Introduction to differential 3-forms

January 7, 2004

These notes should be studied in conjunction with lectures.¹

Figure 1: The parallelepiped spanned by column-vectors $v_1$, $v_2$ and $v_3$ anchored at a point $x \in \mathbb{R}^m$.

1 Orienting a parallelepiped Two ways of ordering the vectors $v_1$, $v_2$ and $v_3$ up to a cyclic permutation correspond to two ways of orienting the parallelepiped they span, see Figure 1. Each of the three orderings: $v_1v_2v_3$, $v_3v_1v_2$, and $v_2v_3v_1$, determines one and the same orientation, while any of the remaining three: $v_1v_3v_2$, $v_3v_2v_1$, or $v_2v_1v_3$, corresponds to the other orientation.

In general, there is no preferred orientation. The situation is, however, different when

$$v_1 = \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix}, \quad v_2 = \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix}, \quad \text{and} \quad v_3 = \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix} \quad (1)$$

¹Abbreviations DCVF, LI and 2F stand for Differential Calculus of Vector Functions, Line Integrals, and Introduction to differential 2-forms, respectively.
are column-vectors in $\mathbb{R}^3$. If the determinant of the $3 \times 3$ matrix formed by column-vectors (1),

$$\omega(x; v_1, v_2, v_3) := \det \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$$

$$:= v_{11}v_{22}v_{33} + v_{12}v_{23}v_{31} + v_{13}v_{21}v_{32} - (v_{11}v_{23}v_{32} + v_{12}v_{21}v_{33} + v_{13}v_{22}v_{31}), \tag{2}$$

is positive then the orientation corresponding to orderings: $v_1v_2v_3$, $v_3v_1v_2$, and $v_2v_3v_1$, is said to be positive, while the orientation corresponding to orderings: $v_1v_3v_2$, $v_3v_2v_1$, and $v_2v_1v_3$, is said to be negative. We reverse this terminology if determinant (2) is negative: the former orientation is then said to be negative and the latter—to be positive.

2 Oriented volume Let us denote by

$$\bigtriangleup_x(v_1, v_2, v_3) \tag{3}$$

the parallelepiped spanned by column-vectors $v_1$, $v_2$ and $v_3$ anchored at point $x \in \mathbb{R}^3$. The absolute value of determinant (2) is equal to the volume of $\bigtriangleup_x(v_1, v_2, v_3)$. It is therefore legitimate to call number $\omega(x; v_1, v_2, v_3)$ in (2) the oriented volume of $\bigtriangleup_x(v_1, v_2, v_3)$.

Exercise 1 Verify that

$$v_1 \cdot (v_2 \times v_3) = \det \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} = (v_1 \times v_2) \cdot v_3. \tag{4}$$

Observations on formula (2):

$3 \times 3$ determinant (2) is the sum of terms $v_{1i}v_{2j}v_{3k}$ with $+$ sign when $ijk$ is one of the three cycles: 123, 231 or 312, and $-$ sign when $ijk$ is one of the three transpositions: 132, 321 or 213. 

Note the following properties of $\omega$:

Note that determinant (2) is also the sum of terms $v_{1i}v_{2j}v_{3k}$ with $+$ sign when $ijk$ is a cycle and $-$ sign when $ijk$ is a transposition.
(a) Linearity in each of its three column-vector variables:

\[
\omega(x; at + bu, v, w) = a\omega(x; t, v, w) + b\omega(x; t, u, v, w) \tag{5}
\]

\[
\omega(x; t, au + bv, w) = a\omega(x; t, u, w) + b\omega(x; t, v, w) \tag{6}
\]

\[
\omega(x; t, u, av + bw) = a\omega(x; t, u, v) + b\omega(x; t, u, w) \tag{7}
\]

(b) Antisymmetry: \(\omega\) changes sign whenever any two of its column-vector arguments are transposed, thus

\[
\omega(x; u, w, v) = -\omega(x; u, v, w), \tag{8}
\]

\[
\omega(x; w, v, u) = -\omega(x; u, v, w), \tag{9}
\]

\[
\omega(x; v, u, w) = -\omega(x; u, v, w) \tag{10}
\]

(t, u, v and w being column-vectors and a and b being scalars).

3 Differential 3-forms Any function

\[
\omega: D \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}
\]

satisfying the above two conditions will be called a differential 3-form on a set \(D \subseteq \mathbb{R}^m\).

Remark:

We have seen so far differential 0-forms (i.e., functions \(D \rightarrow \mathbb{R}\)), 1-forms, 2-forms and 3-forms. A picture that emerges is that differential q-forms are functions of q column-vectors \(v_1, \ldots, v_q\) anchored at a point \(x \in D\), which behave like the oriented volume of the corresponding q-dimensional “parallelepiped” spanned by these q vectors.

Thus, 1-forms are modelled on the oriented length of a line segment, 2-forms are modelled on the oriented area of a parallelogram, and finally 3-forms are modelled on the oriented volume of a parallelepiped.
4 Exterior product of three 1-forms  

Given three differential 1-forms $\varphi_1$, $\varphi_2$ and $\varphi_3$ on $D$, the formula

$$
\omega(x; v_1, v_2, v_3) := \det \begin{pmatrix}
\varphi_1(x; v_1) & \varphi_1(x; v_2) & \varphi_1(x; v_3) \\
\varphi_2(x; v_1) & \varphi_2(x; v_2) & \varphi_2(x; v_3) \\
\varphi_3(x; v_1) & \varphi_3(x; v_2) & \varphi_3(x; v_3)
\end{pmatrix}
$$

(11)

gives us a differential 3-form. We denote it $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ and call it the exterior product of 1-forms $\varphi_1$, $\varphi_2$ and $\varphi_3$.

Note that

$$
\varphi_i \wedge \varphi_j \wedge \varphi_k = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \quad \text{(if } ijk \text{ is a cycle)} 
$$

(12)

$$
= -\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \quad \text{(if } ijk \text{ is a transposition)} .
$$

(13)

This follows from the fact that transposing any two columns in a matrix changes the sign of its determinant.

Exercise 2  Verify that for any differential 1-forms $\varphi$, $\chi$, $\upsilon$, $\vartheta$ and scalars $a$ and $b$, one has:

(a1)  $(a\varphi + b\chi) \wedge \upsilon \wedge \vartheta = a \varphi \wedge \upsilon \wedge \vartheta + b \chi \wedge \upsilon \wedge \vartheta$ ;

(a2)  $\varphi \wedge (a\chi + b\upsilon) \wedge \vartheta = a \varphi \wedge \chi \wedge \vartheta + b \varphi \wedge \upsilon \wedge \vartheta$ ;

(a3)  $\varphi \wedge \chi \wedge (a\upsilon + b\vartheta) = a \varphi \wedge \chi \wedge \upsilon + b \varphi \wedge \chi \wedge \vartheta$ .

5 Exterior product of 1-forms and 2-forms  

Recall that any 1-form $\varphi$ is uniquely represented as

$$
\sum_i f_i \, dx_i
$$

and that any 2-form $\psi$ is uniquely represented as

$$
\sum_{j,k} g_{jk} \, dx_j \wedge dx_k .
$$

(14)

We can define exterior products $\varphi \wedge \psi$ and $\psi \wedge \varphi$ as:

$$
\varphi \wedge \psi := \sum_{i,j,k} f_i g_{jk} \, dx_j \wedge dx_i \wedge dx_k .
$$

(15)
and

\[ \psi \wedge \varphi := \sum_{i,j,k} g_{jk} f_i \, dx_i \wedge dx_j \wedge dx_k \ \text{,} \quad (16) \]

respectively. It follows immediately from definition (11) that

\[ dx_j \wedge dx_k \wedge dx_i = dx_i \wedge dx_j \wedge dx_k \ ; \quad (17) \]

hence,

\[ \psi \wedge \varphi = \varphi \wedge \psi \quad (18) \]

for any 2-form \( \psi \).

**Exercise 3** Verify that for any differential 1-forms \( \varphi, \chi \), differential 2-forms \( \psi, \xi \) and scalars \( a \) and \( b \), one has:

1. \( (b_1) \ (a \varphi + b \chi) \wedge \psi = a \varphi \wedge \psi + b \chi \wedge \psi ; \)
2. \( (b_2) \ \varphi \wedge (a \psi + b \xi) = a \varphi \wedge \psi + b \varphi \wedge \xi . \)

6. **dx \wedge dy \wedge dz** Note that

\[ dx \wedge dy \wedge dz (x; \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \det \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \quad (19) \]

which is the right-hand-side of (2) and, up to a sign, the volume of parallelepiped formed by column-vectors \( \mathbf{v}_1, \mathbf{v}_2 \) and \( \mathbf{v}_3 \) at point \( x \in \mathbb{R}^3 \). We call the differential 3-form on \( \mathbb{R}^3 \), \( dx \wedge dy \wedge dz \), the **oriented volume-element**.

7. **Differential 3-forms on \( \mathbb{R}^3 \)** For any differential 3-forms \( \omega \) on a subset \( D \) of \( \mathbb{R}^3 \), and column-vectors

\[ \mathbf{v}_i = \begin{pmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{pmatrix} ; \quad (i = 1, 2, 3) \ , \quad (20) \]
plugging (20) into $\omega(x; v_1, v_2, v_3)$ and using properties (5)–(10), yields the following simple formula

$$\omega = f \, dx \wedge dy \wedge dz \quad \text{where} \quad f(x) := \omega(x; i, j, k).$$ \hspace{1cm} (21)

In particular, every differential 3-form on a set $D \subseteq \mathbb{R}^3$, is a multiple, with a function-coefficient, of the oriented-volume element, $dx \wedge dy \wedge dz$. Compare this with the situation regarding 1-forms in $\mathbb{R}^1$, and regarding 2-forms in $\mathbb{R}^2$ (see Section 7 of 2F).

The function-coefficient $f$ in (21) is, for obvious reasons, denoted

$$\frac{\omega}{dx \wedge dy \wedge dz}$$ \hspace{1cm} (22)

(compare this with formula (19) in 2F).

Any differential 3-form on a set in one- or two-dimensional Euclidean space is identically zero.

8 Example In Section 4 of 2F we calculated $df_1 \wedge df_2$ for two functions in $\mathbb{R}^2$. Similarly, one can calculate $df_1 \wedge df_2 \wedge df_3$ for three functions in $\mathbb{R}^3$. The formula we obtain is remarkably similar to formula (8) of 2F:

$$df_1 \wedge df_2 \wedge df_3 = (\det J_f(x)) \, dx_1 \wedge dx_2 \wedge dx_3,$$ \hspace{1cm} (23)

where $f := \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ denotes the vector function $D \to \mathbb{R}^3$ having $f_1$, $f_2$ and $f_3$ as its components.

Let us collect various formulae for the determinant of the Jacobi matrix of a vector function $f : D \to \mathbb{R}^d$, whose domain $D$ is a subset of $\mathbb{R}^m$, for three smallest values of dimension

\footnote{Recall that such functions are called in College textbooks of Multivariable Calculus vector fields (on a set $D$); cf. Section 13 of 2F.}
m = 1, 2 and 3:

\[
\det J_f(x) = \frac{df}{dx} \quad \text{(for m = 1)} \\
= \frac{df_1 \wedge df_2}{dx_1 \wedge dx_2} \quad \text{(for m = 2)} \\
= \frac{df_1 \wedge df_2 \wedge df_3}{dx_1 \wedge dx_2 \wedge dx_3} \quad \text{(for m = 3)}.
\]

The determinant of the Jacobi matrix of \( f \) is often referred to as the Jacobian of \( f \).

9 The differential of a 2-form We already know that differential \( df \) of a function (i.e., of a 0-form) is a 1-form and that differential \( d\phi \) of a 1-form is a 2-form (see Section 11 of 2F). Now, it is time to extend this operation to 2-forms. For any differential 2-form \( \psi \) on a set \( D \subseteq \mathbb{R}^n \), which is represented as in (14), we set

\[
d\psi := \sum_{j,k} dg_{jk} \wedge dx_j \wedge dx_k \quad \text{(27)}
\]

\[
= \sum_{i,j,k} \frac{\partial g_{jk}}{\partial x_i} dx_i \wedge dx_j \wedge dx_k. \quad \text{(28)}
\]

10 A calculation: For any function \( f: D \to \mathbb{R} \) and a 2-form \( \psi \) on \( D \), one has

\[
d(f\psi) = df \wedge \psi + f d\psi \quad \text{(29)}
\]

Indeed, it suffices to verify (29) for \( \psi = g \, dx_j \wedge dx_k \):

\[
d(f\psi) = d(fg \, dx_j \wedge dx_k) = d(fg) \wedge dx_j \wedge dx_k = (g df + f dg) \wedge dx_j \wedge dx_k \\
= df \wedge (g \, dx_j \wedge dx_k) + f (dg \wedge dx_j \wedge dx_k) \\
= df \wedge \psi + f d\psi. \quad \text{(30)}
\]
Another calculation: For any 1-forms \( \varphi \) and \( \chi \) on \( D \), one has
\[
d(\varphi \wedge \chi) = d\varphi \wedge \chi - \varphi \wedge d\chi.
\] (31)

Similarly, it suffices to verify (31) for \( \varphi = f \, dx_j \) and \( \chi = g \, dx_k \):
\[
d(\varphi \wedge \chi) = d(f \, dx_j \wedge g \, dx_k) = (df \wedge dx_j) \wedge dx_k + f \, (dg \wedge dx_j) \wedge dx_k
= (df \wedge dx_j) \wedge (dg \wedge dx_k) + f \, (dg \wedge dx_k) \wedge (df \wedge dx_j)
= d\varphi \wedge \chi - \varphi \wedge d\chi.
\] (32)

In the last equality in (32), we have used identity (6) from 2F.

Yet another calculation: If coefficients of a 1-form \( \varphi = f_1 \, dx_1 + \cdots + f_n \, dx_n \) satisfy the condition
\[
\frac{\partial^2 f_i}{\partial x_k \partial x_j} = \frac{\partial^2 f_i}{\partial x_j \partial x_k} \quad \text{(for all } i, j \text{ and } k),
\] (33)
then
\[
d(d\varphi) = 0.
\] (34)

Indeed,
\[
(d \circ d)(\varphi) = d(df_1 \wedge dx_1 + \cdots + df_n \wedge dx_n)
= (d(df_1) \wedge dx_1 + \cdots + d(df_n) \wedge dx_n) - (df_1 \wedge d(dx_1) + \cdots + df_n \wedge d(dx_n))
= 0
\] (35)
in view of formula (31) above and identity (41) in 2F.

The remaining properties of the operation of differential are left to you as an exercise.

Exercise 4 Verify that for any differential 2-forms \( \psi \), \( \xi \), and scalars \( a \) and \( b \), one has:
\((c_1) \ (a \psi + b \xi) = a \psi + b \xi; \)

\((c_2) \ \text{d}(f^* \psi) = f^* \text{d} \psi. \)

Here \(f: E \rightarrow \mathbb{R}^n\) is a vector function sending its domain into \(D\) and the pullback of 3-forms is defined in exactly the same manner as for 1-forms and 2-forms:

\[
(f^* \omega)(x; u, v, w) := \omega(f(x); f'_x(u), f'_x(v), f'_x(w)) \quad \text{. (36)}
\]

13 \ Example: the divergence of a vector field in \(\mathbb{R}^3\) \ Let us calculate the differential of an arbitrary 2-form in \(\mathbb{R}^3\):

\[
d(f_1 \ dx_2 \wedge dx_3 + f_2 \ dx_3 \wedge dx_1 + f_3 \ dx_1 \wedge dx_2)
\]

\[
= df_1 \wedge dx_2 \wedge dx_3 + df_2 \wedge dx_3 \wedge dx_1 + df_3 \wedge dx_1 \wedge dx_2 \\
\]

\[
= \left( \frac{\partial f_1}{\partial x_1} \ dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial f_1}{\partial x_2} \ dx_2 \wedge dx_3 \wedge dx_1 + \frac{\partial f_1}{\partial x_3} \ dx_3 \wedge dx_2 \wedge dx_1 \right) \\
+ \left( \frac{\partial f_2}{\partial x_1} \ dx_1 \wedge dx_3 \wedge dx_1 + \frac{\partial f_2}{\partial x_2} \ dx_2 \wedge dx_3 \wedge dx_1 + \frac{\partial f_2}{\partial x_3} \ dx_3 \wedge dx_1 \wedge dx_1 \\
+ \left( \frac{\partial f_3}{\partial x_1} \ dx_1 \wedge dx_1 \wedge dx_2 + \frac{\partial f_3}{\partial x_2} \ dx_2 \wedge dx_1 \wedge dx_2 + \frac{\partial f_3}{\partial x_3} \ dx_3 \wedge dx_1 \wedge dx_2 \right) \\
\]

\[
= \frac{\partial f_1}{\partial x_1} \ dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial f_2}{\partial x_2} \ dx_2 \wedge dx_3 \wedge dx_1 + \frac{\partial f_3}{\partial x_3} \ dx_3 \wedge dx_1 \wedge dx_2 \\
= \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) \ dx_1 \wedge dx_2 \wedge dx_3 \quad \text{. (37)}
\]

We have used here properties (12) and (13) of the exterior product, and the fact that \(\phi \wedge \phi = 0\) for any 1-form, see (7) of \(2 \Phi\).

The function-coefficient in (37) is known under the name of divergence\(^4\)

\[
\text{div} \ F := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \quad \text{. (38)}
\]

\(^4\)The divergence of \(F\) is often denoted \(\nabla \cdot F\) in Physics textbooks (note the “dot”).
of the vector field

\[
F = \begin{pmatrix}
  f_1 \\
  f_2 \\
  f_3
\end{pmatrix}.
\]

In the language that avoids mentioning differential forms, identity (34) becomes the following statement:

\[
\text{div(curl } F) = 0.
\]

(39)

14 Grand Picture  Let \( \Omega^q_D \) denote the the set of differential \( q \)-forms on a set \( D \subseteq \mathbb{R}^n \). We are already familiar with cases \( q = 0, 1, 2 \) and \( 3 \). It is not difficult to see how to define differential \( q \)-forms also for higher values of \( q \) (make an attempt at such a definition! it's worth it).

Sets of differential forms for different values of \( q \) are related to each other by means of the operation of differential:

\[
\Omega^0_D \xrightarrow{d} \Omega^1_D \xrightarrow{d} \Omega^2_D \xrightarrow{d} \Omega^3_D \xrightarrow{d} \cdots
\]

(40)

so that the composition of two consecutive operations of differential is zero \( d \circ d = 0 \). What you see in (40) is called the Rham\(^5\) complex of set \( D \). Differential forms \( \eta \) whose differential is zero: \( d\eta = 0 \), are called closed forms. Forms \( \eta \) which are equal to \( d\xi \) for some form \( \xi \), are called exact. It follows from what has been just said that

\[
\text{every exact form is closed.}
\]

(41)

De Rham’s lifetime discovery was that

the extent to which closed forms on a given set \( D \) are not exact provides a very precise measure of the geometrical complexity of \( D \)

(42)

This is what is called de Rham’s theory.

\(^5\)Georges de Rham (1903–1990).
One can easily extend our definitions of exterior product to arbitrary forms, so that the product of a $p$-form $\eta$ and a $q$-form $\vartheta$

$$\eta \wedge \vartheta$$

is a $(p + q)$-form. Then

$$d(\eta \wedge \vartheta) = d\eta \wedge \vartheta + (-1)^p \eta \wedge d\vartheta \quad . \quad (43)$$

The $p$-th power of $-1$ in (43) signals that the sign is $+$ for all even values of $p$ and $-$ for all odd values of $p$.

We already know this formula for $p = q = 0$ (this is the derivative-of-the-product formula of Freshman Calculus), $p = 0$ and $q = 1$ (this is formula (b) in Section (14) of 2F), $p = 0$ and $q = 2$ (this is formula (29) above) and $p = q = 1$ (formula (31) above). These formulae are collectively known under the name of Leibniz Rule.

15 Maxwell’s Equations Functions in $\mathbb{R}^3$ which evolve “with time” are profitably thought of as functions on subsets of $\mathbb{R}^4$. We shall denote coordinates in $\mathbb{R}^4$ by $x_0$, $x_1$, $x_2$ and $x_3$.\(^6\) Any 2-form in $\mathbb{R}^4$ can be represented as

$$F = E_1 \, dx_0 \wedge dx_1 + E_2 \, dx_0 \wedge dx_2 + E_3 \, dx_0 \wedge dx_3$$

$$-B_1 \, dx_2 \wedge dx_3 - B_2 \, dx_3 \wedge dx_1 - B_3 \, dx_1 \wedge dx_2$$

for unique function-coefficients $E_1$, $E_2$, $E_3$, $B_1$, $B_2$ and $B_3$.

Similarly, any 3-form in $\mathbb{R}^4$ can be represented as

$$J = \rho \, dx_1 \wedge dx_2 \wedge dx_3 - j_1 \, dx_0 \wedge dx_2 \wedge dx_3 - j_2 \, dx_0 \wedge dx_3 \wedge dx_1 - j_3 \, dx_0 \wedge dx_1 \wedge dx_2$$

(45)

for unique function-coefficients $\rho$, $j_1$, $j_2$ and $j_3$.

In Electrodynamics, the vector functions

$$E := \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

(46)

\(^6\)The physical meaning is $x_0 = ct$, where $t$ stands for time and $c$ denotes the speed of light; $x_1 = x$, $x_2 = y$ and $x_3 = z$ are spatial variables.
are called the **electric** and, respectively, **magnetic** field, the vector function

\[ j = \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} \]  

(47)

is called the electric **current**, and finally, \( \rho \) is a scalar-valued function playing the role of the **density of electric charge**.

It is remarkable that the whole theory of Electrodynamics\(^7\) in the language of differential forms is contained in the following elegant pair of equations:

\[
\begin{align*}
\text{d}F &= 0 \quad \text{and} \quad \text{d}(\ast F) = 4\pi J
\end{align*}
\]

(48)

where \( \ast F \) denotes the 2-form:

\[
\ast F := B_1 dx_0 \wedge dx_1 + B_2 dx_0 \wedge dx_2 + B_3 dx_0 \wedge dx_3 + E_1 dx_2 \wedge dx_3 + E_2 dx_3 \wedge dx_1 + E_3 dx_1 \wedge dx_2
\]

(49)

The closedness of 2-form \( F \) is expressed by the following four equations

\[
\begin{align*}
\frac{\partial E_2}{\partial x_3} - \frac{\partial E_3}{\partial x_2} - \frac{\partial B_1}{\partial x_0} &= 0 \\
\frac{\partial E_3}{\partial x_1} - \frac{\partial E_1}{\partial x_3} - \frac{\partial B_2}{\partial x_0} &= 0 \\
\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} - \frac{\partial B_3}{\partial x_0} &= 0 \\
\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} &= 0
\end{align*}
\]

(50)

\(^7\)in vacuum
while equation \( d(*F) = J \) is equivalent to the following four

\[
\begin{align*}
\frac{\partial B_2}{\partial x_3} - \frac{\partial B_3}{\partial x_2} + \frac{\partial E_1}{\partial x_0} &= -4\pi j_1 \\
\frac{\partial B_3}{\partial x_1} - \frac{\partial B_1}{\partial x_3} + \frac{\partial E_2}{\partial x_0} &= -4\pi j_2 \\
\frac{\partial B_1}{\partial x_2} - \frac{\partial B_2}{\partial x_1} + \frac{\partial E_3}{\partial x_0} &= -4\pi j_3 \\
\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} &= 4\pi \rho .
\end{align*}
\]

Collectively, these eight partial differential equations are called Maxwell's Equations.

Some authors of traditional textbooks of Electrodynamics express these eight equations in the following equivalent form that is more compact:

\[
\begin{align*}
curl \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= 0, \\
\text{div } \mathbf{B} &= 0 \\
\text{and } \\
curl \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c} \mathbf{j}, \\
\text{div } \mathbf{E} &= 4\pi \rho .
\end{align*}
\]

while others prefer to express the same equations by employing an alternative notation for \( \text{curl} \) and \( \text{div} \):

\[
\begin{align*}
\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= 0, \\
\nabla \cdot \mathbf{B} &= 0 \\
\text{and } \\
\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c} \mathbf{j}, \\
\nabla \cdot \mathbf{E} &= 4\pi \rho .
\end{align*}
\]

16 Integration of 3-forms This is done very similarly to how we did that for 2-forms in Sections 16–22 of 2F:

(a) rectangles in \( \mathbb{R}^2 \) are replaced by rectangular boxes;

(b) the area of plane regions is replaced by the volume of space regions;

---

8In these eight equations, James Clerk Maxwell (1831–1879) gave a mathematical formulation to discoveries of Michael Faraday (1791–1867).

Inspired by these equations great physicist Ludwig Boltzmann (1844–1906) exclaimed, in imitation of Romantic poet Goethe, Was it a God who traced these signs?:

13
(c) *double* integrals \( \int \int_D f(x, y) \, dx \, dy \) are replaced by *triple* integrals \( \int \int \int_D f(x, y, z) \, dx \, dy \, dz \); in particular, \( \int \int \int_D dx \, dy \, dz = \text{Vol}(D) \);

(d) the equality
\[
\int \int \int_D f(x, y, z) \, dx \, dy \, dz = \int_D f \, dx \wedge dy \wedge dz \tag{54}
\]
replaces equality (61) from \(2\text{F}\);

(e) the inequality
\[
\left| \int \int \int_D f(x, y, z) \, dx \, dy \, dz \right| \leq M \text{Vol}(D) \tag{55}
\]
replaces inequality (70) from \(2\text{F}\); in particular,
\[
\int \int \int_D f(x, y, z) \, dx \, dy \, dz = 0 \tag{56}
\]
for any bounded function \( f \) on a set \( D \) of zero volume;

(f) “Fubini’s Theorem” for triple integrals
\[
\int \int \int_D f(x, y, z) \, dx \, dy \, dz = \int_{a_1}^{b_1} \left( \int_{a_2}^{b_2} \left( \int_{a_3}^{b_3} f(x, y, z) \, dx \right) \, dy \right) \, dz \tag{57}
\]
replaces “Fubini’s Theorem” for double integrals, see (74) in \(2\text{F}\).

(g) The Change of Variables Formula for Triple Integral:
\[
\int \int \int_D f(u, v, w) \, du \, dv \, dw = \int \int \int_D (f \circ h)(x, y, z) \left| \det J_h(x, y, z) \right| \, dx \, dy \, dz \tag{58}
\]
replaces the corresponding formula for double integrals, see formula (109) in 2F. Here \( h : D \to D' \) is a \textit{diffeomorphism}\(^9\) of three-dimensional region \( D \) onto another region \( D' \).

(h) \textit{Gauß–Ostrogradski’s Theorem:} \(^{10}\)

Let \( D \) be a region in \( \mathbb{R}^3 \) whose boundary, \( \partial D \), is a surface that can be decomposed into \textit{regular patches}, see Sections 33 and 35 of 2F. Let \( \psi \) be a differential 2-form on a region \( D \subseteq \mathbb{R}^3 \).

Then

\[
\int_D d\psi = \int_{\partial D} \psi.
\]

replaces Green’s Theorem (75) of 2F.

Note that the boundary, \( \partial D \), of the \textit{region} \( D \) is \textit{automatically oriented}. Indeed, as was explained in Section 34 of 2F, orienting a patch in \( \mathbb{R}^3 \) is the same as telling which ‘side’ is ‘positive’ and which one is ‘negative’. Thus, we orient the patches which are portions of boundary \( \partial D \), by declaring ‘positive’ the side that faces \textit{outside} \( D \).

17 \textbf{Linking number} An oriented curve in \( \mathbb{R}^3 \) consisting of two \textit{disjoint simple closed curves} \( C_1 \) and \( C_2 \) is called a \textit{link}. \(^{11}\) A link is said to be \textit{trivial} if loop \( C_1 \) is \textit{contractible} in the \textit{complement} to \( C_2 \)

\[ E = \mathbb{R}^3 \setminus C_2, \]

see Section 27 in 2F. This definition does not depend on which of the two closed curves is labelled \( C_1 \) and which is labelled \( C_2 \).

The number of times curve \( C_2 \) is intertwined with curve \( C_1 \) is called the \textit{linking number} and denoted \( \text{Ln}(C_1, C_2) \). In order to determine \( \text{Ln}(C_1, C_2) \), project the link onto a plane \( P \subseteq \mathbb{R}^3 \) such that the ‘shadows’ of constituent curves \( C_1 \) and \( C_2 \) intersect \textit{transversally}, i.e. they intersect at regular points and they are not tangent when they intersect (cf. Section 29 of 2F). Think of the projected curves as being \textit{one-way roads}. When they cross, one

\(^9\)See Section 32 of 2F.

\(^{10}\)\textbf{Johann Carl Friedrich Gauß} (1777–1855); \textbf{Михаил Васильевич Остроградский} (1801–1862).

\(^{11}\)More precisely, a \textit{2-link}. Oriented curves in \( \mathbb{R}^3 \) consisting of \( n \) disjoint simple closed curves are called \( n \)-\textit{links}. 1-links are better known as \textit{knots}. 
of them, the “overpass,” goes over the other one, the “underpass.” Each time they cross

![Figure 2: Simplest links (orientation not indicated).](image)

add 1 if the underpass crosses leftwards and subtract 1 if it crosses rightwards. Since both “roads” are closed, they must cross each other an even number of times. Thus, the total is always an even integer. This integer does not depend on the choice of plane \( P \) onto which we projected the link. By definition,

\[ \text{Ln}(C_2, C_1) = \text{Ln}(C_1, C_2) = \frac{1}{2} \text{ total}. \]

Linking number of a trivial link is zero.

**An alternative definition:**

*Count only those crossings where \( C_1 \) is the overpass and, thus, \( C_2 \) is the underpass. The total obtained equals* \( \text{Ln}(C_1, C_2) \).
Figure 4: At each of eight crossings the \textit{underpass} crosses leftwards, hence the linking number equals

$$\frac{1}{2}(1 + 1 + 1 + 1 + 1 + 1 + 1) = 4.$$ 

Linking number is an analog of \textit{winding number} discussed in Sections 29–30 of 2F. In particular, there is an analog of Index Formula (103) in 2F. Let $\gamma_1: [a, b] \to \mathbb{R}^3$ and $\gamma_2: [c, d] \to \mathbb{R}^3$ be the corresponding parametrizations of $C_1$ and $C_2$, respectively. The function

$$\sigma(t, u) := \gamma_2(u) - \gamma_1(t) \quad (61)$$

is defined on rectangle $[a, b] \times [c, d]$ and its image \textit{does not} contain the origin, 0, because $\gamma_1(t) \neq \gamma_2(u)$ for all $t \in [a, b]$ and $u \in [c, d]$ (curves $C_1$ and $C_2$ are disjoint!). One should think of $\sigma$ as being a parametric surface in $\mathbb{R}^3 \setminus \{0\}$. This surface is \textit{closed}, i.e., does not have a boundary,\footnote{More properly, one should say that the boundary is empty.} since curves $C_1$ and $C_2$ are closed.

18 Linking Number Formula The following is a close relative of Index Formula (108) in 2F:

$$\text{Ln}(C_1, C_2) = \frac{1}{4\pi} \int_\sigma x_1 \, dx_2 \wedge dx_3 + x_2 \, dx_3 \wedge dx_1 + x_3 \, dx_1 \wedge dx_2 \left(\frac{x_1^2 + x_2^2 + x_3^2}{2}\right)^{3/2} \quad (62).$$

This formula can be established similarly to how Index Formula (103) was proved in 2F.

One notes first that the differential 2-form on $\mathbb{R}^3 \setminus \{0\}$:

$$\omega_2 := \frac{1}{4\pi} \frac{x_1 \, dx_2 \wedge dx_3 + x_2 \, dx_3 \wedge dx_1 + x_3 \, dx_1 \wedge dx_2}{\left(\frac{x_1^2 + x_2^2 + x_3^2}{2}\right)^{3/2}} \quad (63)$$
which is sometimes called the Gauß form, is closed, cf. sample problem ?? in Problembook.

Using Gauß’-Ostrogradski’s Theorem, one can show that the integral of a closed 2-form over a closed surface does not change when one continuously deforms the surface—this is exactly analogous to Theorem (101) of 2F (which was established using the parametric form of Stokes’ Theorem, see Section 2.5 in 2F).

Without loss of generality, one can assume that curves $C_1$ and $C_2$ are parametrized by interval $[0, 1]$. Then it can be shown that, if $\ln(C_1, C_2) = m$, then parametric surface $\sigma$ can be deformed in $\mathbb{R}^3 \setminus \{0\}$ to the function

$$
\sigma_1(t, u) := \begin{pmatrix} 
\sin(\pi t) \sin(2\pi mu) \\
\sin(\pi t) \cos(2\pi mu) \\
\cos t
\end{pmatrix}.
$$

which parametrizes unit sphere in $\mathbb{R}^3$ so that every point of sphere, except for the Northern and Southern Poles, is ‘visited’ exactly $m$ times. The integral of $\omega_2$ over $\sigma_1$ is $m$ times the integral of $\omega_2$ over the sphere, i.e., equals $m$ (cf., exercise ?? and sample problem ?? in Problembook).