Some surfaces

February 27, 2003

Many remarkable surfaces in $\mathbb{R}^3$ arise as *zero sets*, i.e. as the level sets

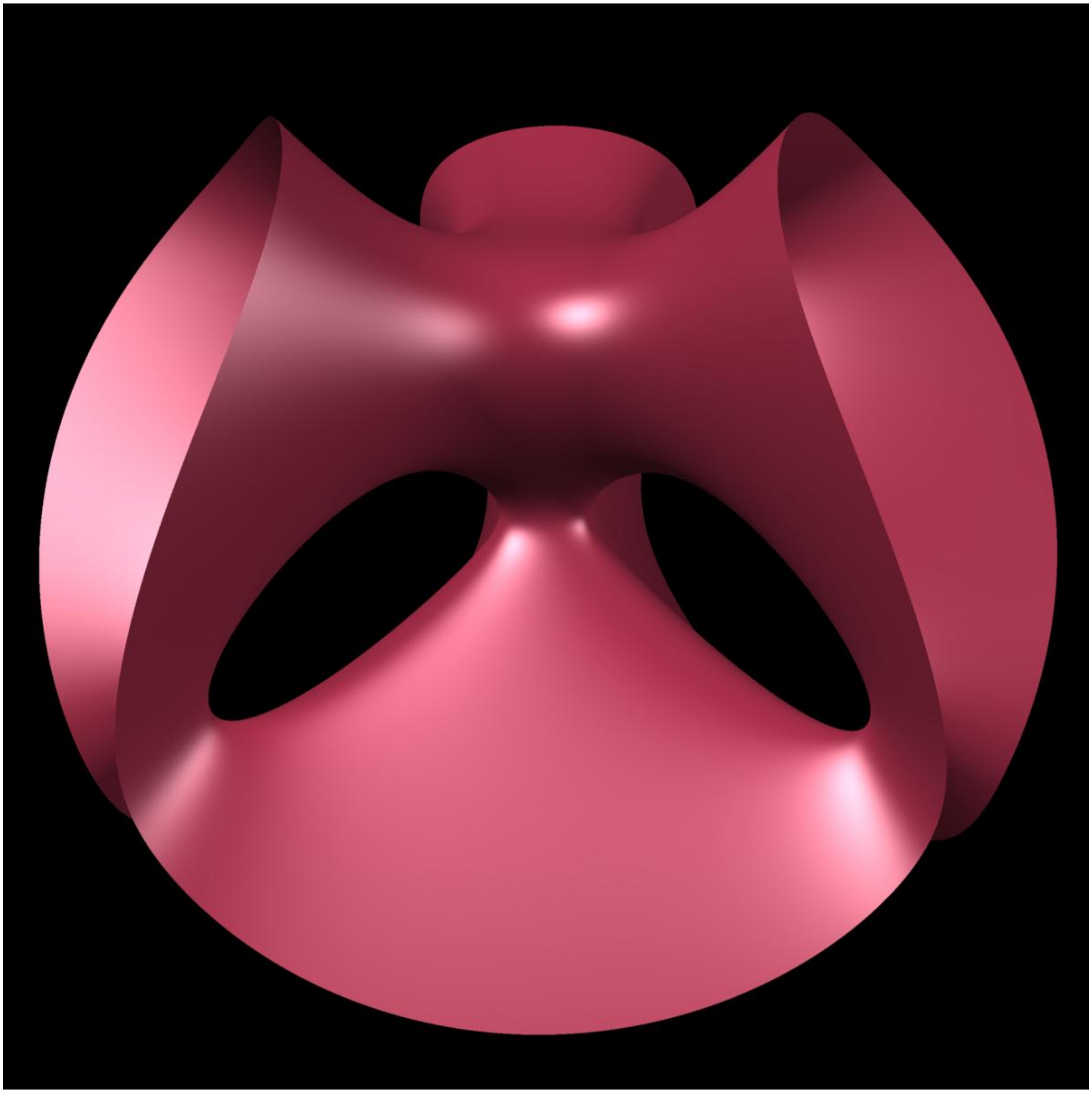
$$\{\mathbf{x} \in \mathbb{R}^3 \mid f(\mathbf{x}) = 0\},$$

for suitable functions $f\colon \mathbb{R}^3 \to \mathbb{R}$. Here are some examples.[1]

1. The **Clebsch diagonal surface** is a smooth cubic surface admitting the symmetry of the tetrahedron. The Clebsch[2] surface contains exactly 27 straight lines. They fall into 9 triplets; each triplet results from intersecting the Clebsch surface with one of 9 special planes.

[1] We shall use the notation $\mathbf{x} = (x, y, z)$.

[2] Rudolf Friedrich Alfred Clebsch (1833–1872)



It is the zero set of the function:

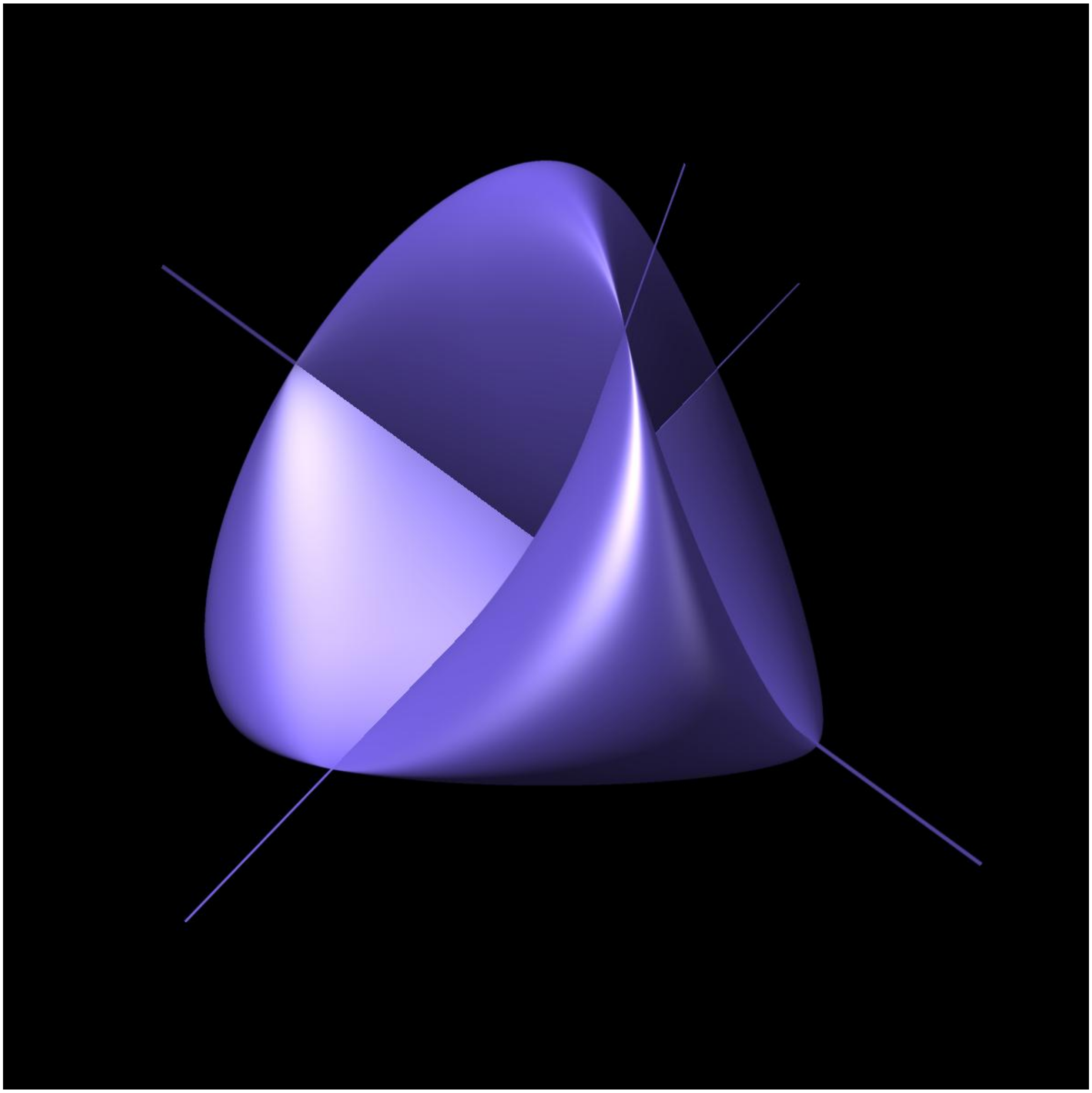
$$f_1(x,y,z) = p^3 + q^3 + r^3 - s^3 - (p+q+r-s)^3 \tag{1}$$

where

$$p = 1 - z - \sqrt{2}x, \qquad q = 1 - z + \sqrt{2}x, \qquad r = 1 + z + \sqrt{2}x, \qquad s = 1 + z - \sqrt{2}x\,.$$

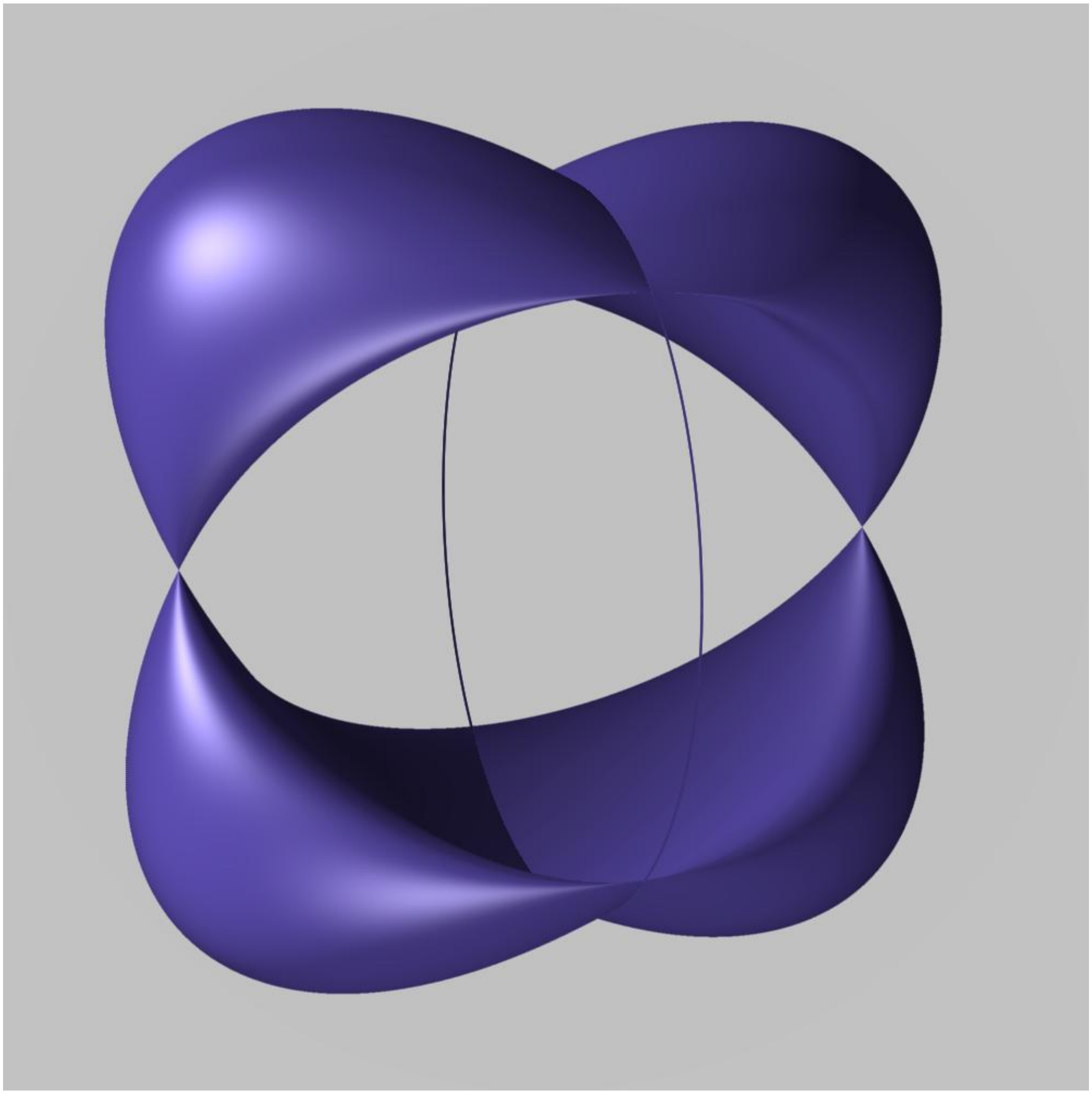
2. The **Steiner Roman surface** is an immersion of the **projective plane** $\mathbb{P}^2(\mathbb{R})$ into $\mathbb{R}^3$. The zero set of the function:

$$f_2(x, y, z) = x^2y^2 + x^2z^2 + y^2z^2 - 17xyz \tag{2}$$



is the union of the Steiner[3] surface and three orthogonal lines.

3. The **cyclide** is a quartic surface with singular conic.

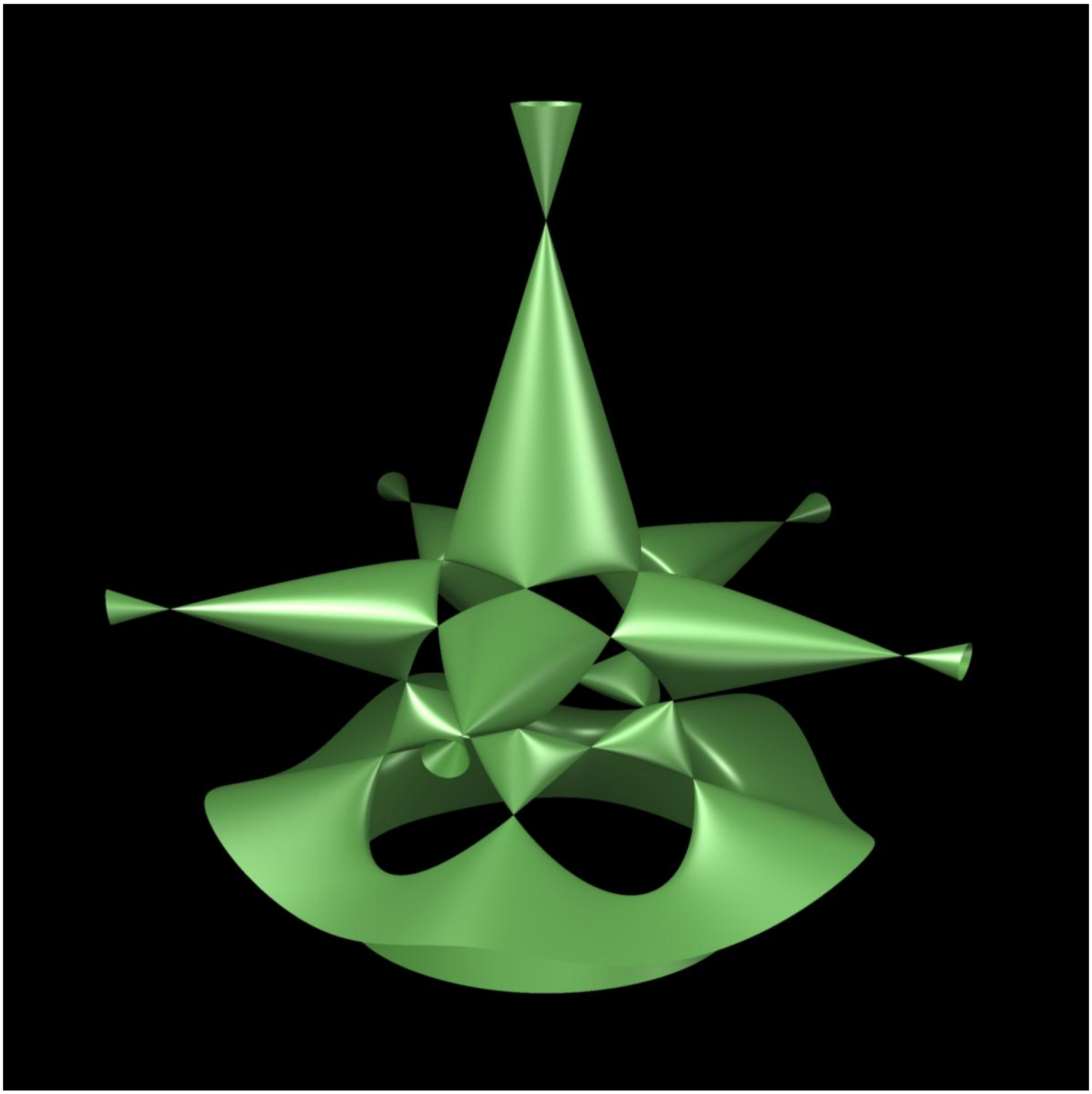


[3] Jakob Steiner (1796–1863)

It is the zero set of the function:

$$f_3(x, y, z) = (x^2 + y^2 - 1)^2 + (3x^2 + y^2 + z^2 - 2)z^2 \,. \tag{3}$$

4. A quintic surface can have at most 31 double points (such a surface is called a **Togliatti surface**).



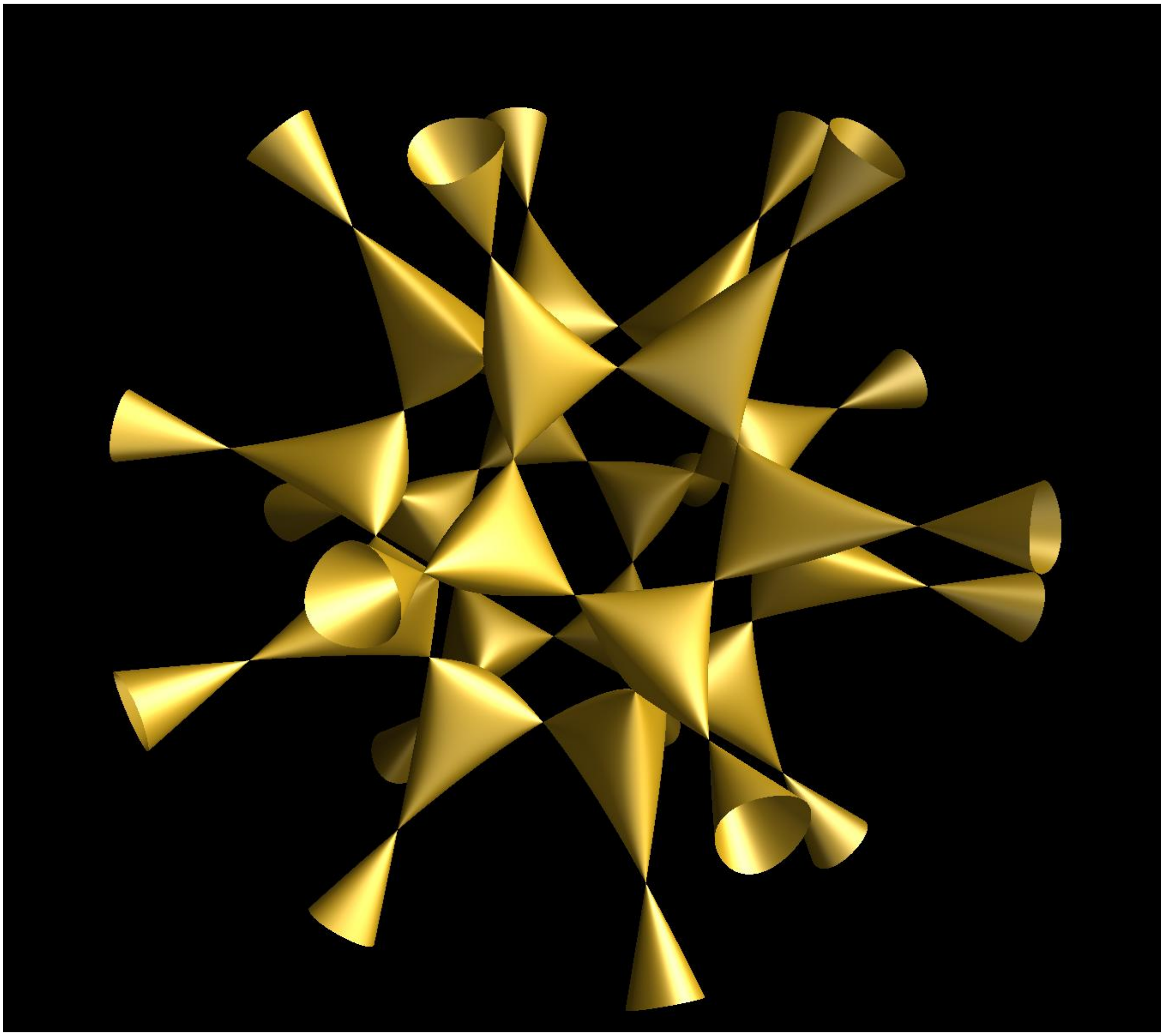
An example is provided by the zero set of the following function:

$$f_4(x,y,z) = -\frac{8}{5}\left(1+\frac{1}{\sqrt{5}}\right)\sqrt{5-\sqrt{5}}\,(x-z)p_2p_4p_6p_8 + \left(1-\frac{\sqrt{5-\sqrt{5}}}{2}z\right)\left(x^2+y^2-1+\frac{1+3\sqrt{5}}{4}z^2\right)^2 \tag{4}$$

where

$$p_n := \cos\frac{n\pi x}{5} - \sin\frac{n\pi y}{5} - z\,.$$

5. The **Barth sextic** is a surface of degree 6 with 65 double points and symmetry of the dodecahedron.



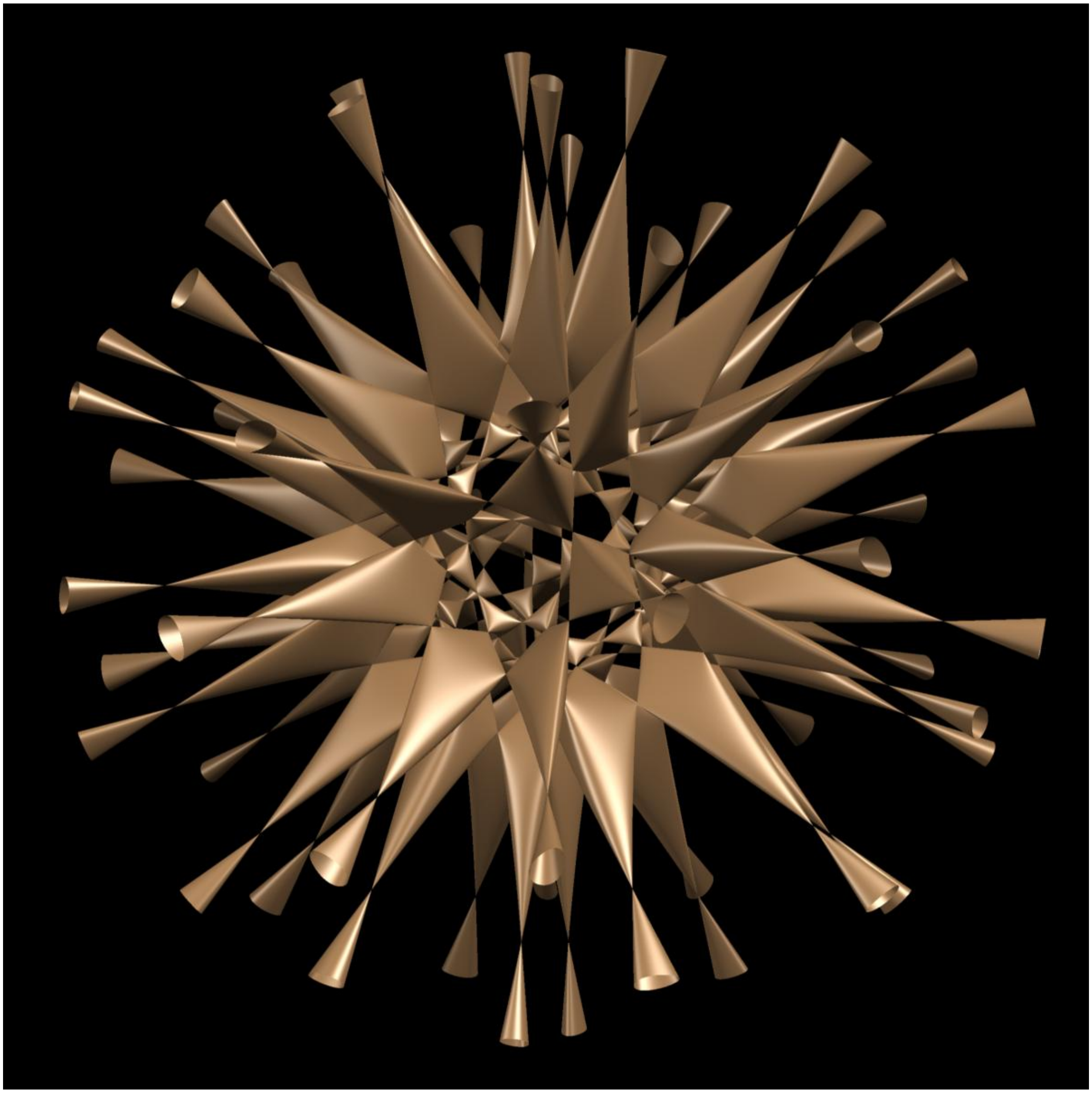
It is the zero set of the function:

$$f_5(x,y,z) = 4g(x,y)g(y,z)g(z,x) - (2+\sqrt{5})\,(x^2+y^2+z^2-1)^2 \tag{5}$$

where

$$g(s,t) := \frac{3+\sqrt{5}}{2}s^2 - t^2\,.$$

6. The **Barth decimic** is a surface of degree 10 with 345 double points.



It is the zero set of the function:

$$f_6(x,y,z) = 8h(x,y)h(y,z)h(z,x)(x^4+y^4+z^4-2x^2y^2-2y^2z^2-2z^2x^2) + \frac{11+5\sqrt{5}}{2}(x^2+y^2+z^2-1)^2(x^2+y^2+z^2-\frac{3-\sqrt{5}}{2})^2 \quad (6)$$

where

$$h(s,t) := s^2 - \frac{7+3\sqrt{5}}{2}t^2 .$$