Introduction to differential 3-forms

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These notes should be studied in conjunction with lectures.[1]

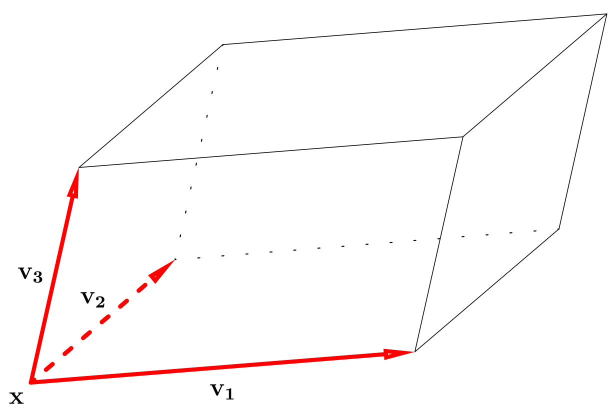


Figure 1: The parallelepiped spanned by column-vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ anchored at a point $\mathbf{x} \in \mathbb{R}^m$.

1 Orienting a parallelepiped *Two* ways of ordering the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ *up to a cyclic permutation* correspond to two ways of orienting the parallelepiped they span, see Figure 1. Each of the three orderings: $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_3\mathbf{v}_1\mathbf{v}_2$, and $\mathbf{v}_2\mathbf{v}_3\mathbf{v}_1$, determines one and the same orientation, while any of the remaining three: $\mathbf{v}_1\mathbf{v}_3\mathbf{v}_2$, $\mathbf{v}_3\mathbf{v}_2\mathbf{v}_1$, or $\mathbf{v}_2\mathbf{v}_1\mathbf{v}_3$, corresponds to the *other* orientation.

In general, there is no preferred orientation. The situation is, however, different when

$$\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix} \tag{1}$$

[1] Abbreviations DCVF, LI and 2F stand for *Differential Calculus of Vector Functions, Line Integrals,* and *Introduction to differential 2-forms*, respectively.

are column-vectors in $\mathbb{R}^3$. If the **determinant** of the 3×3 matrix formed by column-vectors (1),

$$\omega(\mathbf{x}; \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) := \det \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$$

$$:= \quad v_{11}v_{22}v_{33} + v_{12}v_{23}v_{31} + v_{13}v_{21}v_{32} - (v_{11}v_{23}v_{32} + v_{12}v_{21}v_{33} + v_{13}v_{22}v_{31}), \qquad (2)$$

is *positive* then the orientation corresponding to orderings: $\mathbf{v_1v_2v_3}$, $\mathbf{v_3v_1v_2}$, and $\mathbf{v_2v_3v_1}$, is said to be *positive*, while the orientation corresponding to orderings: $\mathbf{v_1v_3v_2}$, $\mathbf{v_3v_2v_1}$, and $\mathbf{v_2v_1v_3}$, is said to be *negative*. We reverse this terminology if determinant (2) is *negative*: the former orientation is then said to be negative and the latter—to be positive.

2 Oriented volume

Let us denote by

$$\Diamond_{\mathbf{x}}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) \qquad (3)$$

the parallelepiped spanned by column-vectors $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ anchored at point $\mathbf{x} \in \mathbb{R}^3$. The absolute value of determinant (2) is equal to the volume of $\Diamond_{\mathbf{x}}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$. It is therefore legitimate to call number $\omega(\mathbf{x}; \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$ in (2) the **oriented volume** of $\Diamond_{\mathbf{x}}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$.

Exercise 1 *Verify that*

$$\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3}) = \det \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} = (\mathbf{v_1} \times \mathbf{v_2}) \cdot \mathbf{v_3}\ . \qquad (4)$$

Observations on formula (2):

> *3×3 determinant (2) is the sum of terms $v_{1i}v_{2j}v_{3k}$ with $+$ sign when ijk is one of the three **cycles**: 123, 231 or 312, and $-$ sign when ijk is one of the three **transpositions**: 132, 321 or 213.* [2]

Note the following properties of ω:

[2] Note that determinant (2) is also the sum of terms $v_{i1}v_{j2}v_{k3}$ with $+$ sign when ijk is a **cycle** and $-$ sign when ijk is a **transposition**.

(a) **Linearity in *each* of its three column-vector variables:**

$$\omega(\mathbf{x}; a\mathbf{t} + b\mathbf{u}, \mathbf{v}, \mathbf{w}) = a\omega(\mathbf{x}; \mathbf{t}, \mathbf{v}, \mathbf{w}) + b\omega(\mathbf{x}; \mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}) \tag{5}$$

$$\omega(\mathbf{x}; \mathbf{t}, a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = a\omega(\mathbf{x}; \mathbf{t}, \mathbf{u}, \mathbf{w}) + b\omega(\mathbf{x}; \mathbf{t}, \mathbf{v}, \mathbf{w}) \tag{6}$$

$$\omega(\mathbf{x}; \mathbf{t}, \mathbf{u}, a\mathbf{v} + b\mathbf{w}) = a\omega(\mathbf{x}; \mathbf{t}, \mathbf{u}, \mathbf{v}) + b\omega(\mathbf{x}; \mathbf{t}, \mathbf{u}, \mathbf{w}) \tag{7}$$

(b) **Antisymmetry:** ω changes sign whenever any two of its column-vector arguments are transposed, thus

$$\omega(\mathbf{x}; \mathbf{u}, \mathbf{w}, \mathbf{v}) = -\omega(\mathbf{x}; \mathbf{u}, \mathbf{v}, \mathbf{w}), \tag{8}$$

$$\omega(\mathbf{x}; \mathbf{w}, \mathbf{v}, \mathbf{u}) = -\omega(\mathbf{x}; \mathbf{u}, \mathbf{v}, \mathbf{w}), \tag{9}$$

$$\omega(\mathbf{x}; \mathbf{v}, \mathbf{u}, \mathbf{w}) = -\omega(\mathbf{x}; \mathbf{u}, \mathbf{v}, \mathbf{w}) \tag{10}$$

($\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ being column-vectors and a and b being scalars).

3 Differential 3-forms

Any function

$$\omega \colon D \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

satisfying the above two conditions will be called a **differential 3-form** on a set $D \subseteq \mathbb{R}^m$.

Remark:

We have seen so far differential 0-forms (i.e., functions $D \to \mathbb{R}$), 1-forms, 2-forms and 3-forms. A picture that emerges is that differential q-forms are *functions of q column-vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_q$ *anchored at a point* $\mathbf{x} \in D$, which behave like the *oriented* **volume** of the corresponding q-dimensional "parallelepiped" spanned by these q vectors.

Thus, 1-forms are modelled on the oriented length of a line segment, 2-forms are modelled on the oriented area of a parallelogram, and finally 3-forms are modelled on the oriented volume of a parallelepiped.

4 Exterior product of three 1-forms Given three differential 1-forms φ_1, φ_2 and φ_3 on D, the formula

$$\omega(x; v_1, v_2, v_3) := \det \begin{pmatrix} \varphi_1(x; v_1) & \varphi_1(x; v_2) & \varphi_1(x; v_3) \\ \varphi_2(x; v_1) & \varphi_2(x; v_2) & \varphi_2(x; v_3) \\ \varphi_3(x; v_1) & \varphi_3(x; v_2) & \varphi_3(x; v_3) \end{pmatrix} \tag{11}$$

gives us a differential 3-form. We denote it $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ and call it the **exterior product** of 1-forms φ_1, φ_2 and φ_2.

Note that

$$\varphi_i \wedge \varphi_j \wedge \varphi_k = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \qquad \text{(if } ijk \text{ is a } \textit{cycle}\text{)} \tag{12}$$

$$= -\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \qquad \text{(if } ijk \text{ is a } \textit{transposition}\text{)} . \tag{13}$$

This follows from the fact that *transposing any two columns* in a matrix changes the sign of its determinant.

Exercise 2 *Verify that for any differential 1-forms* φ, χ, υ, ϑ *and scalars* a *and* b, *one has:*

(a₁) $(a\varphi + b\chi) \wedge \upsilon \wedge \vartheta = a\,\varphi \wedge \upsilon \wedge \vartheta + b\,\chi \wedge \upsilon \wedge \vartheta$;

(a₂) $\varphi \wedge (a\chi + b\upsilon) \wedge \vartheta = a\,\varphi \wedge \chi \wedge \vartheta + b\,\varphi \wedge \upsilon \wedge \vartheta$;

(a₃) $\varphi \wedge \chi \wedge (a\upsilon + b\vartheta) = a\,\varphi \wedge \chi \wedge \upsilon + b\,\varphi \wedge \chi \wedge \vartheta$.

5 Exterior product of 1-forms and 2-forms Recall that any 1-form φ is uniquely represented as

$$\sum_i f_i dx_i$$

and that any 2-form ψ is uniquely represented as

$$\sum_{j,k} g_{jk}\, dx_j \wedge dx_k \, . \tag{14}$$

We can define exterior products $\varphi \wedge \psi$ and $\psi \wedge \varphi$ as:

$$\varphi \wedge \psi := \sum_{i,j,k} f_i g_{jk}\, dx_i \wedge dx_j \wedge dx_k \tag{15}$$

and

$$\psi \wedge \varphi := \sum_{i,j,k} g_{jk} f_i \, dx_j \wedge dx_k \wedge dx_i \quad , \tag{16}$$

respectively. It follows immedietely from definition (11) that

$$dx_j \wedge dx_k \wedge dx_i \ = \ dx_i \wedge dx_j \wedge dx_k \ ; \tag{17}$$

hence,

$$\psi \wedge \varphi \ = \ \varphi \wedge \psi \tag{18}$$

for *any* 2-form ψ.

Exercise 3 *Verify that for any differential 1-forms* φ, χ, *differential 2-forms* ψ, ξ *and scalars* a *and* b, *one has:*

(b₁) $(a\varphi + b\chi) \wedge \psi = a\, \varphi \wedge \psi + b\, \chi \wedge \psi$;

(b₂) $\varphi \wedge (a\psi + b\,\xi) = a\, \varphi \wedge \psi + b\, \varphi \wedge \xi$.

6 $dx \wedge dy \wedge dz$ Note that

$$dx \wedge dy \wedge dz\,(\mathbf{x}; \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \det \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \tag{19}$$

which is the right-hand-side of (2) and, up to a sign, the volume of parallelepiped formed by column-vectors $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ at point $\mathbf{x} \in \mathbb{R}^3$. We call the differential 3-form on $\mathbb{R}^3$, $dx \wedge dy \wedge dz$, the **oriented volume-element.**

7 Differential 3-forms on $\mathbb{R}^3$ For any differential 3-forms ω on a subset D of $\mathbb{R}^3$, and column-vectors

$$\mathbf{v}_i = \begin{pmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{pmatrix} ; \qquad (i = 1,\, 2,\, 3) \ , \tag{20}$$

plugging (20) into $\omega(\mathbf{x}; \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$ and using properties (5)–(10), yields the following simple formula

$$\omega = f\, dx \wedge dy \wedge dz \qquad \text{where} \qquad f(\mathbf{x}) := \omega(\mathbf{x}; \mathbf{i}, \mathbf{j}, \mathbf{k}) \quad . \tag{21}$$

In particular, every differential 3-form on a set $D \subseteq \mathbb{R}^3$, is a multiple, with a function-coefficient, of the oriented-volume element, $dx \wedge dy \wedge dz$. Compare this with the situation regarding 1-forms in $\mathbb{R}^1$, and regarding 2-forms in $\mathbb{R}^2$ (see Section 7 of **2F**).

The function-coefficient f in (21) is, for obvious reasons, denoted

$$\frac{\omega}{dx \wedge dy \wedge dz} \tag{22}$$

(compare this with formula (19) in **2F**).

Any differential 3-form on a set in one- or two-dimensional Euclidean space is identically zero.

8 Example In Section 4 of **2F** we calculated $df_1 \wedge df_2$ for *two* functions in $\mathbb{R}^2$. Similarly, one can calculate $df_1 \wedge df_2 \wedge df_3$ for *three* functions in $\mathbb{R}^3$. The formula we obtain is remarkably similar to formula (8) of **2F**:

$$df_1 \wedge df_2 \wedge df_3 = (\det J_{\mathbf{f}}(\mathbf{x}))\, dx_1 \wedge dx_2 \wedge dx_3\,, \tag{23}$$

where $\mathbf{f} := \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ denotes the vector function $D \to \mathbb{R}^3$ having f_1, f_2 and f_3 as its components.

Let us collect various formulae for the determinant of the Jacobi matrix of a vector function $\mathbf{f}\colon D \to \mathbb{R}^d$, whose domain D is a subset of $\mathbb{R}^m$,[3] for three smallest values of dimension

[3] Recall that such functions are called in College textbooks of Multivariable Calculus *vector fields* (on a set D); cf. Section 13 of **2F**.

$m = 1$, 2 and 3:

$$\det J_f(\mathbf{x}) = \frac{df}{dx} \qquad \text{(for } m = 1\text{)} \tag{24}$$

$$= \frac{df_1 \wedge df_2}{dx_1 \wedge dx_2} \qquad \text{(for } m = 2\text{)} \tag{25}$$

$$= \frac{df_1 \wedge df_2 \wedge df_3}{dx_1 \wedge dx_2 \wedge dx_3} \qquad \text{(for } m = 3\text{)} . \tag{26}$$

The determinant of the Jacobi matrix of f is often referred to as the **Jacobian** of f. ☜

9 The differential of a 2-form We already know that differential df of a function (i.e., of a 0-form) is a 1-form and that differential $d\varphi$ of a 1-form is a 2-form (see Section 11 of 2F). Now, it is time to extend this operation to 2-forms. For any differential 2-form ψ on a set $D \subseteq \mathbb{R}^n$, which is represented as in (14), we set

$$d\psi := \sum_{j,k} dg_{jk} \wedge dx_j \wedge dx_k \tag{27}$$

$$= \sum_{i,j,k} \frac{\partial g_{jk}}{\partial x_i} dx_i \wedge dx_j \wedge dx_k . \tag{28}$$

10 A calculation: For any function $f\colon D \to \mathbb{R}$ and a 2-form ψ on D, one has

$$\boxed{d(f\psi) = df \wedge \psi + f d\psi} . \tag{29}$$

Indeed, it suffices to verify (29) for $\psi = g\, dx_j \wedge dx_k$:

$$d(f\psi) = d(fg\, dx_j \wedge dx_k) = d(fg) \wedge dx_j \wedge dx_k = (gdf + fdg) \wedge dx_j \wedge dx_k$$

$$= df \wedge (g\, dx_j \wedge dx_k) + f(dg \wedge dx_j \wedge dx_k)$$

$$= df \wedge \psi + f d\psi . \tag{30}$$

11 Another calculation: For any 1-forms φ and χ on D, one has

$$\boxed{d(\varphi \wedge \chi) = d\varphi \wedge \chi - \varphi \wedge d\chi} \quad . \tag{31}$$

Similarly, it suffices to verify (31) for $\varphi = f\,dx_j$ and $\chi = g\,dx_k$:

$$\begin{aligned} d(\varphi \wedge \chi) = d(f\,dx_j \wedge g\,dx_k) &= d(fg) \wedge dx_j \wedge dx_k = (gdf + fdg) \wedge dx_j \wedge dx_k \\ &= (df \wedge dx_j) \wedge (g\,dx_k) + f(dg \wedge dx_j \wedge dx_k) \\ &= (df \wedge dx_j) \wedge (g\,dx_k) - (f\,dx_j) \wedge (dg \wedge dx_k) \\ &= d\varphi \wedge \chi - \varphi \wedge d\chi\,. \end{aligned} \tag{32}$$

In the last equality in (32), we have used identity (6) from **2F**.

12 Yet another calculation: If coefficients of a 1-form $\varphi = f_1\,dx_1 + \cdots + f_n\,dx_n$ satisfy the condition

$$\frac{\partial^2 f_i}{\partial x_k \partial x_j} = \frac{\partial^2 f_i}{\partial x_j \partial x_k} \qquad \text{(for all } i,\ j \text{ and } k), \tag{33}$$

then

$$\boxed{d(d\varphi) = 0} \quad . \tag{34}$$

Indeed,

$$\begin{aligned} (d \circ d)(\varphi) &= d(df_1 \wedge dx_1 + \cdots + df_n \wedge dx_n) \\ &= (d(df_1) \wedge dx_1 + \cdots + d(df_n) \wedge dx_n) - (df_1 \wedge d(dx_1) + \cdots + df_n \wedge d(dx_n)) \\ &= 0 \end{aligned} \tag{35}$$

in view of formula (31) above and identity (41) in **2F**.

The remaining properties of the operation of differential are left to you as an exercise.

Exercise 4 *Verify that for any differential 2-forms ψ, ξ and scalars a and b, one has:*

(c_1) $(a\psi + b\xi) = a\,\psi + b\,\xi\,;$

(c_2) $d(\mathbf{f}^*\psi) = \mathbf{f}^* d\psi.$

Here $\mathbf{f}\colon E \to \mathbb{R}^n$ *is a vector function sending its domain into* D *and the pullback of 3-forms is defined in exactly the same manner as for 1-forms and 2-forms:*

$$(\mathbf{f}^*\omega)(\mathbf{x};\mathbf{u},\mathbf{v},\mathbf{w}) := \omega\big(\mathbf{f}(\mathbf{x});\mathbf{f}'_{\mathbf{x}}(\mathbf{u}),\mathbf{f}'_{\mathbf{x}}(\mathbf{v}),\mathbf{f}'_{\mathbf{x}}(\mathbf{w})\big) \quad . \tag{36}$$

13 Example: the divergence of a vector field in $\mathbb{R}^3$ Let us calculate the differential of an arbitrary 2-form in $\mathbb{R}^3$:

$$d(f_1\,dx_2\wedge dx_3 + f_2\,dx_3\wedge dx_1 + f_3\,dx_1\wedge dx_2)$$

$$\begin{aligned}
&= df_1\wedge dx_2\wedge dx_3 + df_2\wedge dx_3\wedge dx_1 + df_3\wedge dx_1\wedge dx_2)\\
&= \left(\frac{\partial f_1}{\partial x_1}dx_1\wedge dx_2\wedge dx_3 + \frac{\partial f_1}{\partial x_2}dx_2\wedge dx_2\wedge dx_3 + \frac{\partial f_1}{\partial x_3}dx_3\wedge dx_2\wedge dx_3\right)\\
&\quad + \left(\frac{\partial f_2}{\partial x_1}dx_1\wedge dx_3\wedge dx_1 + \frac{\partial f_2}{\partial x_2}dx_2\wedge dx_3\wedge dx_1 + \frac{\partial f_2}{\partial x_3}dx_3\wedge dx_3\wedge dx_1\right)\\
&\quad + \left(\frac{\partial f_3}{\partial x_1}dx_1\wedge dx_1\wedge dx_2 + \frac{\partial f_3}{\partial x_2}dx_2\wedge dx_1\wedge dx_2 + \frac{\partial f_3}{\partial x_3}dx_3\wedge dx_1\wedge dx_2\right)\\
&= \frac{\partial f_1}{\partial x_1}dx_1\wedge dx_2\wedge dx_3 + \frac{\partial f_2}{\partial x_2}dx_2\wedge dx_3\wedge dx_1 + \frac{\partial f_3}{\partial x_3}dx_3\wedge dx_1\wedge dx_2\\
&= \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1\wedge dx_2\wedge dx_3\ .
\end{aligned} \tag{37}$$

We have used here properties (12) and (13) of the exterior product, and the fact that $\varphi\wedge\varphi = 0$ for any 1-form, see (7) of **2F**.

The function-coefficient in (37) is known under the name of **divergence**[4]

$$\operatorname{div}\mathbf{F} := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\ . \tag{38}$$

[4]The divergence of $\mathbf{F}$ is often denoted $\nabla\cdot\mathbf{F}$ in Physics textbooks (note the “dot”).

of the vector field

$$\mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

In the language that avoids mentioning differential forms, identity (34) becomes the following statement:

$$\boxed{\operatorname{div}(\operatorname{curl}\ \mathbf{F}) = 0}\ . \qquad (39)$$

14 Grand Picture Let Ω_D^q denote the the set of differential q-forms on a set $D \subseteq \mathbb{R}^n$. We are already familiar with cases $q = 0, 1, 2$ and 3. It is not difficult to see how to define differential q-forms also for higher values of q (make an attempt at such a definition! it's worth it).

Sets of differential forms for different values of q are related to each other by means of the operation of differential:

$$\Omega_D^0 \xrightarrow{d} \Omega_D^1 \xrightarrow{d} \Omega_D^2 \xrightarrow{d} \Omega_D^3 \xrightarrow{d} \cdots \qquad (40)$$

so that the composition of two consecutive operations of differential is zero $d \circ d = 0$. What you see in (40) is called the **Rham**[5] **complex** of set D. Differential forms η whose differential is zero: $d\eta = 0$, are called **closed forms**. Forms η which are equal to $d\xi$ for some form ξ are called **exact**. It follows from what has been just said that

every exact form is closed. (41)

De Rham's lifetime discovery was that

> the extent to which closed forms on a given set D *are not* exact provides a very precise measure of the geometrical complexity of D . (42)

This is what is called **de Rham's theory.**

[5] Georges de Rham (1903–1990).

One can easily extend our definitions of exterior product to arbitrary forms, so that the product of a p-form η and a q-form ϑ

$$\eta \wedge \vartheta$$

is a $(p+q)$-form. Then

$$\boxed{d(\eta \wedge \vartheta) \;=\; d\eta \wedge \vartheta + (-1)^p \, \eta \wedge d\vartheta} \quad . \tag{43}$$

The p-th power of -1 in (43) signals that the sign is $+$ for all *even* values of p and $-$ for all *odd* valuse of p.

We already know this formula for $p = q = 0$ (this is the derivative-of-the-product formula of Freshman Calculus), $p = 0$ and $q = 1$ (this is formula (b) in Section (14) of **2F**), $p = 0$ and $q = 2$ (this is formula (29) above) and $p = q = 1$ (formula (31) above). These formulae are collectively known under the name of **Leibniz Rule.**

15 Maxwell's Equations Functions in $\mathbb{R}^3$ which *evolve* "with time" are profitably thought of as functions on subsets of $\mathbb{R}^4$. We shall denote coordinates in $\mathbb{R}^4$ by x_0, x_1, x_2 and x_3.[6]

Any 2-form in $\mathbb{R}^4$ can be represented as

$$\begin{aligned} F \;=\;& E_1\, dx_0 \wedge dx_1 + E_2\, dx_0 \wedge dx_2 + E_3\, dx_0 \wedge dx_3 \\ & - B_1\, dx_2 \wedge dx_3 - B_2\, dx_3 \wedge dx_1 - B_3\, dx_1 \wedge dx_2 \end{aligned} \tag{44}$$

for unique function-coefficients E_1, E_2, E_3, B_1, B_2 and B_3.

Similarly, *any* 3-form in $\mathbb{R}^4$ can be represented as

$$J = \rho\, dx_1 \wedge dx_2 \wedge dx_3 - j_1\, dx_0 \wedge dx_2 \wedge dx_3 - j_2\, dx_0 \wedge dx_3 \wedge dx_1 - j_3\, dx_0 \wedge dx_1 \wedge dx_2 \tag{45}$$

for unique function-coeeficients ρ, j_1, j_2 and j_3.

In Electrodynamics, the vector functions

$$\mathbf{E} := \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} := \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \tag{46}$$

[6] The physical meaning is $x_0 = ct$, where t stands for *time* and c denotes the speed of light; $x_1 = x$, $x_2 = y$ and $x_3 = z$ are *spatial* variables.

are called the **electric** and, respectively, **magnetic** field, the vector function

$$\mathbf{j} = \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} \tag{47}$$

is called the electric **current**, and finally, ρ is a scalar-valued function playing the role of the **density of electric charge**.

It is remarkable that the whole theory of Electrodynamics[7] in the language of differential forms is contained in the following elegant pair of equations:

$$dF = 0 \qquad \text{and} \qquad d(*F) = 4\pi J \tag{48}$$

where $*F$ denotes the 2-form:

$$\begin{aligned} *F \quad := \quad & B_1\, dx_0 \wedge dx_1 + B_2\, dx_0 \wedge dx_2 + B_3\, dx_0 \wedge dx_3 \\ & + E_1\, dx_2 \wedge dx_3 + E_2\, dx_3 \wedge dx_1 + E_3\, dx_1 \wedge dx_2 \end{aligned} \tag{49}$$

The closedness of 2-form F is expressed by the following four equations

$$\begin{cases} \dfrac{\partial E_2}{\partial x_3} - \dfrac{\partial E_3}{\partial x_2} - \dfrac{\partial B_1}{\partial x_0} = 0 \\[2ex] \dfrac{\partial E_3}{\partial x_1} - \dfrac{\partial E_1}{\partial x_3} - \dfrac{\partial B_2}{\partial x_0} = 0 \\[2ex] \dfrac{\partial E_1}{\partial x_2} - \dfrac{\partial E_2}{\partial x_1} - \dfrac{\partial B_3}{\partial x_0} = 0 \\[2ex] \dfrac{\partial B_1}{\partial x_1} + \dfrac{\partial B_2}{\partial x_2} + \dfrac{\partial B_3}{\partial x_3} = 0 \end{cases} \tag{50}$$

[7] in vacuum

while equation $d(*F) = J$ is equivalent to the following four

$$\begin{cases} \dfrac{\partial B_2}{\partial x_3} - \dfrac{\partial B_3}{\partial x_2} + \dfrac{\partial E_1}{\partial x_0} = -4\pi j_1 \\ \dfrac{\partial B_3}{\partial x_1} - \dfrac{\partial B_1}{\partial x_3} + \dfrac{\partial E_2}{\partial x_0} = -4\pi j_2 \\ \dfrac{\partial B_1}{\partial x_2} - \dfrac{\partial B_2}{\partial x_1} + \dfrac{\partial E_3}{\partial x_0} = -4\pi j_3 \\ \dfrac{\partial E_1}{\partial x_1} + \dfrac{\partial E_2}{\partial x_2} + \dfrac{\partial E_3}{\partial x_3} = 4\pi\rho \ . \end{cases} \tag{51}$$

Collectively, these eight partial differential equations are called **Maxwell's**[8] **Equations.**

Some authors of traditional textbooks of Electrodynamics express these eight equations in the following equivalent form that is more compact:

$$\begin{cases} \operatorname{curl} \mathbf{E} + \dfrac{1}{c}\dfrac{\partial \mathbf{B}}{\partial t} = 0 \\ \operatorname{div} \mathbf{B} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \operatorname{curl} \mathbf{B} - \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} = \dfrac{4\pi}{c}\mathbf{j} \\ \operatorname{div} \mathbf{E} = 4\pi\rho \ , \end{cases} \tag{52}$$

while others prefer to express the same equations by employing an alternative notation for curl and div:

$$\begin{cases} \nabla \times \mathbf{E} + \dfrac{1}{c}\dfrac{\partial \mathbf{B}}{\partial t} = 0 \\ \nabla \cdot \mathbf{B} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \nabla \times \mathbf{B} - \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} = \dfrac{4\pi}{c}\mathbf{j} \\ \nabla \cdot \mathbf{E} = 4\pi\rho \ . \end{cases} \tag{53}$$

16 Integration of 3-forms This is done very similarly to how we did that for 2-forms in Sections 16–22 of **2F**:

(a) *rectangles* in $\mathbb{R}^2$ are replaced by *rectangular boxes;*

(b) the *area* of plane regions is replaced by the *volume* of space regions;

[8]In these eight equations, James Clerk Maxwell (1831–1879) gave a mathematical formulation to discoveries of Michael Faraday (1791–1867).

Inspired by these equations great physicist Ludwig Boltzmann (1844–1906) exclaimed, in imitation of Romantic poet Goethe, *Was it a God who traced these signs?*.

(c) *double* integrals $\iint_D f(x,y)\,dx\,dy$ are replaced by *triple* integrals $\iiint_D f(x,y,z)\,dx\,dy\,dz$; in particular, $\iiint_D dx\,dy\,dz = \mathrm{Vol}(D)$;

(d) the equality

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_D f\,dx\wedge dy\wedge dz \tag{54}$$

replaces equality (61) from **2F**;

(e) the inequality

$$\left|\iiint_D f(x,y,z)\,dx\,dy\,dz\right| \leqslant M\,\mathrm{Vol}(D) \tag{55}$$

replaces inequality (70) from **2F**; in particular,

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = 0 \tag{56}$$

for any bounded function f on a set D of zero volume;

(f) "Fubini's Theorem" for triple integrals

$$\iiint_{\mathcal{J}} f(x,y,z)\,dx\,dy\,dz = \int_{a_3}^{b_3}\left(\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y,z)\,dx\right)dy\right)dz \tag{57}$$

replaces "Fubini's Theorem" for double integrals, see (74) in **2F**.

(g) The Change of Variables Formula for Triple Integral:

$$\iiint_{D'} f(u,v,w)\,du\,dv\,dw = \iiint_D (f\circ \mathbf{h})(x,y,z)\,|\det J_{\mathbf{h}}(x,y,z)|\,dx\,dy\,dz\ . \tag{58}$$

replaces the corresponding formula for double integrals, see formula (109) in **2F**. Here $\mathbf{h}\colon D \to D'$ is a *diffeomorphism*[9] of three-dimensional region D onto another region D'.

(h) **Gauß'–Ostrogradski's Theorem:**[10]

Let D be a region in $\mathbb{R}^3$ whose boundary, ∂D, is a surface that can be decomposed into *regular patches*, see Sections 33 and 35 of **2F**. Let ψ be a differential 2-form on a region $D \subseteq \mathbb{R}^3$. Then

$$\int_D d\psi = \int_{\partial D} \psi \,. \tag{59}$$

replaces Green's Theorem (75) of **2F**.

Note that the boundary, ∂D, of the *region* D is *automatically oriented*. Indeed, as was explained in Section 34 of **2F**, orienting a patch in $\mathbb{R}^3$ is the same as telling which 'side' is 'positive' and which one is 'negative'. Thus, we orient the patches which are portions of boundary ∂D, by declaring 'positive' the side that faces **outside** D.

17 Linking number An oriented curve in $\mathbb{R}^3$ consisting of two *disjoint simple closed curves* C_1 and C_2 is called a **link**.[11] A link is said to be **trivial** if loop C_1 is *contractible* in the *complement* to C_2

$$E = \mathbb{R}^3 \setminus C_2 \,, \tag{60}$$

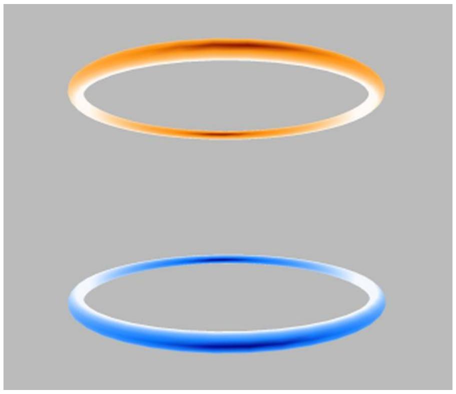
see Section 27 in **2F**. This definition does not depend on which of the two closed curves is labelled C_1 and which is labelled C_2.

The number of times curve C_2 is intertwined with curve C_1 is called the **linking number** and denoted $\mathrm{Ln}(C_1, C_2)$. In order to determine $\mathrm{Ln}(C_1, C_2)$, project the link onto a plane $P \subseteq \mathbb{R}^3$ such that the 'shadows' of constituent curves C_1 and C_2 intersect *transversally*, i.e. they intersect at regular points and they are not tangent when they intersect (cf. Section 29 of **2F**). Think of the projected curves as being *one-way roads*. When they cross, one

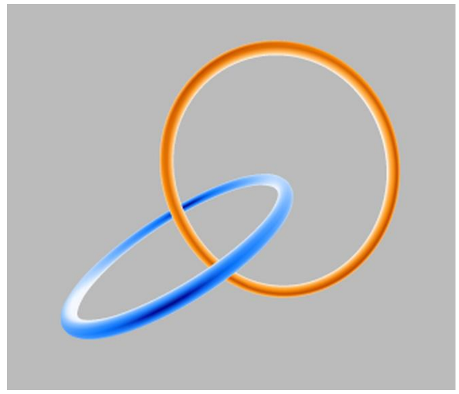
[9] See Section 32 of **2F**.

[10] Johann Carl Friedrich Gauß (1777–1855); Михаил Васильевич Остроградский (1801–1862).

[11] More precisely, a 2-*link*. Oriented curves in $\mathbb{R}^3$ consisting of n disjoint simple closed curves are called n-*links*. 1-links are better known as *knots*.



(a) A trivial link



(b) A nontrivial link

Figure 2: Simplest links (orientation not indicated).

of them, the "*overpass*," goes over the other one, the "*underpass*." Each time they cross

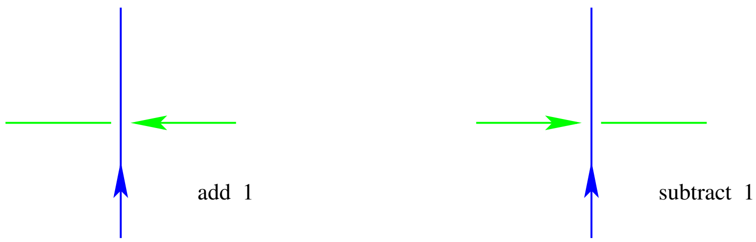


Figure 3: At each crossing *add* 1 when the underpass crosses leftwards and *subtract* 1 when it crossses rightwards.

add 1 if the *underpass* crosses *leftwards* and *subtract* 1, if it crosses *rightwards*. Since both "roads" are closed, they must cross each other an even number of times. Thus, the total is always an even integer. This integer does not depend on the choice of plane P onto which we projected the link. By definition,

$$\mathrm{Ln}(C_2, C_1) = \mathrm{Ln}(C_1, C_2) = \frac{1}{2}\,\mathrm{total}\ .$$

Linking number of a trivial link is zero.

An alternative definition:

Count only those crossings where C_1 *is the overpass and, thus,* C_2 *is the underpass. The total obtained* ***equals*** $\mathrm{Ln}(C_1, C_2)$.

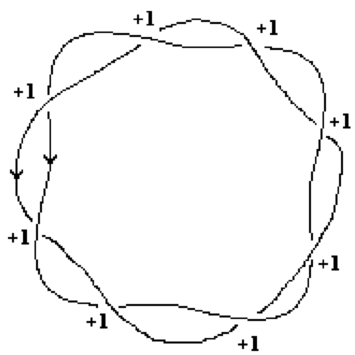


Figure 4: At each of eight crossings the *underpass* crosses leftwards, hence the linking number equals

$$\tfrac{1}{2}(1+1+1+1+1+1+1+1) = 4\,.$$

Linking number is an analog of *winding number* dicussed in Sections 29–30 of **2F**. In particular, there is an analog of Index Formula (103) in **2F**. Let $\gamma_1\colon [a,b] \to \mathbb{R}^3$ and $\gamma_2\colon [c,d] \to \mathbb{R}^3$ be the corresponding parametrizations of C_1 and C_2, respectively. The function

$$\boldsymbol{\sigma}(t,u) := \boldsymbol{\gamma_2}(u) - \boldsymbol{\gamma_1}(t) \tag{61}$$

is defined on rectangle $[a,b]\times[c,d]$ and its image *does not* contain the origin, $\mathbf{0}$, because $\boldsymbol{\gamma_1}(t) \neq \boldsymbol{\gamma_2}(u)$ for all $t\in[a,b]$ and $u\in[c,d]$ (curves C_1 and C_2 are disjoint!). One should think of $\boldsymbol{\sigma}$ as being a parametric surface in $\mathbb{R}^3\setminus\{\mathbf{0}\}$. This surface is *closed*, i.e., does not have a boundary,[12] since curves C_1 and C_2 are closed.

18 Linking Number Formula

The following is a close relative of Index Formula (108) in **2F**:

$$\mathrm{Ln}(C_1,C_2) \;=\; \frac{1}{4\pi}\int_{\boldsymbol{\sigma}} \frac{x_1\,dx_2\wedge dx_3 + x_2\,dx_3\wedge dx_1 + x_3\,dx_1\wedge dx_2}{(x_1^2+x_2^2+x_3^2)^{3/2}}\,. \tag{62}$$

This formula can be established similarly to how Index Formula (103) was proved in **2F**. One notes first that the differential 2-form on $\mathbb{R}^3\setminus\{\mathbf{0}\}$:

$$\omega_2 \;:=\; \frac{1}{4\pi}\,\frac{x_1\,dx_2\wedge dx_3 + x_2\,dx_3\wedge dx_1 + x_3\,dx_1\wedge dx_2}{(x_1^2+x_2^2+x_3^2)^{3/2}}\,, \tag{63}$$

[12] More properly, one should say that the boundary is empty.

which is sometimes called the *Gauß form*, is *closed*, cf. sample problem ?? in **Problembook**.

Using Gauß'-Ostrogradski's Theorem, one can show that the integral of a closed 2-form over a closed surface does not change when one *continuously deforms* the surface—this is exactly analogous to Theorem (101) of **2F** (which was established using the parametric form of Stokes' Theorem, see Section 25 in **2F**).

Without loss of generality, one can assume that curves C_1 and C_2 are parametrized by interval $[0,1]$. Then it can be shown that, if $\mathrm{Ln}(C_1, C_2) = m$, then parametric surface $\boldsymbol{\sigma}$ can be deformed in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ to the function

$$\boldsymbol{\sigma_1}(t,u) := \begin{pmatrix} \sin(\pi t)\sin(2\pi m u) \\ \sin(\pi t)\cos(2\pi m u) \\ \cos t \end{pmatrix}. \tag{64}$$

which parametrizes unit sphere in $\mathbb{R}^3$ so that every point of sphere, except for the Northern and Southern Poles, is 'visited' exactly m times. The integral of ω_2 over $\boldsymbol{\sigma_1}$ is m *times* the integral of ω_2 over the sphere, i.e., equals m (cf., exercise ?? and sample problem ?? in **Problembook**).