# Line Integrals

October 23, 2003

*These notes should be studied in conjunction with lectures.*[1]

**1 Path integrals** Let $\gamma\colon [a, b] \to \mathbb{R}^m$ be a path contained in a subset $D \subseteq \mathbb{R}^m$ and let

$$\varphi\colon D \times \mathbb{R}^m \to \mathbb{R} \tag{1}$$

be a function of two variables: a point $\mathbf{x} \in D$ and a column-vector $\mathbf{v} \in \mathbb{R}^m$.

We shall define the integral $\int_\gamma \varphi$ as the limit

$$\int_\gamma \varphi := \lim_{|\mathscr{P}| \to 0} \sum_{j=1}^{k} \varphi(\gamma(t_j^*); \gamma(t_j) - \gamma(t_{j-1})) \tag{2}$$

where the limit is taken over all *tagged* partitions $\mathscr{P}$ of interval $[a, b]$:

$$a = t_0 < t_1 < \cdots < t_k = b \qquad (t_j^* \in [t_{j-1}, t_j]) \tag{3}$$

while the *mesh* of the partition

$$|\mathscr{P}| := \max(|t_1 - t_0|, \ldots, |t_k - t_{k-1}|) \tag{4}$$

tends to zero. Limit (2), when exists, is called the integral of $\varphi$ along path $\gamma$. ☜

**2 An alternative approach** For functions whose arguments are vectors *anchored* at points of D,

$$\phi\colon \{\overrightarrow{\mathbf{ab}} \mid \mathbf{a} \in D\} \to \mathbb{R}, \tag{5}$$

the definition of integral $\int_\gamma \phi$ is more natural:

[1] Abbreviation DCVF stands for *Differential Calculus of Vector Functions.*

$$\int_\gamma \phi \;:=\; \lim_{|\mathscr{P}|\to 0} \sum_{j=1}^{k} \phi(\overrightarrow{\mathbf{a}_{j-1}\mathbf{a}_j}) \tag{6}$$

where the limit in (6) is taken over all (not tagged) partitions of interval $[a, b]$:

$$a = t_0 < t_1 < \cdots < t_k = b\,, \tag{7}$$

and $\mathbf{a}_j := \boldsymbol{\gamma}(t_j)$. Isn't definition (6) also simpler than (2)?

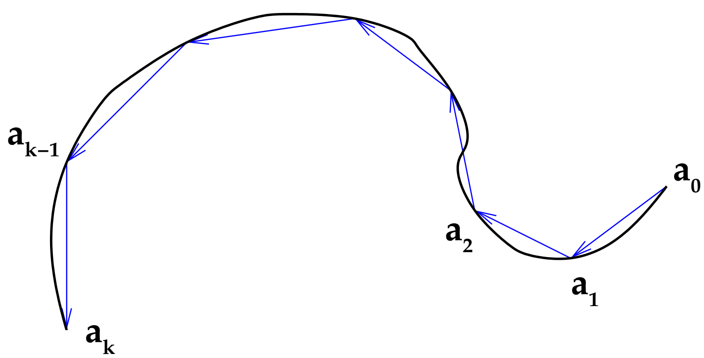



Figure 1: Polygonal approximation of a path $\boldsymbol{\gamma}\colon [a, b] \to \mathbb{R}^m$ associated with a partition, (7), of parameter interval $[a, b]$.

The link between these two definitions reflects, as usual, the connection between *anchored* vectors and *column-vectors*. Recall from Section 9 of **Prelim** that the set of vectors anchored at points $\mathbf{a} \in D$,

$$\{\overrightarrow{\mathbf{ab}} \mid \mathbf{a} \in D\}\,, \tag{8}$$

is naturally identified with the set of ordered pairs

$$D \times \mathbb{R}^m = \{(\mathbf{a}, \mathbf{v}) \mid \mathbf{a} \in D \text{ and } \mathbf{v} \in \mathbb{R}^m\} \tag{9}$$

via correspondnce (23) in **Prelim**. This observation allows us to treat functions (5) as functions $D \times \mathbb{R}^m \to \mathbb{R}$:

$$(\mathbf{a}, \mathbf{v}) \;\mapsto\; \phi(\overrightarrow{\mathbf{ab}}) \qquad \text{where} \qquad \mathbf{b} := \mathbf{a} + \mathbf{v}\,, \tag{10}$$

and *vice-versa*, we are allowed to treat functions (1) as functions of the type (5):

$$\overrightarrow{\mathbf{ab}} \;\mapsto\; \varphi(\mathbf{a}; \mathbf{b} - \mathbf{a})\,. \tag{11}$$

Having these identifications in mind, one now sees that definition of path integral (6) corresponds to definition (2), if one tags each partition $\mathscr{P}$ at the *left ends* of subintervals $[t_{j-1}, t_j]$:

$$t_j^* := t_{j-1} \tag{12}$$

for all $0 \leqslant j \leqslant k$.

**3 Basic properties** Two fundamental properties of path integral follow directly from its definition:

*additivity with respect to integrand*

$$\int_\gamma (\varphi + \psi) = \int_\gamma \varphi + \int_\gamma \psi \tag{13}$$

and

*additivity with respect to path*

$$\int_{\gamma_1 \sqcup \gamma_2} \varphi = \int_{\gamma_1} \varphi + \int_{\gamma_2} \varphi\,. \tag{14}$$

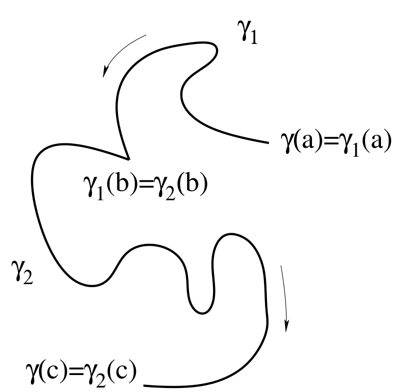
Here $\boldsymbol{\gamma_1}$ is a path $[a, b] \to \mathbb{R}^m$, $\boldsymbol{\gamma_2}$ is a path $[b, c] \to \mathbb{R}^m$ and the endpoint of $\boldsymbol{\gamma_1}$ is supposed to coincide with the beginning of $\boldsymbol{\gamma_2}$:

$$\boldsymbol{\gamma_1}(b) = \boldsymbol{\gamma_2}(b)\,. \tag{15}$$

Such paths can be **concatenated** to form the single path $\boldsymbol{\gamma} = \boldsymbol{\gamma_1} \sqcup \boldsymbol{\gamma_2}$,

$$(\boldsymbol{\gamma_1} \sqcup \boldsymbol{\gamma_2})(t) := \begin{cases} \boldsymbol{\gamma_1}(t) & \text{if } t \in [a, b] \\ \boldsymbol{\gamma_2}(t) & \text{if } t \in [b, c] \end{cases}, \tag{16}$$

as illustrated by the following picture:




(17)

**4 Path length** The function $\lambda : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$:

$$\lambda(\mathbf{a}; \mathbf{v}) := \|\mathbf{v}\| \qquad \text{(the norm of } \mathbf{v}\text{)} \tag{18}$$

corresponds to the function associating with a vector $\overrightarrow{\mathbf{ab}}$ its length $\|\mathbf{b} - \mathbf{a}\|$. In particular, (18) does not depend on a point $\mathbf{a}$; it depends only on $\mathbf{v}$. For any path $\boldsymbol{\gamma}$, the integral

$$\text{Length}(\boldsymbol{\gamma}) := \int_{\boldsymbol{\gamma}} \lambda \tag{19}$$

exists in the sense that it is either finite:

$$\text{Length}(\boldsymbol{\gamma}) < \ ,$$

in this case we say that $\boldsymbol{\gamma}$ is a **rectifiable** path, or

$$\text{Length}(\boldsymbol{\gamma}) = \ ,$$

in which case we say that path $\boldsymbol{\gamma}$ is **nonrectifiable**.

This follows from the fact that $\int_{\boldsymbol{\gamma}} \lambda$ is the limit of lengths

$$\sum_{j=1}^{k} \left\| \overrightarrow{\mathbf{a}_{j-1}\mathbf{a}_j} \right\| \tag{20}$$

of polygonal approximations to path $\boldsymbol{\gamma}$, see Figure 1, and quantity (20) can only increase when we pass to a finer approximation.

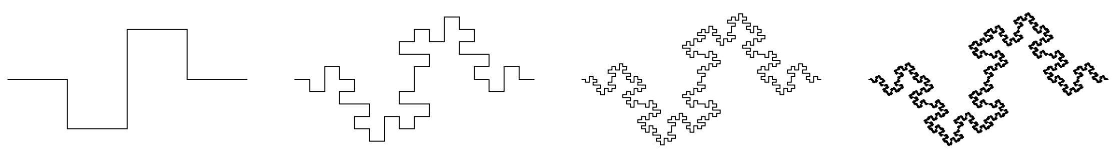


Figure 2: *A simple example of a nonrectifiable path:* the path in question is the limit of rectangular paths whose lengths are $2d$, $4d$, $8d$, $16d$, $\dots$, $2^n d$, $\dots$, where $d$ is the distance between the endpoints.

We shall say that a path

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{pmatrix} : [a, b] \to \mathbb{R}^m$$

is **nondecreasing** if all of its component functions $\gamma_j : [a, b] \to \mathbb{R}$ are nondecreasing.

**5 Theorem** Any nondecreasing path is rectifiable.

Indeed, for such a path, one has an obvious inequality

$$\text{Length}(\gamma) \leqslant \sum_{i=1}^{m} (\gamma_i(b) - \gamma_i(a)) < \ . \tag{21}$$

**Exercise 1** *Explain how to get inequality (21).*

**Exercise 2** *Show that:*

*(a)* $\textit{Length}(\gamma_1 + \gamma_2) \leqslant \textit{Length}(\gamma_1) + \textit{Length}(\gamma_2)$,

*(b)* $\textit{Length}(c\gamma) = |c|\,\textit{Length}(\gamma)$.

It follows from the above exercise that a linear combination of rectifiable paths is rectifiable. In particular, the difference of two *nondecreasing* paths

$$\gamma = \gamma_1 - \gamma_2 \tag{22}$$

is rectifiable.[2] That the reverse is true is a remarkable theorem discovered by French mathematician Marie Ennemond Camille Jordan (1838–1922).

**6 Jordan's Theorem**[3] A path $\gamma$ is rectifiable *if and only if* it can be represented as difference (22) of two nondecreasing paths.

7 Even everywhere differentiable paths need not be rectifiable, but general continuous paths can be truly astounding. A theorem due to Polish mathematician Stefan Mazurkiewicz (1888–1945) and Austrian Hans Hahn (1879–1934) says that

> Any subset $S \subseteq \mathbb{R}^m$ which is **connected, locally connected**[4]**, closed,**[5] and **bounded,**[6] is necessarily a **continuous** image of interval $[0, 1]$.

[2] We define $\gamma_1 - \gamma_2$ by $(\gamma_1 - \gamma_2)(t) := \gamma_1(t) - \gamma_1(t)$.

[3] *Course d'analyse de l'École Polytechnique,* 3 vols, Paris, 1882.

[4] A set is **locally connected** if every point in it has an arbitrarily small connected neighborhood.

[5] A set is **closed** if it contains all its accumulation points.

[6] A set is **bounded** if it it is contained in some ball.

The first example of such a path is the famous **Peano curve,**[7] i.e. a continuous function from $[0, 1]$ *onto* the unit square in the plane. You can learn more about it by visiting the following web sites:

go to: `http://www.math.ohio-state.edu/~fiedorow/math655/Peano.html`
go to: `http://www.cut-the-knot.com/do_you_know/hilbert.shtml`
go to: `http://mmc.et.tudelft.nl/~frits/peanogrow.html`
go to: `http://www.csua.berkeley.edu/~raytrace/java/peano/peano.html`
go to: `http://www-math.uni-paderborn.de/~fazekas/course/peano.html`
go to: `http://www.geom.umn.edu/~dpvc/CVM/1998/01/vsfcf/article/sect8/peano.html`

8 If $f\colon D \to \mathbb{R}$ is a function on $D$ then the **integral** of $f$ **along** path $\boldsymbol{\gamma}$ is defined as the integral

$$\int_{\gamma} f\lambda \tag{23}$$

where $(f\lambda)(\mathbf{x}; \mathbf{v}) = f(\mathbf{x})\|\mathbf{v}\|$. In Section 24 we shall find a method to calculate such integrals.

9 **Differential forms** Among all functions (1) those which are *linear* with respect to the column-vector variable:

$$\boldsymbol{\varphi}(\mathbf{x}; a\mathbf{v} + b\mathbf{w}) = a\boldsymbol{\varphi}(\mathbf{x}; \mathbf{v}) + b\boldsymbol{\varphi}(\mathbf{x}; \mathbf{w}) \qquad (a, b \in \mathbb{R};\ \mathbf{v}, \mathbf{w} \in \mathbb{R}^m) \tag{24}$$

play a particularly important role. They are called **differential forms**[8] on set $D \subseteq \mathbb{R}^m$.

**Exercise 3** *Let* $f\colon D \to \mathbb{R}$ *be a function and* $\boldsymbol{\varphi}\colon D \times \mathbb{R}^m \to \mathbb{R}$ *a differential form. Verify that* $f\boldsymbol{\varphi}$ *is a differential form.*

For any differentiable function $f\colon D \to \mathbb{R}$, its differential:

$$df : D \times \mathbb{R}^m \to \mathbb{R}, \qquad df(\mathbf{x}; \mathbf{v}) := (f'(\mathbf{x}))(\mathbf{v}) \tag{25}$$

(cf. DCVF, p. 17) is a differential form on $D$.

[7] Discovered in 1890 by Italian mathematician Giuseppe Peano (1858–1932).

[8] Or, more precisely, **differential** $1$**-forms**, since we are going to encounter later also $0$-forms, see Section 34, $2$-forms and $3$-forms.

It is customary to denote by $dx_i$ the differential of the $i$-th coordinate function $\pi_i\colon \mathbb{R}^m \to \mathbb{R}$:

$$\pi_i\left(\left(\begin{array}{c} v_1 \\ \vdots \\ v_m \end{array}\right)\right) := v_i \tag{26}$$

Forms $dx_1, \ldots, dx_m$ are often called **basic** differential forms. One reason why are they important is the following fact. ☜

**10 Proposition** Every differential form $\varphi$ on $D \subseteq \mathbb{R}^m$ can be expressed as

$$\boxed{\varphi = f_1 dx_1 + \cdots + f_m dx_m} \tag{27}$$

for unique functions $f_1, \ldots, f_m$ on $D$.

*Indeed, for any vector* $\mathbf{v} \in \mathbb{R}^m$, *one has*

$$\mathbf{v} = \left(\begin{array}{c} v_1 \\ \vdots \\ v_m \end{array}\right) = v_1\mathbf{e_1} + \cdots + v_m\mathbf{e_m}$$

*and hence*

$$\begin{aligned} \varphi(\mathbf{x};\mathbf{v}) &= \varphi(\mathbf{x};\mathbf{e_1})v_1 + \cdots + \varphi(\mathbf{x};\mathbf{e_m})v_m \\ &= \varphi(\mathbf{x};\mathbf{e_1})\, dx_1(\mathbf{x};\mathbf{v}) + \cdots + \varphi(\mathbf{x};\mathbf{e_m})\, dx_m(\mathbf{x};\mathbf{v})\,. \end{aligned} \tag{28}$$

*Thus, if we introduce the functions*

$$f_i(\mathbf{x}) := \varphi(\mathbf{x};\mathbf{e_i})\,, \tag{29}$$

*then identity* (28) *reads*

$$\varphi = f_1 dx_1 + \cdots + f_m dx_m$$

*as desired. To show the uniqueness of representation* (27)*, note that*

$$dx_j(\mathbf{x};\mathbf{e_i}) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}\,. \tag{30}$$

*Hence,*

$$(f_1 dx_1 + \cdots + f_m dx_m)(\mathbf{x};\mathbf{e_i}) = f_i(\mathbf{x})$$

*which shows that the coefficient functions* $f_i$ ***must*** *be given by formula* (29).

11 A differential form is said to be **constant** if its coefficient functions $f_1, \ldots, f_m$ are constant.

12 Identity (27) can be rewritten in abbreviated form as

$$\varphi = \mathbf{F} \cdot dx \tag{31}$$

where $\mathbf{F}\colon D \to \mathbb{R}^m$ is a function[9] whose components are functions $f_1, \ldots, f_m$ :

$$\mathbf{F} := \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

and $dx\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is the *vector valued* form:

$$dx := \begin{pmatrix} dx_1 \\ \vdots \\ dx_m \end{pmatrix}. \tag{32}$$

**Exercise 4** *What is* $dx(\mathbf{x}; \mathbf{v})$ *equal to?*

**Exercise 5** *What is* $\mathbf{F}$ *equal to when* $\varphi = df$ *is the differential of a function* $f\colon D \to \mathbb{R}$ *?*

13 When College Multivariable Calculus textbooks[10] talk about *integrating a vector field* $\mathbf{F}$ along a path $\gamma$ what is meant by that is the integral

$$\int_\gamma \mathbf{F} \cdot dx = \int_\gamma (f_1 dx_1 + \cdots + f_m dx_m).$$

**14 Riemann Integral as a special case of path integral** You should have recognized by now that the definite integral

$$\int_a^b f(t)\, dt$$

[9] In College textbooks of Multivariable calculus such a function is often called a *vector field* on set $D \subseteq \mathbb{R}^m$.

[10] Or, oldfashioned textbooks of Physics.

from Freshman Calculus is the integral of $f\,dt$, considered as a differential form on interval $D = [a, b]$, along the path in $\mathbb{R}$ which traverses interval $[a, b]$ with constant velocity $1$. We shall denote this path by $\iota$:[11]

$$\iota : [a, b] \to \mathbb{R}, \qquad \iota(t) = t. \tag{33}$$

**15 Tangent map** Suppose that a differentiable function $f\colon D \to \mathbb{R}^n$ is given. We can use $\mathbf{f}$ to "transport" any pair $(\mathbf{x}; \mathbf{v}) \in D \times \mathbb{R}^m$ to a pair in $\mathbf{f}(D) \times \mathbb{R}^n$ :

$$(\mathbf{x}; \mathbf{v}) \;\mapsto\; Tf(\mathbf{x}; \mathbf{v}) := (\mathbf{f}(\mathbf{x}); \mathbf{f}'_{\mathbf{x}}(\mathbf{v})) \quad . \tag{34}$$

Correspondence (34) is very important. It is often denoted $Tf$ and called the **tangentt map** of $\mathbf{f}$. ☜

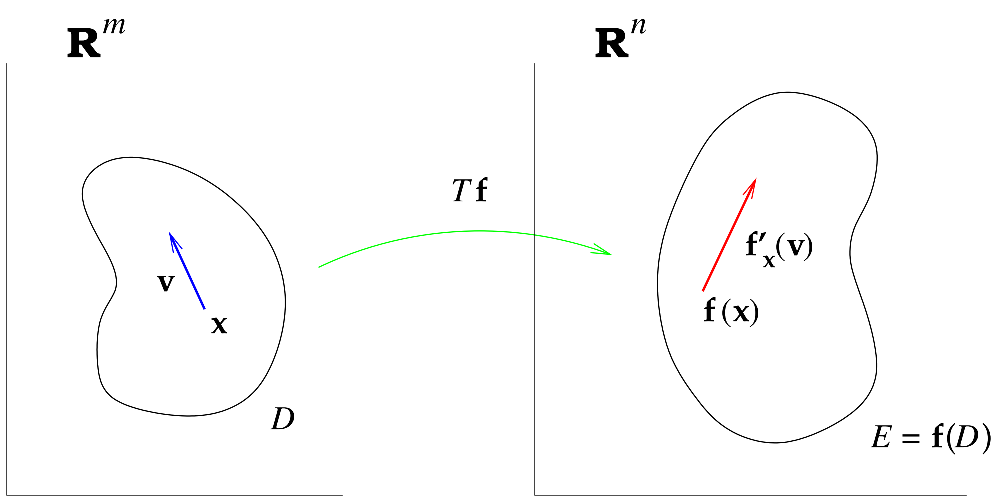



Figure 3: Tangent map $Tf$ sends vector $\mathbf{v}$ anchored at point $\mathbf{x} \in D$ to vector $\mathbf{f}'_{\mathbf{x}}(\mathbf{v})$ anchored at point $\mathbf{f}(\mathbf{x}) \in E$ where $E = \mathbf{f}(D)$ denotes the image of $D$ under $\mathbf{f}$.

[11] Greek letter *iota*.

**16 Pullback** Denote by $E = \mathbf{f}(D)$ the subset of $\mathbb{R}^n$ which is the image of $\mathbf{f}$. Given a function $\varphi\colon E \times \mathbb{R}^n \to \mathbb{R}$, we can define a new function[12] $\chi : D \times \mathbb{R}^m \to \mathbb{R}$ by the formula

$$\chi(\mathbf{x};\mathbf{v}) := (\varphi \circ T\mathbf{f})(\mathbf{x};\mathbf{v}) = \varphi(\mathbf{f}(\mathbf{x}); \mathbf{f}'_{\mathbf{x}}(\mathbf{v})) \quad . \tag{35}$$

**Exercise 6** *Verify that $\chi$ is a differential form if $\varphi$ is one.*

Function $\chi$ is denoted $\mathbf{f}^*\varphi$ and called the **pullback** of $\varphi$ by function $\mathbf{f}$ .

**Exercise 7** *Verify the following properties of pullback:*

*(a)* $\mathbf{f}^*(\varphi_1 + \varphi_2) = \mathbf{f}^*\varphi_1 + \mathbf{f}^*\varphi_2$ *and* $\mathbf{f}^*(\varphi_1\varphi_2) = (\mathbf{f}^*\varphi_1)(\mathbf{f}^*\varphi_2)$;

*(b)* $\mathbf{f}^*\varphi = \varphi \circ \mathbf{f}$ *if* $\varphi$ *does not depend on the column-vector variable (i.e., if* $\varphi$ *is simply a function* $E \to \mathbb{R}$*);*

*(c)* $(\mathbf{f} \circ \mathbf{g})^*\varphi = \mathbf{g}^*(\mathbf{f}^*\varphi)$ . *(Hint: Use the Chain Rule.)*

**17 Calculating pullbacks** 1) On $\mathbb{R}$ we have constant differential form $dt$. If $f\colon D \to \mathbb{R}$ is a differentiable function then the calculation

$$(f^*(dt))(\mathbf{x};\mathbf{v}) = dt(f(\mathbf{x}); f'_{\mathbf{x}}(\mathbf{v})) = f'_{\mathbf{x}}(\mathbf{v}) = df(\mathbf{x};\mathbf{v})$$

demonstrates that

$$f^*(dt) = df\,, \tag{36}$$

i.e., the differential of $f$ is the pullback, by $f$, of the standard differential form $dt$ on $\mathbb{R}$.

2) For a function $g\colon E \to \mathbb{R}$ and a vector-function $\mathbf{f}\colon D \to \mathbb{R}^n$ whose image is contained in set $E$, one has

$$(\mathbf{f}^*dg)(\mathbf{x};\mathbf{v}) = dg(\mathbf{f}(\mathbf{x})); \mathbf{f}'_{\mathbf{x}}(\mathbf{v})) = g'_{\mathbf{f}(\mathbf{x})}(\mathbf{f}'_{\mathbf{x}}(\mathbf{v})) = (g'_{\mathbf{f}(\mathbf{x})} \circ \mathbf{f}'_{\mathbf{x}})(\mathbf{v})$$

which, by the Chain Rule, equals

$$(g \circ \mathbf{f})'_{\mathbf{x}}(\mathbf{v}) = d(g \circ \mathbf{f})(\mathbf{x};\mathbf{v})\,.$$

[12] Greek letter *khi*.

In other words,

$$f^* dg = d(g \circ f)\,. \tag{37}$$

Since $f_i = \pi_i \circ \mathbf{f}$ and $dx_i = d\pi_i$ , formula (37) yields the identity

$$\mathbf{f}^* dx_i = \mathbf{f}^* d\pi_i = (\pi_i \circ \mathbf{f})^* dt = df_i\,. \tag{38}$$

3) Let $\boldsymbol{\varphi} = f_1 dx_1 + \cdots + f_m dx_m$ be a differential form on $D \subseteq \mathbb{R}^m$, and $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{R}^m$ be a path contained in $D$ . In view of formula (38) we have

$$\begin{aligned}
\boldsymbol{\gamma}^*(f_1 dx_1 + \cdots + f_m dx_m) &= (f_1 \circ \boldsymbol{\gamma})\,(\boldsymbol{\gamma}^* dx_1) + \cdots + (f_m \circ \boldsymbol{\gamma})\,(\boldsymbol{\gamma}^* dx_m) \\
&= (f_1 \circ \boldsymbol{\gamma})\, d\gamma_1 + \cdots + (f_m \circ \boldsymbol{\gamma})\, d\gamma_m \\
&= \sum_{i=1}^{m} (f_i \circ \boldsymbol{\gamma})\, \frac{d\gamma_i}{dt} dt = \left( (\mathbf{F} \circ \boldsymbol{\gamma}) \cdot \frac{d\boldsymbol{\gamma}}{dt} \right) dt
\end{aligned} \tag{39}$$

4) Let $x_1, \ldots, x_m$ be coordinates in $\mathbb{R}^m$ and $y_1, \ldots, y_n$ be coordinates in $\mathbb{R}^n$. For a general vector-function $\mathbf{f}\colon D \to \mathbb{R}^n$, one has

$$\begin{aligned}
\mathbf{f}^*(g_1 dy_1 + \cdots + g_n dy_n) &= (g_1 \circ \mathbf{f})\, \mathbf{f}^* dy_1 + \cdots + (g_n \circ \mathbf{f})\, \mathbf{f}^* dy_n \\
&= (g_1 \circ \mathbf{f})\, df_1 + \cdots + (g_n \circ \mathbf{f})\, df_n = (\mathbf{G} \circ \mathbf{f}) \cdot \mathbf{df}
\end{aligned} \tag{40}$$

where

$$\mathbf{G} := \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$$

and

$$\mathbf{df} := \begin{pmatrix} df_1 \\ \vdots \\ df_m \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} dx_1 + \cdots + \dfrac{\partial f_1}{\partial x_m} dx_m \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial f_m}{\partial x_1} dx_1 + \cdots + \dfrac{\partial f_n}{\partial x_m} dx_m \end{pmatrix} = J_{\mathbf{f}}\, \mathbf{dx}\,. \tag{41}$$

Formula (40) can be rewritten as follows:

$$\begin{aligned} f^*(g_1 dy_1 + \cdots + g_n dy_n) &= \sum_{\substack{1\leqslant i\leqslant n \\ 1\leqslant j\leqslant m}} (g_i \circ \mathbf{f}) \frac{\partial f_i}{\partial x_j} dx_j \\ &= (g_1 \circ \mathbf{f} \ \ldots \ g_n \circ \mathbf{f}) J_f \begin{pmatrix} dx_1 \\ \vdots \\ dx_m \end{pmatrix} . \end{aligned} \tag{42}$$

**18 Example: Polar Coordinates** Polar coordinates of a point $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$ are a pair of numbers $(r, \theta)$ such that

$$a_1 = r\cos\theta \qquad \text{and} \qquad a_2 = r\sin\theta\,. \tag{43}$$

From equalities (43) it follows that $|r| = \|\mathbf{a}\|$. Such a pair *is not* unique: if $(r, \theta)$ are polar coordinates of $\mathbf{a}$ then also $((-1)^n r, \theta + n\pi)$ are polar coordinates of $\mathbf{a}$ for any integer $n$. For $\mathbf{a} \neq \mathbf{0}$, there is a unique choice of polar coordinates such that $r > 0$ and $0 \leqslant \theta < 2\pi$. In practice, it is convenient to allow other choices of polar coordinates.

Some plane curves have simple equations in polar coordinates while their equations in Cartesian coordinates are much more complicated, e.g., the *cardioid*, see Figure **??** in **Problembook**.

Let $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function that describes the change from polar to Cartesian coordinates:

$$\mathbf{f}\left(\begin{pmatrix} r \\ \theta \end{pmatrix}\right) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} . \tag{44}$$

Note that the components of a variable column-vector belonging to the domain of $\mathbf{f}$ are *denoted* $r$ and $\theta$, while the components of a column-vector belonging to the target of $\mathbf{f}$ will be denoted $x$ and $y$. This is a natural thing to do, since function $\mathbf{f}$ expresses Cartesian coordinates of a point in $\mathbb{R}^2$ in terms of polar coordinates of the same point.

According to formula (38), we have

$$f^* dx = d(r\cos\theta) = \cos\theta\, dr - r\sin\theta\, d\theta \tag{45}$$

and

$$f^* dy = d(r\sin\theta) = \sin\theta\, dr + r\cos\theta\, d\theta\,. \tag{46}$$

**19 Change of Variables Formula** For a differentiable function $\mathbf{f}\colon D \to \mathbb{R}^n$, a path $\boldsymbol{\gamma}$ contained in $D$, and a differential form $\boldsymbol{\varphi}$ on a set $E \subseteq \mathbb{R}^n$ containing path $\mathbf{f} \circ \boldsymbol{\gamma}$, the following integrals are equal:

$$\boxed{\int_{\mathbf{f}\circ\boldsymbol{\gamma}} \boldsymbol{\varphi} = \int_{\boldsymbol{\gamma}} \mathbf{f}^* \boldsymbol{\varphi}} \quad . \tag{47}$$

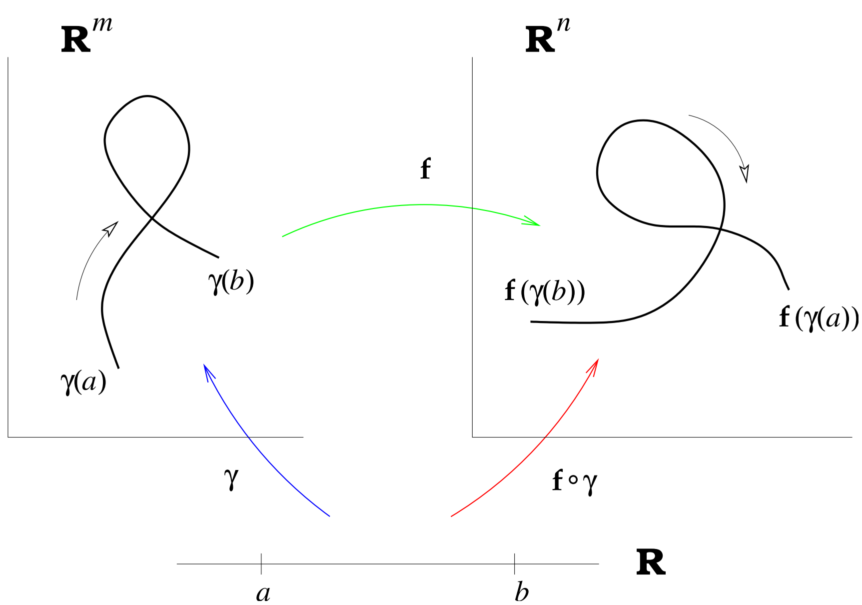



Figure 4: A path in $\mathbb{R}^m$ mapped into $\mathbb{R}^n$ by a function $\mathbf{f}$.

20 Change of Variables Formula has several fundamental applications. For example, a path $\boldsymbol{\gamma}\colon [a, b] \to \mathbb{R}^m$ can be represented as composition $\boldsymbol{\gamma} \circ \iota$ where $\iota : [a, b] \to \mathbb{R}$ is the path introduced in (33). Thus, for any differential form $\boldsymbol{\varphi}$ on a set $D \subseteq \mathbb{R}^m$ containing path $\boldsymbol{\gamma}$, we have

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\varphi} = \int_{\boldsymbol{\gamma}\circ\iota} \boldsymbol{\varphi} = \int_{\iota} \boldsymbol{\gamma}^* \boldsymbol{\varphi} = \int_{\iota} \boldsymbol{\varphi}\left(\boldsymbol{\gamma}(t); \frac{d\boldsymbol{\gamma}}{dt}\right) dt = \int_a^b \boldsymbol{\varphi}\left(\boldsymbol{\gamma}(t); \frac{d\boldsymbol{\gamma}}{dt}\right) dt \,. \tag{48}$$

since, as we noted in Section 14, $\int_\iota f dt$ coincides with familiar Riemann[13] integral $\int_a^b f(t)dt$.

With help of formula (39), we can rewrite this also as follows:

$$\int_\gamma (f_1 dx_1 + \cdots + f_m dx_m) = \int_a^b \left( \sum_{i=1}^m f_i(\gamma(t)) \frac{d\gamma_i}{dt} \right) dt = \int_a^b \left( (\mathbf{F} \circ \gamma) \cdot \frac{d\gamma}{dt} \right) dt \tag{49}$$

The above formula is valid when functions $f_i \circ \gamma$ and $\frac{d\gamma_i}{dt}$ are Riemann integrable on interval $[a, b]$.

Formula (49) reduces calculation of path integrals to Riemann integrals of Freshman Calculus. At the moment, it is your practically only tool for calculation of path integrals.

**21 Example** Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ be two points in $\mathbb{R}^2$ and $\gamma$ be the natural, constant-speed, parametrization of the straight-line-segment connecting $\mathbf{b}$ to $\mathbf{a}$:

$$\gamma(t) = (1-t)\mathbf{a} + t\mathbf{b}. \tag{50}$$

Integral of the differential form $\varphi = x dy$ along path $\gamma$ is calculated with help of formula (49) as follows:

$$\begin{aligned} \int_\gamma x dy &= \int_0^1 ((1-t)a_1 + tb_1) \frac{d((1-t)a_2 + tb_2)}{dt} dt = \int_0^1 ((1-t)a_1 + tb_1)(b_2 - a_2) dt \\ &= \left( \frac{t^2}{2}(b_1 - a_1) + ta_1 \right)(b_2 - a_2) \Big|_0^1 = \frac{(b_1 + a_1)(b_2 - a_2)}{2}. \end{aligned} \tag{51}$$

**22 Fundamental Theorem of Calculus for path integrals** The Fundamental Theorem of Calculus for Riemann Integral simply says that

$$\int_\iota df = \int_a^b f'(t)dt = f(b) - f(a). \tag{52}$$

[13] Georg Friedrich Bernhard Riemann (1826–1866)

Combined with (48) and identity (37), it yields a much more general theorem for path integrals

$$\int_{\gamma} df \;=\; \int_{\iota} \gamma^{*} df \;=\; \int_{\iota} d(f \circ \gamma) \;=\; f(\gamma(b)) - f(\gamma(a)) \quad . \tag{53}$$

**23 Path Length Formula** If we define $\|d\mathbf{x}\| : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ by:

$$\|d\mathbf{x}\|(\mathbf{x};\mathbf{v}) := \sqrt{(dx_1(\mathbf{x};\mathbf{v}))^2 + \cdots + (dx_m(\mathbf{x};\mathbf{v}))^2} = \sqrt{v_1^2 + \cdots + v_m^2} = \|\mathbf{v}\|\,, \tag{54}$$

then we see that $\|d\mathbf{x}\|$ coincides with function $\lambda$ introduced in (18). Function $\|d\mathbf{x}\|$ is often called the **line element**.

*In many College textbooks of Multivariable Calculus (e.g., in Stewart) the line element,* $\|d\mathbf{x}\|$*, is denoted by* $|ds|$*. Such a notation may suggest that there exists some function* $s$ *such that* $\|d\mathbf{x}\| = |ds|$*.* ***This is not so*** *except when* $n = 1$.

**Exercise 8** *The line element,* $\|d\mathbf{x}\|$*, is not a differential form. Explain why?*

Let us calculate the pullback of $\|d\mathbf{x}\|$ by a path $\gamma$:

$$\begin{aligned}
\gamma^{*}\|d\mathbf{x}\| &= \sqrt{(d\gamma_1)^2 + \cdots + (d\gamma_m)^2} \\
&= \sqrt{\left(\frac{d\gamma_1}{dt}dt\right)^2 + \cdots + \left(\frac{d\gamma_m}{dt}dt\right)^2} \\
&= \sqrt{\left(\frac{d\gamma_1}{dt}\right)^2 + \cdots + \left(\frac{d\gamma_m}{dt}\right)^2}\;|\,dt\,| \\
&= \left\|\frac{d\gamma}{dt}\right\|\;|\,dt\,|\,.
\end{aligned} \tag{55}$$

Even though $\|dx\|$ is not a differential form, yet, armed with formula (55), one can show in the same manner as (47) that

$$\text{Length}(\gamma) = \int_\gamma \|dx\| = \int_\iota \gamma^* \|dx\| = \int_a^b \left\| \frac{d\gamma}{dt} \right\| \mid dt \mid = \int_a^b \left\| \frac{d\gamma}{dt} \right\| dt \quad . \tag{56}$$

24 If $f\colon D \to \mathbb{R}$ is a function and D contains path $\gamma$ then we also obtain a similar formula for line integral (23):

$$\int_\gamma f\|dx\| = \int_a^b (f \circ \gamma) \left\| \frac{d\gamma}{dt} \right\| dt \quad . \tag{57}$$

25 **Example** A segment of a plane curve given by polar equation $r = f(\theta)$ has a natural parametrization $\gamma\colon [\alpha, \beta] \to \mathbb{R}^2$, $\gamma(\theta) = \begin{pmatrix} f(\theta)\cos\theta \\ f(\theta)\sin\theta \end{pmatrix}$. The length of $\gamma$ can be calculated by recourse to formula (56)

$$\begin{aligned} \text{Length}(\gamma) &= \int_\alpha^\beta \sqrt{((f(\theta)\cos\theta)')^2 + ((f(\theta)\sin\theta)')^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta))^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f(\theta)^2 + f'(\theta)^2}\, d\theta \end{aligned} \tag{58}$$

26 **Path reparametrization** We would like to examine what happens to path integral $\int_\gamma \varphi$ when one *reparametrizes* path $\gamma\colon [a, b] \to \mathbb{R}^m$, or, more precisely, when one replaces $\gamma$ by the path

$$\gamma \circ h\colon [c, d] \to \mathbb{R}^m$$

where $h\colon [c, d] \to [a, b]$ is a **diffeomorphism** of interval $[c, d]$ onto interval $[a, b]$. This means that function h is everywhere differentiable, *one-to-one*, *onto*, and has *no critical* points. The latter means that $dh/dt \neq 0$ everywhere on $[c,d]$. This can happen only when either $dh/dt > 0$ *everywhere* on $[c,d]$, or $dh/dt < 0$ *everywhere* on $[c,d]$.

*The proof of this fact is remarkably simple. Suppose, to the contrary that*

$$\frac{dh}{dt}(\alpha) > 0 \quad \textit{and} \quad \frac{dh}{dt}(\beta) < 0$$

*for some points $\alpha < \beta$ belonging to interval $[c,d]$. Function $h$ being everywhere differentiable is continuous on $[\alpha,\beta]$ and therefore attains its maximum value at some point $\tau \in [\alpha,\beta]$. Now, $h$ is **strictly increasing at the left end**, since $\frac{dh}{dt}(\alpha) > 0$, and is **strictly decreasing at the right end**, since $\frac{dh}{dt}(\beta) < 0$. Thus, none of the two endpoints of interval $[\alpha,\beta]$ is even a local maximum of $h$ on $[\alpha,\beta]$. It follows that $\tau$ is not an endpoint. Thus, $\tau \in (\alpha,\beta)$ and then $\frac{dh}{dt}(\tau) = 0$ by the oft quoted Fermat's Theorem. This contradiction proves our assertion (the case when $\frac{dh}{dt}(\alpha) < 0$ and $\frac{dh}{dt}(\beta) > 0$ is treated similarly by considering a point $\tau'$ where $h$ attains its minimum on $[\alpha,\beta]$).*

If everywhere $dh/dt > 0$, then we say that $h$ is an orientation *preserving* diffeomorphism. If everywhere $dh/dt < 0$, then we say that $h$ is an orientation *reversing* diffeomorphism.

**27 Behavior of path integral with respect to reparametrization** We shall now examine what happens to $\int_{\boldsymbol{\gamma}} (f_1 dx_1 + \cdots + f_m dx_m)$ when we replace path $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{R}^m$ by the reparametrized path $\boldsymbol{\gamma} \circ h\colon [c,d] \to \mathbb{R}^m$ where $h\colon [c,d] \to [a,b]$ is a diffeomorphism.

In view of formula (49), we have

$$\begin{aligned} \int_{\boldsymbol{\gamma}\circ h} f_i dx_i &= \int_c^d f_i(\boldsymbol{\gamma}(h(t))) \frac{d(\gamma_i \circ h)}{dt} dt \\ &= \int_c^d (f_i \circ \boldsymbol{\gamma})(h(t)))\, \frac{d\gamma_i}{du}\bigg|_{u=h(t)} \frac{dh}{dt} dt\,. \end{aligned} \tag{59}$$

Recall now the Change of Variable Formula in Riemann Integral from Freshman Calculus:[14]

$$\int_c^d g(h(t)) \frac{dh}{dt} dt = \begin{cases} \int_a^b g(u)du & \text{if } h(c) = a \text{ and } h(d) = b \\ \int_b^a g(u)du & \text{if } h(c) = b \text{ and } h(d) = c \end{cases}. \tag{60}$$

When $h$ is a diffeomorphism, the first case occurs if everywhere $dh/dt > 0$ and the second if everywhere $dh/dt < 0$.

[14] See, e.g., boxed formula (6) on p. 414 in Stewart, and recall the convention about Riemann integral: $\int_b^a f(t)dt = -\int_a^b f(t)dt$. Looking at Riemann integrals from the more general standpoint of line integrals allows you to see why such a convention makes sense.

By applying formula (60) to the function

$$g(u) := (f_i \circ \boldsymbol{\gamma})(u))\,\frac{d\gamma_i}{du}\,,$$

we infer that the last integral in (59) equals

$$\int_a^b (f_i \circ \boldsymbol{\gamma})(u))\,\frac{d\gamma_i}{du} du = \int_\gamma f_i dx_i \tag{61}$$

if $h$ is **orientation preserving,** and equals

$$-\int_a^b (f_i \circ \boldsymbol{\gamma})(u))\,\frac{d\gamma_i}{du} du = -\int_\gamma f_i dx_i \tag{62}$$

if $h$ is **orientation reversing.** Hence, for any differential form $\boldsymbol{\varphi} = f_1 dx_1 + \cdots + f_m dx_m$, we have the following identity

$$\int_{\gamma \circ h} \boldsymbol{\varphi} = \pm \int_\gamma \boldsymbol{\varphi} \tag{63}$$

with:

**plus** sign: when $dh/dt > 0$ everywhere on $[c, d]$, $(63^+)$

**minus** sign: when $dh/dt < 0$ everywhere on $[c, d]$. $(63^-)$

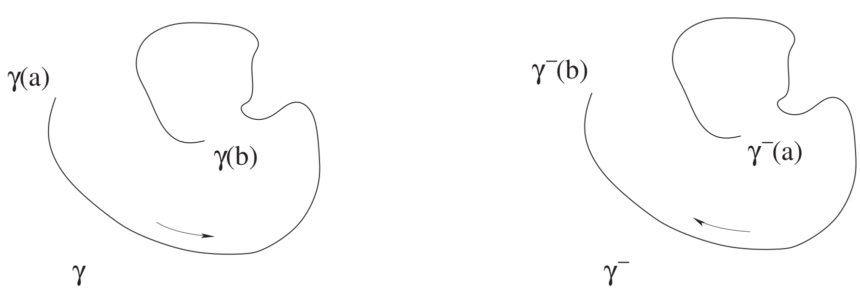
28 **Reverse path** For any path $\boldsymbol{\gamma}\colon [a, b] \to \mathbb{R}^m$, we define the **reverse** path

$$\boldsymbol{\gamma}^- : [a, b] \to \mathbb{R}^m, \qquad \boldsymbol{\gamma}^-(t) := \boldsymbol{\gamma}(a + b - t)\,. \tag{64}$$

Note that $\boldsymbol{\gamma}^-(a) = \boldsymbol{\gamma}(b)$, $\boldsymbol{\gamma}^-(b) = \boldsymbol{\gamma}(a)$ and $\boldsymbol{\gamma}^- = \boldsymbol{\gamma} \circ h$ where $h\colon [a, b] \to [a, b]$ is given by $h(t) = a + b - t$, and the velocity vector of $\boldsymbol{\gamma}^-$ equals the *minus* velocity vector of $\boldsymbol{\gamma}$:

$$\frac{d\boldsymbol{\gamma}^-}{dt}(t) = -\frac{d\boldsymbol{\gamma}}{du}(a + b - t)$$

since $dh/dt = -1$.




(65)

Thus, the second case of identity (63) applies here and we obtain:

$$\int_{\gamma^-} \varphi = -\int_{\gamma} \varphi \,. \tag{66}$$

**29 Equivalent parametrizations** We shall say that two parametrizations $\gamma_1 \colon [c, d] \to \mathbb{R}^m$ and $\gamma_2 \colon [a, b] \to \mathbb{R}^m$ of a curve $C$ are **equivalent** if

$$\gamma_2 = \gamma_1 \circ h$$

for a suitable *orientation preserving diffeomorphism* $h \colon [c, d] \to [a, b]$, cf. condition ($63^+$). From (63) we know that

$$\int_{\gamma_1} = \int_{\gamma_2} \tag{67}$$

for equivalent parametrizations.

**30 Regular arcs** Recall from Section 24 in **DCVF**, that a path $\gamma \colon [a, b] \to \mathbb{R}^m$ is **regular** if function $\gamma$ has no critical points. This is equivalent to saying that the velocity-vector function $\frac{d\gamma}{dt}$ nowhere vanishes, see Section 26, Case $m = 1$, in **DCVF**.

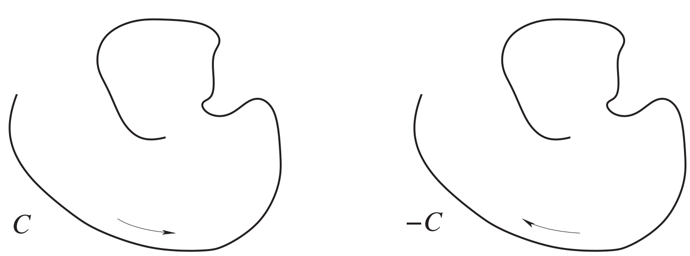
☞ A curve $C$ is a **regular arc** if it admits a one-to-one and onto parametrization by a regular path. One can then demonstrate that, for any two such parametrizations $\gamma_1 \colon [a, b] \to \mathbb{R}^m$ and $\gamma_2 \colon [c, d] \to \mathbb{R}^m$, either

$$\text{the starting points of } \gamma_1 \text{ and } \gamma_2 \text{ coincide and } \gamma_2 \text{ is equivalent to } \gamma_1 \tag{68}$$

or

$$\begin{array}{l}\text{the starting point of } \gamma_2 \text{ is the endpoint of } \gamma_2 \\ \text{and } \gamma_2 \text{ is equivalent to the reverse path, } \gamma_1^- \,.\end{array} \tag{69}$$

Thus, a regular arc $C$ has, up to equivalence, only two kinds of good parametrizations and the choice between the two strictly corresponds to the choice of **orienta-tion** of $C$.




(70)

The latter corresponds to choosing which "end" of $C$ is the starting point and which is the end point. If that choice is made, then one is dealing with an **oriented regular arc** $C$. For such an arc we set

$$\int_C \varphi := \int_\gamma \varphi \tag{71}$$

and

$$\int_{-C} \varphi := \int_{\gamma^-} \varphi \tag{72}$$

where $\gamma$ is *any*, one-to-one and onto, regular parametrization of oriented arc $C$. The main point is that **this definition does not depend on the choice of regular parametrization $\gamma$.**

**31 Oriented chains** The proper setting for line integrals now reveals itself: we

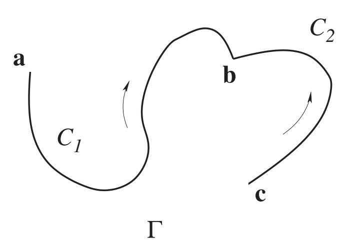
**integrate differential forms over *"objects"* $\Gamma$ which can be *decomposed* into a finite number of regular oriented arcs $C_1, \ldots, C_k$.** (73)

☞ Such objects are called **oriented chains**. Thanks to additivity of path integral, cf. (14), the quantity

$$\int_\Gamma \varphi := \int_{C_1} \varphi + \cdots + \int_{C_k} \varphi \tag{74}$$

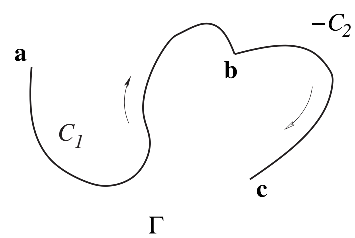
depends only on $\Gamma$ and not on *how* $\Gamma$ is decomposed into oriented arcs $C_1, \ldots, C_k$.

**32 Boundary of an oriented chain** For any chain $\Gamma$, we shall defineits **boundary**, $\partial\Gamma$, as ☜
a *set-with-multiplicities*, i.e., as a set whose elements are "tagged" by integers. As a set, $\partial\Gamma$ consists of all the "ends" of constituent arcs $C_1, \ldots, C_k$. Every such point is counted as many times as it occurs among the ends of arcs $C_1, \ldots, C_k$: each time it occurs as the *end* point we *add* 1, each time it is the *starting* point we *substract* 1. This is how we determine
☞ its **multiplicity**. The following two examples illustrate this definition:




*$\partial\Gamma$ consists of point* **a** *with multiplicity* $-1$*, point* **b** *with multiplicity* 2 *and point* **c** *with multiplicity* $-1$

(75)




*$\partial\Gamma_1$ consists of point* **a** *with multiplicity* $-1$ *and point* **c** *with multiplicity* $1$ *(point* **b** *does not contribute to* $\partial\Gamma$*, since its multiplicity equals* $1-1=0$*)* (76)

Then, the Fundamental Theorem of Calculus for path integrals, see Section 22, combined with additivity of path integral, gives us our final result here

**33 Fundamental Theorem of Calculus for line integrals** For any oriented chain $\Gamma$ contained in the domain of a function $f\colon D \to \mathbb{R}$, one has

$$\boxed{\int_\Gamma df = \sum_{\mathbf{a}\in\partial\Gamma} mult(\mathbf{a})\, f(\mathbf{a})} \tag{77}$$

where $mult(\mathbf{a})$ denotes the *multiplicity* of a point $\mathbf{a} \in D$ with which it is counted in $\partial\Gamma$.

**34** Formula (77) suggests that we should treat ordinary functions $f\colon D \to \mathbb{R}$ as **0-forms.** ☜ Such forms are *naturally* integrated over finite *sets-with-multiplicities*:

$$\int_S f := \sum_{\mathbf{a}\in S} mult(\mathbf{a})\, f(\mathbf{a})\,. \tag{78}$$

Then, the Fundamental Theorem of Calculus for line integrals acquires the following beautiful and compact form:

$$\boxed{\int_\Gamma df = \int_{\partial\Gamma} f}\,. \tag{79}$$

☞ This is the one-dimensional case of **Stokes' Theorem**. Soon, we shall also encounter the two- and three-dimensional cases of this theorem.

**35 Cycles** An oriented chain $\Gamma$ is called a **cycle** if its boundary $\partial\Gamma$ is empty. The Fundamental Theorem of Calculus *for cycles* becomes:

$$\int_\Gamma df = 0 \quad . \tag{80}$$

☞ We say that a differential form $\varphi$ on $D$ is **exact** if $\varphi = df$ for a certain function $f\colon D \to \mathbb{R}$. In this case, function $f$ is called the **primitive** of $\varphi$.[15] It follows from (80) that ☜

$$\int_\Gamma \varphi = 0 \quad \text{for } any \text{ cycle } \Gamma \text{ and } any \text{ exact form } \varphi \quad . \tag{81}$$

From Freshman Calculus we know that any *continuous* differential form $\varphi$ on an interval of real line is exact. Indeed, $\varphi = g\,dt$ for a suitable continuous function $g$ and, therefore, $\varphi = df$ for the following *antiderivative* of $g$:

$$f(x) := \int_a^x g(t)dt\,. \tag{82}$$

The situation is very different, however, in higher dimensions.

**36 Loop integrals and existence of the primitive** A path $\gamma\colon [a,b] \to \mathbb{R}^m$ that ends where it starts:

$$\gamma(a) = \gamma(b) \tag{83}$$

is called a **loop**. It follows from the Fundamental Theorem of Calculus for Path Integrals, (53), that

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)) = 0 \tag{84}$$

for *any* loop $\gamma$, contained in the domain of function $f$, along which differential form $df$ is integrable.

[15] In traditional Physics courses, a function $\mathbf{F}\colon D \to \mathbb{R}^3$, on a subset of $\mathbb{R}^3$, is said to be a **conservative vector field** on $D$ if differential form $\mathbf{F}\cdot d\mathbf{x}$ is *exact*. A function $f\colon D \to \mathbb{R}$ such that $df = \mathbf{F}\cdot d\mathbf{x}$ (i.e., such that $\nabla f = \mathbf{F}$) is then called a **potential** for $\mathbf{F}$. ☜ ☞

Now, let $\boldsymbol{\varphi} = f_1 dx_1 + \cdots + f_m dx_m$ be any differential form on a set $D \subseteq \mathbb{R}^m$ whose coefficient functions $f_1, \ldots, f_m$ are continuous. Such a form is integrable along any rectifiable path in $D$. Suppose that

$$\int_{\gamma} \boldsymbol{\varphi} = 0 \qquad \text{for any rectifiable loop } \boldsymbol{\gamma} \text{ in } D. \tag{85}$$

Pick a point $\mathbf{a} \in D$. Then for any two paths $\boldsymbol{\gamma_0}$ and $\boldsymbol{\gamma_1}$ contained in $D$, which connect a point $\mathbf{x} \in D$ with $\mathbf{a}$, the integrals of $\boldsymbol{\varphi}$ coincide:

$$\int_{\gamma_0} \boldsymbol{\varphi} = \int_{\gamma_1} \boldsymbol{\varphi}\,. \tag{86}$$

Indeed, without loss of generality we can assume that path $\boldsymbol{\gamma_0}$ is parametrized by interval $[0,1]$ while $\boldsymbol{\gamma_1}$ is parametrized by interval $[1,2]$ (we can choose equivalent parametrizations without changing the values of corresponding integrals, see Section 29). Then

$$\int_{\gamma_0} \boldsymbol{\varphi} - \int_{\gamma_1} \boldsymbol{\varphi} = \int_{\gamma_0} \boldsymbol{\varphi} + \int_{\gamma_1^-} \boldsymbol{\varphi} = \int_{\gamma_0 \sqcup \gamma_1^-} \boldsymbol{\varphi} = 0\,. \tag{87}$$

Here $\boldsymbol{\gamma_0} \sqcup \boldsymbol{\gamma_1^-} \colon [0,2] \to D$ is the loop starting and ending at $\mathbf{a}$ which traverses path $\boldsymbol{\gamma_0}$ for $0 \leqslant t \leqslant 1$ and traverses path $\boldsymbol{\gamma_1}$ *in reverse* for $1 \leqslant t \leqslant 2$.

Thus, for differential forms satisfying condition (85), integral $\int_{\gamma} \boldsymbol{\varphi}$ dependends only on the path endpoints. This allows us to introduce the notation

$$\int_{\mathbf{a}}^{\mathbf{x}} \boldsymbol{\varphi} := \int_{\gamma} \boldsymbol{\varphi} \tag{88}$$

where $\boldsymbol{\gamma}$ denotes *any* rectifiable path with starting point $\mathbf{a}$ and endpoint $\mathbf{x}$. (Beware, however, notation (88) makes sense *only* for forms satisfying condition (85).)

Suppose, for a moment, that every point $\mathbf{x} \in D$ can be connected by a (rectifiable) path with point $\mathbf{a}$ (we say, in this case, that set $D$ is **path-connected**). Then, integral (88) defines ☜ a function of point $\mathbf{x} \in D$ whose differential equals $\boldsymbol{\varphi}$ (we omit verification of this fact).

We have established the following important theorem *characterizing* exact differential forms:

> A differential form $\varphi$ on a set $D$ is *exact if and only if* it satisfies condition (85). Its primitive is then given by
>
> $$f(\mathbf{x}) = \int_{\mathbf{a}}^{\mathbf{x}} \varphi \tag{89}$$
>
> where $\mathbf{a}$ is a fixed point of $D$ and $\mathbf{x}$ is any point of $D$ that can be connected with $\mathbf{a}$.

If set $D \subseteq \mathbb{R}^m$ is the disjoint union of path-connected components $D_1, D_2, \dots$, then we pick a 'reference' point $\mathbf{a_i} \in D_i$ in each path-connected component $D_i$. This provides a primitive for $\varphi$ which is defined *everywhere* on $D$:

$$f(\mathbf{x}) = \int_{\mathbf{a_i}}^{\mathbf{x}} \varphi \qquad (\text{for } \mathbf{x} \in D_i). \tag{90}$$

**37 Example** Consider the form

$$\varphi \;=\; \frac{x\,dy - y\,dx}{x^2 + y^2} \tag{91}$$

on the "punctured" plane, i.e., on the Euclidean plane with the origin removed:

$$D \;:=\; \left\{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} \neq \mathbf{0}\right\}. \tag{92}$$

Let $C_r$ be the counterclockwise oriented circle of radius $r$ with center at the origin:

$$C_r \;:=\; \left\{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = r\right\}. \tag{93}$$

Circle $C_r$ can be decomposed into any number of arcs, each naturally parametrized by angle $\theta$:

$$\boldsymbol{\gamma}(\theta) \;:=\; \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \tag{94}$$

By using formula (49), we thus obtain:

$$\int_{C_r} \frac{x\,dy - y\,dx}{x^2 + y^2} = \int_0^{2\pi} \frac{r\cos\theta(r\cos\theta) - r\sin\theta(-r\sin\theta)}{r^2}\,d\theta = \int_0^{2\pi} d\theta = 2\pi \tag{95}$$

Note that the result does not depend on radius $r$: it has the same value $2\pi$ for circles arbitrary small or arbitrary large. In particular, form (91) is not exact.

**38 Appendix: Proof of Formula (49)** It is enough to show that, for any function $f\colon D \to \mathbb{R}$ and integer $1 \leqslant i \leqslant n$, one has

$$\int_\gamma f dx_i = \int_a^b f(\boldsymbol{\gamma}(t)) \frac{d\gamma_i}{dt} dt\,. \tag{96}$$

Integral $\int_\gamma f dx_i$ is the limit, when that limit exists, of finite sums

$$\sum_{j=1}^{k} (f dx_i)(\boldsymbol{\gamma}(t_j^*);\, \boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})) \;=\; \sum_{j=1}^{k} f(\boldsymbol{\gamma}(t_j^*))\,(\gamma_i(t_j) - \gamma_i(t_{j-1})) \tag{97}$$

that we associate with *tagged* partitions $\mathscr{P}$ of interval $[a, b]$, see (3). This means that numbers (97) approach $\int_\gamma f dx_i$ when the *mesh* of $\mathscr{P}$, see (4), tends to zero.

Let us make a crucial assumption:

$$\textit{we assume that functions } \gamma_i\colon [a, b] \to \mathbb{R} \textit{ are differentiable}\,. \tag{98}$$

Then, by Mean-Value Theorem of Freshman Calculus,[16]

$$\gamma_i(t_j) - \gamma_i(t_{j-1}) \;=\; \frac{d\gamma_i}{dt}(t_j^*)\,(t_j - t_{j-1}) \tag{99}$$

for some point $t_j^* \in [t_{j-1}, t_j]$. Let us use these points $t_j^*$ to *tag* a given partition

$$a = t_0 < t_1 < \cdots < t_k = b\,. \tag{100}$$

Then

$$\sum_{j=1}^{k} f(\boldsymbol{\gamma}(t_j^*))\,(\gamma_i(t_j) - \gamma_i(t_{j-1})) \;=\; \sum_{j=1}^{k} f(\boldsymbol{\gamma}(t_j^*))\,\frac{d\gamma_i}{dt}(t_j^*)\,(t_j - t_{j-1})\,. \tag{101}$$

The sums on the right-hand-side of (101) converge to the Riemann integral

$$\int_a^b f(\boldsymbol{\gamma}(t)) \frac{d\gamma_i}{dt} dt \tag{102}$$

if they converge at all. But they do, because sums (97) have a limit, namely $\int_\gamma f dx_i$. Thus, we succeeded establishing two things. First, Riemann integral $\int_a^b f(\boldsymbol{\gamma}(t)) \frac{d\gamma_i}{dt} dt$ exists if

[16] Cf. e.g., Stewart, Section 4.2.

path integral $\int_{\gamma} f dx_i$ exists and if, of course, function $\gamma_i$ is differentiable on interval $[a, b]$. Second, these two integrals are equal.

Remarkably, the general Change of Variables Formula, (47), follows easily from (49) with help of pullback formula (42). Indeed, on one hand,

$$\begin{aligned}\int_{\gamma} \mathbf{g}^*(f dx_i) &= \int_{\gamma} (f \circ \mathbf{g})\, dg_j = \int_{\gamma} (f \circ \mathbf{g}) \sum_{j=1}^{m} \frac{\partial g_i}{\partial x_j} dx_j \\ &= \int_a^b f(\mathbf{g}(\boldsymbol{\gamma}(t))) \sum_{j=1}^{m} \frac{\partial g_i}{\partial x_j}(\boldsymbol{\gamma}(t)) \frac{d\gamma_j}{dt}(t)\, dt \qquad \textit{by } (49) \\ &= \int_a^b f(\mathbf{g}(\boldsymbol{\gamma}(t))) \left( \nabla g_i(\boldsymbol{\gamma}(t)) \cdot \frac{d\boldsymbol{\gamma}}{dt}(t) \right) dt \end{aligned} \tag{103}$$

which, in view of the Chain Rule, see (23) in **DCVF**, equals

$$\int_a^b f(\mathbf{g}(\boldsymbol{\gamma}(t))) \frac{d((\mathbf{g} \circ \boldsymbol{\gamma})_i)}{dt}(t)\, dt = \int_a^b f((\mathbf{g} \circ \boldsymbol{\gamma})(t)) \frac{d((\mathbf{g} \circ \boldsymbol{\gamma})_i)}{dt}(t)\, dt\,. \tag{104}$$

This last integral equals $\int_{\mathbf{g} \circ \boldsymbol{\gamma}} f dx_i$ by formula (96), as desired.

*Note that we established the Change of Variables Formula under assumption that path $\boldsymbol{\gamma}$ is differentiable.*