|\,\|_{m(\psi)} = \|\,\|_{m(\phi)}$. The sequence ϕ is quasiconcave; cf. (35). If ψ is quasiconcave then $\phi(\|\,\|_{m(\psi)}) = \psi$; hence the fundamental sequence of $\|\,\|_{m(\psi)}$ is the unique quasiconcave sequence for which the associated Marcinkiewicz norm coincides with $\|\,\|_{m(\psi)}$. If both ψ and ψ' are quasiconcave then $m(\psi) = m(\psi')$ if and only if $\psi \asymp \psi'$. This follows from Lemma 2.11.

Since $1/(\phi\omega)$ is quasiconcave, we infer that $1/(\phi\omega) \asymp \sigma(\pi)$ for some $\pi \in c_0^{\star}$. It follows that $m^{\star}(\psi) = m^{\star}(\phi) = (\mathcal{O}_\pi)^-$, the *am*-closure of the principal characteristic set $\mathcal{O}_\pi$; cf. Section 2.8.

2.34. Proposition. *The Boyd and the fundamental* α*-indices of* $m^{\star}(\psi)$ *coincide*:

$$\alpha(m^{\star}(\psi)) = \alpha(\phi(\|\,\|_{m(\psi)})).$$

Moreover,

$$\alpha(m^{\star}(\psi)) = 1 - \beta(\sigma(\pi)),$$

where $\pi \in c_0^{\star}$ *is any sequence such that*

$$\pi_a \asymp \frac{1}{\phi(\|\,\|_{m(\psi)})} \tag{50}$$

and $\beta(\sigma(\pi))$ *is the Matuszewska index* (20b) *of the sequence of partial sums of* π.

Proof. For any $\lambda \in c_0^{\star}$, one has

$$\|D_m\lambda\|_{m(1/\pi_a)} = \sup \frac{\sigma(\lambda^{\oplus m})}{\sigma(\pi)} = \sup \frac{m\sigma(\lambda)}{m^{\bullet}\sigma(\pi)}.$$

Hence

$$\begin{aligned}\frac{\|D_m\lambda\|_{m(1/\pi_a)}}{\|\lambda\|_{m(1/\pi_a)}} &\leqslant \sup\left(\frac{m\sigma(\lambda)}{m^{\bullet}\sigma(\pi)}\Big/\frac{\sigma(\lambda)}{\sigma(\pi)}\right) = \sup \frac{m\sigma(\pi)}{m^{\bullet}\sigma(\pi)} \\ &= \frac{m}{\inf m^{\bullet}\sigma(\pi)/\sigma(\pi)} = \overline{\phi}_m(\|\,\|_{m(1/\pi_a)}),\end{aligned}$$

since $\phi(||\,||_{m(1/\pi_a)}) = 1/\pi_a$. In general, $m(\psi) = m(1/\pi_a)$ for $\pi \in c_0^{\star}$ satisfying (50). The Boyd and the fundamental α-indices coincide with those of $m(1/\pi_a)$ and $||\,||_{m(1/\pi_a)}$, respectively. $\square$

The following proposition is rather well-known though not, perhaps, the proof given below. Compare this proposition with Proposition 2.31.

2.35. Proposition. *Let* $||\,|| : S \to [0, \infty)$ *be a symmetric norm on a symmetric solid sequence space* $S \subseteq c_0$. *Then,*

$$||\,||_{m(\phi)} \leqslant ||\,|| \tag{51}$$

on S *where* $\phi = \phi(||\,||)$ *is the fundamental sequence of* $||\,||$.

Proof. Let $\mathscr{E}_n$ denote the set of all bijections $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $f(m) = m$ for $m > n$. Then,

$$\frac{1}{n!} \sum_{f \in \mathscr{E}_n} f_*(\lambda \mathbb{1}_n) = (\lambda_a)_n \mathbb{1}_n$$

for any $\lambda \in c_0^{\star}$ and $n \in \mathbb{Z}_+$. It follows that

$$(\lambda_a)_n ||\mathbb{1}_n|| \leqslant \frac{1}{n!} \sum_{f \in \mathscr{E}_n} ||f_*(\lambda \mathbb{1}_n)|| = \frac{1}{n!} \sum_{f \in \mathscr{E}_n} ||\lambda \mathbb{1}_n|| = ||\lambda \mathbb{1}_n|| \leqslant ||\lambda||. \qquad \square$$

We are now sufficiently equipped to establish the following important result.

2.36. Theorem. *A countably generated symmetric solid sequence space* $S \subset c_0$ *is complete with respect to some symmetric norm if and only if* $S = m(1/\pi_a) = \ell_\infty(1/\pi)$ *for some* $\pi \in c_0^{\star}$ *such that* $\pi \asymp \pi_a$.

Proof. If S admits a symmetric complete norm $||\,||$ then, by Lemma 2.21 above, $S = \ell_\infty(1/\pi)$ for some $\pi \in c_0^{\star}$. By Corollary 2.12, the norm $||\,||$ is equivalent to the c-norm $||\,||_{\infty,1/\pi}$, introduced in Section 2.20, i.e.,

$$(1/K)||\,||_{\infty,1/\pi} \leqslant ||\,|| \leqslant K'||\,||_{\infty,1/\pi} \tag{52}$$

for suitable constants $K, K' > 0$. By combining inequality (52) with inequality (51), we obtain the inequality

$$||\,||_{m(\varphi)} \leqslant K'||\,||_{\infty,1/\pi},$$

where $\varphi = \phi(||\,||)$ is the fundamental sequence of the norm $||\,||$. Since

$$1/\pi = \phi(||\,||_{\infty,1/\pi}) \leqslant K\varphi,$$

we deduce that

$$\sup \frac{\pi_a}{\pi} \leqslant K \sup(\pi_a \varphi) = K||\pi||_{m(\varphi)} \leqslant KK'||\pi||_{\infty,1/\pi} = KK' < \infty. \qquad \square$$

2.37. Orlicz characteristic sets. The *Orlicz class*

$$\not{h}_M := \{\alpha \in c_0 \mid \rho_M(\alpha) := \sum_{n=1}^{\infty} M(|\alpha_n|) < \infty\}$$

of a nondecreasing function $M \in [0, \infty]^{[0,\infty)}$ such that $M(0) = 0$, generates two sequence spaces:

$$(\textit{Orlicz sequence space}) \quad \ell_M := \bigcup_{t>0} t \not{h}_M$$

and

$$(\textit{small Orlicz sequence space}) \quad \ell_M^{(0)} := \bigcap_{t>0} t \not{h}_M.$$

The Minkowski functional $\alpha \mapsto ||\alpha||_{\ell_M} := \inf\{c > 0 \mid \rho_M(\alpha/c) \leqslant 1\}$ is a norm if M is convex. In particular, ℓ_M and $\ell_M^{(0)}$ become symmetric Banach sequence spaces (up to equivalent norms) if M is equivalent to a convex function. In general, $|| \, ||_{\ell_M}$ is only a gauge on $\ell_M^{\star}$.

The following concepts and the subsequent lemma play a significant role in the theory of Orlicz sequence spaces:

(i) the Δ_2-*condition at* 0:

$$\overline{\lim_{t \to 0^+}} \; M(2t)/M(t) < \infty;$$

(ii) two relations on $[0, \infty]^{[0,\infty)}$:

$$M \prec_0 N \quad \text{if} \quad \overline{\lim_{t \to 0^+}} \; M(t)/N(ct) < \infty \quad \text{for some } c > 0$$

and

$$M \sim_0 N \quad \text{if } M \prec_0 N \text{ and } N \prec_0 M;$$

(iii) *the Matuszewska indices at* 0 of a monotonic function $M \in (0, \infty)^{(0,\infty)}$:

$$\begin{aligned} \alpha_0(M) := \inf_{t<1} \frac{\log \underline{M}_0(t)}{\log t} &= \lim_{t \to 0^+} \frac{\log \underline{M}_0(t)}{\log t} \\ = \inf_{t>1} \frac{\log \overline{M}_0(t)}{\log t} &= \lim_{t \to \infty} \frac{\log \overline{M}_0(t)}{\log t} \end{aligned} \tag{53}$$

and

$$\beta_0(M) := \sup_{t<1} \frac{\log \overline{M}_0(t)}{\log t} = \lim_{t \to 0^+} \frac{\log \overline{M}_0(t)}{\log t}$$
$$= \sup_{t>1} \frac{\log \underline{M}_0(t)}{\log t} = \lim_{t \to \infty} \frac{\log \underline{M}_0(t)}{\log t}, \tag{54}$$

where $\underline{M}_0(t) := \underline{\lim}_{u \to 0^+} M(tu)/M(u)$ and $\overline{M}_0(t) := \overline{\lim}_{u \to 0^+} M(tu)/M(u)$.

One has the inequality

$$0 \leqslant \beta_0(M) \leqslant \alpha_0(M) \leqslant \infty$$

and $\alpha_0(M) < \infty$ precisely when M satisfies the Δ_2-condition at 0. If $M \sim_0 N$ then their α_0- and β_0-indices coincide. One has $\beta_0(M) \geqslant 1$ for functions that are convex in a neighborhood of 0. It follows that $\beta_0(M) \geqslant 1$ if $M \sim_0 N$ for some function N convex in a neighborhood of 0. If $\beta_0(M) > 1$ the reverse is true (see [1, Lemma 1 and Corollary 1, p. 15]).

It is convenient to extend Matuszewska's indices to monotonic functions $M \in [0, \infty]^{(0,\infty)}$. Formulae (53)–(54) are applicable if $0 < M(t) < \infty$ on a neighborhood of 0^+. If this is not so, then either $M(t) = \infty$ for $t > 0$ or $M(t) = 0$ on a neighborhood of 0^+. In the former case $\ell_M = \{0\}$ and in the latter $\ell_M = c_0$. In either of these cases we simply set

$$\alpha_0(M) = \beta_0(M) = \infty$$

which agrees with definitions (53)–(54) provided one uses functions $\underline{M}_0$ and $\overline{M}_0$ only for $t < 1$.

The following useful lemma is inspired by certain results of Matuszewska ([46], 2.25 and 2.28).

2.38. Lemma. *Let* $(M_i)_{i \in \mathbb{Z}_+}$ *and* $(N_j)_{j \in \mathbb{Z}_+}$ *be two sequences of nondecreasing functions* $[0, \infty) \to [0, \infty]$ *sending* 0 *to* 0. *Then,*

$$\bigcap_{i=1}^{\infty} \ell_{M_i} \subseteq \bigcup_{j=1}^{\infty} \ell_{N_j} \tag{55}$$

if and only if

$$\overline{\lim_{t \to 0^+}} \frac{N_\ell(t)}{(M_1 \vee \cdots \vee M_k)(t)} < \infty \tag{56}$$

for some $k, \ell \in \mathbb{Z}_+$. *Here* $(M_1 \vee \cdots \vee M_k)(t) := \max(M_1(t), \ldots, M_k(t))$.

In particular, (55) *is equivalent to the existence of such positive integers k and ℓ that*

$$\bigcap_{i=1}^{k} \not{p}_{M_i} \subseteq \not{p}_{N_\ell}.$$

Proof. Since $\not{p}_{M_1 \vee \cdots \vee M_k} = \bigcap_{i=1}^{k} \not{p}_{M_i}$, we replace functions M_i, $i \geqslant 1$, by $L_i \coloneqq M_1 \vee \cdots \vee M_i$. Suppose that $\overline{\lim}_{t \to 0^+} N_\ell(t)/L_k(t) = \infty$ for all $k, \ell \in \mathbb{Z}_+$. Then there exists a double sequence $\alpha \in [0, \infty)^{\mathbb{Z}_+ \times \mathbb{Z}_+}$ such that

$$N_\ell(\alpha_{k\ell}) > (k\ell)^2 L_k(\alpha_{k\ell})$$

and

$$L_k(\alpha_{k\ell}) \leqslant \frac{1}{(k\ell)^2}$$

for all $k, \ell \in \mathbb{Z}_+$.

Let

$$m_{k\ell} \coloneqq \begin{cases} \lceil \frac{1}{(k\ell)^2 L_k(\alpha_{k\ell})} \rceil & \text{if } L_k(\alpha_{k\ell}) > 0, \\ \lceil \frac{1}{N_k(\alpha_{k\ell})} \rceil & \text{if } L_k(\alpha_{k\ell}) = 0 \end{cases}$$

and let λ be the monotonic rearrangement of the "double sequence" obtained from α by *skipping* all the terms $\alpha_{k\ell}$, for $k < \ell$, and *repeating* $m_{k\ell}$ times each term $\alpha_{k\ell}$, for $k \geqslant \ell$. In view of the inequalities $L_1 \leqslant L_2 \leqslant \cdots$, we have

$$\begin{aligned} \rho_{L_i}(\lambda) &= \sum_{1 \leqslant \ell \leqslant k < i} m_{k\ell} L_i(\alpha_{k\ell}) + \sum_{\substack{1 \leqslant \ell \leqslant k < \infty \\ i \leqslant k}} m_{k\ell} L_i(\alpha_{k\ell}) \\ &\leqslant \sum_{1 \leqslant \ell \leqslant k < i} m_{k\ell} L_i(\alpha_{k\ell}) + \sum_{k,\ell=1}^{\infty} m_{k\ell} L_k(\alpha_{k\ell}) < \infty, \end{aligned} \tag{57}$$

since the first sum in (57) has finitely many terms and the second sum admits an estimate $\sum_{k,\ell=1}^{\infty} 2/k^2\ell^2 = 2\zeta(2)^2$. Therefore $\lambda \in \bigcap_{i=1}^{\infty} \not{p}_{L_i}$.

On the other hand, for any $p \geqslant \ell$,

$$\rho_{N_\ell}(\lambda) \geqslant \sum_{k=\ell}^{p} m_{k\ell} N_\ell(\alpha_{k\ell}) \geqslant \sum_{k=\ell}^{p} 1 = p - \ell + 1,$$

i.e., $\rho_{N_\ell}(\lambda) = \infty$ and therefore $\lambda \notin \bigcup_{j=1}^{\infty} \not{p}_{N_j}$. This proves that condition (55) implies condition (56). The reverse is obvious. $\square$

2.39. Corollary. *For any pair* $M, N \in [0, \infty]^{[0,\infty)}$ *of nondecreasing and vanishing at* 0 *functions, the following conditions are equivalent*:

(a) $M \prec_0 N$,
(b) $\ell_N \subseteq \ell_M$,
(c) $\ell_N^{(0)} \subseteq \ell_M$.

If $M, N \in [0, \infty)^{[0,\infty)}$ *then any of the above is equivalent to the condition*:

(d) $\ell_N^{(0)} \subseteq \ell_M^{(0)}$.

2.40. Corollary. *For any pair* $M, N \in [0, \infty]^{[0,\infty)}$ *of nondecreasing and vanishing at* 0 *functions, the following conditions are equivalent*:

(a) $M \sim_0 N$,
(b) $\ell_N = \ell_M$.

If $M, N \in [0, \infty)^{[0,\infty)}$ *then either one is equivalent to the condition*:

(c) $\ell_N^{(0)} = \ell_M^{(0)}$.

We note here also

2.41. Lemma. *If* M *is convex then Orlicz class* $\wp_M$ *has the following property*:

$$\lambda \in \wp_M \quad \textit{whenever} \quad \lambda_a \leqslant \mu_a \quad \textit{and} \quad \mu \in \wp_M \quad (\lambda, \mu \in c_0^{\star}).$$

Proof. The inequality $\rho_M(\lambda) \leqslant \rho_M(\mu)$ was proved by Tomić [61] and Weyl [69] (note that M is nondecreasing). It can be also deduced from the Hardy–Littlewood–Pólya Inequality [36] (see [51]). □

2.42. Corollary. *If* M *is convex then the characteristic sets* $\ell_M^{\star}$ *and* $(\ell_M^{(0)})^{\star}$ *are am-closed.*

3. Certain inequalities

The linear operators $D_m : c_0 \to c_0$ are isometries. Therefore, the operators $m - D_m$, $m > 1$, are invertible in the Banach algebra $\mathscr{B}(c_0)$, the inverses being given by the absolutely convergent series

$$(m - D_m)^{-1} = \frac{1}{m}\sum_{i=0}^{\infty}\left(\frac{D_m}{m}\right)^i. \tag{58}$$

The relation with the operator of arithmetic mean is established by the following double inequality.

3.1. Proposition. *For any $\lambda \in c_0^{\star}$ and any integer $m > 1$, one has the double inequality*

$$\frac{1}{m(m-1)}\lambda_a \leqslant (m - D_m)^{-1}\lambda \leqslant \frac{1}{m-1}\lambda_a, \tag{59}$$

the constants $\frac{1}{m(m-1)}$ and $\frac{1}{m-1}$ being the best possible.

Proof. Since each $\lambda \in c_0^{\star}$ is represented by the absolutely convergent (in c_0) series $\lambda = \sum_{\ell=1}^{\infty}(\Delta_\ell^- \lambda)\mathbb{1}_\ell$ with $\Delta_\ell^-\lambda \geqslant 0$, cf. (15), the linearity and continuity of the operators $(m - D_m)^{-1}$ and $\alpha \mapsto \alpha_a$ reduces the proof of (59) to the case $\lambda = \mathbb{1}_\ell$, $\ell \in \mathbb{Z}_+$. By using representation (58), we deduce that

$$[(m - D_m)^{-1}(\mathbb{1}_\ell)]_n = \frac{1}{m^k(m-1)}, \tag{60}$$

where $k = 0$ if $1 \leqslant n \leqslant \ell$; for $n > \ell$ the value of k is determined from the inequality $\ell m^{k-1} < n \leqslant \ell m^k$. By combining (60) with the formula

$$[(\mathbb{1}_\ell)_a]_n = \begin{cases} 1 & \text{for } 1 \leqslant n \leqslant \ell, \\ \frac{\ell}{n} & \text{for } \ell \leqslant n \end{cases}$$

we obtain, for a given ℓ and m, that

$$\sup \frac{(m - D_m)^{-1}\mathbb{1}_\ell}{(\mathbb{1}_\ell)_a} = \sup_{k \in \mathbb{N}} \frac{1}{m^k(m-1)}\Big/\frac{\ell}{\ell m^k} = \frac{1}{m-1},$$

and the infimum

$$\inf \frac{(m - D_m)^{-1}\mathbb{1}_\ell}{(\mathbb{1}_\ell)_a}$$

is the smaller one of the two numbers: $\frac{1}{m-1}$ and

$$\inf_{k \in \mathbb{Z}_+} \frac{1}{m^k(m-1)}\Big/\frac{\ell}{\ell m^{k-1}+1} = \frac{1}{m(m-1)} \inf_{k \in \mathbb{Z}_+}\left(1 + \frac{1}{\ell m^{k-1}}\right) = \frac{1}{m(m-1)}. \qquad \square$$

Note that either of the two inequalities in (59) is sharp for *each* of the characteristic sequences $\mathbb{1}_\ell$. By comparing (58) and (59) we obtain

3.2. Corollary. *Suppose that*

$$D_{m^i}\lambda \leqslant \frac{Km^i}{(1+i)^s}\mu \quad (i \in \mathbb{N})$$

for some $\lambda, \mu \in c_0^{\star}$, *an integer* $m > 1$, *and some* $s \in (1, \infty)$. *Then*

$$\lambda_a \leqslant K(m-1)\zeta(s)\mu,$$

where $\zeta(s)$ *denotes the value of Riemann's zeta function at* s.

3.3. Example. One has $D_\ell\omega \leqslant \ell\omega$ and $\omega_a \sim \omega \log$ where the sequence $\omega \log^s$ is defined by the formula $(\omega \log^s)_n = (\log n)^s/n$ except that when $s \leqslant 0$ the first term is either ignored or set, e.g., to be equal 1. In particular, $\omega_a \notin \mathcal{O}_{\omega \log^s}$ for any $s < 1$. This shows that λ_a need not be dominated by μ if the hypothesis of Corollary 3.2 is satisfied for some $s < 1$.

The following useful inequality is a cousin of a certain inequality due to Tetsuya Shimogaki (cf. [58, proof of Theorem 1]).

3.4. Proposition. *For any* $\lambda \in c_0^{\star}$ *and* $m \in \mathbb{Z}_+$, *one has the inequality*

$$(D_m\lambda)_a \leqslant \frac{1}{(\omega_a)_m}(\lambda_a)_a \tag{61}$$

and the constant

$$\frac{1}{(\omega_a)_m} = \frac{m}{1 + \frac{1}{2} + \cdots + \frac{1}{m}},$$

which is the harmonic mean of numbers $1, \ldots, m$, *is the best possible.*

Proof. It suffices to prove inequality (61) only for $\lambda = \mathbb{1}_\ell$, $\ell \in \mathbb{Z}_+$. The reason is the same as in the proof of Proposition 3.1.

For $n \leqslant \ell m$, the numerator of the nth term of the ratio-sequence

$$(D_m\mathbb{1}_\ell)_a/(\mathbb{1}_\ell)_{a^2} \tag{62}$$

equals 1 while the denominator does not increase. Accordingly, the ratio does not decrease. For $n \geqslant \ell m$, the nth term of the ratio equals

$$\frac{\ell m}{\ell + \frac{\ell}{\ell+1} + \cdots + \frac{\ell}{n}} = \frac{m}{1 + \frac{1}{\ell+1} + \cdots + \frac{1}{n}},$$

hence the ratio decreases in this region and

$$\sup \frac{(D_m \mathbb{1}_\ell)_a}{(\mathbb{1}_\ell)_{a^2}} = \frac{m}{1 + \frac{1}{\ell+1} + \cdots + \frac{1}{\ell m}}, \tag{63}$$

which is the ℓmth term of the ratio sequence (62).

For a fixed m, the right-hand side of (63), viewed as a function of ℓ, decreases. Consequently, it is majorized by its first term and

$$\sup_{\ell \in \mathbb{Z}_+} \sup \frac{(D_m \mathbb{1}_\ell)_a}{(\mathbb{1}_\ell)_{a^2}} = \sup \frac{(D_m \mathbb{1})_a}{\mathbb{1}_{a^2}} = \sup \frac{(\mathbb{1}_m)_a}{\omega_a} = \frac{1}{(\omega_a)_m}. \qquad \square$$

3.5. Theorem. *Suppose that a characteristic set Σ is e-complete for some $e > 0$, cf. Definition 2.13. Then, for any $s > 0$, the following conditions are equivalent:*

(a) $(\Sigma^s)_a = \Sigma^s$,
(b) *the Boyd index $\alpha(\Sigma)$ is less than $1/s$,*
(c) *there exists $\varepsilon > 0$ such that $||D_m|| < m^{(1/s)-\varepsilon}$ for all $m \gg 0$,*
(d) *there exists $m_0 \in \mathbb{Z}_+$ such that $||D_{m_0}|| < m_0^{1/s}$.*

Proof. The equivalence of (b)-(d) is a direct consequence of the definition of Boyd α-index (29).

Proposition 3.4 produces the inequality

$$D_m\lambda \leqslant \frac{1}{((\omega_a)^{1/s})_m}((\lambda^s)_{a^2})^{1/s} \quad (\lambda \in c_0^{\star}, s > 0)$$

which, for any gauge ϕ on Σ, implies the inequality

$$\phi(D_m\lambda) \leqslant \frac{1}{((\omega_a)^{1/s})_m}\phi(((\lambda^s)_{a^2})^{1/s}). \tag{64}$$

If $(\Sigma^s)_a = \Sigma^s$ then Σ is invariant under the operator

$$A_s : \lambda \mapsto ((\lambda^s)_{a^2})^{1/s}.$$

The latter, being homogeneous and order preserving, is bounded on Σ, by Lemma 2.11. Inequality (64) therefore implies that

$$||D_m|| \leqslant \frac{||A_s||}{((\omega_a)^{1/s})_m} = O\left(\left(\frac{m}{\log m}\right)^{1/s}\right),$$

which in turn implies (d).

In order to show that (c) implies (a) we first observe that, in view of Proposition 2.15, we can assume that $e \leqslant s$. Then, we note that:

(i) Σ^e is a complete characteristic set with gauge given by formula (24) where $p = 1/e$.
(ii) Condition (c) for Σ is equivalent to the similar condition for Σ^e with s replaced by s/e. This follows from the norm equality (30).
(iii) $\Sigma^s = (\Sigma^e)^{s/e}$.

Thus, by replacing Σ by Σ^e and s by s/e, we reduce the general case to the case when Σ is complete and $s \geqslant 1$.

Condition (c) yields, for $m \gg 0$, the estimates

$$\sum_{i=0}^{\infty} \mathcal{q}\left(\left(\frac{D_m}{m^{1/s}}\right)^i \lambda\right) \leqslant \sum_{i=0}^{\infty} \frac{||D_{m^i}||}{m^{i/s}} \mathcal{q}(\lambda) \leqslant \left(\sum_{i=0}^{\infty} \frac{1}{m^{i\varepsilon}}\right) \mathcal{q}(\lambda) = \frac{m^{\varepsilon}}{m^{\varepsilon}-1} \mathcal{q}(\lambda),$$

which, combined with the completeness of Σ, show that

$$\sum_{i=0}^{\infty} \left(\frac{D_m}{m^{1/s}}\right)^i \lambda$$

belongs to Σ for all $\lambda \in \Sigma$. On the other hand, inequality (59) implies that

$$((\lambda^s)_a)^{1/s} \leqslant \left((m-1) \sum_{i=0}^{\infty} \left(\frac{D_m}{m}\right)^i \lambda^s\right)^{1/s} = (m-1)^{1/s} \left(\sum_{i=0}^{\infty} \left(\left(\frac{D_m}{m^{1/s}}\right)^i \lambda\right)^s\right)^{1/s}$$

the last sequence does not exceed, for $s \geqslant 1$, the sequence

$$(m-1)^{1/s} \sum_{i=0}^{\infty} \left(\frac{D_m}{m^{1/s}}\right)^i \lambda. \qquad \square$$

Theorem 3.5 combined with results of Chapter 2 allows us to establish the following important result that was announced in the introduction; cf. (4).

3.6. Theorem. *Let $S \subseteq c_0$ be a symmetric solid sequence space and $\Sigma = S^{\star}$ be the corresponding characteristic set. Then, Σ is e-complete for some $e > 0$ if and only if a certain power S^t, $t > 0$, admits a complete rearrangement invariant norm (which makes it an r.i.* BK-*space). This is so, in fact, for any*

$$t \in (0, e] \cap (0, 1/\alpha(\Sigma)). \tag{65}$$

Proof. Let us assume Σ to be e-complete and t to satisfy (65). In view of Proposition 2.15 combined with Proposition 2.14, the characteristic set Σ^t admits a complete

c-norm $\mathfrak{q}$. Then $(\Sigma^t)_a = \Sigma^t$ by part (a) of Theorem 3.5, and the arithmetic mean operator $\lambda \mapsto \lambda_a$ on Σ^t is bounded with respect to the c-norm $\mathfrak{q}$ in view of Lemma 2.11. In particular, the rearrangement invariant norm (38) is equivalent to $\mathfrak{q}$. □

3.7. Corollary. *Any e-complete gauge on a characteristic set Σ is equivalent to the c-norm $\lambda \mapsto ||\lambda^t||^{1/t}$ for some $t > 0$ and some rearrangement invariant norm $||\,||$ on the sequence space S^t.*

The following corollaries of Theorem 3.5 refer to the Lorentz characteristic set $\ell^{\star}(\varphi)$ and the Marcinkiewicz characteristic set $m^{\star}(\varphi)$, both of which are complete.

3.8. Corollary. *Let φ be a nondecreasing positive sequence satisfying the Δ_2-condition* (22) *and $s > 0$. Then $(\ell^{\star}(\varphi)^s)_a = \ell^{\star}(\varphi)^s$ if and only if $\alpha(\varphi) < 1/s$.*

3.9. Corollary. *For any $\psi \in (0, \infty)^{\mathbb{Z}_+}$ and $s > 0$, the following conditions are equivalent*:

(a) $(m^{\star}(\psi)^s)_a = m^{\star}(\psi)^s$,
(b) $\alpha(\phi) < 1/s$ *where* $\phi = \phi(||\,||_{m(\psi)})$ *is the quasiconcave sequence* (49),
(c) $\beta(\sigma(\pi)) > 1 - 1/s$ *where* $\pi \in c_0^{\star}$ *is defined by* (50).

Rather unexpectedly, Theorem 3.5 and Corollary 3.9 combined allow us to obtain a simple proof of the following result about monotonic sequences.

3.10. Theorem. *For any nonzero sequence $\pi \in c_0^{\star}$, the following conditions are equivalent*:

(a) $\pi \asymp \pi_a$,
$(a)_{k\ell}$ $\pi_{a^k} \asymp \pi_{a^\ell}$ *for natural numbers* $k < \ell$,
(b) $\beta(\pi) > -1$,
(b)$'$ *there exists* $c > 0$ *and an integer* $m_0 \geqslant 1$ *such that*

$$\pi_n \leqslant m^{1-c}\pi_{nm}$$

for all $m \geqslant m_0$ *and all* $n \in \mathbb{Z}_+$,
(c) $\beta(\sigma(\pi)) > 0$,
(c)$'$ *there exists* $d > 0$ *and an integer* $m_0 \geqslant 1$ *such that*

$$\sigma_{mn}(\pi) \geqslant m^d \sigma_n(\pi) \tag{66}$$

for all $m \geqslant m_0$ *and all* $n \in \mathbb{Z}_+$.

Proof. (a)$\Leftrightarrow$(b) The condition $\pi \asymp \pi_a$ is essentially a restatement of the fact that $(\mathcal{O}_\pi)_a = \mathcal{O}_\pi$, since $\pi_a = O(\pi^{\oplus m})$ implies that $m^\bullet\pi_a = O(\pi)$, and $m^\bullet\pi_a \asymp \pi_a$. In this case $\mathcal{O}_\pi = \ell_\infty^{\star}(1/\pi)$ is a complete characteristic set and the condition $(\mathcal{O}_\pi)_a = \mathcal{O}_\pi$ is equivalent to condition (b) by virtue of Theorem 3.5 combined with Lemma 2.24.

Equivalences (b)$\Leftrightarrow$(b)$'$ and (c)$\Leftrightarrow$(c)$'$ follow directly from Definition (20b) of the Matuszewska β-index, and implications (a) $\Rightarrow$ $(\mathrm{a})_{k\ell}$ are clear.

(c)$'$ $\Rightarrow$ (a) Choose one $m > 1$ such that inequality (66) holds. Then

$$(m-1)n\pi_{n+1} \geqslant \sigma_{mn}(\pi) - \sigma_n(\pi) \geqslant (m^d - 1)\sigma_n(\pi).$$

In particular,

$$\pi \leqslant \pi_a \leqslant \frac{m-1}{m^d-1}\pi,$$

i.e., $\pi \asymp \pi_a$.

$(\mathrm{a})_{12}$$\Leftrightarrow$(c) Condition $\pi_a \asymp \pi_{a^2}$ is equivalent to $\pi_a \in {}_a\{\pi_a\}$ or, using equality (48), to

$$(m^{\star}(1/\pi_a))_a = m^{\star}(1/\pi_a).$$

This last equality is equivalent to condition (c) in view of Corollary 3.9.

By combining the last two proven implications we obtain implication $(\mathrm{a})_{12} \Rightarrow$ (a) *for any* $\pi \in c_0^{\star}$. By applying this to π_{a^k} we obtain implications $(\mathrm{a})_{k+1,k+2} \Rightarrow (\mathrm{a})_{k,k+1}$ for all $k \in \mathbb{N}$. Since implication $(\mathrm{a})_{k\ell} \Rightarrow (\mathrm{a})_{k,k+1}$ is obvious, implication $(\mathrm{a})_{k\ell} \Rightarrow$ (a) follows. $\square$

3.11. Remark. The equivalence of (a) and $(\mathrm{a})_{12}$ is implicit among the results of Várga (cf. [62], Theorem IRR; his ideal $\mathcal{M}(A)$ coincides with the Marcinkiewicz ideal $\mathcal{M}(1/s(A)_a)$). All the elementary proofs of this fact which are known to us are quite delicate.

Equivalence (a)$\Leftrightarrow$(b) in the function case seems to have been first established by Aljančić and Aranđelović in their important article [1] (cf. their Theorem 3). The sequence case can be deduced, of course, from the function case.

The following theorem characterizes the condition $\Sigma^s = (\Sigma^s)_a$ in terms not requiring Σ to be e-complete, and thus applies to *all* characteristic sets. Note that conditions (d) and (e) below are "local" which contrasts with the "global" character of conditions (c) and (d) in the statement of Theorem 3.5.

3.12. Theorem. *For any characteristic set* $\Sigma \subseteq c_0^{\star}$ *and* $s \in (0, \infty)$, *the following conditions are equivalent*:

(a) $\Sigma^s = (\Sigma^s)_a$;

(b) *there exists an integer* $m_0 > 1$ *such that* $(m_0 - D_{m_0})^{-1}\Sigma^s \subseteq \Sigma^s$;

(c) *for every integer* $m > 1$, $(m - D_m)^{-1}\Sigma^s \subseteq \Sigma^s$;

(d) *there exists* $t_0 > 0$ *such that, for any* $\lambda \in \Sigma$, *there exist an integer* $\ell > 1$ *and a sequence* $\mu \in \Sigma$ *with the property*:

$$\sup \frac{D_{\ell^i}\lambda}{\mu} = O\left(\frac{\ell^{i/s}}{i^{t_0}}\right) \quad (i \in \mathbb{Z}_+);$$

(e) *for every* $\lambda \in \Sigma$, *every real number* $t > 0$ *and every integer* $m > 1$, *there exists* $\mu \in \Sigma$ *such that*

$$\sup \frac{D_{m^i}\lambda}{\mu} = O\left(\frac{m^{i/s}}{i^{t}}\right) \quad (i \in \mathbb{Z}_+); \tag{67}$$

(f) *there exists* $t_0 > 0$ *such that, for any* $\lambda \in \Sigma$, *there is* $\mu \in \Sigma$ *with the property that*

$$\sup \frac{D_m\lambda}{\mu} = O\left(\frac{m^{1/s}}{(\log m)^{t_0}}\right) \quad (m > 1);$$

(g) *for every* $\lambda \in \Sigma$ *and every real number* $t > 0$, *there exists* $\mu \in \Sigma$ *such that*

$$\sup \frac{D_m\lambda}{\mu} = O\left(\frac{m^{1/s}}{(\log m)^{t}}\right) \quad (m > 1). \tag{68}$$

Proof. The equivalence of (a)–(c) is a direct consequence of Proposition 3.1.

(a) ⇒ (f) Inequality (61) combined with the identity $D_m\lambda^s = (D_m\lambda)^s$ produce the double inequality

$$D_m\lambda \leqslant \frac{1}{(\omega_a)_m^{1/s}}((\lambda^s)_{a^2})^{1/s} < \frac{m^{1/s}}{(\log m)^{1/s}}((\lambda^s)_{a^2})^{1/s} \quad (m > 1),$$

and $\mu := ((\lambda^s)_{a^2})^{1/s} \in \Sigma$ in view of the hypothesis.

(f) ⇒ (d) Trivial.

(d) ⇒ (g) Let $\lambda \in \Sigma$ and $t > 0$. By a repeated use of hypothesis (d), we construct three sequences: of integers $\ell_1, \ell_2, \ldots$ greater than 1, of positive real numbers $K_1, K_2, \ldots$, and of elements $\mu_0 := \lambda, \mu_1, \ldots$ of Σ, such that

$$D_{(\ell_j)^i}\mu_{j-1} \leqslant \frac{K_j(\ell_j)^{i/s}}{i^{t_0}}\mu_j \quad (i \in \mathbb{Z}_+). \tag{69}$$

Put $r = \lceil t/t_0 \rceil$, $\ell = \ell_1 \cdots \ell_r$ and $K = K_1 \cdots K_r$. Since each operator $D_{(\ell_j)^i}$ preserves the partial ordering of $c_0^{\star}$, inequalities (69) combined together produce the inequality

$$D_{\ell^i}\lambda \leqslant \frac{K\ell^{i/s}}{i^{rt_0}}\mu_r \leqslant \frac{K\ell^{i/s}}{i^t}\mu_r \quad (i \in \mathbb{Z}_+),$$

where we have used the identity $D_{\ell_1^i} \circ \cdots \circ D_{\ell_r^i} = D_{\ell^i}$. For an integer $m > 1$, let $\kappa = \lceil \frac{\log m}{\log \ell} \rceil$. Then

$$D_m\lambda \leqslant D_{\ell^\kappa}\lambda \leqslant \frac{K\ell^{\kappa/s}}{\kappa^t}\mu_r \leqslant \frac{K' m^{1/s}}{(\log m)^t}\mu_r$$

for $K' = K\ell^{1/s}(\log \ell)^t$.

(g) $\Rightarrow$ (e) is trivial and (e) $\Rightarrow$ (a) follows from inequalities (59) combined with identity (58):

$$\begin{aligned}(\lambda^s)_a &\leqslant m(m-1)(m-D_m)^{-1}\lambda^s = (m-1)\sum_{i=0}^{\infty}\frac{D_{m^i}\lambda^s}{m^i} \\ &\leqslant (m-1)\left(\lambda^s + \sum_{i=1}^{\infty}\frac{1}{i^2}\mu^s\right)\end{aligned}$$

for some $\mu \in \Sigma$. It suffices to choose $t = 2/s$ in (67). $\square$

3.13. Example. Consider the sequence $\omega \log^s$, $s > 0$, mentioned in Example 3.3 above. This sequence is monotonic except for finitely many initial terms and, for any $t > s$, there is a constant $K = K(s,t)$ such that

$$\sup \frac{D_m(\omega \log^s)^{\star}}{(\omega \log^t)^{\star}} \leqslant K \sup_{\ell > 1} \frac{(\log \ell)^s}{\ell} \bigg/ \frac{(\log(\ell m))^t}{\ell m} < \frac{Km}{(\log m)^{t-s}}$$

for all $m > 1$. Therefore, the characteristic sets $\Sigma_\rho := \bigcup_{s<\rho} \mathcal{O}_{\omega \log^s}$, $\rho > 0$, have the property:

for every $\lambda \in \Sigma_\rho$, there exist $s_0 > 0$ and $\mu \in \Sigma_\rho$ such that

$$\sup \frac{D_m\lambda}{\mu} = O\left(\frac{m}{(\log m)^{s_0}}\right) \quad (m > 1)$$

yet Σ_ρ is not equal to

$$(\Sigma_\rho)_a = \begin{cases} \mathcal{O}_\omega & \text{if } \rho \leqslant -1, \\ \bigcup_{s<\rho+1} \mathcal{O}_{(\omega \log^s)^{\star}} & \text{if } \rho > -1. \end{cases}$$

Now we come to the following central inequality.

3.14. Proposition. *For any* $\lambda \in c_0^{\star}$,

$$\lambda \diamond \omega \leqslant \lambda_a \leqslant 2\lambda \diamond \omega. \tag{70}$$

More precisely, for a given $n \in \mathbb{Z}_+$,

$$\sup_{\lambda \in c_0^{\star}} \left(\frac{\lambda_a}{\lambda \diamond \omega} \right)_n = 2 - \frac{1}{n} \tag{71}$$

and

$$\sup_{\lambda \in c_0^{\star}} \left(\frac{\lambda \diamond \omega}{\lambda_a} \right)_n = 1. \tag{72}$$

3.15. Remark. The double inequality (70) was originally discovered by one of us in 1994 in the form:

$$\lambda \diamond \omega \leqslant \lambda_a \leqslant 2\lambda \diamond \omega',$$

where $\omega' = (1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$. More than a year later, Christian Valqui pointed out how to strengthen it to (71).

Proof. For a given $\lambda \in c_0^{\star}$, choose an injective map $\varphi : \mathbb{Z}_+ \to \mathbb{Z}_+ \times \mathbb{Z}_+$, $n \mapsto (i_n, j_n)$, such that $(\lambda \diamond \omega)_n = \lambda_{i_n}/j_n$.

Fix $n \in \mathbb{Z}_+$. The subset $S_n \coloneqq \varphi(\{1, \dots, n\}) \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ has the following property:

$$\text{if } i' \leqslant i \text{ and } j' \leqslant j \text{ then } (i,j) \in S_n \text{ implies } (i',j') \in S_n$$

(i.e., S_n is a Young diagram). Let $r \coloneqq \max\{i \mid (i,1) \in S_n\}$ denote the number of rows in S_n and $j(i) \coloneqq \max\{j \mid (i,j) \in S_n\}$ the number of elements of S_n in the ith row. Note that $r \leqslant n = \sum_{i=1}^{r} j(i)$ and

$$(\lambda \diamond \omega)_n \leqslant \frac{\lambda_i}{j(i)} \quad (1 \leqslant i \leqslant r) \tag{73}$$

as well as

$$\frac{\lambda_i}{j(i) + 1} \leqslant (\lambda \diamond \omega)_n \quad (i \in \mathbb{Z}_+). \tag{74}$$

We obtain from inequalities (73) that

$$(\lambda \diamond \omega)_n = \frac{1}{n} \sum_{i=1}^{r} j(i)(\lambda \diamond \omega)_n \leqslant \frac{1}{n} \sum_{i=1}^{r} \lambda_i \leqslant (\lambda_a)_n,$$

while inequalities (74) yield

$$\sum_{i=1}^{n} \lambda_i = \lambda_{i_n} + \sum_{\substack{i=1 \\ i \neq i_n}}^{n} \lambda_i \leqslant \left(j(i_n) + \sum_{\substack{i=1 \\ i \neq i_n}}^{n} (j(i) + 1) \right) (\lambda \otimes \omega)_n$$
$$= \left(\sum_{j=1}^{n} j(i) + n - 1 \right) (\lambda \otimes \omega)_n = (2n-1)(\lambda \otimes \omega)_n$$

if one notes that $(\lambda \otimes \omega)_n = \lambda_{i_n}/j(i_n)$.

This gives the $\leqslant$ inequalities in (72) and (71), respectively. That these inequalities are, in fact, equalities follows from considering $\lambda = \mathbb{1}_{n-1} + \mathbb{1}_n$ in (71) and $\lambda = \mathbb{1}_n$ in (72). $\square$

3.16. Corollary. *For any $\lambda \in c_0^{\star}$ and $p \in (0, \infty)$, one has*

$$\lambda \otimes \omega^{1/p} \leqslant ((\lambda^p)_a)^{1/p} \leqslant 2^{1/p} \lambda \otimes \omega^{1/p}. \tag{75}$$

Particularly important is the following.

3.17. Corollary. *For any characteristic set $\Sigma \subseteq c_0^{\star}$, one has*

$$\Sigma_a = \Sigma \otimes \mathcal{O}_\omega. \tag{76}$$

Note that identity (76) is equivalent to the identity

$$\Sigma_a \otimes \Sigma' = \Sigma \otimes \Sigma'_a, \tag{77}$$

valid for any Σ, Σ'. Indeed, aided by the associativity and the commutativity of the $\otimes$-product on the lattice of characteristic sets, identity (76) implies that

$$\Sigma_a \otimes \Sigma' = \Sigma \otimes \mathcal{O}_\omega \otimes \Sigma' = \Sigma \otimes \Sigma'_a.$$

Conversely, identity (77) gives

$$\Sigma_a = \Sigma_a \otimes c_{\mathrm{f}}^{\star} = \Sigma \otimes (c_{\mathrm{f}}^{\star})_a = \Sigma \otimes \mathcal{O}_\omega.$$

The next proposition collects some useful information about sequences ω^s, $s \in (0, \infty)$.

3.18. Proposition. *One has the following asymptotic relations*:

$$(\omega^s)_a \sim \begin{cases} \frac{\omega^s}{1-s} & \text{if } 0 < s < 1, \\ \omega \log & \text{if } s = 1, \\ \zeta(s)\omega & \text{if } s > 1 \end{cases} \tag{78}$$

and

$$\omega^s \circledast \omega \sim \begin{cases} \zeta(1/s)^s \omega^s & \text{if } 0 < s < 1, \\ \omega \log & \text{if } s = 1, \\ \zeta(s)\omega & \text{if } s > 1. \end{cases} \tag{79}$$

Proof. For a given $s > 0$, consider the number $N(s,x)$ of pairs (i,j) of positive integers such that $i^s j \leqslant x$. The identity

$$N(s,x) = \sum_{i=1}^{\infty} \left[\frac{x}{i^s}\right] = \sum_{1 \leqslant i \leqslant x^{1/s}} \left[\frac{x}{i^s}\right] = \sum_{1 \leqslant i \leqslant x^{1/s}} \frac{x}{i^s} + O(x^{1/s})$$

implies that

$$N(s,x) \sim \zeta(s)x \quad \text{as } x \to \infty$$

if $s > 1$ and

$$N(1,x) = x \log x + O(x),$$

from which the set of asymptotics, (79), follows. The other set, (78), is even more elementary. $\square$

3.19. Corollary. *For any* $s > 0$, *one has*

$$\lim \frac{(\omega^s)_a}{\omega^s \circledast \omega} = c(s),$$

where

$$c(s) = \begin{cases} \dfrac{1}{(1-s)\zeta(1/s)^s} & \text{for } 0 < s < 1, \\ 1 & \text{for } s \geqslant 1. \end{cases}$$

The function $c(s)$ is real analytic except at $s = 1$ where it is only continuous. Its one-sided derivatives at 1 equal $-\infty$ and 0, respectively. The maximum value

$$\max_{0 < s < \infty} c(s) = 1.59137\ 07384\ 08698\ldots$$

is attained at $s_{\max} = 0.60917\ 92260\ 28796\ldots$

The following construction produces an example of a sequence $\lambda \in c_0^{\star}$ such that

$$\underline{\lim} \frac{\lambda_a}{\lambda \circledast \omega} = 1 \tag{80}$$

and

$$\overline{\lim}\,\frac{\lambda_a}{\lambda \circledast \omega} = 2. \tag{81}$$

3.20. Proposition. *There is a sequence $\lambda \in c_0^{\star}$ such that any $x \in [1,2]$ is a limit point of the ratio sequence $\lambda_a/\lambda \circledast \omega$.*

Proof. Let

$$\lambda = (\underbrace{1,\dots,1}_{m_0},\underbrace{2^{-1},\dots,2^{-1}}_{m_1},\dots,\underbrace{2^{-k},\dots,2^{-k}}_{m_k},\dots),$$

with $m_k \in \mathbb{Z}_+$ to be chosen. Then letting $q_k = m_0 + \cdots + m_k$ we have

$$(\lambda_a)_n = \frac{\sum_{j=0}^{k-1} 2^{-j} m_j + 2^{-k}(n - q_{k-1})}{n} \quad (q_{k-1} < n \leqslant q_k,\ k \geqslant 1)$$

and

$$\lambda \circledast \omega = (\underbrace{1,\dots,1}_{\ell_1},\underbrace{2^{-1},\dots,2^{-1}}_{\ell_2},\dots,\underbrace{k^{-1},\dots,k^{-1}}_{\ell_k},\dots),$$

where $\ell_k = m_0 + \cdots + m_r$ with r being the number of times that 2 divides k. Thus the first term of $\lambda \circledast \omega$ that is equal to 2^{-k} is the n_kth, where

$$\begin{aligned} n_k &= 1 + 2^{k-1} m_0 + 2^{k-2}(m_0 + m_1) + \cdots + 2(m_0 + \cdots + m_{k-1}) \\ &= 1 + \sum_{j=0}^{k-1} (2^{k-j} - 1) m_j. \end{aligned}$$

Therefore choosing m_k large enough, namely

$$m_k \geqslant 1 + \sum_{j=0}^{k-1} (2^{k-j} - 2) m_j, \tag{82}$$

ensures $q_{k-1} < n_k \leqslant q_k$ and hence

$$\frac{(\lambda_a)_{n_k}}{(\lambda \circledast \omega)_{n_k}} = 2^k (\lambda_a)_{n_k} = \frac{\sum_{j=0}^{k-1} 2^{k-j} m_j + 1 + \sum_{j=0}^{k-1} (2^{k-j} - 2) m_j}{1 + \sum_{j=0}^{k-1} (2^{k-j} - 1) m_j} = 2 - \frac{1}{n_k}.$$

Thus any choice of $m_0, m_1, \dots$ growing fast enough so that (82) holds yields λ satisfying (81).

But if m_k satisfies (82) then $(\lambda \circledast \omega)_{q_k} = 2^{-k}$. If $m_0, \dots, m_{k-1}$ have been chosen then taking m_k large enough will force $(\lambda_a)_{q_k}$ to be as close to 2^{-k} as desired, which will force the ratio

$$\frac{(\lambda_a)_{q_k}}{(\lambda \otimes \omega)_{q_k}}$$

to be as close to 1 as desired.

Since the ratio sequence $\rho := \lambda_a/(\lambda \otimes \omega)$ is bounded and $\rho_{n+1}/\rho_n \to 1$ as $n \to \infty$, any $x \in [1,2]$ is a limit point of ρ. $\square$

Orlicz sequence spaces are the subject of the following.

3.21. Theorem. *For any nondecreasing function $M \in [0,\infty)^{[0,\infty)}$ which vanishes at 0 and for any real number $s \in (0,\infty)$, the following conditions are equivalent*:

(a) $((\ell_M^{(0)})^s)_a \subseteq (\ell_M)^s$,
(b) $((\ell_M^{(0)})^s)_a = (\ell_M^{(0)})^s$,
(c) $((\ell_M)^s)_a = (\ell_M)^s$,
(d) *the Matuszewska β-index at zero of M, cf.* (54), *is greater than s*:

$$\beta_0(M) > s,$$

(e) *there exist constants $\delta, \varepsilon, K > 0$ such that*

$$M(tu) \leqslant KM(u)t^{s+\varepsilon} \quad (0 < t, u \leqslant \delta). \tag{83}$$

The proof is based on the following.

3.22. Proposition. *Let $\mu \in c_0^{\star}$ and $M \in [0,\infty]^{(0,\infty)}$ be a nondecreasing and vanishing at 0 function. If*

$$\ell_M^{(0)} \otimes \mathcal{O}_\mu \subseteq \ell_M \tag{84}$$

and $\mu \neq O(\omega)$ then $\beta_0(M) > 1$.

Proof. Inclusion (84) is equivalent to the inclusion

$$\ell_M^{(0)} \subseteq \ell_{M\#\mu}, \tag{85}$$

where $M\#\mu \in [0,\infty]^{(0,\infty)}$ is defined as follows:

$$M\#\mu(t) = \rho_M(t\mu) \equiv \sum_{n=1}^{\infty} M(t\mu_n) \quad (0 \leqslant t < \infty).$$

By Corollary 2.39, inclusion (85) is equivalent to the condition

$$M\#\mu \prec_0 M;$$

i.e., there exist constants $K, c, \delta > 0$ such that

$$M\#\mu(t) \leqslant KM(ct) \quad (0 < t < \delta). \tag{86}$$

The obvious inequality in $c_0(\mathbb{Z}_+ \times \mathbb{Z}_+)$:

$$\mu_m(\lambda \otimes \mathbb{1}_m) \leqslant \lambda \otimes \mu, \quad (m \in \mathbb{Z}_+, \lambda \in c_0^{\star})$$

translates into the following inequality in $c_0^{\star}$:

$$\mu_m D_m \lambda \leqslant \lambda \circledast \mu. \tag{87}$$

Note that μ_m is the best possible constant. By combining inequalities (86) and (87) for $\lambda = (u/c)\mathbb{1}$, we obtain inequality

$$mM\left(\frac{\mu_m u}{c}\right) \leqslant M\#\mu(u/c) \leqslant KM(u) \quad (m \in \mathbb{Z}_+, 0 < u < c\delta).$$

It follows that

$$\overline{M}_0\left(\frac{\mu_m}{c}\right) \leqslant \frac{K}{m} \quad (m \in \mathbb{Z}_+).$$

Suppose that $\mu \neq O(\omega)$. Then there exists $m_0 \in \mathbb{Z}_+$ such that

$$\frac{K}{m_0} < \frac{\mu_{m_0}}{c} < 1$$

and, hence,

$$\beta_0(M) \geqslant \frac{\log \overline{M}_0(\mu_{m_0}/c)}{\log(\mu_{m_0}/c)} \geqslant \frac{\log(K/m_0)}{\log(\mu_{m_0}/c)} > 1. \qquad \square$$

Proof of Theorem 3.21. Let $M^{(s)}(x) := M(x^s)$. Since

$$\ell_{M^{(s)}}^{(0)} = (\ell_M^{(0)})^{1/s}, \quad \ell_{M^{(s)}} = (\ell_M)^{1/s} \tag{88}$$

and

$$\beta_0(M^{(s)}) = s\beta_0(M), \tag{89}$$

it suffices to prove the equivalence of conditions (a)–(e) for $s = 1$. (a) $\Rightarrow$ (d). Inclusion $(\ell_M^{(0)})_a \subseteq \ell_M$ is seen to be equivalent, with help of Corollary 3.17, to the inclusion $\ell_M^{(0)} \subseteq \ell_{M\#\omega}$. The latter is equivalent, in view of Corollary 2.39, to inclusion

$\ell_M \subseteq \ell_{M\#\omega}$ which is itself equivalent to $\ell_M \circledast \mathcal{O}_\omega \subseteq \ell_M$. It follows that

$$\ell_M^{(0)} \circledast \mathcal{O}_{\omega^{\circledast 2}} = \ell_M^{(0)} \circledast \mathcal{O}_\omega \circledast \mathcal{O}_\omega \subseteq \ell_M \circledast \mathcal{O}_\omega \subseteq \ell_M$$

and, since $\omega^{\circledast 2} \neq O(\omega)$, Proposition 3.22 can be applied.

(d) $\Rightarrow$ (e). Let $0 < \varepsilon < \beta_0(M) - 1$. Then there exists $0 < \tau < 1$ such that

$$\overline{M}_0(\tau) < \tau^{1+\varepsilon}.$$

In particular,

$$M(\tau u) \leqslant M(u)\tau^{1+\varepsilon} \quad (0 < u \leqslant \upsilon)$$

for some $\upsilon > 0$. It follows that

$$M(tu) \leqslant M(\tau^n u) \leqslant M(u)\tau^{n(1+\varepsilon)} \quad (0 < u \leqslant \upsilon) \tag{90}$$

if $t \leqslant \tau^n$. For $n = [\frac{\log t}{\log \tau}]$, inequality (90) combined with the inequality $\tau^n < t/\tau$ yields the following inequality:

$$M(tu) \leqslant KM(u)t^{1+\varepsilon} \quad (0 < t \leqslant \tau; 0 < u \leqslant \upsilon),$$

where $K \coloneqq \tau^{-1-\varepsilon}$.

If condition (e) holds then

$$\sum_{n \geqslant 1/\delta} M\left(\frac{t}{n}\right) \leqslant \sum_{n \geqslant 1/\delta} \frac{KM(t)}{n^{1+\varepsilon}} \quad (0 < t \leqslant \delta)$$

and

$$M\#\omega(t) \leqslant K'M(t) \quad (0 < t \leqslant \delta) \tag{91}$$

for some constant $K' < 1/\delta + K\zeta(1+\varepsilon)$. Thus $M\#\omega \prec_0 M$.

The obvious inequality

$$M\#\omega(t) \leqslant \sum_{n=1}^{m-1} M\left(\frac{t}{n}\right) + \#\omega\left(\frac{t}{m}\right),$$

which holds for any $t > 0$ and $m \in \mathbb{Z}_+$, when combined with inequality (91) for $m \coloneqq \lceil t/\delta \rceil$ proves that $M\#\omega(t) < \infty$ for any $t > 0$. Therefore Corollary 2.39 implies also the inclusions $\ell_M^{(0)} \subseteq \ell_{M\#\omega}^{(0)}$ and $\ell_M \subseteq \ell_{M\#\omega}$ which are equivalent to (b) and (c) of Theorem 3.21, respectively.

Implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are clear. $\square$

Note that the proof of implication (d) $\Rightarrow$ (e) demonstrates the existence, for any choice of $\varepsilon \in (0, \beta_0(M) - s)$, of constants $\delta, K > 0$ for which inequality (83) is satisfied.

Theorem 3.21 combined with Theorem 3.5 yields the following interesting result.

3.23. Corollary. *For any nondecreasing function $M \in [0, \infty)^{[0,\infty)}$ which vanishes at* 0 *the following conditions are equivalent*:

(a) $(\ell_M^{(0)})^{\star}$ *is e-complete for some* $e > 0$,
(b) $\ell_M^{\star}$ *is e-complete for some* $e > 0$,
(c) $\beta_0(M) > 0$.

If so, then $\alpha((\ell_M^{(0)})^{\star}) = \alpha(\ell_M^{\star}) = 1/\beta_0(M)$. *In fact, for any* $t \in (0, \beta_0(M)]$, *one has* $(\ell_M^{(0)})^t = \ell_N^{(0)}$ *and* $(\ell_M)^t = \ell_N$ *for a suitable convex Orlicz function N.*

Proof of Corollary 3.23. Equalities (88) and (89) show that, for any $t \in (0, \beta_0(M)]$ the powers $(\ell_M^{(0)})^t$ and $(\ell_M)^t$ are the corresponding Orlicz sequence spaces whose Matuszewska's β_0-index is greater than 1. In this case, there exists an equivalent convex Orlicz function; cf. the remark on p. 33.

If $\ell_M^{\star}$ is e-complete for some $e > 0$, then the comparison of the equivalence (a)$\Leftrightarrow$(b) of Theorem 3.5 with the equivalence (c)$\Leftrightarrow$(d) of Theorem 3.21 demonstrates that $\beta_0(M) = 1/\alpha(\ell_M^{\star}) > 0$. A similar argument applies also when $(\ell_M^{(0)})^{\star}$ is e-complete. □

4. Spectral description of ideals in $\mathscr{B}(H)$

4.1. Singular numbers. From now on H denotes a separable infinite dimensional Hilbert space which is tacitly identified with $\ell_2(\mathbb{Z}_+)$ each time we mention the matrix representation of a bounded linear operator $A \in \mathscr{B}(H)$.

Let $Gr_n(H)$ denote the set of vector subspaces $V \subset H$ of finite dimension n, and $Gr^n(H)$ denote the set of closed vector subspaces $W \subset H$ of codimension n. The correspondence $V \leftrightarrow V^{\perp}$ provides an obvious bijection $Gr_n(H) \leftrightarrow Gr^n(H)$.

For any $A \in \mathscr{B}(H)$, the two types of s-numbers

$$\inf_{V \in Gr_{n-1}(H)} ||H \xrightarrow{A} H \overset{\pi_V}{\twoheadrightarrow} H/V||, \tag{92}$$

where $\pi_V : H \twoheadrightarrow H/V$ is the quotient map, and

$$\inf_{W \in Gr^{n-1}(H)} ||W \overset{i_W}{\hookrightarrow} H \xrightarrow{A} H||, \tag{93}$$

where $i_W : W \hookrightarrow H$ is the inclusion map, are equal to the distance from A to the set $\mathscr{F}_{n-1}$ of operators whose rank does not exceed $n-1$. This is well-known and very easy to see: if P_W denotes the orthogonal projection onto $W \subset H$ and $P^\perp := I - P$, then

$$||W \overset{i_W}{\hookrightarrow} H \overset{A}{\to} H|| = ||AP_W|| = ||A - AP_W^\perp|| \geqslant \mathrm{dist}(A, \mathscr{F}_{\mathrm{codim}\ W})$$

and, for every $K \in \mathscr{F}_{n-1}$,

$$||A - K|| \geqslant ||(A-K)_{|\mathrm{Ker}\ K}|| = ||A_{|\mathrm{Ker}\ K}|| = ||\mathrm{Ker}\ K \hookrightarrow H \overset{A}{\to} H||.$$

This demonstrates the equality of $\mathrm{dist}(A, \mathscr{F}_{n-1})$ and quantity (93). By considering A^* one obtains the equality of $\mathrm{dist}(A, \mathscr{F}_{n-1}) = \mathrm{dist}(A^*, \mathscr{F}_{n-1})$ and quantity (92). Their common value is denoted $s_n(A)$ and called the *nth singular number* of a bounded linear operator A (for an exhaustive treatment of various scales of s-numbers the reader is referred to Chapter 2 of Pietsch's book [50]; cf. also Chapter II, Sections 1–2 of [33]). The *infimum* in (92) and (93) may be replaced by *minimum*.

For a compact operator $T \in \mathscr{B}(H)$, one has also $s_n(T) = \lambda_n(|T|)$, the nth eigenvalue of $|T| = (T^*T)^{1/2}$ listed with multiplicity and in nonincreasing order (essentially, due to Ernst Fischer [31]; cf., e.g., [54, Section I.5]).

Since every proper ideal $J \subsetneq \mathscr{B}(H)$ is contained in the ideal $\mathscr{K} = \mathscr{K}(H)$ of compact operators, the set of sequences of singular numbers $s(T) := (s_1(T), s_2(T), \dots)$ forms the subset

$$\Sigma(J) := \{s(T) \mid T \in J\} \tag{94}$$

of $c_0^{\star}$. It is easy to verify that $\Sigma(J)$ is a characteristic set and that $J(\Sigma) := \{T \in \mathscr{B}(H) \mid s(T) \in \Sigma\}$ is an ideal for a given characteristic set. This leads to the following spectral description of the lattice of proper ideals in the ring $\mathscr{B}(H)$ which is equivalent to Theorem 12 in Section I.7 of [54]. Schatten's theorem is closely related to Theorem 1.6 on p. 844 of [19].

4.2. Proposition. *The correspondence $J \mapsto \Sigma(J)$ and its inverse $\Sigma \mapsto J(\Sigma)$ establish an isomorphism between the lattice of proper ideals in the ring $\mathscr{B}(H)$ and the lattice Λ of characteristic subsets of $c_0^{\star}$.*

4.3. Recall that the lattice of characteristic sets is isomorphic also to the lattice of symmetric solid subspaces $S \subseteq c_0(\Gamma)$, where Γ is any countably infinite set, via the correspondence $S \mapsto S^{\star}$ of Section 2.7. In particular, $S(J) \subseteq c_0$ will denote the symmetric sequence space corresponding to an ideal $J \subsetneq \mathscr{B}(H)$.

It seems convenient to treat these three canonically isomorphic lattices as three equivalent realizations of a single lattice which from now on will be denoted by Λ. Any operation or relation on Λ which is introduced using one of the three descriptions will then produce the corresponding operation or relation in terms of the remaining two. For example, the operations on characteristic sets which were introduced in Section 2.8 give rise to the corresponding operations on ideals: IJ, $I \diamondtimes J$, $J^{\times}$, ${}_aJ$, J_a, J^s $(s > 0)$, J° (the *am*-interior of J) and J^- (the *am*-closure of J). Note that only sporadically do we have $I \diamondtimes J = I \otimes_R J$ for some ring $R \subseteq \mathscr{B}(H)$. This explains why we decided to use a distinct notation for the internal tensor product.

In particular, we will say that an ideal J is *am-closed* if $J^- = J$, cf. Section 2.8 above. This property plays a special role in the last chapter devoted to single commutator space $[\mathscr{B}(H), J]_1$.

4.4. In order to keep notation simple we shall denote by $(\pi_1, \pi_2, \dots)$ the ideal whose characteristic set is generated by sequences $\pi_1, \pi_2, \dots$. In particular, (π) will denote the principal ideal $(\mathrm{Diag}(\pi))$.

4.5. We shall say that J is a Banach ideal if it is equipped with a *complete* norm $\|\ \|$ such that the trilinear structural map $\mathscr{B}(H) \times J \times \mathscr{B}(H) \to \mathscr{B}(H), (A, T, B) \mapsto ATB$, is bounded. Any such norm is equivalent to the symmetric norm given by

$$\|T\|_{\mathrm{sym}} := \sup_{\substack{A,B \in \mathscr{B}(H) \\ \|A\|_{\mathscr{B}}, \|B\|_{\mathscr{B}} \leqslant 1}} \|ATB\|. \tag{95}$$

Namely, $\|\ \| \leqslant \|\ \|_{\mathrm{sym}} \leqslant K\|\ \|$ where

$$K := \|\mathscr{B}(H) \widehat{\otimes}_{\pi} J \widehat{\otimes}_{\pi} \mathscr{B}(H) \to J\|.$$

In particular, any Banach ideal becomes a complete *uniform-cross-norm* ideal in the terminology of Section V.1 of [54], or a *symmetrically normed ideal* in the later terminology of Gokhberg and Kreĭn [33].

4.6. ***e*-complete ideals.** In Section 2.13 we introduced the concept of an e-complete radial set. We will call the ideal associated with an e-complete characteristic set an *e-complete ideal*. Thus, all finitely generated ideals $(T_1, \dots, T_m) = (|T_1| + \cdots + |T_m|)$ with the property that the sequence $1/\sum_{i=1}^m s(T_i)$ satisfies the Δ_2-condition (22) are, in our terminology, e-complete for every $e > 0$; cf. Corollary 2.23 above. The *Boyd index* $\alpha(J)$ of an e-complete ideal J is defined as the Boyd index of its characteristic set $\Sigma(J)$; cf. Section 2.17 above. We record here also the following corollary of Theorem 3.6:

e-complete ideals coincide with the powers J^s, $s > 0$, of symmetrically normed ideals.

4.7. Lorentz, Marcinkiewicz and Orlicz characteristic sets, which were discussed in Sections 2.25–42, define the corresponding operator ideals:

- Lorentz ideals: $\mathscr{L}(\varphi) := J(\ell^{\star}(\varphi))$ and $\mathscr{L}_p(\varphi) := \mathscr{L}(\varphi)^{1/p}$,
- Marcinkiewicz ideals: $\mathscr{M}(\psi) := J(m^{\star}(\psi))$ and $\mathscr{M}_p(\psi) := \mathscr{M}(\psi)^{1/p}$,
- Orlicz ideals: $\mathscr{L}_M^{(0)} := J((\ell_M^{(0)})^{\star})$ and $\mathscr{L}_M := J(\ell_M^{\star})$.

Here p is a positive real number. These ideals are Banach (and, simultaneously, *am*-closed) if:

(a) $p \geqslant 1$ and φ is concave $(\mathscr{L}_p(\varphi))$,
(b) $p \geqslant 1$ $(\mathscr{M}_p(\psi))$,
(c) M is convex $(\mathscr{L}_M^{(0)}$ and $\mathscr{L}_M)$.

Lorentz ideals $\mathscr{L}_p(\varphi)$ and Marcinkiewicz ideals $\mathscr{M}_p(\psi)$ are e-complete for any $e \leqslant p$.

4.8. A few words are due in order to describe certain special types of ideals $J \subsetneq \mathscr{B}(H)$ already present in the literature.

The ideal of compact operators $\mathscr{K}$ corresponds to $c_0^{\star}$ while the ideal of finite rank operators $\mathscr{F}$ to $c_{\mathrm{f}}^{\star}$. Omnipresent Schatten ideals $\mathscr{L}_p = (\mathscr{L}_1)^{1/p}$, $p \in (0, \infty)$, correspond to the symmetric sequence spaces ℓ_p. They appeared in print for the first time apparently in [55]; cf. Remark 4.1 on p. 580. They are also denoted $\mathfrak{S}_p$ in [33] and $\mathscr{C}_p$ in [17]. (This last notation seems to have resulted from confusing Gothic letters.)

4.9. Two types of ideals studied by Gokhberg and Kreĭn in their book [33] are special instances of Lorentz and Marcinkiewicz ideals:

$$\mathfrak{S}_\pi = \mathscr{L}(\sigma(\pi)) \quad \text{and} \quad \mathfrak{S}_\Pi = \mathscr{M}(1/\pi_a).$$

Section III.4 of their book is a classic exposition of the more general theory of ideals $\mathfrak{S}_\Phi$ introduced by Schatten ([54, Sections V.5–9]; Schatten uses a different notation). Each $\mathfrak{S}_\Phi$ is associated with a *symmetric norming function* Φ. This class of Banach ideals contains Lorentz ideals $\mathscr{L}_p(\varphi)$, for $p \geqslant 1$ and φ concave, Marcinkiewicz ideals $\mathscr{M}_p(\psi)$, for $p \geqslant 1$, and Orlicz ideals $\mathscr{L}_M$, for convex M. Every ideal $\mathfrak{S}_\Phi$ is *am*-closed. This readily follows from a theorem of Ky Fan [30] (cf. also [33, Section III.3, Lemma 3.1]).

4.10. The Macaev ideal $\mathfrak{S}_\omega$, cf. [43], which coincides with the Lorentz ideal $\mathscr{L}(\log)$, and its square-root $(\mathfrak{S}_\omega)^{1/2}$ appear in the work of Alain Connes on θ-summable Fredholm modules; see [23]. The celebrated Köthe dual $\mathfrak{S}_\Omega = (\mathfrak{S}_\omega)^\times$ which owes its current vogue[6] to Connes' work and the fact that it supports Dixmier traces (see [24] and the references therein) coincides with the Marcinkiewicz ideal $\mathscr{M}(\frac{1}{\omega \log}) = \mathscr{M}(1/\omega_a)$. Connes' own preference is to denote $\mathfrak{S}_\omega$ by $\mathscr{L}^{(\infty,1)}$ and $\mathfrak{S}_\Omega$ by $\mathscr{L}^{1+}$, $\mathscr{L}^1_{\text{weak}}$ or $\mathscr{L}^{(1,\infty)}$; see [24, Section IV.2.α]. His logarithmic integral ideal $\mathscr{Li}$, cf. [23] and Section IV.8.α of [24], coincides with the Marcinkiewicz ideal $\mathscr{M}(\log)$ which is, incidentally, equal to the principal ideal $(1/\log)$.

[6] Cf., e.g., [56].

4.11. Lorentz ideals $\mathscr{L}_{pq}$. The classical Lorentz sequence spaces ℓ_{pq}, $0 < p < \infty$, are defined, to use our notation, as

$$\ell_{pq} := \begin{cases} \ell(\frac{1}{\omega^{q/p}})^{1/q} & \text{if } 0 < q < \infty, \\ \mathcal{O}_{\omega^{1/p}} = (\mathcal{O}_{\omega})^{1/p} & \text{if } q = \infty; \end{cases}$$

cf., e.g., [10, Section 1.3, p. 8] or [9, Definition 4.4.1 on p. 216].

The corresponding Lorentz ideals $\mathscr{L}_{pq}$ coincide, for $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, with the interpolation ideals $\mathscr{L}^{(p,q)}$ discussed by Connes [24], Section IV.2.α. Earlier, ideals $\mathscr{L}^{(p,1)}$ played an important role in the work of D. V. Voiculescu on quasi-central approximate units [63–65].

Connes does not define $\mathscr{L}^{(1,q)}$ except for $\mathscr{L}^{(1,\infty)} := \mathfrak{S}_{\Omega} \equiv \mathscr{M}(1/\omega_a)$; cf. Section IV.2.$\beta$ of his book. Let us finally mention that $\mathscr{L}_{pp} = \mathscr{L}_p$, $0 < p < \infty$, and that $\mathscr{L}_1 = \mathscr{M}(1/\omega)$.

Note that the standard gauge on $\ell_{pq}^{\star}$ does not induce a rearrangement invariant norm on ℓ_{pq} if $q > p$. Nevertheless, there exists an equivalent rearrangement invariant norm on ℓ_{pq} for $p > 1$, This is well known (cf., e.g., [9, p. 218]) and agrees with the fact that the Boyd index $\alpha(\ell_{pq}) = 1/p$ is less than 1 (cf. Theorem 3.6 above).

4.12. Ideals $S(f)$ and $D(f)$ studied in [17,53,57] and to some extent also in [2,48] coincide with Orlicz ideals $\mathscr{L}inf > M$ and $\mathscr{L}_M^{(0)}$, respectively, where $M = f$.

5. Sums of commutators

5.1. We describe in this chapter the commutator space $[I, J]$, introduced in (1) on p. 1, when at least one ideal is proper. Key to this is the action of the monoid $\mathrm{Emb}(\Gamma)$, introduced in Section 1.1, on the Banach space $\mathscr{B}(\ell_2(\Gamma))$:

$$A \mapsto U_f A U_{f^\dagger} = U_f A U_f^* \quad (A \in \mathscr{B}(\ell_2(\Gamma)),\ f \in \mathrm{Emb}(\Gamma)). \tag{96}$$

The partial isometries U_f which are defined on basis vectors $e_\gamma \in \ell_2(\Gamma)$ by

$$U_f e_\gamma := \begin{cases} e_{f(\gamma)} & \text{if } \gamma \in \mathrm{Dom} f, \\ 0 & \text{otherwise.} \end{cases} \tag{97}$$

constitute a $*$-representation of $\mathrm{Emb}(\Gamma)$ on the Hilbert space $\ell_2(\Gamma)$:

$$(U_f)^* = U_{f^\dagger}.$$

Here $f \mapsto f^\dagger$ is the antipode operation on $\mathrm{Emb}(\Gamma)$, introduced in Section 1.1. The embedding

$$\ell_\infty(\Gamma) \hookrightarrow \mathscr{B}(\ell_2(\Gamma)), \quad \alpha \mapsto \mathrm{Diag}(\alpha), \tag{98}$$

is a morphism of representations of $\mathrm{Emb}(\Gamma)$ if we recall that $\ell_\infty(\Gamma)$ is equipped with a natural action of $\mathrm{Emb}(\Gamma)$, cf. (7).

The action (96) is *not* via (nonunital) algebra endomorphisms unless we restrict it to the submonoid $\mathscr{E}_\Gamma$ of self-injections $\Gamma \rightarrowtail \Gamma$; see Section 1.1. The reason is that $\mathscr{E}_\Gamma$ coincides with the submonoid of left-invertible elements of $\mathrm{Emb}(\Gamma)$, and $f^\dagger f = \mathrm{id}_\Gamma$ precisely for $f \in \mathscr{E}_\Gamma$.

5.2. An action of a monoid M on a k-module V is equivalent to making V a module over the monoid k-algebra of M

$$kM := \left\{ \sum_{m \in M} a_m m \mid a_m \in k, \text{ all but finitely many zero} \right\}.$$

The module of coinvariants V_M, the largest trivial quotient representation of M, equals $V/\mathscr{I}_M V$ where $\mathscr{I}_M$ is the augmentation ideal of kM, i.e., the kernel of the k-algebra homomorphism

$$kM \to k, \quad \tau = \sum_{m \in M} a_m m \mapsto \deg \tau := \sum_{m \in M} a_m.$$

The submodule $\mathscr{I}_M V$ is the union of the sets

$$(\mathscr{I}_M V)_r := \left\{ \sum_{i=1}^r (m_i - m_i')v_i \mid m_i, m_i' \in M; v_i \in V \right\},$$

since any $\sum_{i=0}^q a_i f_i \in \mathscr{I}_M$ equals $\sum_{i=1}^q a_i(f_i - f_0)$.

We begin the description of the commutator structure of operator ideals from the following identity.

5.3. Lemma. *For any* $\beta', \beta'' \in \mathbb{C}^\Gamma$ *and* $f, g \in \mathscr{E}_\Gamma$, *one has*

$$\mathrm{Diag}((f-g)_*\beta) = [U_f \,\mathrm{Diag}(\beta') U_{g^\dagger}, U_g \,\mathrm{Diag}(\beta'') U_{f^\dagger}], \tag{99}$$

where $\beta := \beta'\beta''$.

From now on we assume that $\Gamma = \mathbb{Z}_+$, $H = \ell_2(\mathbb{Z}_+)$ and $\mathscr{E} = \mathscr{E}_{\mathbb{Z}_+}$.

Since every ideal in $\mathscr{B}(H)$ is the filtered union of principal ideals, Lemma 5.3 implies

5.4. Corollary. *For any pair of ideals in* $\mathscr{B}(H)$, *at least one proper*, *one has*

$$(\mathscr{I}_{\mathscr{E}} S(IJ))_r \overset{(99)}{\hookrightarrow} [I, J]_r, \tag{100}$$

where $S(IJ) \subseteq c_0$ *is the symmetric sequence space associated with the ideal* IJ (*cf. Section* 4.3).

Another identity involves the unilateral shift $s : \mathbb{Z}_+ \to \mathbb{Z}_+$, $n \mapsto n+1$, and the sequence of partial sums:

$$\alpha = (\mathrm{id} - s)_* \sigma(\alpha) \quad (\alpha \in \mathbb{C}^\Gamma). \tag{101}$$

5.5. Consider the adjoint action of the unitary group $\mathscr{U} = \mathscr{U}(H)$ on the set of compact normal operators $\mathscr{K}_{nl}$. Each orbit $\mathscr{U}T := \{UTU^* \mid U \in \mathscr{U}\}$ is contained in the set $\mathscr{U}\mathrm{Diag}(Q) := \bigcup_{\lambda \in Q} \mathscr{U}\mathrm{Diag}(\lambda)$ for a unique quasiorbit $Q \subset c_0$, cf. (9). This unique quasiorbit will be denoted $\llbracket T \rrbracket$ and it consists of all rearrangements of the sequence of eigenvalues of T, each nonzero eigenvalue occurring as many times as its multiplicity, and taken in any order.

For any subset $\Sigma \subseteq c_0^{\star}$, we form the set

$$\llbracket T \rrbracket_\Sigma := \{\alpha \in \llbracket T \rrbracket \mid \mathrm{uni}(|\alpha|) \in \Sigma\} \tag{102}$$

(the upper nonincreasing envelope was defined in (18)).

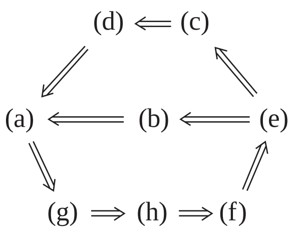
If Σ is the characteristic set of an ideal $J \subsetneq \mathscr{B}(H)$ then $\llbracket T \rrbracket_\Sigma \neq \varnothing$ if and only if $T \in J$. Indeed, $\llbracket T \rrbracket_\Sigma$ contains in this case the set $\{\alpha \in \llbracket T \rrbracket \mid |\alpha| \in c_0^{\star}\}$ of sequences of eigenvalues of T listed in the nondecreasing order of their absolute values (this leaves only nonzero eigenvalues unless T has finite rank).

The main result of this chapter and one of the main results of the whole article is the following.

5.6. Theorem. *Let I and J be ideals in $\mathscr{B}(H)$, at least one of them proper. For any normal operator $T \in IJ$, the following conditions are equivalent*:

(a) $T \in [I, J]$,
(b) $T \in [I, J]_3$,
(c) $\llbracket T \rrbracket \cap \mathscr{I}_{\mathscr{E}} S(IJ) \neq \varnothing$,
(d) $\llbracket T \rrbracket \subseteq \mathscr{I}_{\mathscr{E}} S(IJ)$,
(e) $\llbracket T \rrbracket \subseteq (\mathscr{I}_{\mathscr{E}} S(IJ))_3$,
(f) $\llbracket T \rrbracket_{\Sigma(IJ)} \cap \{\alpha \in c_0 \mid \alpha_a \in S(IJ)\} \neq \varnothing$,
(g) $\llbracket T \rrbracket_{\Sigma(IJ)} \subseteq \{\alpha \in c_0 \mid \alpha_a \in S(IJ)\}$,
(h) *there exists $\alpha \in \llbracket T \rrbracket$ such that $\alpha_a \in S(IJ)$ and $|\alpha|$ is monotonic.*

The following diagram shows the organization of the proof



The principal implications are (a) $\Rightarrow$ (g) and (f) $\Rightarrow$ (e). Implication (c) $\Rightarrow$ (d) follows from the fact that $\mathscr{I}_{\mathscr{E}}S(IJ)$ is a vector space and any two elements in $[\![T]\!]$ are of the form $f_*\alpha$ and $g_*\alpha$, for some $f, g \in \mathscr{E}$ and a particular element α of $[\![T]\!]$; hence their difference belongs to $\mathscr{I}_{\mathscr{E}}S(IJ)$. Implications (e) $\Rightarrow$ (b) and (d) $\Rightarrow$ (a) are special cases of Corollary 5.4. The remaining ones are obvious.

The proof of implication (a) $\Rightarrow$ (g) is based on

5.7. Proposition. *Suppose that $T = \sum_{i=1}^{r}[A_i, B_i]$ for some $T, A_i, B_i \in \mathscr{B}(H)$. Then, for any projection $P \in \mathscr{B}(H)$ of rank $p < \infty$, one has the inequality*

$$\frac{|\mathrm{Tr}\, PTP|}{p} \leqslant (8r+2)\sum_{i=1}^{r} s_{p+1}(A_i)s_{p+1}(B_i) + 4r\|P^{\perp}TP^{\perp}\|. \tag{103}$$

Proof. According to the definition of singular numbers, cf. (92)–(93), there exist projections E_{ij}, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant 4$, of rank at most p such that

$$\|E_{i1}^{\perp}A_i\| = s_{p+1}(A_i), \quad \|A_iE_{i2}^{\perp}\| = s_{p+1}(A_i^*) = s_{p+1}(A_i) \tag{104}$$

and

$$\|E_{i3}^{\perp}B_i\| = s_{p+1}(B_i), \quad \|B_iE_{i4}^{\perp}\| = s_{p+1}(B_i^*) = s_{p+1}(B_i). \tag{105}$$

Denote by E the projection onto

$$\mathrm{Im}\, P + \sum_{i=1}^{r}\sum_{j=1}^{4}\mathrm{Im}\, E_{ij}.$$

Its rank does not exceed $p + 4rp$ and

$$\begin{aligned}
|\mathrm{Tr}\, ETE| &= \left|\sum_{i=1}^{r}\mathrm{Tr}([EA_i, EB_i]) + \sum_{i=1}^{r}(\mathrm{Tr}\, EA_iE^{\perp}B_i - \mathrm{Tr}\, EB_iE^{\perp}A_i)\right| \\
&\leqslant \sum_{i=1}^{r}(\|A_iE^{\perp}\|\,\|E^{\perp}B_i\| + \|B_iE^{\perp}\|\,\|E^{\perp}A_i\|)\,\mathrm{rank}\, E \\
&\leqslant (8r+2)p\sum_{i=1}^{r} s_{p+1}(A_i)s_{p+1}(B_i)
\end{aligned}$$

in view of equalities (104)–(105) and the definition of E. On the other hand, since $E \geqslant P$ and $P^{\perp} \geqslant E - P$, one has

$$\begin{aligned}
\mathrm{Tr}\, PTP &= \mathrm{Tr}\, ETE - \mathrm{Tr}(E-P)T(E-P) \\
&= \mathrm{Tr}\, ETE - \mathrm{Tr}(E-P)(P^{\perp}TP^{\perp})(E-P)
\end{aligned}$$

which implies that

$$|\mathrm{Tr}\, PTP| \leqslant |\mathrm{Tr}\, ETE| + ||P^{\perp} TP^{\perp}||\, \mathrm{rank}(E-P)$$
$$\leqslant (8r+2)p \sum_{i=1}^{r} s_{p+1}(A_i)s_{p+1}(B_i) + 4rp||P^{\perp}TP^{\perp}||. \qquad \square$$

For a compact normal operator T and $\lambda \in [\![T]\!]$, let $u = (u_n)$ be the corresponding orthogonal sequence of eigenvectors: $Tu_n = \lambda_n u_n$, $n \in \mathbb{Z}_+$. Set P_n to be the projection onto the linear span of $u_1, \dots, u_n$. Then

$$\frac{\mathrm{Tr}\, P_n T P_n}{n} = \frac{\lambda_1 + \cdots + \lambda_n}{n} = (\lambda_a)_n.$$

If $T = \sum_{i=1}^{r} [A_i, B_i]$ for some $A_i \in I$, $B_i \in J$, then

$$|\lambda_a| \leqslant (8r+2) \sum_{i=1}^{r} s(A_i)s(B_i) + 4r\, \mathrm{uni}(|\lambda|)$$

in view of Proposition 5.7. Property (g) follows by choosing $\lambda \in [\![T]\!]_{\Sigma(IJ)}$.

Now we prove implication (f) $\Rightarrow$ (e). Consider the function $e : \mathbb{Z}_+ \to \mathbb{Z}_+$, $e(n) = 2^{[\log_2 n]+1}$. For all $n \in \mathbb{Z}_+$, $\frac{1}{2} e(n) \leqslant n < e(n)$.

For a given $\lambda \in c_0$, set $\beta_n = \frac{1}{e(n)} \sigma_{e(n)-1}(\lambda)$. The estimate

$$|\beta_n| \leqslant \frac{1}{e(n)} \left(\left| \sum_{1 \leqslant i \leqslant n} \lambda_i \right| + \left| \sum_{n < i < e(n)} \lambda_i \right| \right)$$
$$\leqslant \frac{1}{n} \left| \sum_{i=1}^{n} \lambda_i \right| + \frac{1}{2} \max\{ |\lambda_i| \mid n < i < e(n) \}$$

(when $n = e(n) - 1$ the term involving *maximum* disappears) yields the inequality

$$|\beta| \leqslant |\lambda_a| + \frac{1}{2} \mathrm{uni}(|\lambda|).$$

In particular, $\beta \in S(IJ)$ provided $\lambda_a \in S(IJ)$ and $\mathrm{uni}(|\lambda|) \in \Sigma(IJ)$. Let $f' : \mathbb{Z}_+ \to \mathbb{Z}_+$, $n \mapsto 2n$, and $f'' : \mathbb{Z}_+ \to \mathbb{Z}_+$, $n \mapsto 2n+1$, and

$$\mu := (2 \cdot \mathrm{id} - f' - f'')_* \beta = (\mathrm{id} - f')_* \beta + (\mathrm{id} - f'')_* \beta. \qquad (106)$$

By noting that $e(2n) = e(2n+1) = 2e(n)$ for all $n \in \mathbb{Z}_+$, we have

$$\mu_{2n} = 2\beta_{2n} - \beta_n = \frac{1}{e(n)}(\sigma_{2e(n)-1}(\lambda) - \sigma_{e(n)-1}(\lambda))$$
$$= \frac{1}{e(n)} \sum_{i=e(n)}^{2e(n)-1} \lambda_i = 2\beta_{2n+1} - \beta_n = \mu_{2n+1}$$

and $\mu_1 = 2\beta_1 = \lambda_1$. In particular,

$$\mu_{2^k} = \cdots = \mu_{2^{k+1}-1} = \frac{1}{2^k} \sum_{i=2^k}^{2^{k+1}-1} \lambda_i.$$

Thus, for any natural number k, we have

$$\sum_{i=2^k}^{2^{k+1}-1} (\mu_i - \lambda_i) = 0. \tag{107}$$

For every $i \in \{2^k, \dots, 2^{k+1}-1\}$, one has the estimate

$$|\mu_i - \lambda_i| \leqslant |\mu_i| + |\lambda_i| \leqslant 2\,\mathrm{uni}_{2^k}(|\lambda|). \tag{108}$$

At this point we need the following well-known result, for the case $E = \mathbb{C}$.

5.8. Steinitz' Lemma. *For every finite dimensional real normed vector space E, there exists $C > 0$ with the property that for every finite collection of vectors $v_1, \dots, v_m \in E$ of norm not exceeding 1 and $\sum_{i=1}^m v_i = 0$, there exists a permutation $g \in S_m$ such that*

$$\left\| \sum_{i=1}^{k} v_{g(i)} \right\| \leqslant C \quad (k = 1, \dots, m).$$

5.9. Remark. The smallest value of C is denoted $S(E)$ and called the *Steinitz' constant* of the normed space E. Ernst Steinitz himself proved that $S(E) \leqslant 2 \dim E$; cf. [60]. This has been improved to $S(E) \leqslant \dim E$ (cf. [34]) and $S(\ell_2^n) \leqslant \sqrt{\frac{3n^2+9n-20}{8}}$ (cf. [7, p. 199], without proof). One has $S(\mathbb{R}) = 1$ (trivial) and $S(\mathbb{C}) = \sqrt{5}/2 = 1.1180\dots$ (cf. [6,8]).

In view of equality (107) and inequality (108) combined with Steinitz' Lemma, there exist permutations g_k of sets $\{2^k, \dots, 2^{k+1}-1\}$ such that the estimates

$$\left| \sum_{i=2^k}^{n} (\mu_{g_k(i)} - \lambda_{g_k(i)}) \right| \leqslant 2S(\mathbb{C})\mathrm{uni}_{2^k}(|\lambda|) \tag{109}$$

hold for $n = 2^k, \dots, 2^{k+1} - 1$ and any $k \in \mathbb{Z}_+$. Assemble all g_k's into a single permutation g of $\mathbb{Z}_+$. Note that $g(1) = 1$.

By noticing that, for $n \in \{2^k, \dots, 2^{k+1} - 1\}$,

$$\mathrm{uni}_{2^k}(|\lambda|) = (\mathrm{uni}(|\lambda|)^{\oplus 2})_{2^{k+1}} \leqslant (\mathrm{uni}(|\lambda|)^{\oplus 2})_n$$

we deduce from (109) the inequality

$$|\sigma(g_*^{-1}(\mu - \lambda))| \leqslant 2S(\mathbb{C})\mathrm{uni}(|\lambda|)^{\oplus 2}.$$

In particular, $\sigma(g_*^{-1}(\mu - \lambda)) \in S(IJ)$ if $\lambda \in [\![T]\!]_{\Sigma(IJ)}$. Recall that

$$g_*^{-1}(\mu - \lambda) = \sigma(g_*^{-1}(\mu - \lambda)) - s_*\sigma(g_*^{-1}(\mu - \lambda));$$

cf. identity (101). Hence

$$\lambda - \mu = (gs - g)_*\gamma, \tag{110}$$

where $\gamma := \sigma(g_*^{-1}(\mu - \lambda))$. By combining identities (110) with (106), we arrive at the representation

$$\lambda = (\mathrm{id} - f')_*\beta + (\mathrm{id} - f'')_*\beta + (gs - g)_*\gamma$$

with $\beta, \gamma \in S(IJ)$ if $\lambda_a \in S(IJ)$ and $\mathrm{uni}(|\lambda|) \in \Sigma(IJ)$. This ends the proof of implication (f) $\Rightarrow$ (e) and hence of Theorem 5.6.

Theorem 5.6 leads to a simple characterization of the commutator space $[I, J]$. This is so, because an operator $T = X + iY$, where $X = X^*$ and $Y = Y^*$, belongs to $[I, J]$ precisely when X and Y do.

What follows is the applications of Theorem 5.6. Some are merely straightforward corollaries, and some depend additionally on deeper results established in Sections 2 and 3.

5.10. Theorem. *For any ideals $I, J \subseteq \mathscr{B}(H)$, one has $[\mathscr{B}(H), IJ] = [I, J]$ and*

$$[I, J] = [I, J]_4. \tag{111}$$

Proof. Let $T = X + iY \in [I, J]$ as above, and let U, V be the corresponding unitaries that diagonalize X and Y, respectively:

$$UXU^* = \mathrm{Diag}(\lambda) \quad \text{and} \quad VYV^* = \mathrm{Diag}(\mu)$$

for some $\lambda, \mu \in S(IJ)$. Then

$$T = U^*\,\mathrm{Diag}(\lambda + i\mu)U + i(V^*\,\mathrm{Diag}(\mu)V - U^*\,\mathrm{Diag}(\mu)U).$$

The normal operator $U^*\,\mathrm{Diag}(\lambda + i\mu)U$ is, according to parts (e) and (b) of Theorem 5.6, the sum of three commutators in $[\mathscr{B}(H), IJ]$ and, alternatively, also of three in $[I, J]$. On the other hand,

$$V^* \operatorname{Diag}(\mu) V - U^* \operatorname{Diag}(\mu) U = [V^* U, U^* \operatorname{Diag}(\mu) V] \quad (\in [\mathscr{B}(H), IJ])$$

$$= [V^* \operatorname{Diag}(\mu_1) U, U^* \operatorname{Diag}(\mu_2) V] \quad (\in [I, J])$$

for any representation $\mu = \mu_1 \mu_2$ with $\mu_1 \in S(I)$ and $\mu_2 \in S(J)$. $\square$

In Sections 6 and 7, we address the question of the minimum number of commutators needed in commutator representations of various types.

5.11. Theorem. (i) *For any operator* $T \in \mathscr{B}(H)$, *the following conditions are equivalent*:

(a) $(T) \subseteq [I, J]$,
(b) $|T| \in [I, J]$.

(ii) *For any ideals* I, J *and* L *in* $\mathscr{B}(H)$, *at least one of them being proper, the following conditions are equivalent*:

(a) $I \subseteq [J, L]$,
(b) $I_a \subseteq JL$,
(c) $I \otimes (\omega) \subseteq JL$.

(iii) $I \subseteq \mathscr{L}_1$ *then the following conditions are equivalent*:

(a) $\{T \in I \mid \operatorname{Tr} T = 0\} \subseteq [J, L]$,
(b) $\{\lambda_{a_\infty} \mid \lambda \in \Sigma(I)\} \subseteq \Sigma(JL)$, *cf.* (16).

Proof. (i) For any positive $S \in (T)$, its eigenvalue sequence $\lambda(S)$ is $O(s(T)^{\oplus m})$ for some $m \geqslant 1$. In view of parts (a) and (h) of Theorem 5.6, the condition $|T| \in [I, J]$ is equivalent to $s(T)_a \in \Sigma(IJ)$. However, $(s(T)^{\oplus m})_a \leqslant m s(T)_a$. Thus $\lambda(S)_a \in \Sigma(IJ)$ and $S \in [I, J]$ by Theorem 5.6 again. A general operator $S \in (|T|)$ is a $\mathbb{C}$-linear combination of positive ones.

(ii) Implication (b) $\Rightarrow$ (a) follows from Theorem 5.6 combined with part (i), the reverse implication (a) $\Rightarrow$ (b)—from Theorem 5.6 alone. Equality $I_a = I \otimes (\omega)$ follows from Corollary 3.17.

(iii) Elements of $\{T \in I \mid \operatorname{Tr} T = 0\}$ are linear combinations of operators

$$U \begin{pmatrix} -\sum_{i=1}^{\infty} \lambda_i & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \ddots \end{pmatrix} U^*,$$

where $\lambda \in \Sigma(I)$ and U is an isometry. Hence the equivalence of conditions (a) and (b) follows from the equivalence of conditions (a) and (g) of Theorem 5.6. $\square$

In Section 6 we prove that any of conditions (a)–(c) of Part (ii) is equivalent to the condition

$$I \subseteq [J, L]_3.$$

Let us return now to the $\mathscr{E}$-equivariant embedding (98) which produces $\mathscr{E}$-equivariant embeddings $S(J) \hookrightarrow J$ for all ideals $J \subsetneq \mathscr{B}(H)$ and the induced maps of modules of coinvariants

$$S(J)_{\mathscr{E}} \to J_{\mathscr{E}} \tag{112}$$

(cf. Section 5.2 above). Since $\mathscr{I}_{\mathscr{E}} J \subseteq [\mathscr{B}(H), J]$, we have also the quotient map

$$J_{\mathscr{E}} \twoheadrightarrow J/[\mathscr{B}(H), J]. \tag{113}$$

5.12. Theorem. *For every ideal $J \subseteq \mathscr{B}(H)$, the composition of maps* (112) *and* (113) *produces an isomorphism*

$$S(J)_{\mathscr{E}} \xrightarrow{\sim} J/[\mathscr{B}(H), J]. \tag{114}$$

Proof. Map (114) is surjective, since every $T = X + iY \in J$ equals $\mathrm{Diag}(\lambda + i\mu)$ modulo $[\mathscr{B}(H), J]$ where $\lambda \in [\![X]\!]$ and $\mu \in [\![Y]\!]$. If $\mathrm{Diag}(\alpha) \in [\mathscr{B}(H), J]$ then $\alpha \in \mathscr{I}_{\mathscr{E}} S(J)$ by implication (a) $\Rightarrow$ (d) of Theorem 5.6. □

We close this chapter by characterizing the condition $J = [\mathscr{B}(H), J]$, i.e., the absence of nonzero traces on J, for e-complete, principal, Lorentz and, respectively, Orlicz ideals. Sections 5.15–20 are devoted to principal ideals alone.

By combining Theorems 5.11(ii) and 3.5, we obtain the following "index theorem".

5.13. Theorem. *If an ideal J is e-complete for some $e > 0$ (cf. Section* 4.6), *then for any $s > 0$, the following conditions are equivalent*:

(a) $J^s = [\mathscr{B}(H), J^s]$,
(b) *the Boyd index $\alpha(\Sigma(J))$ is less than $1/s$.*

Since the Boyd α-index of a Banach sequence space is less or equal than 1, we infer that a Banach ideal J admits a nonzero trace *if and only if* its Boyd α-index equals 1. It follows that no power J^s, $s > 1$, of such a Banach ideal J is Banach, cf. (6).

5.14. Remark. The importance of knowing for which $s > 0$ one has $J^s = [\mathscr{B}(H), J^s]$ is due to the following fact proved in [72] (cf. also [70, Theorem 7 and Corollary of

Theorem 8]):

> *If $J^p = [\mathscr{B}(H), J^p]$ then the relative cyclic homology groups $HC_n(\mathscr{B}(H), J)$ vanish for all $n < 2p$ and $HC_{2p}(\mathscr{B}(H), J)$ is canonically isomorphic to*$J^{p+1}/[J, J^p] = J^{p+1}/[\mathscr{B}(H), J^{p+1}]$.

By combining Theorems 5.11(ii) and 3.10, we obtain

5.15. Theorem. *For any nonzero sequence $\pi \in c_0^{\star}$ and every real number $s > 0$, the following conditions are equivalent*:

(a) $(\pi)^s = [\mathscr{B}(H), (\pi)^s]$,
(b) *the Matuszewska β-index* (20b) *of π is greater than* $-1/s$:

$$\beta(\pi) > -1/s,$$

(b)′ *there exists $c > 0$ and an integer $m_0 \geqslant 1$ such that*

$$\pi_n \leqslant m^{1/s-c}\pi_{nm}$$

for all $m \geqslant m_0$ and all $n \in \mathbb{Z}_+$,
(c) $\pi^s \asymp (\pi^s)_a$.

5.16. Corollary. *For any nonzero sequence $\pi \in c_0^{\star}$, the following conditions are equivalent*:

(a) $(\pi)^s = [\mathscr{B}(H), (\pi)^s]$ *for all $s > 0$,*
(b) $\beta(\pi) = 0$,
(b)′ *for any $\varepsilon > 0$ there is an integer $m_0 \geqslant 1$ such that*

$$\pi_n \leqslant m^{\varepsilon}\pi_{nm}$$

for all $m \geqslant m_0$ and all $n \in \mathbb{Z}_+$.

The Matuszewska β-index vanishes, for example, for any slowly varying sequence.[7] Hence

5.17. Corollary. *For any slowly varying sequence $\pi \in c_0^{\star}$ and $s > 0$, one has* $(\pi)^s = [\mathscr{B}(H), (\pi)^s]$.

[7] Several equivalent definitions of slowly varying sequences are reviewed in Section 1.9 of book [11]; see the references given there, particularly, [12].

5.18. Remark. This corollary has far reaching consequences. With the help of Remark 5.14 we infer that the relative cyclic homology groups $HC_q(\mathscr{B}(H),(\pi))$ vanish in all dimensions if π is slowly varying. This, in turn, implies that the relative algebraic K-groups $K_q((\mathscr{B}(H),(\pi))$ are in all dimensions isomorphic to the topological K-groups $K_q^{\text{top}}(\mathbb{C})$; cf. [70]. Recall that the latter are isomorphic to $\mathbb{Z}$ when q is even, and vanish when q is odd.

All of the above applies, in particular, to any power of the logarithmic integral ideal $\mathscr{L}i$. Recall that the ideal $\mathscr{L}i^{\frac{1}{2}}$ plays an important role in Connes' theory of θ-summable Fredholm modules; cf. [23] and Section IV.8.α of [24].

The following corollary of Theorem 5.11 describes the commutator space $[\mathscr{B}(H),(\pi)]$ for π summable.

5.19. Corollary. *For a summable sequence* $\pi \in c_0^{\star}$, *the following conditions are equivalent*:

(a) $[\mathscr{B}(H),(\pi)] = \{T \in (\pi) \mid \operatorname{Tr} T = 0\}$,
(b) $\pi_{a_\infty} = O(\pi^{\oplus m})$ *for some* $m \in \mathbb{Z}_+$.

By combining Theorems 2.36 and 5.15 we obtain

5.20. Theorem. *For any sequence* $\pi \in c_0^{\star}$, *the following conditions are equivalent*:

(a) *the ideal* (π) *admits a complete symmetric norm*,
(b) $\pi \asymp \pi_a$,
(c) *the Marcinkiewicz ideal* $\mathscr{M}(1/\pi_a)$ *is principal*,
(d) $\mathscr{M}(1/\pi_a) = (\pi)$,
(e) $(\pi) = [\mathscr{B}(H),(\pi)]$,
(f) $\mathscr{M}(1/\pi_a) = [\mathscr{B}(H),\mathscr{M}(1/\pi_a)]$.

Proof. Theorem 2.36 provides implication (a) $\Rightarrow$ (b), while Theorem 5.15 supplies the equivalence of (b) and (e). If $\mathscr{M}(1/\pi_a) = (v)$ for some sequence $v \in c_0^{\star}$, then $v \asymp v_a$ by Theorem 2.36 and condition (f) follows in view of the equivalence of (b) and (e). Finally, in view of Theorem 5.11(ii), condition (f) implies that $(\pi_a)_a = O(\pi_a)$ which is equivalent to condition (b) by Theorem 3.10. □

By combining Theorem 5.11(ii) with Corollary 3.8, we obtain the following result for Lorentz ideals $\mathscr{L}_p(\varphi)$; cf. Section 4.7.

5.21. Theorem. *Let* φ *be a nondecreasing sequence satisfying the* Δ_2*-condition* (22) *and let* $p > 0,\ s > 0$. *Then* $\mathscr{L}_p(\varphi)^s = [\mathscr{B}(H),\mathscr{L}_p(\varphi)^s]$ *if and only if* $\alpha(\varphi) < p/s$.

5.22. Corollary. $(\mathfrak{S}_\omega)^s = [\mathscr{B}(H),(\mathfrak{S}_\omega)^s]$ *for all* $s > 0$.

Indeed, $\mathfrak{S}_\omega = \mathscr{L}(\log)$, cf. 4.10, and $\alpha(\log) = 0$, since the log sequence is slowly varying.

For classical Lorentz ideals $\mathscr{L}_{pq}$ $(0 < p < \infty,\ 0 < q \leqslant \infty)$, cf. Section 4.11, Theorems 5.21 and 5.15 combined together yield

5.23. Corollary. *$\mathscr{L}_{pq} = [\mathscr{B}(H), \mathscr{L}_{pq}]$ if and only if $p > 1$.*

The analogue of Theorem 5.21 for Marcinkiewicz ideals $\mathscr{M}_p(\psi)$, cf. Section 4.7, uses instead Corollary 3.9. In view of the discussion in Section 2.33, one can assume without loss of generality that the sequence ψ is quasiconcave.

5.24. Theorem. *Let ψ be a quasiconcave positive sequence and $p > 0$. Then $\mathscr{M}_p(\psi) = [\mathscr{B}(H), \mathscr{M}_p(\psi)]$ if and only if $\alpha(\psi) < p$.*

Finally, we come to Orlicz ideals; cf. Section 4.7. By combining Theorems 5.11(ii) and 3.21, we obtain

5.25. Theorem. *For any nondecreasing function $M \in [0, \infty)^{[0,\infty)}$ which vanishes at 0 and for any real number $s \in (0, \infty)$, the following conditions are equivalent:*

(a) $(\mathscr{L}_M^{(0)})^s \subseteq [\mathscr{B}(H), (\mathscr{L}_M)^s]$,
(b) $(\mathscr{L}_M^{(0)})^s = [\mathscr{B}(H), (\mathscr{L}_M^{(0)})^s]$,
(c) $(\mathscr{L}_M)^s = [\mathscr{B}(H), (\mathscr{L}_M)^s]$,
(d) *the Matuszewska β-index at zero of M, cf.* (54), *is greater than s:*

$$\beta_0(M) > s,$$

(e) *there exist constants $\delta, \varepsilon, K > 0$ such that*

$$M(tu) \leqslant KM(u)t^{s+\varepsilon} \quad (0 < t, u \leqslant \delta). \tag{115}$$

Theorem 5.11(ii) and Corollary 3.23 combined yield

5.26. Theorem. *For any nondecreasing function $M \in [0, \infty)^{[0,\infty)}$ which vanishes at 0, the following conditions are equivalent:*

(a) $\mathscr{L}_M^{(0)}$ *is e-complete for some $e > 0$,*
(a)′ $(\mathscr{L}_M^{(0)})^s = [\mathscr{B}(H), (\mathscr{L}_M^{(0)})^s]$ *for some $s > 0$,*
(b) $\mathscr{L}_M$ *is e-complete for some $e > 0$,*
(b)′ $(\mathscr{L}_M)^s = [\mathscr{B}(H), (\mathscr{L}_M)^s]$ *for some $s > 0$,*
(c) $\beta_0(M) > 0$.

5.27. Remarks. (1) By constructing a family of positive and continuous (with respect to the Marcinkiewicz norm $T \mapsto \|s(T)\|_{m(1/\pi_a)}$) traces, Dixmier [26] proved that the closure of $[\mathscr{B}(H), \mathscr{M}(\pi_a)]$ has an infinite codimension in $\mathscr{M}(\pi_a)$ whenever $\pi = o(\pi_a)$. All the same, this provided the first examples of ideals $J \nsubseteq \mathscr{L}_1$ with $J \neq [\mathscr{B}(H), J]$.

(2) A complete characterization of sequences π for which the principal ideal (π) and the Marcinkiewicz ideal $\mathscr{M}(1/\pi_a)$ admit a nonzero positive trace was given by Várga ([62, Theorem IRR]) and by one of the authors [71]. It is notable that characterization coincides with our characterization of the condition $J \neq [\mathscr{B}(H), J]$ for these ideals, which is of course the same as the existence of some (not necessarily positive, or continuous) nonzero trace on J. Thus we conclude that the existence of a nonzero trace on any ideal of these two types implies also the existence of a trace which is positive (and, simultaneously, continuous with respect to the Marcinkiewicz norm $T \mapsto ||s(T)||_{m(1/\pi_a)}$).

We emphasize, however, that no such implication holds in general. For example, while there may be many traces on an ideal of the form $J\mathscr{K}$, where $\mathscr{K}$ denotes the ideal of compact operators, no positive trace exists on $J\mathscr{K}$ if $J \nsubseteq \mathscr{L}_1$. An example: the ideal $\{T \in \mathscr{K} \mid s(T) = o(\omega)\} = (\omega)\mathscr{K}$.

In the same article [62], Várga also proved that the ideal (π) supports no positive trace precisely when (π) is a complete normed ideal. This result is contained in our Theorem 5.20.

(3) Equality $\mathscr{L}_p = [\mathscr{L}_{p/2}, \mathscr{L}_{p/2}]$ $(p > 1)$ was proved by Pearcy and Topping [49]; equality $\{T \in \mathscr{L}_p \mid \operatorname{Tr} T = 0\} = [\mathscr{B}(H), \mathscr{L}_p]$ $(p < 1)$—by Anderson [4]. The fact that $[\mathscr{B}(H), \mathscr{L}_1]$ does not coincide with the space $\{T \in \mathscr{L}_1 \mid \operatorname{Tr} T = 0\}$ was originally established in [66–68] (more precisely, it was proven that the largest ideal $I \subseteq \mathscr{L}_1$ such that *all* of its trace-zero operators belong to $[\mathscr{B}(H), \mathscr{L}_1]$ coincides with the Lorentz ideal $\mathscr{L}(\varphi)$ where $\varphi_n = 1 + n \log n$).

(4) Kalton proved [37] (cf. his Corollary 7), a result which is essentially equivalent to our Corollary 5.19.

(5) In another article [38], Kalton gave the first ever description of the commutator space $[\mathscr{B}(H), \mathscr{L}_1]$. He characterized membership in $[\mathscr{B}(H), \mathscr{L}_1]$ *in purely spectral terms*:

$$[\mathscr{B}(H), \mathscr{L}_1] = \{T \in \mathscr{L}_1 \mid \lambda(T)_a \in \ell_1\}.$$

(Here $\lambda(T)$ denotes the sequence of eigenvalues of T, which may be finite or empty, each value counted according to the dimension of the generalized eigenspace $\bigcup_{n=1}^{\infty} \operatorname{Ker}(T - \lambda I)^n$ and listed in order of decreasing absolute value).

The natural question is, of course: *for which ideals J does membership in $[\mathscr{B}(H), J]$ depend solely on the spectrum of an operator?* This question achieved a partial resolution in two articles [28,39] which depend on some of the main results of the present article.

6. The minimum number of commutators

The following result can be obtained independently of Theorem 5.6 and with considerably less effort.

6.1. Theorem. *For any ideals I and J in* $\mathscr{B}(H)$:

$$[I, J] = [I, J]_2 + [\mathscr{B}(H), IJ]_1. \tag{116}$$

In particular,

$$[\mathscr{B}(H), J] = [\mathscr{B}(H), J]_3. \tag{117}$$

The proof is based on three observations.

6.2. Lemma. *For any unital ring R, one has the following identity holding in the matrix ring* $M_n(R)$:

$$\begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_n \end{pmatrix} = \left[\begin{pmatrix} 0 & \sigma_1 & & & \\ & 0 & \sigma_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & \sigma_{n-1} \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \right]$$

where $\sigma_j \coloneqq r_1 + \cdots + r_j$, *provided* $\sigma_n = 0$.

6.3. Lemma. *For any ideals I and J in* $\mathscr{B}(H)$, *every element* $T \in IJ$ *is a single product* $T = RS$ *for some* $R \in I$ *and* $S \in J$.

Proof. Suppose $T = \sum_{i=1}^m R_i S_i$ for $R_i \in I$ and $S_i \in J$. Choose an isomorphism $\Phi : H \xrightarrow{\sim} H^{\oplus m}$. Then the composite map

$$H \xrightarrow{\Delta} H^{\oplus m} \xrightarrow{\begin{pmatrix} S_1 & & \\ & \ddots & \\ & & S_m \end{pmatrix}} H^{\oplus m} \xrightarrow{\Phi^{-1}} H$$

is the desired S while R is the composite map

$$H \xrightarrow{\Phi} H^{\oplus m} \xrightarrow{\begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_m \end{pmatrix}} H^{\oplus m} \xrightarrow{\Delta^*} H,$$

where $\Delta(v) = \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} \in H^{\oplus m}$ and $\Delta^* \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = v_1 + \cdots + v_m$. $\square$

6.4. Lemma. *Suppose $T \in [I, J]$ has an infinite dimensional reducing null subspace H_0, i.e., $H = H_0 \oplus H_1$ with $T_{|H_0} = 0$ and $T(H_1) \subseteq H_1$. Then*

$$T = [R, S] + [N, X] \tag{118}$$

for some $R \in I,\ S \in J,\ X \in IJ$ and $N \in \mathscr{B}(H)$ where $N^2 = X^2 = 0$ if $I = \mathscr{B}(H)$ and $N^3 = X^3 = 0$ in general. In particular,

$$T \in [I, J]_1 + [\mathscr{B}(H), IJ]_1.$$

Proof. Suppose $T = \sum_{i=1}^m [R_i, S_i]$. In view of the hypothesis, there exists an isomorphism $\Phi : H \to H^{\oplus m}$ such that

$$\Phi T \Phi^{-1} = \begin{pmatrix} \sum_{i=1}^m [R_i, S_i] & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

in $M_m(\mathscr{B}(H))$. By Lemma 6.2,

$$\Phi T \Phi^{-1} - \begin{pmatrix} [R_1, S_1] & & \\ & \ddots & \\ & & [R_m, S_m] \end{pmatrix}$$
$$= \left[\begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & X_1 & & & \\ & 0 & X_2 & & \\ & & & \ddots & \\ & & & 0 & X_{m-1} \\ & & & & 0 \end{pmatrix} \right],$$

where $X_j = -\sum_{i=j+1}^k [R_i, S_i]$. At the same time,

$$\begin{pmatrix} [R_1, S_1] & & \\ & \ddots & \\ & & [R_m, S_m] \end{pmatrix} = \left[\begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_m \end{pmatrix}, \begin{pmatrix} S_1 & & \\ & \ddots & \\ & & S_m \end{pmatrix} \right].$$

This shows $T = [R, S] + [N, X]$ with $R \in I,\ S \in J,\ X \in IJ,\ N \in \mathscr{B}(H)$ and $N^m = X^m = 0$. However, if $A \in \mathscr{B}(H),\ T' \in I,\ T'' \in J$, then

$$[A, T'T''] = [AT', T''] - [T', T''A],$$

so

$$[\mathscr{B}(H), IJ]_1 \subseteq [I, J]_2. \tag{119}$$

Thus $T \in [I,J]_1 + [\mathscr{B}(H), IJ]_1$ which equals $[\mathscr{B}(H), J]_2$ if $I = \mathscr{B}(H)$ and is contained in $[I,J]_3$ in the general case. By repeating the first part of the proof with $m = 2$ (case $I = \mathscr{B}(H)$) and $m = 3$ (the general case) we obtain a representation of the form (118). $\square$

Choose any isometry V of H onto a subspace VH of infinite dimension and codimension. Suppose $T = T'T''$. Then

$$T = [T'V^*, VT''] + VT''T'V^*$$

and $VT''T'V^*$ has $(VH)^\perp$ as a reducing subspace on which it is zero. Lemmas 6.3 and 6.4 combined then give Theorem 6.1.

Now, combining Theorem 6.1 with inclusion (119) yields

6.5. Corollary. *For any ideals I and J in* $\mathscr{B}(H)$,

$$[I,J] = [I,J]_4. \tag{120}$$

Another proof of equality (120) has been given in Section 5; cf. Theorem 5.10.

Equalities (116) and (120) provide our best estimates of the necessary number of commutators *which hold in general.* If, however, $|T| \in [I,J]$, then operator T is the sum of fewer commutators, as the following theorem and its corollaries show.

6.6. Theorem. *For any ideals I and J in* $\mathscr{B}(H)$, *if* $|T| \in [I,J]$, *then*

$$T \in [I,J]_1 + [\mathscr{B}(H), IJ]_1. \tag{121}$$

6.7. Corollary. *For any ideal* $I \subseteq [\mathscr{B}(H), J]$, *one has*

$$I \subseteq [\mathscr{B}(H), J]_2. \tag{122}$$

By combining Corollary 6.7 with Theorem 5.11, we obtain

6.8. Corollary. *For any ideals I and J in* $\mathscr{B}(H)$, *the following conditions are equivalent*:

(a) $I \circledast (\omega) \subseteq J$,
(b) $I \subseteq [\mathscr{B}(H), J]_2$.

In particular, $\mathscr{L}_p = [\mathscr{B}(H), \mathscr{L}_p]_2$, for $p > 1$, and yet

$$\mathscr{F} \cap [\mathscr{B}(H), \mathscr{L}_p]_1 = \mathscr{F}^0 := \{T \in \mathscr{F} \mid \operatorname{Tr} T = 0\},$$

for $1 < p \leqslant 2$, as we shall see in the next chapter. This example shows that I may not be contained in $[\mathscr{B}(H), J]_1$, and therefore the double commutator set in (122) cannot be replaced by the single commutator set, in general.

In view of inclusion (119), we also have the following corollary involving three ideals in $\mathscr{B}(H)$.

6.9. Corollary. *For any ideals I, J and L in $\mathscr{B}(H)$, if $I \subseteq [J, L]$ then*

$$I \subseteq [J, L]_3. \qquad (123)$$

Unlike (122), it seems plausible that inclusion (123) can be improved to $I \subseteq [J, L]_2$ which then must be the best possible.

The proof of Theorem 6.6 is based on two lemmas.

6.10. Lemma. *Let n be a nilpotent in a unital ring R which commutes with an invertible element $u \in R$. Then for every ideal $I \subset R$ the map $\phi_{n,u} : I \to I$ given by*

$$\phi_{n,u}(t) = ut + [n, t] \quad (t \in I)$$

is bijective.

Proof. Let $m \in \mathbb{Z}_+$ be such that $n^m = 0$. Denote by ρ, L, and R the endomorphisms of the additive group of I defined by left multiplication by u, left multiplication by n and right multiplication by n, respectively. They commute with each other and $\phi_{n,u} = \rho + L - R$. Since $L^m = R^m = 0$ we have $(R - L)^{2m} = 0$ and hence

$$\sum_{i=0}^{2m-1} \rho^{-i-1}(R - L)^i$$

is the required inverse map $\phi_{n,u}^{-1}$. $\square$

6.11. Lemma. *Let I and J be ideals in $\mathscr{B}(H)$ and let $T \in IJ$. Suppose there is an identification of H with $H^{\oplus k}$ for some $k \in \mathbb{Z}_+$ such that the diagonal entries T_{ii}, $1 \leqslant i \leqslant k$, of $T = (T_{ij})_{1 \leqslant i, j \leqslant k}$ viewed as an element of $M_k(\mathscr{B}(H))$, belong to $[I, J]$ and have infinite dimensional reducing null subspaces. Then $T \in [I, J]_1 + [\mathscr{B}(H), IJ]_1$.*

Proof. From Lemma 6.4, $T_{ii} = [R_i, S_i] + [N, X_i]$ for $R_i \in I$, $S_i \in J$, $X_i \in IJ$ and $N \in \mathscr{B}(H_i)$, where we can take the same N for each i and $N^3 = 0$. Let $X_{ii} = X_i$. Of course, $T_{ij} \in IJ$ for each i,j. By Lemma 6.10 applied to the matrix ring $R = M_k(\mathscr{B}(H))$, for every pair $i \neq j$, there is $X_{ij} \in IJ$ such that $T_{ij} = (j - i)X_{ij} + [N, X_{ij}]$. Letting $X = (X_{ij})_{1 \leqslant i, j \leqslant k}$, we obtain the following commutator representation of T:

$$T = \left[\begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_k \end{pmatrix}, \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_k \end{pmatrix} \right]$$

$$+ \left[\begin{pmatrix} 1+N & & & \\ & 2+N & & \\ & & \ddots & \\ & & & k+N \end{pmatrix}, X \right]. \qquad \square$$

Proof of Theorem 6.6. According to Anderson and Stampfli (cf. [5, Theorem 2, and the proof of Theorem 3]), T is similar, after we choose an identification $H \simeq H^{\oplus 4}$, to an operator $T' \in M_4(\mathscr{B}(H))$, whose diagonal entries T_{ii}', $1 \leqslant i \leqslant 4$, have infinite dimensional reducing null subspaces. Each entry T_{ij}' belongs to the principal ideal (T), which is contained in $[I, J]$ in view of the hypothesis $|T| \in [I, J]$ and Theorem 5.11(i). Thus, Lemma 6.11 applies and inclusion (121) follows. $\square$

7. Single commutators

We open this section by stating the main results first.

7.1. Theorem. *For any compact operator $T \in \mathscr{B}(H)$, one has*

$$T \in [\mathscr{B}(H), (T) \otimes (\omega^{\frac{1}{2}})]_1.$$

This provides a sufficient condition for an operator T to be representable as a commutator $[A, S]$ for some bounded operator A and an operator S from a given ideal J, namely: $s(T) \otimes \omega^{\frac{1}{2}} \in \Sigma(J)$. Thus, we have the following corollary.

7.2. Corollary. *For any ideals $I, J \subseteq \mathscr{B}(H)$, if $I \otimes (\omega^{\frac{1}{2}}) \subseteq J$ then $I \subseteq [\mathscr{B}(H), J]_1$.*

In particular:

the single commutator space $[\mathscr{B}(H), J]_1$ contains
all finite rank operators provided $\omega^{\frac{1}{2}} \in \Sigma(J)$. (124)

Note that the space of finite rank operators of trace zero coincides with $[\mathscr{B}(H), \mathscr{F}]_1$. This follows readily from Shōda's Theorem which says that any matrix of trace zero is similar to a matrix having zeros on the diagonal [59]. Condition (124) is also nearly necessary as is demonstrated by our next result.

(It is *precisely* necessary when ideal J is *am*-closed; cf. Section 4.3. See Corollaries 7.10 and 7.11.)

7.3. Theorem. *For any operators $A \in \mathscr{B}(H)$ and $Q \in \mathscr{K}$, if their commutator $[A, Q]$ has finite rank and nonzero trace then $\omega^{\frac{1}{2}} = O(s(Q) \otimes \omega)$.*

Equivalently, if $[I, J]_1$ contains a finite rank operator whose trace is nonzero then $\omega^{\frac{1}{2}}$ belongs to the am-closure $\Sigma(I)^-$, defined in Section 2.8, of the characteristic set of I:

$$\omega^{\frac{1}{2}} \in \Sigma(I)^-,$$

and the same holds for J.

This last statement, of course, follows from the fact that $(\omega^{\frac{1}{2}})_a \sim 2\omega^{\frac{1}{2}}$; cf. Proposition 3.18 above.

Proof of Theorem 7.1. The standard rank one projection

$$P = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}$$

admits the commutator representation $P = [A, Q]$ in terms of block tri-diagonal matrices

$$A = \begin{pmatrix} 0 & A_1^+ & & \\ A_1^- & 0 & A_2^+ & \\ & A_2^- & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & Q_1^+ & & \\ Q_1^- & 0 & Q_2^+ & \\ & Q_2^- & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

where A_n^+ and Q_n^+ are the $n \times (n+1)$ matrices

$$A_n^+ = \frac{1}{n}\begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & n \end{pmatrix} \quad \text{and} \quad Q_n^+ = -\frac{1}{n}\begin{pmatrix} 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

while A_n^- and Q_n^- are the $(n+1)\times n$ matrices

$$A_n^- = \frac{1}{n+1}\begin{pmatrix} n & & & \\ 0 & n-1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \quad \text{and} \quad Q_n^- = \frac{1}{n+1}\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & 0 \\ & & & 1 \end{pmatrix}.$$

7.4. Lemma. *One has* $(A) = \mathscr{B}(H)$ *and* $(Q) = (\omega^{\frac{1}{2}})$.

Proof of Lemma 7.4. Operator Q is the sum of two operators Q^+ and Q^- of the form $U^\pm \mathrm{Diag}(q^\pm)V^\pm$ for suitable sequences $q^\pm \in c_0$ and partial isometries $U^\pm, V^\pm$. Thus

$$q^+ = -(1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \dots) \sim -\tfrac{1}{\sqrt{2}}\omega^{\frac{1}{2}}$$

and, similarly, $q^- \sim \frac{1}{\sqrt{2}}\omega^{\frac{1}{2}}$. This shows that $s(Q) = O(\omega^{\frac{1}{2}})$.

Let Γ_n be the set of the basis vectors of H on which Q_n^+ does not vanish, and let E be the projection onto the Hilbert subspace of H generated by the set

$$\bigcup_{n \equiv 1,2 \pmod 4} \Gamma_n.$$

The sequence of singular numbers of EQE majorizes the sequence

$$(1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{5}(5 \ \textit{times}), \tfrac{1}{6}(6 \text{ times}), \tfrac{1}{9}(9 \ \textit{times}), \dots) \sim \tfrac{1}{2}\omega^{\frac{1}{2}}$$

Therefore $\omega^{\frac{1}{2}} = O(s(Q))$.

Operator A is the sum of two operators A^+ and A^-, each having norm 1. It is clear that A is not compact. $\square$

Lemma 7.4 combined with the identity $T\otimes P = [\mathrm{id}_H \otimes A, T \otimes Q]$ proves the assertion of Theorem 7.1 for operators having an infinite dimensional reducing subspace on which they vanish.

In the general case, we recall from the proof of Theorem 6.6 that any compact operator T is similar to an operator $T' \in M_4((T))$, whose diagonal entries T'_{ii} have infinite dimensional reducing null subspaces.

Note that

$$(s(T) \circledast \omega^{\frac{1}{2}})_a \overset{\text{Prop.3.14}}{\asymp} (s(T) \circledast \omega^{\frac{1}{2}}) \circledast \omega \overset{\text{Prop.3.18}}{\asymp} s(T) \circledast \omega^{\frac{1}{2}}.$$

Hence $(T) \circledast (\omega^{\frac{1}{2}}) = (s(T) \circledast \omega^{\frac{1}{2}})$ coincides with the Marcinkiewicz ideal

$$\mathcal{M} = \mathcal{M}(\frac{1}{s(T) \circledast \omega^{\frac{1}{2}}}),$$

which, we recall, is a Banach ideal; cf. Section 4.7.

Let us split T' into its diagonal and off-diagonal parts: $T' = T'_d + T'_o$. In view of the already proven assertion,

$$T'_d = \begin{pmatrix} [A_1, S_{11}] & & \\ & \ddots & \\ & & [A_4, S_{44}] \end{pmatrix} \tag{125}$$

for suitable $A_i \in \mathcal{B}(H)$ and $S_{ii} \in \mathcal{M}$, $1 \leqslant i \leqslant 4$. We can assume that A_i's have nonoverlapping spectra; otherwise, we replace A_i by $\lambda_i I + A_i$ for appropriately chosen $\lambda_i \in \mathbb{C}$.

To complete the proof of Theorem 7.1 we need the following lemma generalizing Theorem 3.1 of [52].

7.5. Lemma. *Let M be a Banach bimodule over a unital Banach algebra B. If $\mathrm{Sp}_B a \cap \mathrm{Sp}_B b = \emptyset$ then the operator $L_a - R_b \in \mathcal{B}(M)$, $m \mapsto am - mb$, is invertible.*

Lemma 7.5 applied to $B = \mathcal{B}(H)$, $a = A_i$, $b = A_j$, $i \neq j$, and $M = \mathcal{M}$ produces the unique $S_{ij} \in \mathcal{M}$ such that $T_{ij}' = A_i S_{ij} - S_{ij} A_j$. By combining this with equality (125), we obtain the single commutator representation:

$$T' = \left[\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_4 \end{pmatrix}, S \right],$$

where $S \in M_4(\mathcal{M})$.

Proof of Lemma 7.5. Let $\mathcal{A} \subset \mathcal{B}(M)$ be the saturated commutative Banach subalgebra containing operators L_a and R_b. Then

$$\mathrm{Sp}_{\mathcal{A}}(L_a) = \mathrm{Sp}_{\mathcal{B}(M)}(L_a) \subseteq \mathrm{Sp}_B(a)$$

and

$$\mathrm{Sp}_{\mathcal{A}}(R_b) = \mathrm{Sp}_{\mathcal{B}(M)}(R_b) \subseteq \mathrm{Sp}_{B^{\mathrm{op}}}(b) = \mathrm{Sp}_B(b).$$

The joint spectrum $\mathrm{Sp}_{\mathcal{A}}(L_a, R_b) \subset \mathbb{C}^2$ is contained in $\mathrm{Sp}_{\mathcal{A}}(L_a) \times \mathrm{Sp}_{\mathcal{A}}(R_b)$. Therefore, by applying the Spectral Mapping Theorem (cf. [13, Chapter I.4.7, Theorem 2]) to the function $f(z_1, z_2) = z_1 - z_2$, we obtain

$$\begin{aligned} \mathrm{Sp}_{\mathscr{B}(M)}(L_a - R_b) &= f(\mathrm{Sp}_{\mathscr{A}}(L_a, R_b)) \subseteq \mathrm{Sp}_{\mathscr{A}}(L_a) - \mathrm{Sp}_{\mathscr{A}}(R_b) \\ &\subseteq \mathrm{Sp}_B(a) - \mathrm{Sp}_B(b) = \{\lambda - \mu \mid \lambda \in \mathrm{Sp}_B(a), \mu \in \mathrm{Sp}_B(b)\} \end{aligned} \tag{126}$$

and the largest set in (126) does not contain 0. □

Note that $I \diamondsuit (\omega^{\frac{1}{2}}) = ((I^2)_a)^{\frac{1}{2}}$ by Corollary 3.17. Therefore Corollary 7.2 can be rephrased, in view of Theorem 5.11, as asserting the implication

$$I^2 \subseteq [\mathscr{B}(H), J^2] \;\Rightarrow\; I \subseteq [\mathscr{B}(H), J]_1. \tag{127}$$

We do not know how to prove this implication directly. Combined with Theorem 5.13, implication (127) results in the following

7.6. Corollary. *If the Boyd index $\alpha(J)$ of an e-complete ideal J, cf. Section* 4.6, *is less than* 1/2 *then* $J = [\mathscr{B}(H), J]_1$.

Similar results for principal, Lorentz, Marcinkiewicz, as well as for Orlicz ideals (if one replaces the Boyd α-index by the Matuszewska β-index at zero), are direct consequences of implication (127) and of Theorems 5.15, 5.21, 5.24 and 5.25, respectively. It would be tedious to formulate all of these results here, so we leave this to the reader.

7.7. Remark. Matrices A and $-Q$ coincide with the limits at $t = 0$ of certain matrices Z_t and C_t, $0 < t < 1$, considered by Anderson in [3]. A small modification of the argument from the proof of Lemma 7.4 shows that

$$(C_t) = (\omega^{t/2}) \quad \text{and} \quad (Z_t) = (\omega^{(1-t)/2}).$$

Combined with the argument from the proof of his Theorem 1 in [3], this leads to the following result.

7.8. Proposition. *Let I and J be proper ideals in $\mathscr{B}(H)$ and suppose $T \in IJ$ has an infinite dimensional reducing null subspace. Then for every* $0 < t < 1$,

$$T \in [L_{t/2}, L_{(1-t)/2}]_1,$$

where $L_s := (I + J) \diamondsuit \omega^s$.

Proof of Theorem 7.3. If $F = [A, Q]$ is an operator of finite rank, then a lemma due to Brown [18] provides a sequence of nonzero mutually orthogonal projections

$P_1, P_2, P_3, \ldots$, satisfying the following conditions:

(a) $F = P_1 F P_1$,
(b) $P_n A P_m = 0 = P_n Q P_m$ for $n > m + 1$,
(c) rank $P_n = O(n)$.

In particular, operators $E_m \coloneqq P_1 + \cdots + P_m$ are projections whose rank r_m grows no faster than lm^2 for some integer l.

As in [18], the identity

$$F = P_1[A, Q]P_1 = (P_1AP_1)(P_1QP_1) - (P_1QP_1)(P_1AP_1) + (P_1AP_2)(P_2QP_1) - (P_1QP_2)(P_2AP_1)$$

combined with further identities

$$\begin{aligned} 0 = P_k[A, Q]P_k = &(P_kAP_{k-1})(P_{k-1}QP_k) - (P_kQP_{k-1})(P_{k-1}AP_k) \\ &+ (P_kAP_k)(P_kQP_k) - (P_kQP_k)(P_kAP_k) \\ &+ (P_kAP_{k+1})(P_{k+1}QP_k) - (P_kQP_{k+1})(P_{k+1}AP_k), \end{aligned}$$

for $k > 1$, leads to the following trace identities:

$$\operatorname{Tr} F = \sum_{k=1}^{n} \operatorname{Tr}(P_k[A, Q]P_k) = \operatorname{Tr}(P_nAP_{n+1}QP_n) - \operatorname{Tr}(P_nQP_{n+1}AP_n) \quad (n \in \mathbb{Z}_+)$$

which, in turn, yield the sequence of inequalities

$$|\operatorname{Tr} F| \leqslant ||A||(||P_{n+1}QP_n||_1 + ||P_nQP_{n+1}||_1) \quad (n \in \mathbb{Z}_+), \tag{128}$$

where $||X||_1$ denotes the nuclear norm $\operatorname{Tr}(|X|)$. At this point, we need the following.

7.9. Lemma. *Let Q be a compact operator and $P_1, P_2, \ldots$ be a sequence of mutually orthogonal projections of finite rank. Then, for any $k \in \mathbb{Z}_+$, there is a partial isometry, $V \in \mathscr{B}(H)$, such that*

$$P_nVQP_n = |P_{n+k}QP_n| \quad (n \in \mathbb{Z}_+).$$

Proof of Lemma 7.9. The polar decomposition yields $P_{n+k}QP_n = V_n|P_{n+k}QP_n|$, where V_n is a partial isometry on H such that $V_n = P_{n+k}V_nP_n$. Let $V \in \mathscr{B}(H)$ be a partial isometry such that

$$P_nVP_m = \begin{cases} V_n^* & \text{if } m = n + k, \\ 0 & \text{otherwise} \end{cases}$$

for all $n, m \in \mathbb{Z}_+$. Then

$$P_n VQP_n = V_n^* P_{n+k} QP_n = V_n^* V_n |P_{n+k} QP_n| = |P_{n+k} QP_n|. \qquad \square$$

Lemma 7.9 supplies partial isometries $V, W \in \mathscr{B}(H)$ such that

$$P_n VQP_n = |P_{n+1} QP_n| \quad \text{and} \quad P_n WQ^* P_n = |P_{n+1} Q^* P_n| \tag{129}$$

for all $n \in \mathbb{Z}_+$. Consider the operator

$$T = \frac{||A||}{|\mathrm{Tr}\, F|}(VQ + WQ^*).$$

Its sequence of singular numbers is dominated by $s(Q)^{\oplus 2}$:

$$s(T)_a = O(s(Q)_a), \tag{130}$$

since $(s(Q)^{\oplus 2})_a \leqslant 2s(Q)_a$.

It follows from (129) that

$$P_n TP_n = \frac{||A||}{|\mathrm{Tr}\, F|}(|P_{n+1} QP_n| + |P_{n+1} Q^* P_n|) \geqslant 0$$

and, from (128), that $\mathrm{Tr}(P_n TP_n) \geqslant 1$. This, combined with Hermann Weyl's Inequality (cf. [69]), gives the estimates

$$\sum_{i=1}^{lm^2} s_i(T) \geqslant \sum_{i=1}^{r_m} s_i(E_m TE_m) \geqslant |\mathrm{Tr}(E_m TE_m)| = \sum_{n=1}^{m} \mathrm{Tr}(P_n TP_n) \geqslant m,$$

where E_m are the projections defined above, whence we deduce the inequality

$$\sum_{i=1}^{n} s_i(T) \geqslant [\sqrt{n/l}] \quad (n \in \mathbb{Z}). \tag{131}$$

By comparing inequality (131) with (130), we conclude that

$$\omega^{\frac{1}{2}} = O(s(Q)_a),$$

or, equivalently (cf. Proposition 3.14), that $\omega^{\frac{1}{2}} = O(s(Q) \diamondtimes \omega)$. This completes the proof of Theorem 7.3.

The following corollary combines Theorem 7.3 with Corollary 7.2 (note that $\mathscr{L}_2 \diamondtimes (\omega^{\frac{1}{2}}) = (\omega^{\frac{1}{2}})$).

7.10. Corollary. *For any am-closed ideal J, cf. Section* 4.3, *the following conditions are equivalent*:

(a) $\mathscr{F} \cap [\mathscr{B}(H), J]_1$ *contains an operator whose trace is not zero*,
(b) $\mathscr{F} \subseteq [\mathscr{B}(H), J]_1$,
(c) $\mathscr{L}_2 \subseteq [\mathscr{B}(H), J]_1$,
(d) $(\omega^{\frac{1}{2}}) \subseteq J$.

The class of *am*-closed ideals includes Lorentz ideals $\mathscr{L}_p(\varphi)$ $(p \geqslant 1,\ \varphi$ concave), Marcinkiewicz ideals $\mathscr{M}_p(\psi)$ $(p \geqslant 1)$, and Orlicz ideals $\mathscr{L}_M$ and $\mathscr{L}_M^{(0)}$ (M convex); cf. Section 4.7. Recall from Section 4.9 that all the symmetrically normed ideals $\mathfrak{S}_\Phi$ are *am*-closed.

Thus we have

7.11. Corollary. *For the ideal $J = \mathfrak{S}_\Phi$, where Φ is any symmetric norming function, each of the conditions* (a)–(d) *above is equivalent to the following condition*:

(e) $\Phi(1, 1/\sqrt{2}, 1/\sqrt{3}, \ldots) < \infty$.

In particular, for Schatten ideals $\mathscr{L}_p$ and for Lorentz ideals $\mathscr{L}_{pq}$, these are equivalent to the inequality $p > 2$.

7.12. Remarks. (1) The last assertion complements a theorem of Brown [18], who proved that $\mathscr{F} \cap [\mathscr{L}_p, \mathscr{L}_q]_1$ contains no operator whose trace *is not* zero if $\frac{1}{p} + \frac{1}{q} \geqslant 2$. The converse was earlier proved by Anderson [3].

(2) We do not know whether any of the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) holds without the hypothesis of *am*-closedness. Neither do we know whether the reverse of implication (127) holds, in general.

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Index