Introduction to differential 2-forms

January 7, 2004

These notes should be studied in conjunction with lectures.[1]

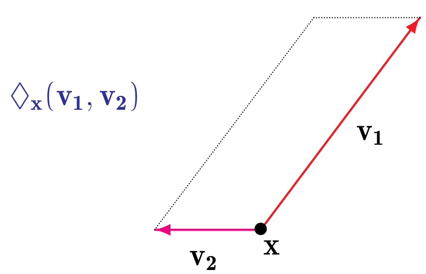
1 Oriented area Consider two column-vectors

$$\mathbf{v_1} = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} \qquad \text{and} \qquad \mathbf{v_2} = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} \tag{1}$$

anchored at a point $\mathbf{x} \in \mathbb{R}^2$. The determinant

$$\psi(\mathbf{x}; \mathbf{v_1}, \mathbf{v_2}) := \det \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = v_{11}v_{22} - v_{21}v_{12} \tag{2}$$

equals, up to a sign, the area of the parallelogram spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$. We will denote this parallelogram by $\lozenge_{\mathbf{x}}(\mathbf{v_1}, \mathbf{v_2})$ and call quantity (2) its **oriented area.**



two ways of ordering a pair of vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ correspond to two ways of orienting parallelogram $\lozenge_{\mathbf{x}}(\mathbf{v_1}, \mathbf{v_2})$: if $\mathbf{v_1}$ comes first then one traverses the boundary of $\lozenge_{\mathbf{x}}(\mathbf{v_1}, \mathbf{v_2})$ by following the direction of $\mathbf{v_1}$; if $\mathbf{v_2}$ comes first then one follows the direction of $\mathbf{v_2}$.

Note the following properties of ψ:

(a) **Linearity in each of its two column-vector variables:**

$$\psi(\mathbf{x}; a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = a\psi(\mathbf{x}; \mathbf{u}, \mathbf{w}) + b\psi(\mathbf{x}; \mathbf{v}, \mathbf{w}) \tag{3}$$

$$\psi(\mathbf{x}; \mathbf{u}, a\mathbf{v} + b\mathbf{w}) = a\psi(\mathbf{x}; \mathbf{u}, \mathbf{v}) + b\psi(\mathbf{x}; \mathbf{u}, \mathbf{w}) \tag{4}$$

[1] Abbreviations DCVF and LI stand for *Differential Calculus of Vector Functions* and *Line Integrals, respectively.*

(b) **Antisymmetry:** $\psi(\mathbf{x};\mathbf{v},\mathbf{u}) = -\psi(\mathbf{x};\mathbf{u},\mathbf{v})$,

($\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ being column-vectors and a and b being scalars).

2 Differential 2-forms Any function $\psi\colon D \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ satisfying the above two conditions will be called a **differential 2-form** on a set $D \subseteq \mathbb{R}^m$. By contrast, differential forms of LI will be called from now on *differential 1-forms*.

3 Exterior product Given two differential 1-forms $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ on D, the formula

$$\psi(\mathbf{x};\mathbf{v}_1,\mathbf{v}_2) := \det\begin{pmatrix} \boldsymbol{\varphi}_1(\mathbf{x};\mathbf{v}_1) & \boldsymbol{\varphi}_1(\mathbf{x};\mathbf{v}_2) \\ \boldsymbol{\varphi}_2(\mathbf{x};\mathbf{v}_1) & \boldsymbol{\varphi}_2(\mathbf{x};\mathbf{v}_2) \end{pmatrix} \tag{5}$$

gives us a differential 2-form. We denote it $\boldsymbol{\varphi}_1 \wedge \boldsymbol{\varphi}_2$ and call it the **exterior product** of 1-forms $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$.

Note that

$$\boxed{\boldsymbol{\varphi}_2 \wedge \boldsymbol{\varphi}_1 = -\boldsymbol{\varphi}_1 \wedge \boldsymbol{\varphi}_2} \quad . \tag{6}$$

Indeed,

$$\begin{aligned} (\boldsymbol{\varphi}_2 \wedge \boldsymbol{\varphi}_1)(\mathbf{x};\mathbf{v}_1,\mathbf{v}_2) &= \det\begin{pmatrix} \boldsymbol{\varphi}_2(\mathbf{x};\mathbf{v}_1) & \boldsymbol{\varphi}_2(\mathbf{x};\mathbf{v}_2) \\ \boldsymbol{\varphi}_1(\mathbf{x};\mathbf{v}_1) & \boldsymbol{\varphi}_1(\mathbf{x};\mathbf{v}_2) \end{pmatrix} \\ &= -\det\begin{pmatrix} \boldsymbol{\varphi}_1(\mathbf{x};\mathbf{v}_1) & \boldsymbol{\varphi}_1(\mathbf{x};\mathbf{v}_2) \\ \boldsymbol{\varphi}_2(\mathbf{x};\mathbf{v}_1) & \boldsymbol{\varphi}_2(\mathbf{x};\mathbf{v}_2) \end{pmatrix} = -(\boldsymbol{\varphi}_1 \wedge \boldsymbol{\varphi}_2)(\mathbf{x};\mathbf{v}_1,\mathbf{v}_2) \,. \end{aligned}$$

In particular, for *any* 1-form $\boldsymbol{\varphi}$ one has

$$\boxed{\boldsymbol{\varphi} \wedge \boldsymbol{\varphi} = 0} \quad . \tag{7}$$

Exercise 1 *Verify that for any differential 1-forms $\boldsymbol{\varphi}$, χ, υ and*[2] *scalars a and b, one has:*

(a_1) $(a\boldsymbol{\varphi} + b\chi) \wedge \upsilon = a\,\boldsymbol{\varphi} \wedge \upsilon + b\,\chi \wedge \upsilon$;

(a_2) $\boldsymbol{\varphi} \wedge (a\chi + b\upsilon) = a\,\boldsymbol{\varphi} \wedge \chi + b\,\boldsymbol{\varphi} \wedge \upsilon$.

[2]Greek letter χ is called *khee* while letter υ is called *ypsilon*.

4 Example Let us calculate $df_1 \wedge df_2$ where f_1 and f_2 are two functions $D \to \mathbb{R}$ on a subset of $\mathbb{R}^2$. We have

$$\begin{aligned} df_1 &= \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 \\ df_2 &= \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 \end{aligned}$$

(it is more instructive to use notation x_1 and x_2 instead of x and y), and

$$\begin{aligned} df_1 \wedge df_2 &= \left(\frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2\right) \wedge \left(\frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2\right) \\ &= \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} dx_1 \wedge dx_2 + \frac{\partial f_2}{\partial x_1}\frac{\partial f_1}{\partial x_2} dx_2 \wedge dx_1 \qquad (\text{since } dx_i \wedge dx_i = 0) \\ &= \left(\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\right) dx_1 \wedge dx_2 \qquad (\text{since } dx_2 \wedge dx_1 = -dx_1 \wedge dx_2) \\ &= (\det J_{\mathbf{f}}(\mathbf{x}))\, dx_1 \wedge dx_2 \,. \end{aligned} \tag{8}$$

where $\mathbf{f} := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ denotes the vector function $D \to \mathbb{R}^2$ having f_1 and f_2 as its components.

5 $dx \wedge dy$ Note that

$$dx \wedge dy\,(\mathbf{x}; \mathbf{v_1}, \mathbf{v_2}) = \det \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \tag{9}$$

which is the right-hand-side of (2) and, up to a sign, the area of parallelogram formed by column-vestors $\mathbf{v}_1$ and $\mathbf{v}_2$ at point $\mathbf{x} \in \mathbb{R}^2$. We call the differential 2-form on $\mathbb{R}^2$, $dx \wedge dy$, the **oriented-area element.**

6 Basic differential forms $dx_i \wedge dx_j$ Differential forms $dx_i \wedge dx_j$, $i \neq j$, on $\mathbb{R}^m$ are called **basic differential 2-forms.** What is their meaning?

If $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$, then

$$dx_i \wedge dx_j\,(\mathbf{x}; \mathbf{u}, \mathbf{v}) = \det \begin{pmatrix} u_i & v_i \\ u_j & v_j \end{pmatrix} \tag{10}$$

which is the (oriented) area of the parallelogram

$$\lozenge_{\bar{\mathbf{x}}}(\bar{\mathbf{u}}, \bar{\mathbf{v}}) \tag{11}$$

where the column-vectors

$$\bar{\mathbf{u}} := \begin{pmatrix} u_i \\ u_j \end{pmatrix} \qquad \text{and} \qquad \bar{\mathbf{v}} := \begin{pmatrix} v_i \\ v_j \end{pmatrix}, \tag{12}$$

and the point

$$\bar{\mathbf{x}} := \begin{pmatrix} x_i \\ x_j \end{pmatrix} \tag{13}$$

are projections of column-vectors $\mathbf{u}$ and $\mathbf{v}$, and point $\mathbf{x}$, respectively, onto the plane $\mathbb{R}^2_{x_i x_j}$ spanned by x_i- and x_j-axes.[3]

7 2-forms on $\mathbb{R}^2$ Let ψ be *any* differential 2-form on a set $D \subseteq \mathbb{R}^2$. For a pair of column-vectors

$$\mathbf{v_1} = v_{11}\,\mathbf{i} + v_{12}\,\mathbf{j} \tag{14}$$

$$\mathbf{v_2} = v_{21}\,\mathbf{i} + v_{22}\,\mathbf{j} \tag{15}$$

to calculate value $\psi(\mathbf{x}; \mathbf{v}_1, \mathbf{v}_2)$ we plug first (14) and use Property (a_1) from Exercise 1:

$$\psi(\mathbf{x}; \mathbf{v}_1, \mathbf{v}_2) = \psi(\mathbf{x}; v_{11}\,\mathbf{i} + v_{12}\,\mathbf{j}, \mathbf{v}_2) = v_{11}\psi(\mathbf{x}; \mathbf{i}, \mathbf{v}_2) + v_{12}\psi(\mathbf{x}; \mathbf{j}, \mathbf{v}_2), \tag{16}$$

and then plug (15) into the right-hand-side of (16) and use Property (a_2) from the same exercise:

$$\begin{aligned}
&= v_{11}(v_{21}\psi(\mathbf{x}; \mathbf{i}, \mathbf{i}) + v_{22}\psi(\mathbf{x}; \mathbf{i}, \mathbf{j})) + v_{12}(v_{21}\psi(\mathbf{x}; \mathbf{j}, \mathbf{i}) + v_{22}\psi(\mathbf{x}; \mathbf{j}, \mathbf{j})) \\
&= (v_{11}v_{22} - v_{21}v_{12})\,\psi(\mathbf{x}; \mathbf{i}, \mathbf{j}) \\
&= \psi(\mathbf{x}; \mathbf{i}, \mathbf{j})\,(dx \wedge dy)(\mathbf{x}; \mathbf{v}_1, \mathbf{v}_2) \\
&= (\psi(\mathbf{x}; \mathbf{i}, \mathbf{j})\,dx \wedge dy)(\mathbf{x}; \mathbf{v}_1, \mathbf{v}_2)\,.
\end{aligned} \tag{17}$$

[3] In other words, parallelogram $\lozenge_{\bar{\mathbf{x}}}(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ is obtained by *projecting* parallelogram $\lozenge_{\mathbf{x}}(\mathbf{u}, \mathbf{v})$ onto $x_i\,x_j$-plane.

In other words, any differential 2-form ψ on a subset of $\mathbb{R}^2$ can be represented as a multiple of the oriented-area element:

$$\boxed{\psi = f\, dx \wedge dy \qquad \text{where} \qquad f(\mathbf{x}) := \psi(\mathbf{x}; \mathbf{i}, \mathbf{j})} \quad . \tag{18}$$

The function-coefficient f in (18) is, for obvious reasons, denoted

$$\frac{\psi}{dx \wedge dy} \,. \tag{19}$$

8 2-forms on $\mathbb{R}^3$ A similar, completely straightforward, calculation shows that any 2-form on a subset $D \subseteq \mathbb{R}^3$ can be represented as

$$\boxed{\psi = f_1\, dy \wedge dz + f_2\, dz \wedge dx + f_3\, dx \wedge dy} \tag{20}$$

or,

$$\psi = f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2 \tag{21}$$

if one uses notation x_1, x_2, x_3 instead of x, y, z.

Exercise 2 *Verify that*

$$f_1(\mathbf{x}) = \psi(\mathbf{x}; \mathbf{j}, \mathbf{k}), \qquad f_2(\mathbf{x}) = \psi(\mathbf{x}; \mathbf{k}, \mathbf{i}) \qquad \textit{and} \qquad f_3(\mathbf{x}) = \psi(\mathbf{x}; \mathbf{i}, \mathbf{j}) \,. \tag{22}$$

A very important observation follows from formulae (20–21):

> on subsets of $\mathbb{R}^3$, *and of* $\mathbb{R}^3$ *only,* both differential 1-forms and differential 2-forms are given in terms of *three* function-coefficients f_1, f_2 and f_3 . (23)

9 Area element The function that associates with a pair of column-vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ anchored at apoint $\mathbf{x} \in \mathbb{R}^m$, the *area* of parallelogram $\Diamond_{\mathbf{x}}(\mathbf{v_1}, \mathbf{v_2})$ will be called the **area element** and denoted α.

We already know that in $\mathbb{R}^2$ the area element coincides with the absolute value of basic differential 2-form

$$\alpha = |dx_1 \wedge dx_2| = |dx \wedge dy| \,. \tag{24}$$

In general, for vectors in Euclidean space $\mathbb{R}^m$, we have the formula

$$\begin{aligned} \alpha(\mathbf{x}; \mathbf{v_1}, \mathbf{v_2}) &= \|\mathbf{v_1}\|\,\|\mathbf{v_2}\| \left|\sin\left(\measuredangle^{\mathbf{v_2}}_{\mathbf{v_1}}\right)\right| = \sqrt{(\|\mathbf{v_1}\|\,\|\mathbf{v_2}\|)^2\left(1-\cos^2\left(\measuredangle^{\mathbf{v_2}}_{\mathbf{v_1}}\right)\right)} \\ &= \sqrt{(\|\mathbf{v_1}\|\,\|\mathbf{v_2}\|)^2 - (\mathbf{v_1}\cdot\mathbf{v_2})^2} \end{aligned} \tag{25}$$

Let us see what does this formula look like in $\mathbb{R}^3$. We have:

$$(\|\mathbf{v_1}\|\,\|\mathbf{v_2}\|)^2 = (v_{11}^2 + v_{12}^2 + v_{13}^2)(v_{21}^2 + v_{22}^2 + v_{23}^2) \tag{26}$$

and

$$(\mathbf{v_1}\cdot\mathbf{v_2})^2 = (v_{11}v_{21} + v_{12}v_{22} + v_{13}v_{23})^2 \,. \tag{27}$$

After expanding the right-hand side of (27) and subtracting it from (26), we get the following formula for the area element in $\mathbb{R}^3$:

$$\alpha(\mathbf{x}; \mathbf{v_1}, \mathbf{v_2}) = \sqrt{\left(\det\begin{pmatrix} v_{21} & v_{32} \\ v_{31} & v_{22}\end{pmatrix}\right)^2 + \left(\det\begin{pmatrix} v_{31} & v_{12} \\ v_{11} & v_{32}\end{pmatrix}\right)^2 + \left(\det\begin{pmatrix} v_{11} & v_{22} \\ v_{21} & v_{12}\end{pmatrix}\right)^2} \tag{28}$$

Recognizing that the 2×2 determinants are just the values of basic forms $dx_2 \wedge dx_3$, $dx_2 \wedge dx_3$ and $dx_2 \wedge dx_3$, we can rewrite (28) in more legible (as well as more easily memorizable!) form:

$$\boxed{\alpha = \sqrt{(dx_2 \wedge dx_3)^2 + (dx_3 \wedge dx_1)^2 + (dx_1 \wedge dx_2)^2}} \,. \tag{29}$$

This is **Pythagoras' Theorem**[4] for the area function, since identity (29) can be expressed

[4] ΠΥΘΑΓΟΡΑΣ (6th Century BC), one of the most mysterious and influential figures in Greek, and therefore also our, intellectual history. He was born in *Samos* in the mid-6th century BC and migrated to *Croton*

also as saying:

> The square of the area of a parallelogram is the sum of the squares of areas of *orthogonal* projections of that parallelogram onto *all* coordinate planes. (30)

As stated, Theorem (30) holds for any n. For $n = 2$, formula (29) reduces to formula (24). For $n = 3$, the coordinate planes are $\mathbb{R}^3_{x_2x_3}$, $\mathbb{R}^3_{x_3x_1}$ and $\mathbb{R}^3_{x_1x_2}$, respectively.

10 Example: cross-product of vectors in $\mathbb{R}^3$ Let us calculate the exterior product of two 1-forms on $\mathbb{R}^3$

$$(a_1dx_1 + a_2dx_2 + a_3dx_3) \wedge (b_1dx_1 + b_2dx_2 + b_3dx_3) \tag{31}$$

with *constant* coefficients a_1, a_2, a_3, b_1, b_2, b_3. The result is the sum of $3 \times 3 = 9$ forms $a_ib_jdx_i \wedge dx_j$. However, three of them are zero, since $dx_i \wedge dx_i = 0$. For the remaining six, one has $a_ib_jdx_i \wedge dx_j = -b_ja_idx_j \wedge dx_i$, so the final result is the following combination of three basic 2-forms on $\mathbb{R}^3$:

$$(a_2b_3 - a_3b_2)dx_2 \wedge dx_3 + (a_3b_1 - a_1b_3)dx_3 \wedge dx_1 + (a_1b_2 - a_2b_1)dx_1 \wedge dx_2 \tag{32}$$

in around 530 BC. There he founded the sect or society that bore his name, and that seems to have played an important role in the political life of *Magna Graecia* for several generations. Pythagoras himself is said to have died as a refugee in *Metapontum*. Pythagorean political influence is attested well into the 4th century, with *Archytas of Tarentum*.

The name of Pythagoras is connected with two parallel traditions, one religious and one scientific. Pythagoras is said to have introduced the doctrine of transmigration of souls into Greece, and his religious influence is reflected in the cult organization of the Pythagorean society, with periods of initiation, secret doctrines and passwords (*akousmata* and *symbola*), special dietary restrictions, and burial rites. Pythagoras seems to have become a legendary figure in his own lifetime and was identified by some with the *Hyperborean Apollo*. His supernatural status was confirmed by a golden thigh, the gift of bilocation, and the capacity to recall his previous incarnations. Classical authors imagine him studying in Egypt; in the later tradition he gains universal wisdom by travels in the east. Pythagoras becomes the pattern of the 'divine man': at once a sage, a seer, a teacher, and a benefactor of the human race.

The scientific tradition ascribes to Pythagoras a number of important discoveries, including the famous geometric theorem that still bears his name. Even more significant for Pythagorean thought is the discovery of the musical consonances: the ratios $2:1$, $3:2$, and $4:3$ representing the length of strings corresponding to the octave and the basic harmonies (the fifth and the fourth). These ratios are displayed in the *tetractys*, an equilateral triangle composed of 10 dots; the Pythagoreans swear an oath by Pythagoras as author of the *tetractys*. The same ratios are presumably reflected in the music of the spheres, which Pythagoras alone was said to hear. (*Quoted from* ***The Oxford Classical Dictionary***, 3rd edition, Oxford: 1996).

The column-vector made of coefficients of 2-form (32) is known under the name of **cross-product** of column-vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:

$$\boxed{\mathbf{a} \times \mathbf{b} := \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}} \quad . \tag{33}$$

Exterior product of 1-forms is the best way to understand the peculiar character of cross-product (unlike dot-product, it exists *only* in 3-dimensional Euclidean space), and its properties:

a) $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$;

b) $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ (and thus also to the plane they span);

c) $\|\mathbf{a} \times \mathbf{b}\| = \mathcal{A}rea(\Diamond(\mathbf{a}, \mathbf{b}))$ where $\Diamond(\mathbf{a}, \mathbf{b})$ denotes the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ (this follows from formulae (28) and (33)).

11 Differential The operation that associates with a function $f\colon D \to \mathbb{R}$ its differential, $f \mapsto df$, defines a correspondence

$$d\colon \{0\text{-forms on } D\} \to \{1\text{-forms on } D\} \tag{34}$$

(we think of functions as differential 0-forms).[5]

We can similarly produce differential 2-forms from 1-forms:

$$d\colon \{1\text{-forms on } D\} \to \{2\text{-forms on } D\} \tag{35}$$

if $\boldsymbol{\varphi} = f_1 dx_1 + \cdots + f_m dx_m$ then its differential $d\boldsymbol{\varphi}$ is defined as the 2-form

$$\boxed{d\boldsymbol{\varphi} := df_1 \wedge dx_1 + \cdots + df_m \wedge dx_m} \quad . \tag{36}$$

[5]Note that (34) is a *function* from the set of 0-forms to the set of 1-forms.

By plugging $df_i = \frac{\partial f_i}{\partial x_1}dx_1 + \cdots + \frac{\partial f_i}{\partial x_n}dx_m$ into the right-hand-side of formula (36) and taking into account the properties of exterior product, we obtain the representation of $d\varphi$ in terms of basic 2-forms:

$$d\varphi = \sum_{1\leqslant i<j\leqslant n} \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \right) dx_i \wedge dx_j \quad . \tag{37}$$

This formula is particularly simple in $\mathbb{R}^2$:

$$d(f_1 dx + f_2 dy) = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx \wedge dy \quad . \tag{38}$$

or,

$$d(f_1 dx_1 + f_2 dx_2) = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 \wedge dx_2 \, . \tag{39}$$

if one uses notation x_1, x_2 instead of x, y.

12 Calculation: $d \circ d = 0$

Calculation of $(d \circ d)f = d(df)$ is a simple exercise:

$$\begin{aligned} d(df) &= d\left(\frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_m \right) \\ &= d\left(\frac{\partial f}{\partial x_1} \right) \wedge dx_1 + \cdots + d\left(\frac{\partial f}{\partial x_n} \right) \wedge dx_m \\ &= \sum_{1\leqslant i,j\leqslant n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \\ &= \sum_{1\leqslant i<j\leqslant n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right) dx_i \wedge dx_j \end{aligned} \tag{40}$$

The classical Clairaut's Theorem, cf. Section 21 in DCVF, says that the mixed partial derivatives

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x_j \partial x_i}$$

are equal if they are *continuous*. Thus, for functions f which have continuous second order partial derivatives, one has the following fundamental identity

$$\boxed{d(df)=0}\quad . \tag{41}$$

13 Example: curl of a vector field In College textbooks of Multivariable Calculus, a function

$$\mathbf{F}=\begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix}:D\to\mathbb{R}^3 \tag{42}$$

defined on a subset $D\subseteq\mathbb{R}^3$, is often called a "vector field" on D. Properly speaking, a **vector field** on a set $D\subseteq\mathbb{R}^m$ is a family of vectors, $\overrightarrow{\mathbf{ab}}$, *one per each point* $\mathbf{a}\in D$. In this case, the correspondence

$$\text{point }\mathbf{a}\quad\mapsto\quad\text{column-vector }\mathbf{b}-\mathbf{a},$$

which sends each point $\mathbf{a}\in D$ to the column-vector $\mathbf{b}-\mathbf{a}$, becomes a function $D\to\mathbb{R}^m$.

In 3-dimensional space the formula for the differential of a 2-form, (37), acquires the form:

$$d(f_1dx+f_2dy+f_3dz)=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)dy\wedge dz+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)dz\wedge dx+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)dx\wedge dy \tag{43}$$

or, preferably,

$$d(f_1dx_1+f_2dx_2+f_3dx_3)=\left(\frac{\partial f_3}{\partial x_2}-\frac{\partial f_2}{\partial x_3}\right)dx_2\wedge dx_3+\left(\frac{\partial f_1}{\partial x_3}-\frac{\partial f_3}{\partial x_1}\right)dx_3\wedge dx_1+\left(\frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}\right)dx_1\wedge dx_2\,. \tag{44}$$

if one uses notation x_1, x_2, x_3 instead of x, y and z.

The vector function made of function-coefficients of 2-form (43) is known under the name of **curl**[6] of vector function $\mathbf{F}$:

$$\operatorname{curl}\mathbf{F} := \begin{pmatrix} \dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_1}{\partial z} - \dfrac{\partial f_3}{\partial x} \\ \dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f_3}{\partial x_2} - \dfrac{\partial f_2}{\partial x_3} \\ \dfrac{\partial f_1}{\partial x_3} - \dfrac{\partial f_3}{\partial x_1} \\ \dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_2} \end{pmatrix} \tag{45}$$

Formula for the differential of a 2-form in $\mathbb{R}^3$, (43), is the real reason why curl $\mathbf{F}$ is important.

In the language that avoids mentioning differential forms, identity (41) becomes the following statement:

$$\boxed{\operatorname{curl}\nabla f = 0} \quad . \tag{46}$$

14 Properties of the operation of differential

For any function $f\colon D \to \mathbb{R}$, differential 1-forms φ and χ on D, and scalars a and b, one has:

(a) $d(a\varphi + b\chi) = a d\varphi + b d\chi$; *Linearity*

(b) $d(f\varphi) = df \wedge \varphi + f d\varphi$; *the Leibniz Rule*

(c) $d(\mathbf{f}^*\varphi) = \mathbf{f}^*(d\varphi)$. *the Chain Rule*

Here $\mathbf{f}\colon E \to \mathbb{R}^m$ is a vector function sending its domain into D and the pullback of 2-forms is defined in exactly the same manner as for 1-forms:

$$\boxed{(\mathbf{f}^*\psi)(\mathbf{x};\mathbf{u},\mathbf{v}) := \psi(\mathbf{f}(\mathbf{x});\mathbf{f}'_{\mathbf{x}}(\mathbf{u}),\mathbf{f}'_{\mathbf{x}}(\mathbf{v}))} \quad . \tag{47}$$

Exercise 3 *Verify the following properties of pullback of 2-forms:*

(a) $\mathbf{f}^*(a\psi_1 + b\psi_2) = a\,\mathbf{f}^*\psi_1 + b\,\mathbf{f}^*\psi_2$; $(a, b \in \mathbb{R})$

[6] The curl of $\mathbf{F}$ is also denoted $\nabla \times \mathbf{F}$.

(b) $\mathbf{f}^*(\varphi \wedge \chi) = \mathbf{f}^*\varphi \wedge \mathbf{f}^*\chi$;

(c) $(\mathbf{f} \circ \mathbf{g})^*\psi = \mathbf{g}^*(\mathbf{f}^*\psi)$. *(Hint: Use the Chain Rule.)*

15 Example Let $D \subseteq \mathbb{R}^2$ and $\mathbf{f}\colon D \to \mathbb{R}^2$ be a function. Pullback $\mathbf{f}^*(dx \wedge dy)$ is a 2-form on a subset of $\mathbb{R}^2$ and, therefore, is a multiple of the basic form $dx \wedge dy$, see Section 7. We shall find the ratio

$$\frac{\mathbf{f}^*(dx \wedge dy)}{dx \wedge dy}\,. \tag{48}$$

One has

$$\begin{aligned}
\mathbf{f}^*(dx \wedge dy) &= (f^*dx) \wedge (\mathbf{f}^*dy) = df_1 \wedge df_2 \\
&= \left(\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy\right) \wedge \left(\frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy\right) \\
&= \left(\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}\right) dx \wedge dy \qquad (49) \\
&= (\det J_{\mathbf{f}})\, dx \wedge dy\,. \qquad (50)
\end{aligned}$$

Thus we obtain the following beautiful formula:

$$\boxed{\frac{\mathbf{f}^*(dx \wedge dy)}{dx \wedge dy} = \det J_{\mathbf{f}}}\,. \tag{51}$$

For a function $\mathbf{f}\colon D \to \mathbb{R}^m$, defined on a subset of $\mathbb{R}^m$, the determinant, $\det J_{\mathbf{f}}(\mathbf{a})$, of the Jacobi matrix of $\mathbf{f}$ at point $\mathbf{a} \in D$ plays a fundamental role in Multivariable Calculus. It is usually referred to as the **Jacobian** of $\mathbf{f}$ at point $\mathbf{a}$.

16 Riemann sums and integral of 2-forms in $\mathbb{R}^2$ For any plane rectangle

$$\mathcal{I} = \left\{\mathbf{x} \in \mathbb{R}^2 \mid a_1 \leqslant x \leqslant b_1\,,\ a_2 \leqslant y \leqslant b_2\right\} \tag{52}$$

its *diameter* $\|\mathbf{b} - \mathbf{a}\|$ will be denoted $|\mathcal{I}|$.

A **partition** of $\mathcal{I}$ consists of a finite family $\mathscr{P} = \{J\}$ of closed rectangles with *non-overlapping* interiors, whose union equals $\mathcal{I}$.

Tagging a partition is the same as choosing for each member $J \in \mathscr{P}$ a point $\mathbf{x}_J^*$. Tagged members of $\mathscr{P}$ will be called *cells*.

Each cell J defines two vectors $\mathbf{u}_J$ and $\mathbf{v}_J$ (see Figure 1).

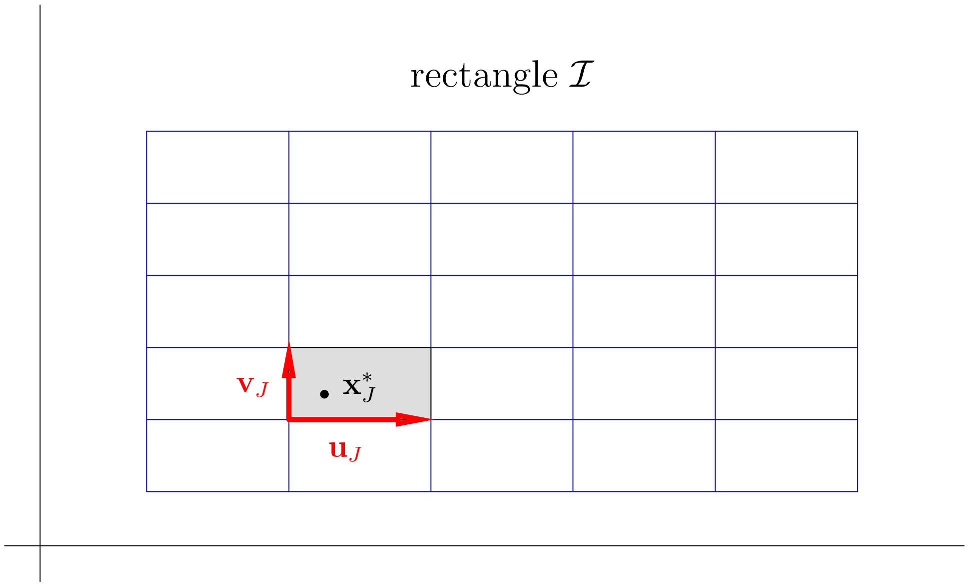


Figure 1: *A tagged partition of* a rectangle $\mathcal{I}$; shown one cell.

With any differential 2-form ψ on $\mathcal{I}$ and any tagged partition $\mathscr{P}$ we can associate the so called **Riemann sum**:

$$S_{\mathscr{P}}(\psi) := \sum_{J \in \mathscr{P}} \psi(\mathbf{x}_J^*; \mathbf{u}_J, \mathbf{v}_J) \ . \tag{53}$$

The **Riemann integral** of ψ *over* $\mathcal{I}$ is defined as the limit of Riemann sums when the mesh of partition $\mathscr{P}$:

$$\|\mathscr{P}\| := \max_{J \in \mathscr{P}} |J| \tag{54}$$

approaches zero:

$$\int_{\mathcal{I}} \psi := \lim_{\|\mathscr{P}\| \to 0} S_{\mathscr{P}}(\psi) \ . \tag{55}$$

Riemann integral over more general *bounded* sets $D \subseteq \mathbb{R}^2$ is introduced as follows. Take *any* closed rectangle $\mathcal{I}$ *containing* D and extend ψ to a form on $\mathcal{I}$ by the formula

$$\bar{\psi}(\mathbf{x};\mathbf{u},\mathbf{v}) = \begin{cases} \psi(\mathbf{x};\mathbf{u},\mathbf{v}) & \text{if } \mathbf{x} \in D \\ 0 & \text{otherwise}. \end{cases} \tag{56}$$

This is called **extension by zero.** Then, the Riemann itegral of form ψ over D is defined as the integral of $\bar{\psi}$ over rectangle $\mathcal{I}$:

$$\int_D \psi := \int_{\mathcal{I}} \bar{\psi}\,. \tag{57}$$

The result does not depend on the choice of rectangle containing D.

17 Double integrals

For any function $f\colon D \to \mathbb{R}$ on a rectangle $\mathcal{I}$, its Riemann integral

$$\iint\limits_{\mathcal{I}} f\,dx\,dy\,, \tag{58}$$

which is also denoted

$$\iint\limits_{\mathcal{I}} f(x,y)\,dx\,dy\,, \tag{59}$$

is defined as the limit of the Riemann sums:

$$S_{\mathscr{P}}(f\,|dx \wedge dy|) = \sum_{J \in \mathscr{P}} f(\mathbf{x}_J^*)\,|dx \wedge dy|(\mathbf{x}_J^*;\mathbf{u}_J,\mathbf{v}_J) = \sum_{J \in \mathscr{P}} f(\mathbf{x}_J^*)\ \text{Area}(J)\,. \tag{60}$$

Traditional notation (58)–(59) explains why it is called the **double integral** of f. A more accurate notation

$$\int_{\mathcal{I}} f\,|dx \wedge dy|$$

is also used.

Double integral over more general bounded regions $D \subseteq \mathbb{R}^2$ is defined as described above in the case of differential forms.

From this definition it is clear that for plane regions

$$\iint\limits_{D} f(x,y)\,dx\,dy = \int_D f\,dx \wedge dy\,. \tag{61}$$

When D is a rectangle $\mathcal{I}$, like in (52), double integral $\iint_{\mathcal{I}} f(x,y)\,dx\,dy$ should not be confused with **iterated intregrals**

$$\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y)dx\right)dy \qquad \text{and} \qquad \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)dy\right)dx\,. \tag{62}$$

That all three are in fact equal is a nontrivial fact.

18 Area of a bounded set in the plane By definition, the area of a bounded subset $D \subseteq \mathbb{R}^2$ equals

$$\mathrm{Area}(D) \;=\; \iint_D dx\,dy\,. \tag{63}$$

Integral (63) exists precisely when the *characteristic function* of set D

$$\chi_D(\mathbf{x}) := \begin{cases} 1 & \text{if } \mathbf{x} \in D \\ 0 & \text{otherwise} \end{cases} \tag{64}$$

is Riemann integrable. In order for characteristic function χ_D *not to be* Riemann integrable the boundary of D:

$$\partial D := \left\{\mathbf{x} \in \mathbb{R}^2 \mid \text{any neighborhood of } \mathbf{x} \text{ contains both points from } D \textit{ and } \text{not from } D\right\} \tag{65}$$

must be "massive."

We say that a subset $B \subseteq \mathbb{R}^2$ has **measure zero** if, for *any* given number $\epsilon > 0$, there exists a (possibly infinite) sequence of squares $\square_m$ such that

$$B \;\subseteq\; \square_1 \cup \square_2 \cup \ldots \qquad \text{and} \qquad \sum_{n=1} \mathrm{Area}(\square_n) < \epsilon\,. \tag{66}$$

Any countable set $D = \{\mathbf{x_1},\ \mathbf{x_2},\ \ldots\}$ and any rectifiable curve have measure zero; in fact, most 'fractal' curves in the plane (e.g. the curve in figure 2 in LI) have measure zero.[7]

[7] A discipline of Mathematics called *Measure Theory* was developed in the first half of XX-th century; the definition of integral based on it (*Lebesgue integral*) is today a standard anybody who is serios about Mathematics must know. Modern Probability Theory and Mathematical Statistics are largely formulated in the language of Measure Theory.

One can indeed show that

$$\boxed{\text{a bounded subset } D \subseteq \mathbb{R}^2 \text{ has a well defined area (i.e., integral (63) exists) \textit{if and only if} boundary } \partial D \text{ has measure zero}}\ . \tag{67}$$

19 An integral inequality A function $f\colon D \to \mathbb{R}$ is said to be **bounded** if there exists a number $M > 0$ such

$$|f(\mathbf{x})| \leqslant M \qquad \text{for all } \mathbf{x} \in D\,. \tag{68}$$

Any such number is then called a **bound** for f. For example, the function

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1 - \sin x_2$$

is bounded on the rectangle

$$D = \left\{\mathbf{x} \in \mathbb{R}^2 \mid -3 \leqslant x_1 \leqslant 2\,,\ 4 \leqslant x_2 \leqslant 10\right\}, \tag{69}$$

since

$$|f(\mathbf{x})| = |x_1 - \sin x_2| \leqslant |x_1| + |\sin x_2| \leqslant 3 + 1 = 4$$

for $\mathbf{x} \in D$. In particular, any $M \geqslant 4$ is a bound for f.

For a bounded function $f\colon D \to \mathbb{R}$ which is integrable over D, the important inequality

$$\boxed{\left|\iint_D f(x,y)\,dx\,dy\right| \leqslant M\ \mathrm{Area}(D)} \tag{70}$$

follows directly from the following obvious inequality satisfied by Riemann sums (60):

$$\left|\,S_{\mathscr{P}}(f\,|dx \wedge dy|)\right| = \left|\sum_{J\in\mathscr{P}} f(\mathbf{x}_J^*)\ \mathrm{Area}(J)\right| \leqslant \sum_{J\in\mathscr{P}} \left|\, f(\mathbf{x}_J^*)\right| \mathrm{Area}(J) \tag{71}$$

$$\leqslant \sum_{J\in\mathscr{P}} M\ \mathrm{Area}(J) = M\ \mathrm{Area}(\mathcal{I})\,. \tag{72}$$

In particular, for *any* bounded function on a set $D \subseteq \mathbb{R}^2$ of zero area, one has

$$\iint_D f(x,y)\,dx\,dy = 0\,. \tag{73}$$

20 The so called "Fubini's Theorem" When the double integral exists, then it is equal to either of the two iterated integrals

$$\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y)\,dx\right)dy \;=\; \iint_{\mathcal{J}} f(x,y)\,dx\,dy \;=\; \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\,dy\right)dx\,. \tag{74}$$

This theorem, in fact, has been proved in 1882 by Paul David Gustav du Bois-Reymond (1831-1889).[8] Italian Guido Fubini (1879-1943) in 1907 proved a much more general theorem and his name has been often attached even to the special case we encounter here.

Fubini's theorem reduces computation of double integrals to integrals of Freshman Calculus.

21 Green's Theorem A *loop* $\gamma\colon [a,b] \to \mathbb{R}^m$, cf. Section 36 of **LI**, is said to be **simple** if it is one-to-one on interval $[a,b)$. A curve C parametrized by simple loop will be called a **simple closed curve**. A simple closed curve is a continuous one-to-one image of a circle.

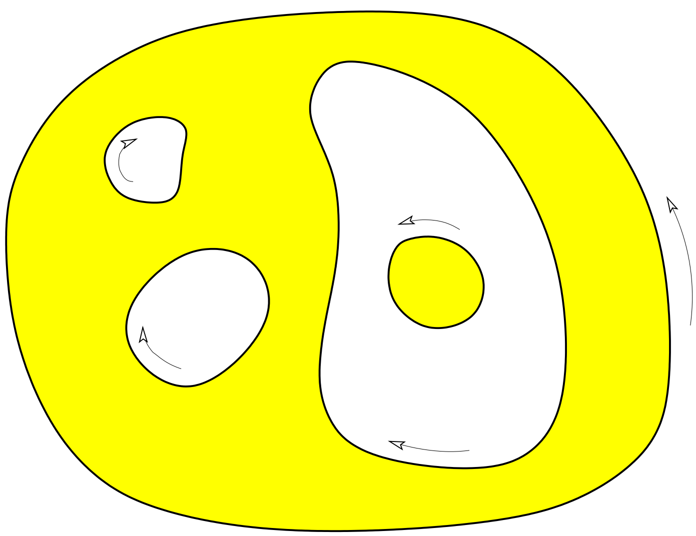


Figure 2: An example of a *simple plain region* D: it has two *connected components* and its boundary ∂D consists of five *simple closed curves* with *induced orientation* indicated by arrows.

[8] Ueber das Doppelintegral, *Journal für Mathematik (Crelle)*, **94** (1883), 273–290.

Let D be a region in $\mathbb{R}^2$ whose boundary is the *disjoint* union of simple closed curves. We shall call such sets *simple (plane) regions*. The orientation of the boundary, ∂D, which leaves set D *leftwards* when we traverse the boundary, will be called the *induced orientation* (or, *positive* with respect to set D).

The following theorem, due to George Green (1793–1841), expresses a fundamental link between the theories of integration of differential 1- and 2-forms. It could be called the *Fundamental Theorem of Calculus for 1-forms.*

> Let φ be a differential 1-form on a simple plane region D. If the components of the boundary, ∂D, of D are rectifiable simple closed curves, then
> $$\int_D d\varphi = \int_{\partial D} \varphi \,. \tag{75}$$

One should understand Green's Theorem (75) as saying that the two integrals are equal when they exist. Each component of the boundary of D is supposed to be positively oriented (with respect to D).

22 Parametric surfaces A continuous function $\boldsymbol{\sigma}\colon D \to \mathbb{R}^m$, where D is a subset of $\mathbb{R}^2$, will be called a **parametric surface.** Set D will be referred to as the *parameter* set of $\boldsymbol{\sigma}$.

We shall say that parametric surface $\boldsymbol{\sigma}$ is contained in set $E \subseteq \mathbb{R}^m$ if $\boldsymbol{\sigma}(\tau) \in E$ for any point $\tau \in D$.

Suppose ψ is differential 2-form on E. Were we to define integral of ψ along parametric surface $\boldsymbol{\sigma}$ by analogy with our definition of path integrals in Section 1 of **LI**, we would have encountered serious difficulties.[9]

Recall that path integrals reduce to Riemann integrals thanks to the Change of Variables Formula, see (47) and (49) in **LI**, we could have *defined* path integral $\int_\gamma \varphi$ of a 1-form φ as the integral

$$\int_{[a,b]} \gamma^*\varphi = \int_a^b \gamma^*\varphi \,.$$

[9] See, e.g. an elementary article by Tibor Radó, What is the area of a surface?, *Amer. Math. Monthly,* 50 (1943), 139–141.

Guided by this, we will define integral $\int_{\boldsymbol{\sigma}} \psi$ as

$$\boxed{\int_{\boldsymbol{\sigma}} \psi := \iint\limits_{D} \boldsymbol{\sigma}^* \psi} \,. \tag{76}$$

The Change of Variables Formula:

$$\boxed{\int_{\mathbf{f}\circ\boldsymbol{\sigma}} \psi = \int_{\boldsymbol{\sigma}} \mathbf{f}^* \psi} \tag{77}$$

is then built-in into our definition.

23 Example: integrating a 2-form over a cone A slice of height r, cut out by planes $z = 0$ and $z = r$, of the cone in $\mathbb{R}^3$ given by the equation

$$z^2 = x^2 + y^2, \tag{78}$$

admits a simple and nearly one-to-one parametrization

$$\boldsymbol{\sigma}\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) = \left(\begin{pmatrix} t_2 \cos t_1 \\ t_2 \sin t_1 \\ t_2 \end{pmatrix}\right) \qquad (0 \leqslant t_1 \leqslant 2\pi,\ 0 \leqslant t_2 \leqslant r)\,. \tag{79}$$

by the rectangle

$$\mathcal{I} = \left\{\tau = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, 0 \leqslant t_1 \leqslant 2\pi,\ 0 \leqslant t_2 \leqslant r\right\}. \tag{80}$$

For any function f whose domain contains this slice of the cone surface, we have

$$\begin{aligned}\int_{\boldsymbol{\sigma}} f\,dx\wedge dy &= \int_{\mathcal{J}} \boldsymbol{\sigma}^*(f\,dx\wedge dy) = \int_{\mathcal{J}} f\left(\begin{pmatrix} t_2\cos t_1\\ t_2\sin t_1\\ t_2\end{pmatrix}\right) d\sigma_1\wedge d\sigma_2\\ &= \int_{\mathcal{J}} f\left(\begin{pmatrix} t_2\cos t_1\\ t_2\sin t_1\\ t_2\end{pmatrix}\right)\frac{d\sigma_1\wedge d\sigma_2}{dt_1\wedge dt_2}\,dt_1\wedge dt_2\\ &= \int_0^r\left(\int_0^{2\pi} f\left(\begin{pmatrix} t_2\cos t_1\\ t_2\sin t_1\\ t_2\end{pmatrix}\right)\det\begin{pmatrix}\dfrac{\partial\sigma_1}{\partial t_1} & \dfrac{\partial\sigma_1}{\partial t_2}\\ \dfrac{\partial\sigma_2}{\partial t_1} & \dfrac{\partial\sigma_2}{\partial t_2}\end{pmatrix} dt_1\right) dt_2\,. \end{aligned} \tag{81}$$

Here

$$\det\begin{pmatrix}\dfrac{\partial\sigma_1}{\partial t_1} & \dfrac{\partial\sigma_1}{\partial t_2}\\ \dfrac{\partial\sigma_2}{\partial t_1} & \dfrac{\partial\sigma_2}{\partial t_2}\end{pmatrix} = \det\begin{pmatrix}\cos t_1 & -t_2\sin t_1\\ \sin t_1 & t_2\cos t_1\end{pmatrix} = t_2. \tag{82}$$

Thus,

$$\int_{\boldsymbol{\sigma}} f\,dx\wedge dy = \int_0^r\left(\int_0^{2\pi} f\left(\begin{pmatrix} t_2\cos t_1\\ t_2\sin t_1\\ t_2\end{pmatrix}\right) t_2 dt_1\right) dt_2\,. \tag{83}$$

For example,

$$\begin{aligned}\int_{\boldsymbol{\sigma}} x^2 z\,dx\wedge dy &= \int_0^r\left(\int_0^{2\pi}(t_2\cos t_1)^2 t_2\cdot t_2 dt_1\right) dt_2\\ &= \left(\int_0^{2\pi}\cos^2 t_1\,dt_1\right)\left(\int_0^r t_2^4 dt_2\right) = \frac{\pi r^5}{5}\,. \end{aligned} \tag{84}$$

24 Area of a parametric surface The area of a parametric surface $\boldsymbol{\sigma}\colon D\to\mathbb{R}^m$ is *defined* as the integral over $\boldsymbol{\sigma}$ of the area element α introduced in Section 9:[10]

$$\mathrm{Area}(\boldsymbol{\sigma}) := \int_{\boldsymbol{\sigma}}\alpha = \int_D \boldsymbol{\sigma}^*\alpha\,. \tag{85}$$

[10]The difficulties with introducing the area as the limit of the areas of polyhedral approximations, have been signalled in a footnote on page 18.

For parametric surfaces in two or three dimensional space, we have explicit formulae for the area element, namely (24) and (28), respectively. Thus,

$$\mathrm{Area}(\boldsymbol{\sigma}) \overset{(\mathrm{in}\,\mathbb{R}^2)}{=} \iint_D \left|\det J_{\boldsymbol{\sigma}}(t,u)\right| dt\,du \tag{86}$$

$$\overset{(\mathrm{in}\,\mathbb{R}^3)}{=} \iint_D \sqrt{(\det J_{\boldsymbol{\sigma}_{\hat{1}}}(t,u))^2 + (\det J_{\boldsymbol{\sigma}_{\hat{2}}}(t,u))^2 + (\det J_{\boldsymbol{\sigma}_{\hat{3}}}(t,u))^2}\, dt\,du \tag{87}$$

where $\boldsymbol{\sigma}_{\hat{i}}\colon D \to \mathbb{R}^2$, $i = 1, 2, 3$, are the functions:

$$\boldsymbol{\sigma}_{\hat{1}} := \begin{pmatrix} \sigma_2 \\ \sigma_3 \end{pmatrix}, \qquad \boldsymbol{\sigma}_{\hat{2}} := \begin{pmatrix} \sigma_3 \\ \sigma_1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\sigma}_{\hat{3}} := \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}. \tag{88}$$

25 Stokes' Theorem (parametric form) Suppose that parameter set D satisfy hypothesis of Green's Theorem, see Section 21.

Then we have

$$\begin{aligned} \int_{\boldsymbol{\sigma}} d\boldsymbol{\varphi} &= \int_D \boldsymbol{\sigma}^* d\boldsymbol{\varphi} && \text{by Definition (76)} \\ &= \int_D d(\boldsymbol{\sigma}^*\boldsymbol{\varphi}) && \text{by Property 14(c)} \\ &= \int_{\partial D} \boldsymbol{\sigma}^*\boldsymbol{\varphi} && \text{by Green's Theorem} \end{aligned} \tag{89}$$

We shall call the equality

$$\boxed{\int_{\boldsymbol{\sigma}} d\boldsymbol{\varphi} = \int_{\partial D} \boldsymbol{\sigma}^*\boldsymbol{\varphi}} \tag{90}$$

the *parametric* form of Stokes' Theorem. We shall now give a remarkable application of (90).

26 Homotopic paths Let E be a subset of $\mathbb{R}^m$. Suppose we are given two paths in E, $\boldsymbol{\gamma}_0\colon [a,b] \to E$ and $\boldsymbol{\gamma}_1\colon [a,b] \to E$, having the same starting point

$$\boldsymbol{\gamma}_0(a) = \boldsymbol{\gamma}_1(a) = \mathbf{a}$$

and the same endpoint

$$\gamma_0(b) = \gamma_1(b) = \mathbf{b}\,.$$

We say that these paths are **homotopic** (**in set** E) if there exists a continuous function

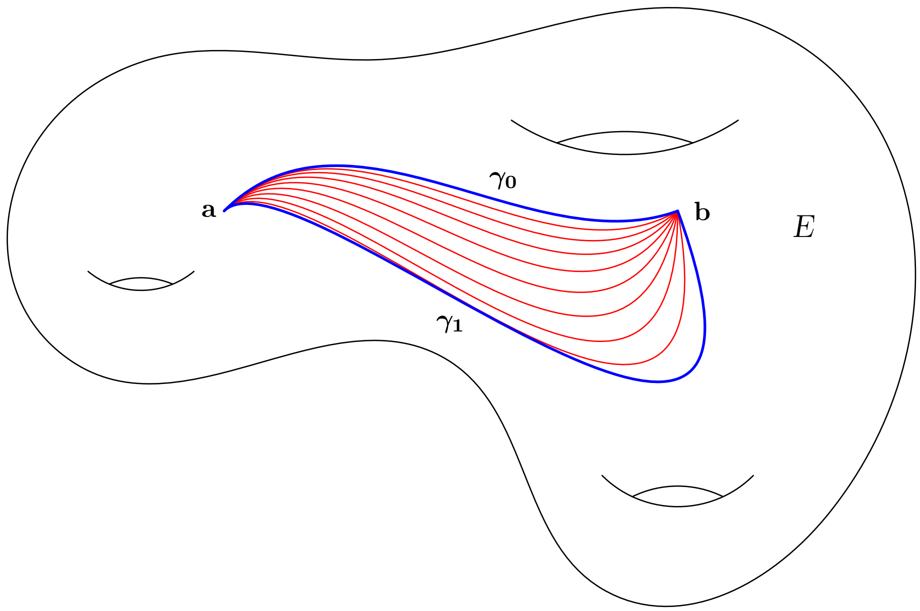


Figure 3: An example of a homotopy between paths.

$H\colon \mathcal{I} \to E$ defined on the rectangle

$$\mathcal{I} := [a, b] \times [0, 1] = \left\{ \begin{pmatrix} t \\ u \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, a \leqslant t \leqslant b\,,\ 0 \leqslant u \leqslant 1 \right\} \tag{91}$$

such that

$$H\left(\begin{pmatrix} t \\ 0 \end{pmatrix}\right) = \gamma_0(t) \qquad \text{and} \qquad H\left(\begin{pmatrix} t \\ 1 \end{pmatrix}\right) = \gamma_1(t) \tag{92}$$

for *all* $t \in [a, b]$, and

$$H\left(\begin{pmatrix} a \\ u \end{pmatrix}\right) = \mathbf{a} \qquad \text{and} \qquad H\left(\begin{pmatrix} b \\ u \end{pmatrix}\right) = \mathbf{b} \tag{93}$$

for *all* $u \in [0, 1]$. Function H in this case is called a **homotopy** *between* γ_0 and γ_1. If the paths are differentiable and homotopic, then one can always find a differentiable homotopy.

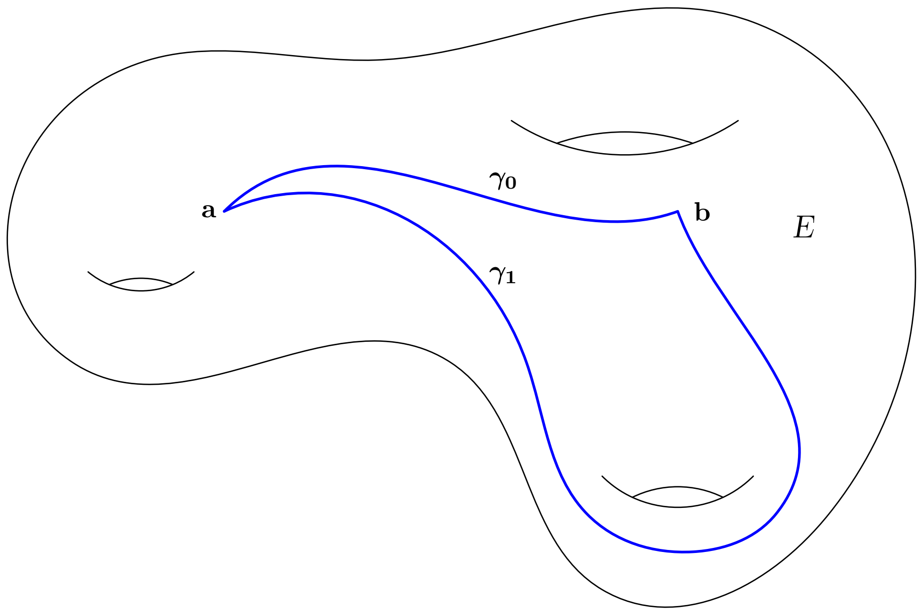


Figure 4: An example of *nonhomotopic* paths: there is no homotopy between these two path in set E.

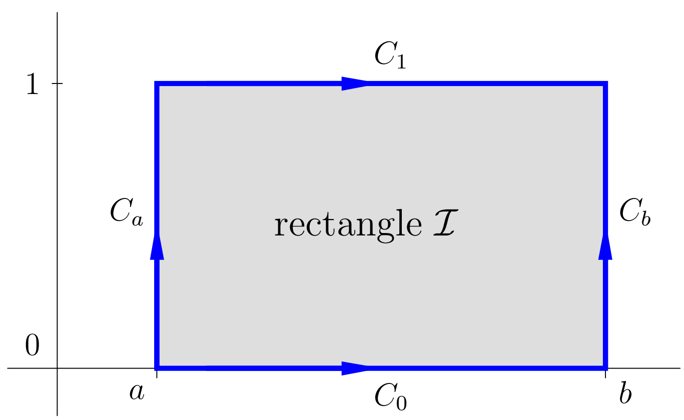


Figure 5: The *positively* oriented boundary of rectangle $\mathcal{I}$ is the union of C_0, C_b, $-C_1$ and $-C_a$.

A homotopy between paths is a *parametric surface*. Stokes' Theorem (90) thus gives us, for any 1-form $\boldsymbol{\varphi}$ on E, the equality:

$$\begin{aligned}\int_H d\boldsymbol{\varphi} = \int_{\partial\mathcal{I}} H^*\boldsymbol{\varphi} &= \int_{C_0} H^*\boldsymbol{\varphi} + \int_{C_b} H^*\boldsymbol{\varphi} + \int_{-C_1} H^*\boldsymbol{\varphi} + \int_{-C_a} H^*\boldsymbol{\varphi} \\ &= \int_{C_0} H^*\boldsymbol{\varphi} + \int_{C_b} H^*\boldsymbol{\varphi} - \int_{C_1} H^*\boldsymbol{\varphi} - \int_{C_a} H^*\boldsymbol{\varphi}\,. \end{aligned} \tag{94}$$

Obvious parametrizations of horizontal segments C_0 and C_1 are:

$$\boldsymbol{\delta_0}\colon [a,b] \to \mathbb{R}^2, \qquad \boldsymbol{\delta_0}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$$

and

$$\boldsymbol{\delta_1}\colon [a,b] \to \mathbb{R}^2, \qquad \boldsymbol{\delta_1}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$$

respectively. Similarly, obvious parametrizations of vertical segments C_0 and C_1 are:

$$\boldsymbol{\delta_a}\colon [0,1] \to \mathbb{R}^2, \qquad \boldsymbol{\delta_a}(u) = \begin{pmatrix} a \\ u \end{pmatrix}$$

and

$$\boldsymbol{\delta_b}\colon [0,1] \to \mathbb{R}^2, \qquad \boldsymbol{\delta_b}(u) = \begin{pmatrix} b \\ u \end{pmatrix}$$

respectively. Thus, the right-hand side of (94) equals

$$\int_a^b (H\circ\boldsymbol{\delta}_0)^*\boldsymbol{\varphi} + \int_0^1 (H\circ\boldsymbol{\delta}_b)^*\boldsymbol{\varphi} - \int_a^b (H\circ\boldsymbol{\delta}_1)^*\boldsymbol{\varphi} - \int_0^1 (H\circ\boldsymbol{\delta}_a)^*\boldsymbol{\varphi}\,. \tag{95}$$

Now,

$$(H\circ\boldsymbol{\delta}_0)(t) = H\left(\begin{pmatrix} t \\ 0 \end{pmatrix}\right) = \boldsymbol{\gamma_0}(t) \qquad \text{and} \qquad (H\circ\boldsymbol{\delta}_1)(t) = H\left(\begin{pmatrix} t \\ 0 \end{pmatrix}\right) = \boldsymbol{\gamma_1}(t)\,,$$

while $(H\circ\boldsymbol{\delta}_a)$ and $(H\circ\boldsymbol{\delta}_b)$ are degenerate one-point paths:

$$(H\circ\boldsymbol{\delta}_0)(u) = H\left(\begin{pmatrix} a \\ u \end{pmatrix}\right) = a \qquad \text{and} \qquad (H\circ\boldsymbol{\delta}_b)(u) = H\left(\begin{pmatrix} b \\ u \end{pmatrix}\right) = b\,.$$

In particular, $(H\circ\boldsymbol{\delta}_a)^*\boldsymbol{\varphi} = (H\circ\boldsymbol{\delta}_b)^*\boldsymbol{\varphi} = 0$ for *any* form $\boldsymbol{\varphi}$, and (95) equals

$$\int_a^b (H\circ\boldsymbol{\delta}_0)^*\boldsymbol{\varphi} - \int_a^b (H\circ\boldsymbol{\delta}_1)^*\boldsymbol{\varphi} = \int_{\boldsymbol{\gamma_0}} \boldsymbol{\varphi} - \int_{\boldsymbol{\gamma_1}} \boldsymbol{\varphi} \tag{96}$$

by Change of Variables Formula (49) in **LI**.

By combining, (94), (95) and (96), we obtain the following important identity:

$$\boxed{\int_{\gamma_0} \varphi - \int_{\gamma_1} \varphi = \int_H d\varphi\,.} \tag{97}$$

27 (Freely) homotopic loops A path $\gamma\colon [a,b] \to \mathbb{R}^m$ which ends where it starts:

$$\gamma(a) = \gamma(b) \tag{98}$$

is called a **loop**. For loops, one can relax the definition of *homotopy*. For two loops $\gamma_0\colon [a,b] \to E$ and $\gamma_1\colon [a,b] \to E$, a function $H\colon \mathcal{I} \to E$ defined on rectangle (91), which satisfies condition (92) and the following condition

$$H\left(\begin{pmatrix} a \\ u \end{pmatrix}\right) = H\left(\begin{pmatrix} b \\ u \end{pmatrix}\right) \qquad (\text{for } \textit{all } u \in [0,1]), \tag{99}$$

which is weaker[11] than condition (93), will be called a **free homotopy** between loops γ_0 and γ_1.

A loop $\gamma\colon [a,b] \to E$ is said to be **contractible** if it is homotopic in E to a constant loop

$$\gamma_0(t) = \mathbf{a}, \qquad (\text{for all } t \in [a,b]), \tag{100}$$

i.e., a loop that stays at a point $\mathbf{a} \in E$. Examples of noncontactible loops in subsets of $\mathbb{R}^3$ are provided by *links*, see Section 17 in **3F** (and illustrations to Problems **??** and **??**–**??** in **Problembook**).

When H is a free homotopy between two loops, integrals $\int_0^1 (H\circ\delta_a)^*\varphi$ and $\int_0^1 (H\circ\delta_b)^*\varphi$ in formula (95) do not necessarily vanish. Instead, they cancel each other, because now $\delta_a = \delta_b$. In particular, identity (97) holds also for *free homotopies* between loops.

[11] Condition (99) means that each path $\gamma_u\colon [a,b] \to \mathbb{R}^m$, $u \in [0,1]$, where $\gamma_u(t) := H\left(\begin{pmatrix} t \\ u \end{pmatrix}\right)$, is a *loop*.

28 Closed 1-forms A 1-form φ is said to be **closed** if $d\varphi = 0$. According to Theorem (41), any *exact* form[12] is closed.

For closed forms, identity (97) yields the following fundamental fact.

Let γ_0 and γ_1 be two homotopic paths in $E \subseteq \mathbb{R}^m$, or two (freely) homotopic loops. Then, for any closed 1-form φ on E,

$$\int_{\gamma_0} \varphi = \int_{\gamma_1} \varphi \,.$$

Since the integral of *any* form over constant loop (100) is zero, we deduce that (101)

$$\int_{\gamma} \varphi = 0$$

for any *closed* 1-form φ and any loop γ that is *contractible in the domain of* φ.

In other words, for any *closed* differential 1-form φ on a set $E \subseteq \mathbb{R}^m$, integral $\int_\gamma \varphi$ depends *only* on the homotopy class of γ.

29 Winding number of an oriented closed curve Let $\mathbf{a}$ be a point belonging to the *complement* of C:

$$\mathbb{R}^2 \setminus C := \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} \notin C\} \tag{102}$$

of an *oriented* closed[13] curve C in the plane. The number of times curve C winds itself around point $\mathbf{a}$ is called the **winding number** of curve C about point $\mathbf{a}$ or, the **index** of point $\mathbf{a}$ with respect to curve C. To determine it, draw a *ray* ρ from point $\mathbf{a}$ to infinity which is **transversal** to curve C. This means that ρ is allowed to intersect C only at *regular* points[14] of C, and that ρ is *not tangent* to C at any point of intersection. Two examples of transversal rays are shown in Figure 6.

[12]Recall from **LI**, Section 35, that a 1-form φ on set D is exact if $\varphi = df$ for some function on D.
[13]We shall be content here with the following *working* definition of a closed curve: *a continuous image of a finite number of circles*. Orienting such a curve is the same as orienting *each* constituent loop.
[14]A point $\mathbf{x} \in C$ is **regular** if in its vicinity C is a *regular arc*, see Section 30 of **LI**.

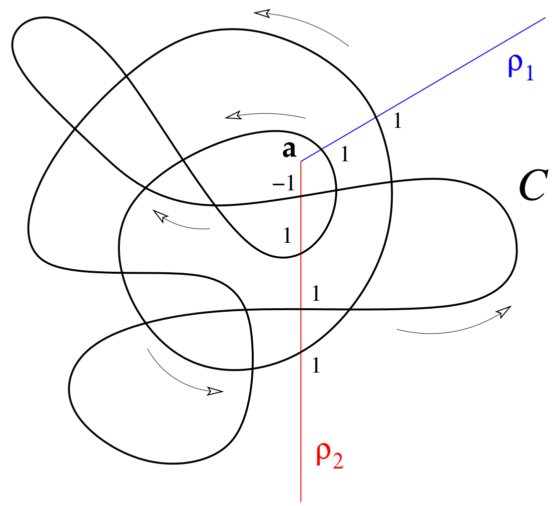


Figure 6: The winding number of an oriented closed curve C about a point $\mathbf{a} \in \mathbb{R}^2 \setminus C$ does not depend on the choice of transversal ray: it equals $1 + 1 = 2$ along ray ρ_1 and $-1 + 1 + 1 + 1 = 2$ along ray ρ_2.

Then, moving *outwards* along ρ from point $\mathbf{a}$, each time you cross C you *add* 1, if at that point curve C moves *leftwards*, and you *subtract* 1 if curve C moves *rightwards*. The result does not depend on the choice of ray (as illustrated by Figure 6). Thus determined number will be denoted $\operatorname{ind}_C(\mathbf{a})$.

Let us make some simple observations:

(a) The index is always an integer. Changing orientation of curve C reverses the sign of index of all points in $\mathbb{R}^2 \setminus C$.

(b) Any integer can occur as index: for example, point $\mathbf{a}$ in Figure 7 has index -5.

(c) The index does not change if one replaces point $\mathbf{a}$ by points sufficiently close to it. This, in turn, implies that *all* points belonging to the same *connected component* of complement $\mathbb{R}^2 \setminus C$ have the same index. I have illustrated this in Figure 8, by indicating, for every connected component of $\mathbb{R}^2 \setminus C$, the index of points belonging to it.
(d) Among connected components of complement $\mathbb{R}^2 \setminus C$, there is exactly one which is *unbounded*. For every point $\mathbf{a}$ in the unbounded component, $\operatorname{ind}_C(\mathbf{a}) = 0$.

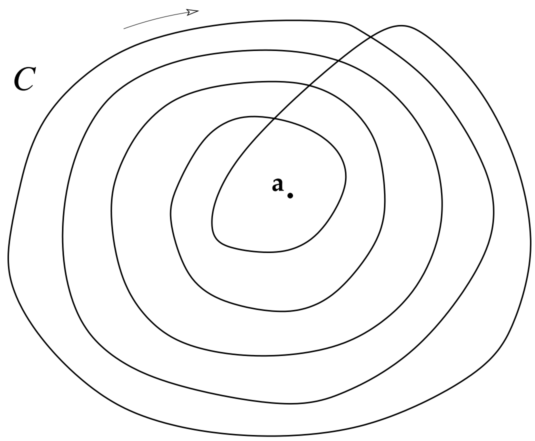


Figure 7: Curve C winds itself 5 times in clockwise (i.e., negative) direction around point $\mathbf{a}$, hence $\operatorname{ind}_C(\mathbf{a}) = -5$.

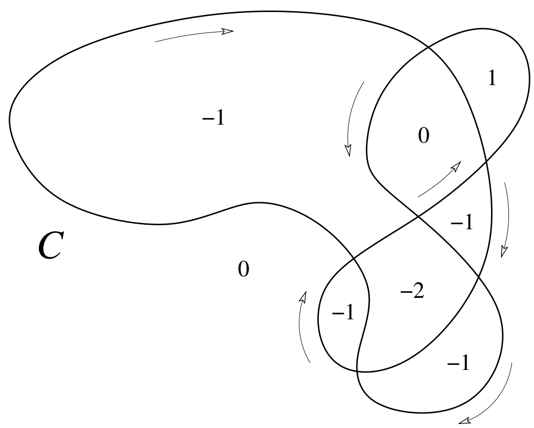


Figure 8: Index $\operatorname{ind}_C(\mathbf{a})$ is constant on each connected component of complement $\mathbb{R}^2 \setminus C$.

30 Index Formula We shall now establish the following remarkable formula:

$$\boxed{\operatorname{ind}_C(\mathbf{a}) = \frac{1}{2\pi}\int_C \frac{(x_1 - a_1)dx_2 - (x_2 - a_2)dx_1}{\|\mathbf{x} - \mathbf{a}\|^2}} \quad . \tag{103}$$

Let $E := \mathbb{R}^2 \setminus \{\mathbf{a}\}$ be the plane with point $\mathbf{a}$ removed. It is sufficient to prove Index Formula (103) for curves parametrized by a single loop $\boldsymbol{\gamma}\colon [0,1] \to E$. If $\operatorname{ind}_C(\mathbf{a}) = m$, then $\boldsymbol{\gamma}$ is *(freely) homotopic* in E to the path $\boldsymbol{\gamma}_1\colon [0,1] \to E$ which traverses m times the unit circle with center at point $\mathbf{a}$:

$$\boldsymbol{\gamma}_1(t) := \mathbf{a} + \begin{pmatrix} \cos(2m\pi t) \\ \sin(2m\pi t) \end{pmatrix} . \tag{104}$$

Since the differential 1-form on set E:

$$\frac{1}{2\pi}\frac{(x_1-a_1)dx_2-(x_2-a_2)dx_1}{\|\mathbf{x}-\mathbf{a}\|^2} \tag{105}$$

is closed, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_C \frac{(x_1-a_1)dx_2-(x_2-a_2)dx_1}{\|\mathbf{x}-\mathbf{a}\|^2} \\
&\quad = \frac{1}{2\pi}\int_{\gamma_1} \frac{(x_1-a_1)dx_2-(x_2-a_2)dx_1}{\|\mathbf{x}-\mathbf{a}\|^2} \qquad \text{(in view of Theorem (101))} \\
&\quad = \frac{1}{2\pi}\int_0^1 \frac{\cos(2m\pi t)(2m\pi\cos^2(2m\pi t)) - \sin(2m\pi t)(-2m\pi\sin^2(2m\pi t))}{\cos^2(2m\pi t)+\sin^2(2m\pi t)}\,dt \\
&\quad = \frac{2m\pi}{2\pi}\int_0^1 dt \;=\; m
\end{aligned} \tag{106}$$

which proves Index Formula (103) (compare the computation of $\displaystyle\int_{\gamma_1}\frac{(x_1-a_1)dx_2-(x_2-a_2)dx_1}{\|\mathbf{x}-\mathbf{a}\|^2}$ to computation (95) in LI).

31 Remark Let

$$\omega_1 \;:=\; \frac{1}{2\pi}\frac{x_1\,dx_2 - x_2\,dx_1}{x_1^2+x_2^2} \tag{107}$$

be a differential 1-form on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. Except for the factor $1/2\pi$, this is differential form (91) which we encountered in Section 37 of LI. Let us denote by $C - \mathbf{a}$ the curve obtained by *translating* curve C by column-vector $-\mathbf{a}$. If function $\boldsymbol{\gamma}$ parametrizes C then function $\boldsymbol{\gamma} - \mathbf{a}$, defined as $(\boldsymbol{\gamma}-\mathbf{a})(t) := \boldsymbol{\gamma}(t) - \mathbf{a}$, parametrizes $C - \mathbf{a}$. Note that Index Formula (103) can now be written as follows:

$$\boxed{\operatorname{ind}_C(\mathbf{a}) \;=\; \frac{1}{2\pi}\int_{C-\mathbf{a}} \omega_1} \;. \tag{108}$$

Note that the *integrand* in formula (108) depends *neither* on curve C *nor* on point $\mathbf{a}$.

32 Change of Variables Formula for Double Integral A differentiable function $\mathbf{h}\colon D \to D'$ is called a **diffeomorphism** if there exists a differentiable function $\mathbf{g}\colon D' \to D$ such that $\mathbf{g} \circ \mathbf{h}$ is the *identity* function on D and $\mathbf{h} \circ \mathbf{g}$ is the *identity* function on D'.[15]

Diffeomorphisms form a very important class of functions.

Suppose that f is an integrable function on set $D' \subseteq \mathbb{R}^2$ and $\mathbf{h}\colon D \to D'$ is a diffeomorphism. The following formula is a 2-dimensional analog of the Change of Variables Formula from Freshman Calculus:

$$\boxed{\iint_{D'} f(u,v)\,du\,dv = \iint_{D} (f \circ \mathbf{h})(x,y)\,\left|\det J_{\mathbf{h}}(x,y)\right|\,dx\,dy} \quad . \tag{109}$$

Formula (109) has many applications. For example, if $\boldsymbol{\sigma}_1\colon D_1 \to \mathbb{R}^m$ and $\boldsymbol{\sigma}_2\colon D_2 \to \mathbb{R}^m$ are two parametrizations of one and the same surface S such that

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2 \circ \mathbf{h} \tag{110}$$

for some *diffeomorphism* $\mathbf{h}\colon D_1 \to D_2$ of their respective parameter spaces, then

$$\boxed{\int_{\sigma_1} \psi = \pm \int_{\sigma_2} \psi} \tag{111}$$

with:

plus sign:	when $\det J_{\mathbf{h}} > 0$ everywhere on D_1 ,	(111^+)
minus sign:	when $\det J_{\mathbf{h}} < 0$ everywhere on D_1 .	(111^-)

In the first case, we say that diffeomorphism $\mathbf{h}$ **preserves orientation**, in the second case we say that $\mathbf{h}$ **reverses orientation**.

[15] In other words, function $\mathbf{h}$ has an inverse and that inverse is differentaible too.

Formula (111) is nothing but a 2-dimensional version of formula (63) for path integrals, see **LI**.

We can say, as we did in Section 29 of **LI**, that parametrizations $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are *equivalent* if equality (110) holds for a diffeomorphism satisfying condition (111^+). Then, identity (111) says that

$$\int_{\boldsymbol{\sigma}_1} \psi = \int_{\boldsymbol{\sigma}_2} \psi \tag{112}$$

for equivalent parametrizations.

33 Integrating 2-forms over oriented patches Particularly important are one-to-one regular parametrizations by *simple regions*, see Section 21. For the sake of these notes, surfaces admitting such parametrizations will be called **regular patches.**

Note that regular pathches are 2-dimensional versions of *regular arcs*, introduced in Section 30. All regular parametrizations of a connected patch S define one of the *two* possible orientations of its boundary ∂S, which is a closed curve in $\mathbb{R}^m$. **Choosing between these two possible orientations will be refered to as orienting the patch.** ☜

Then, for an *oriented* patch S and a 2-form ψ on S, we define the integral $\int_S \psi$ as

$$\boxed{\int_S \psi := \int_{\boldsymbol{\sigma}} \psi} \tag{113}$$

where $\boldsymbol{\sigma}$ is *any* regular parametrization of S which is compatible with the given orientation of S.

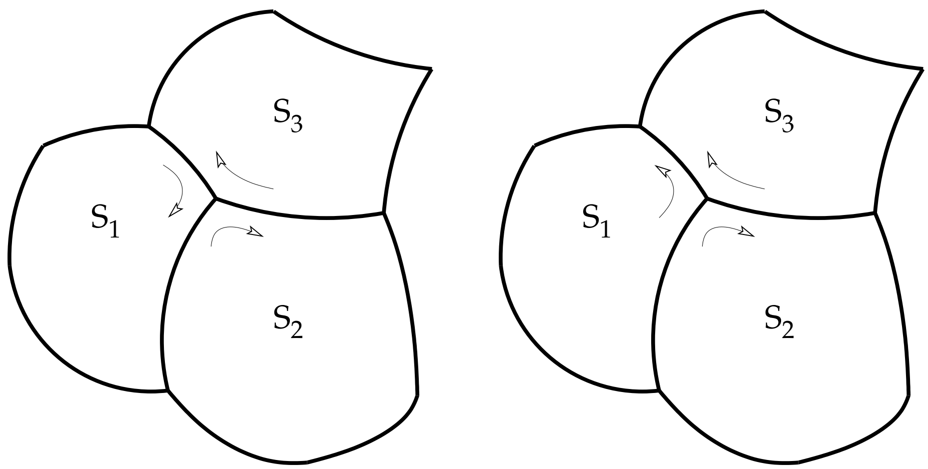
From formula (109) we know that the result does not depend on the particular choice of parametrization $\boldsymbol{\sigma}$.

34 Orienting regular patches in $\mathbb{R}^3$ When one looks at a small neighborhood of an internal point on a regular patch S embedded in $\mathbb{R}^3$, then points *not belonging* to the patch are on *one* or *another* 'side' of the patch. Orienting a patch embedded in $\mathbb{R}^3$ is the same as telling which 'side' is "in" and which one is "out": **when one looks at the patch from the side that is *in*, the positively oriented simple loops on S are the ones that are traversed** ☜ ***counterclockwise***.

The situation is entirely different for regular patches in $\mathbb{R}^m$ when dimension $m > 3$. The concept of 'side' does not make sense in that case.[16]

35 Orientable surfaces Suppose that S is a surface in $\mathbb{R}^m$ which is decomposed into a number of nonoverlapping *oriented* regular patches $S_1, \ldots, S_k$.

When we look at arcs of the common boundary between two adjacent patches, they receive orientation from either of the two patches. When these two orientations are opposite to each other, we say that the patches are *compatibly* oriented.



(a) *Compatibly* oriented patches

(b) *Incompatibly* oriented patches

Figure 9: On the left, all the patches have the *same* orientation; on the right, patches S_2 and S_3 have the *same*, and patch S_1 has the *opposite* orientation.

A surface S that can be decomposed into compatibly oriented regular patches will be called *orientable*. The choice of orientation of S is the same as the choice of *compatible* orientation *for* all patches.

Not every surface is orientable. The simplest example is provided by Möbius surface,[17] see

[16] We shall explain this by analogy with arcs. An arc *in* $\mathbb{R}^2$ has two 'sides,' and orienting an arc in $\mathbb{R}^2$ is the same as telling which 'side' is 'positive' and which one is 'negative'. The situation is entirely different for arcs *in* $\mathbb{R}^m$ when dimension $m > 2$. Look, for example at the case of an arc in $\mathbb{R}^3$.

[17] August Ferdinand Möbius (1790–1868)

figure 10.

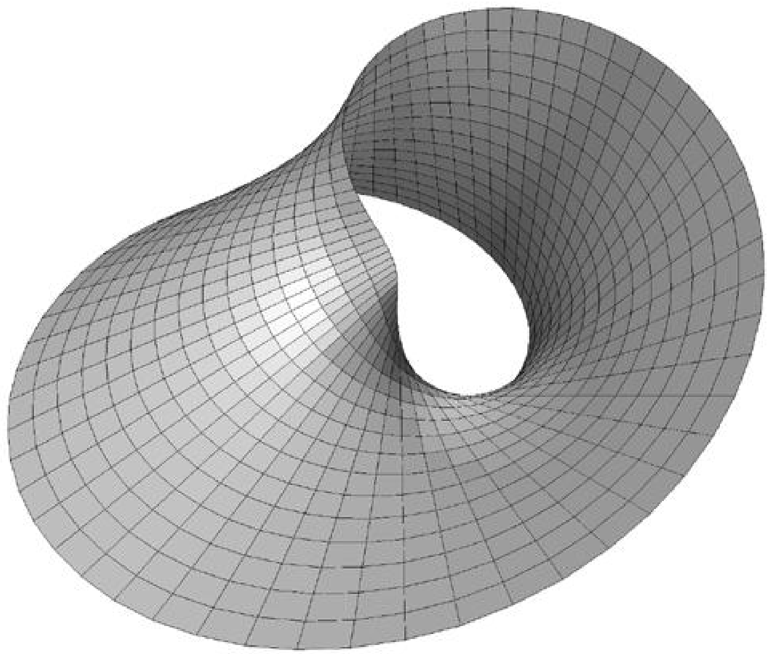


Figure 10: Möbius's surface.

For an *oriented* surface S and a differential 2-form ψ on S, we define the integral of ψ over S as the sum of integrals of ψ over all patches:

$$\int_S \psi := \int_{S_1} \psi + \cdots + \int_{S_k} \psi \tag{114}$$

The right-hand-side does not depend on the decomposition, but changes sign if we reverse all patch orientations.

36 Remark Some oriented surfaces *which are not patches,* e.g. spheres, ellipsoids, tori,[18] *etc.*, admit a parametrization $\boldsymbol{\sigma}\colon D \to S$ by a simple region D of plane $\mathbb{R}^2$ which is regular and one-to-one *except* for a subset $D_{bad} \subseteq D$ of *area zero*. One can show without effort that in this situation

$$\int_\sigma \psi \;=\; \int_S \psi \tag{115}$$

[18] A *torus* is the surface of a donut; *tori* is the plural.

for any differential 2-form ψ.[19]

37 An application: computation of the Gaußian integral For any real number $R > 0$, let $D_R \subset \mathbb{R}^2$ denote the disk of radius R with center at the origin, and let $\square_R$ denote the square $[-R, R] \times [-R, R]$. Consider the positive function on $\mathbb{R}^2$:

$$\mathbf{x} \mapsto e^{-\|\mathbf{x}\|^2} . \tag{116}$$

Since, $D_R \subseteq \square_R \subseteq D_{\sqrt{2}R}$, we have the double inequality:

$$\iint\limits_{D_R} e^{-\|\mathbf{x}\|^2} dx\,dy < \iint\limits_{\square_R} e^{-\|\mathbf{x}\|^2} dx\,dy < \iint\limits_{D_{\sqrt{2}R}} e^{-\|\mathbf{x}\|^2} dx\,dy . \tag{117}$$

Disk D_R has the polar-coordinates parametrization:

$$\mathbf{h}\colon [0, R] \times [0, 2\pi] \to D_R, \qquad \mathbf{h}\left(\begin{pmatrix} r \\ \theta \end{pmatrix}\right) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \tag{118}$$

which is a diffeomorphism (preserving orientation) except for the subset of $[0, R] \times [0, 2\pi]$:

$$\left\{ \begin{pmatrix} r \\ \theta \end{pmatrix} \mid r = 0 \text{ or } \theta = 0 \text{ or } 2\pi \right\} \tag{119}$$

which is of area zero. Thus, by Remark 36,

$$\iint\limits_{B_R} e^{-\|\mathbf{x}\|^2} dx\,dy \;=\; \int_{\mathbf{h}} e^{-\|\mathbf{x}\|^2} dx \wedge dy \tag{120}$$

$$= \iint\limits_{[0,R]\times[0,2\pi]} e^{-(r\cos^2\theta + r\sin^2\theta)} \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} dr\,d\theta \tag{121}$$

$$= \int_0^R e^{-r^2} r\,dr \cdot \int_0^{2\pi} d\theta \qquad \text{(by Fubini's Theorem)} \tag{122}$$

$$= \frac{1}{2}(1 - e^{-R^2}) \cdot 2\pi \;=\; \pi(1 - e^{-R^2}) . \tag{123}$$

[19] At least when the function-coefficients of ψ are *bounded*; otherwise, integral $\int_\sigma \psi$ may exist while $\int_S \psi$ may not.

By Fubini's Theorem, also

$$\iint_{\square_R} e^{-\|x\|^2} dx\,dy \;=\; \iint_{\square_R} e^{-x^2} e^{-y^2} dx\,dy \;=\; \int_{-R}^{R} e^{-x^2} dx \cdot \int_{-R}^{R} e^{-y^2} dy \;=\; \left(\int_{-R}^{R} e^{-x^2} dx \right)^2 . \tag{124}$$

Therefore, double inequality (117) yields the estimates

$$\sqrt{\pi}\sqrt{1-e^{-R^2}} < \int_{-R}^{R} e^{-x^2} dx < \sqrt{\pi}\sqrt{1-e^{-2R^2}} , \tag{125}$$

and, in the limit $R \to$,

$$\sqrt{\pi} \leqslant \int_{-} e^{-x^2} dx \leqslant \sqrt{\pi} . \tag{126}$$

It follows that the value of the **Gaußian integral** is:

$$\boxed{\int_{-} e^{-x^2} dx \;=\; \sqrt{\pi}} . \tag{127}$$

This is a fundamentally important result (especially for Probability Theory and Physics). You were told when learning Freshman Calculus that the anti-derivative of the function $x \mapsto e^{-x^2}$ cannot be expressed in terms of elementary functions of Calculus. Thus, what you have learned in Freshman Calculus did not allow you to compute Gaußian integral $\int_{-} e^{-x^2} dx$.

38 Stokes' Theorem (general case) For any differential 1-form φ on an oriented surface S, one has

$$\boxed{\int_S d\varphi = \int_{\partial S} \varphi} \tag{128}$$

Indeed, if S is a regular patch, then equality (128) follows from the respective definition of both integrals in (128) and from the parametric form of Stokes Theorem, see (90).

In general case, if S decomposes into patches $S_1, \ldots, S_k$, then

$$\int_S d\varphi = \int_{S_1} d\varphi + \cdots + \int_{S_k} d\varphi = \int_{\partial S_1} \varphi + \cdots + \int_{\partial S_k} \varphi \tag{129}$$

in view of Stokes' Theorem applied to each patch S_i separately.

Notice that for each arc C in the boundary of a given patch S_i, C *either* belongs also to the boundary of some other patch, say S_j, in which case its contributions to $\int_{S_i} \varphi$ and $\int_{S_j} \varphi$ are with *opposite* signs (and therefore cancel each other!), *or* C is "solitary", in which case it is an arc of the boundary of surface S (see, e.g., Figure 9(a)). Therefore, the sum

$$\int_{\partial S_1} \varphi + \cdots + \int_{\partial S_k} \varphi$$

coincides with the sum over all arcs of the boundary of S, i.e.,

$$\int_{\partial S_1} \varphi + \cdots + \int_{\partial S_k} \varphi = \int_{\partial S} \varphi,$$

and equality (128) has been proved.