The language of categories

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1 Universal constructions

1.1 Initial and final objects

1.1.1 Initial objects

An object i of a category $\mathcal{C}$ is said to be *initial* if for any object $c \in \operatorname{Ob}\mathcal{C}$, there exists a *unique* morphism $i \to c$. Any two initial objects are isomorphic and the isomorphism is unique.

1.1.2 Final objects

An object f of a category $\mathcal{C}$ is said to be *final* if for any object $c \in \operatorname{Ob}\mathcal{C}$, there exists a *unique* morphism $c \to f$. Any two final objects are isomorphic and the isomorphism is unique.

1.1.3 Example: the category of sets

In the category of sets, denoted hereafter **Set**, the empty set $\emptyset$ is the unique initial object. A set is a final object in **Set** precisely when it has one element.

The same holds in the category of topological spaces **Top**.

1.1.4 Example: a partially ordered set

Given a partially ordered set $(S, \leq)$, consider the category $\mathcal{S}$ whose objects are elements of S and, for any pair $s, t \in S$, there is exactly one morphism $s \to t$ if $s \leq t$, and $\operatorname{Hom}_{\mathcal{S}}(s, t) = \emptyset$ otherwise. In what follows we shall not distinguish a partially ordered set and the corresponding category.

The empty set, $0 = \emptyset$, corresponds to the empty category:

$$\mathrm{Ob}\, 0 = \emptyset. \tag{1}$$

An element $s \in S$ is an initial object of $\mathcal{S}$, if $s = \min S$; it is final, if $s = \max S$. In particular, the initial object is unique when it exists. The same applies also to the final object.

1.1.5 Example: the category of all (small) categories

A category Γ is said to be *small* if its objects and arrows form sets.

Small categories form a category denoted **Cat** and functors $\Gamma \rightsquigarrow \Gamma'$ are morphisms from Γ to Γ':

$$\mathrm{Hom}_{\mathbf{Cat}}(\Gamma, \Gamma') = \mathrm{Funct}(\Gamma, \Gamma') =: \left(\Gamma'\right)^{\Gamma}.$$

The empty category, 0, is the unique initial object in **Cat**. Category 1 is a final object.

1.1.6 Skeletal categories

The category $\mathcal{S}$ associated with an arbitrary partially ordered set $(S, \leq)$ is an example of a skeletal category: a category is said to be *skeletal* if there is at most one morphism between any pair of objects.

The definition of $\mathcal{S}$ only requires that the relation on set S be reflexive and transitive. Any small skeletal category arises this way and small skeletal categories form a subcategory of **Cat** which is isomorphic to the category of sets equipped with a relation that is both reflexive and transitive.

1.1.7 Discrete categories

A category $\mathcal{C}$ is said to be *discrete* if the only morphisms in $\mathcal{C}$ are the identity morphisms

$$\mathrm{id}_c \qquad (c \in \mathrm{Ob}\, \mathcal{C}).$$

Small discrete categories form a subcategory in **Cat** which is isomorphic to the category of sets, **Set**. Note that the initial and final objects of **Set** are also initial and, respectively, final objects in **Cat**.

1.1.8 Example: the category of R-modules

An R-module is an initial as well as a final object in the category of R-modules precisely when it has only one element. We speak of this module as *the* zero module even though it is not unique: any one-element set can be made into an R-module.

This applies both to the category of left R-modules, which is denoted R**-mod**, and to the category of right R-modules which is denoted **mod-**R.

A special case is provided by the category of vector spaces, $\mathbf{Vect}_K$, over a field K.

1.2 Tensor product

1.2.1 Modules over k-algebras

Let k be a commutative unital ring and A be k-algebra. A (left) A-module structure on a k-module M is a k-algebra homomorphism

$$\lambda : A \longrightarrow \operatorname{End}_{k\text{-}\mathbf{mod}} M, \qquad a \mapsto \lambda_a.$$

It is customary to denote $\lambda_a(m)$ by am.

1.2.2 Unitary modules

If A is a unital ring, we say that M is *unitary* A-module if λ is a *unital* homomorphism, i.e., $\lambda_{1_A} = \mathrm{id}_M$.

1.2.3 *Left* versus *right* module structures

A *right* A-module structure on M is the same as an *anti*homomorphism of k-algebras

$$\rho : A \longrightarrow \operatorname{End}_{k\text{-}\mathbf{mod}} M, \qquad a \mapsto \rho_a.$$

It is customary to denote $\rho_a(m)$ by ma.

Right A-module structures on M are identified with left A^{op}-module structures via:

$$a^{\mathrm{op}} m := ma \qquad (a \in A, m \in M).$$

1.2.4 Bimodules over k-algebras

A k-module M equipped with a structure of a left module over a k-algebra A, and with a structure of a right module over a k-algebra B is said to be an (A, B)-*bimodule*, if the two module structures commute:

$$[\lambda_a, \rho_b] = 0 \qquad (a \in A, b \in B).$$

1.2.5

A left A-module is the same as an (A, k)-bimodule and, likewise, a right B-module is the same as a (k, B)-bimodule.

1.2.6 Balanced bilinear maps

Let A, B, and C be unital k-algebras. Let M be an (A, B)-bimodule, N be a (B, C)-bimodule, and Q be an (A, C)-bimodule—all three assumed to be unitary.

We say that a *pairing*

$$\phi : M \times N \longrightarrow Q \tag{2}$$

is (A, C)-*bilinear*, if it is A-linear in the left argument and C-linear in the right argument.

We say that (2) is B-*balanced* if

$$\phi(mb, n) = \phi(m, bn) \qquad (m \in M, b \in B, n \in N). \tag{3}$$

1.2.7 Tensor product of bimodules

For a given (A, B)-bimodule M and a (B, C)-bimodule N, consider the category $\mathcal{B}(M, N)$ whose objects are balanced bilinear maps (2), and morphisms $\phi \to \phi'$ are (A, C)-bimodule maps $f : Q \to Q'$ such that

$$\phi' = f \circ \phi,$$

i.e., such that the following diagram commutes:

$$\begin{tikzcd} & M \times N \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ Q \arrow[rr, dotted, "f"'] & & Q' \end{tikzcd}$$

Lemma 1.1 *Category $\mathcal{B}(M, N)$ possesses an initial object.*

Proof. Consider the set $M \times N$. Left multiplication by elements $a \in A$ and right multiplication by elements $c \in C$,

$$a(m, n) := (am, n) \qquad (m, n)c := (m, nc),$$

define a left action of the multiplicative monoid $A^\times$ and with a right action of the multiplicative monoid $C^\times$. The two actions commute with each other since they are applied to different factors of the Cartesian product $M \times N$.

In particular, the free abelian group $F = \mathbb{Z}^{(M \times N)}$ with basis $M \times N$ inherits these two actions. Note that the subset of F

$$\{(m_1 + m_2, n) - (m_1, n) - (m_2, n) \mid m_1, m_2 \in M; n \in N\} \tag{4}$$

is invariant under both actions so is the subgroup $\mathfrak{a}_1 \subseteq F$ it generates. In particular, the quotient $F/\mathfrak{a}_1$ becomes a left module over *ring* A.

Likewise, the subset of F

$$\{(m, n_1 + n_2) - (m, n_1) - (m, n_2) \mid m \in M; n_1, n_2 \in N\} \tag{5}$$

is invariant under both actions, and so is the subgroup $\mathfrak{a}_2 \subseteq F$ it generates. In particular, the quotient $F/\mathfrak{a}_2$ becomes a right module over *ring* C.

Exercise 1 *Show that the left A-module $F/\mathfrak{a}_1$ is canonically isomorphic to the direct sum of left A-modules M*

$$\bigoplus_{n \in N} M$$

whereas the right C-module $F/\mathfrak{a}_2$ is canonically isomorphic isomorphic to the direct sum of right C-modules N

$$\bigoplus_{m \in M} N.$$

If we divide abelian group F by the subgroup $\mathfrak{a}_1 + \mathfrak{a}_2$, then the quotient $F/(\mathfrak{a}_1 + \mathfrak{a}_2)$ becomes an (A, C)-bimodule where A and C are treated as rings (i.e., as $\mathbb{Z}$-algebras).

Note that the canonical inclusion map composed with the canonical quotient map

$$M \times N \hookrightarrow k^{(M \times N)} \twoheadrightarrow F/(\mathfrak{a}_1 + \mathfrak{a}_2) \tag{6}$$

is (A, C)-bilinear.

Next, observe that the subset of F

$$\{(mb, n) - (m, bn) \mid m \in M, b \in B, n \in N\} \tag{7}$$

is invariant under the actions of multiplicative monoids $A^\times$ and $C^\times$, and so is the subgroup $\mathfrak{b} \subseteq F$ it generates.

It follows that the quotient group

$$F/(\mathfrak{a}_1 + \mathfrak{b} + \mathfrak{a}_2) \tag{8}$$

is an (A, C)-bimodule where A and C are now treated as k-algebras. We denote the equivalence class of a pair (m, n) in $F/(\mathfrak{a}_1 + \mathfrak{b} + \mathfrak{a}_2)$ by $m \otimes n$, and bimodule (8)—by $M \otimes_B N$. The subscript B is often omitted when B is clear from the context.

By construction, the pairing

$$\otimes : M \times N \longrightarrow M \otimes_B N, \qquad (m, n) \mapsto m \otimes n, \tag{9}$$

is (A, C)-bilinear and B-balanced.

Any pairing $\phi : M \times N \longrightarrow Q$ with values in an abelian group gives rise to a unique homomorphism of abelian groups

$$\phi' : \mathbb{Z}^{(M \times N)} \longrightarrow Q.$$

Homomorphism ϕ' annihilates $\mathfrak{a}_1$, if it is linear in the first argument, annihilates $\mathfrak{a}_2$, if it is linear in the second argument, and annihilates $\mathfrak{b}$, if it is balanced. Hence, ϕ' induces a homomorphism of abelian groups

$$\tilde{\phi} : M \otimes_B N \longrightarrow Q, \qquad m \otimes n \mapsto \phi(m, n).$$

It is clear that $\tilde{\phi}$ is a morphism of (A, C)-bimodules:

$$\tilde{\phi}(am \otimes n) = \phi(am, n) = a\phi(m, n) = a\tilde{\phi}(m \otimes n)$$

and

$$\tilde{\phi}(m \otimes nc) = \phi(m, nc) = \phi(m, n)c = \tilde{\phi}(m \otimes n)c.$$

The morphism we constructed is the unique homomorphism of abelian groups $f : M \otimes_B N \longrightarrow Q$ such that $\phi = f \circ \otimes$. Indeed, $M \otimes_B N$ as additively generated by the subset
$$(M \otimes_B N)_1 := \{m \otimes n \mid m \in M, n \in N\},$$
whose elements are referred to as *rank one* tensors, and on rank one tensors the value of f is predetermined by the value of ϕ. This completes the proof that the *tensor-product pairing* (9) is an initial object in category $\mathcal{B}(M, N)$.

□

2 Direct and inverse limits

2.1 Diagrams

2.1.1

Let Γ be a small category. For any category $\mathcal{C}$, functors $\Gamma \rightsquigarrow \mathcal{C}$ are often called Γ-diagrams (in $\mathcal{C}$). This usage is particularly frequent when Γ has very few objects and morphisms.

2.1.2 Example: the empty diagram

For any category $\mathcal{C}$, there is just one 0-diagram in $\mathcal{C}$, where 0 denotes the empty category, cf. (1). We shall refer to it as the *empty* diagram.

2.1.3 Example: an object

An object in a category $\mathcal{C}$ is the same as a Γ-diagram in $\mathcal{C}$ where Γ is the partially ordered set 2^0 (the set of all subsets of $0 = \emptyset$).

2.1.4 Example: an arrow

An arrow in a category $\mathcal{C}$ is the same as a Γ-diagram in $\mathcal{C}$ where Γ is the linearly ordered set 2^1 (the set of all subsets of $1 = \{0\}$).

2.1.5 Example: a commutative square

A commutative square in a category $\mathcal{C}$

$$\begin{tikzcd} c_{00} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & c_{01} \arrow[d] \\ c_{10} \arrow[r] & c_{11} \end{tikzcd} \tag{10}$$

is the same as a Γ-diagram in $\mathcal{C}$ where Γ is the partially ordered set 2^2 (the set of all subsets of $2 = \{0, 1\}$).

2.1.6 Example: a composable pair of arrows

A composable pair of arrows in a category $\mathcal{C}$

$$c_0 \longrightarrow c_1 \longrightarrow c_2 \tag{11}$$

is the same as a Γ-diagram in $\mathcal{C}$ where Γ is the linearly ordered set $3 = \{0, 1, 2\}$.

2.1.7 Example: a parallel pair

Consider the category with just two objects, s and t, and two morphisms $\gamma, \gamma' : s \to t$. We shall denote this category by $\rightrightarrows$. A $\rightrightarrows$-diagram in a category $\mathcal{C}$ is the same as a pair of morphisms in $\mathcal{C}$ with the common source and target

$$c \underset{g}{\overset{f}{\rightrightarrows}} c' \, . \tag{12}$$

2.1.8 Example: a G-object

The category of monoids **Mon** is naturally isomorphic with the full subcategory of **Cat** consisting of categories with a single object:

$$G \leftrightsquigarrow \Gamma, \qquad \mathrm{End}_\Gamma(*) = \mathrm{Hom}_\Gamma(*, *) = G,$$

where $*$ denotes the only object of Γ.

From now on we shall identify monoids with categories having a single object. Given a monoid G and a category $\mathcal{C}$, a G-diagram in C is the same as an object $c \in \mathrm{Ob}\,\mathcal{C}$ equipped with the *action* of G on c, i.e., with a homomorphism of monoids

$$G \to \mathrm{End}_{\mathcal{C}}(c).$$

We shall refer to it as a G*-object*.

In the case of $\mathcal{C} = \mathbf{Set}$, $\mathbf{Top}$, or $\mathbf{Vect}_K$, we talk of G-sets, G-spaces and, respectively, (K-linear) representations of G.

2.2 The category of Γ-diagrams

2.2.1

Since Γ-diagrams in $\mathcal{C}$ are just functors,

$$\Gamma \rightsquigarrow \mathcal{C},$$

Γ-diagrams in $\mathcal{C}$ form a category

$$\mathcal{C}^{\Gamma} := \mathrm{Funct}(\Gamma, \mathcal{C})$$

with morphisms being *natural transformations of functors*.

2.2.2 The diagonal embedding $\mathcal{C} \hookrightarrow \mathcal{C}^{\Gamma}$

There is a canonical embedding of $\mathcal{C}$ onto a full subcategory of $\mathcal{C}^{\Gamma}$ which sends $c \in \mathcal{C}$ to the *constant* diagram Δ_c:

$$\Delta_c(g) = c \qquad \text{for any } g \in \mathrm{Ob}\,\Gamma \tag{13}$$

and

$$\Delta_c(\gamma) = \mathrm{id}_c \qquad \text{for any } \gamma \in \mathrm{Arr}\,\Gamma \tag{14}$$

In what follows we shall identify $\mathcal{C}$ with its image in $\mathcal{C}^{\Gamma}$ under the diagonal embedding.

2.2.3 Direct limits

For a given Γ-diagram $D : \Gamma \rightsquigarrow \mathcal{C}$ in a category $\mathcal{C}$, consider the category $\mathcal{C}^D$ whose objects are families of morphisms

$$\phi = \{\phi_g : D(g) \to c\}_{g \in \mathrm{Ob}\,\Gamma}$$

with common target $c \in \mathrm{Ob}\,\mathcal{C}$ which are compatible with the diagram, i.e., such that

$$\phi_{g'} \circ D(\gamma) = \phi_g \qquad (\gamma \in \mathrm{Hom}_\Gamma(g, g')).$$

Morphisms $\phi \to \phi'$ in category $\mathcal{C}^D$ are defined as morphisms $\alpha : c \to c'$ between the targets such that

$$\alpha \circ \phi_g = \phi'_g \qquad (g \in \mathrm{Ob}\,\Gamma).$$

An initial object in $\mathcal{C}^D$ is called the *direct limit* of diagram D.[1] The direct limit of D is denoted

$$\varinjlim D.$$

In use there are also terms: the *inductive limit* of D and the *push-out* of D. The latter usage is generally confined to diagrams with few vertices.

Exercise 2 *Denote by $\mathbb{D}$ the partially ordered set of positive integers ordered by the divisibility relation:*

$$m \leq n \qquad \textit{if} \qquad m|n.$$

Consider the following $\mathbb{D}$-diagram in the category of abelian groups **Ab**

$$D_n = \mathbb{Z}, \qquad D(m|n) := \textit{multiplication by } \frac{n}{m}, \qquad (m, n \in \mathbb{Z}_+).$$

Show that

$$\varinjlim D = \mathbb{Q}$$

with the morphisms $D_n \to \mathbb{Q}$ being the homomorphisms of abelian groups

$$\mathbb{Z} \longrightarrow \mathbb{Q}, \qquad i \mapsto \frac{i}{n} \qquad (i \in \mathbb{Z}).$$

2.2.4 Inverse limits

For a given Γ-diagram $D : \Gamma \rightsquigarrow \mathcal{C}$ in a category $\mathcal{C}$, consider the category $D^{\mathcal{C}}$ whose objects are families of morphisms

$$\psi = \{\psi_g : c \to D(g)\}_{g \in \mathrm{Ob}\,\Gamma}$$

[1] Note that we are using the definite article even though *direct limit* is usually not unique; it is unique only up to a *unique* isomorphism.

with common source $c \in \operatorname{Ob} \mathcal{C}$ which are compatible with the diagram, i.e., such that i.e., such that

$$D(\gamma) \circ \psi_g = \psi_{g'} \qquad (\gamma \in \operatorname{Hom}_\Gamma(g, g')).$$

Morphisms $\psi \to \psi'$ in category $D^{\mathcal{C}}$ are defined as morphisms $\alpha : c \to c'$ between the sources such that

$$\alpha \circ \phi_g = \phi'_g \qquad (g \in \operatorname{Ob} \Gamma).$$

A final object in $\mathcal{C}^D$ is called the *inverse limit* of diagram D. The inverse limit of D is denoted

$$\varprojlim D.$$

In use there are also terms: the *projective limit* of D and the *pull-back* of D. The latter usage is generally confined to diagrams with few vertices whereas the former one is more often applied to diagrams with infinitely many vertices.

2.2.5 Limits and colimits

Another usage gaining popularity is: *limits*—for inverse limits, and *colimits*—for direct limits.

Exercise 3 *Show that $\varprojlim \emptyset$ is a final object of $\mathcal{C}$, while $\varinjlim \emptyset$ is an initial object of $\mathcal{C}$.*

Exercise 4 *Let $\mathbb{N}$ be the set of natural numbers with the reverse linear order $\geq$ and let p be a prime. Consider the following $\mathbb{N}$-diagram in the category of rings*

$$D_n = \mathbb{Z}/p^n\mathbb{Z}, \qquad D(m \geq n) := \textit{the canonical epimorphism } \mathbb{Z}/p^m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z},$$

where $m, n \in \mathbb{Z}_+$. Show that the inverse limit of D is the ring of p-adic numbers,

$$\varprojlim D = \mathbb{Z}_p,$$

with the morphisms $\mathbb{Z}_p \to D_n$ being the quotient maps

$$\mathbb{Z}_p \longrightarrow \mathbb{Z}_p/p^n\mathbb{Z}_p.$$

Exercise 5 *Denote by $\mathbb{D}$ the partially ordered set introduced in Exercise 2. Consider the following $\mathbb{D}^{\mathrm{op}}$-diagram in the category of rings*

$$D_n = \mathbb{Z}/n\mathbb{Z}, \qquad D\left((m|n)^{\mathrm{op}}\right) := \textit{the canonical epimorphism } \mathbb{Z}/m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z},$$

where $m, n \in \mathbb{Z}_+$. Show that

$$\varprojlim D = \prod_{p \text{ prime}} \mathbb{Z}_p \tag{15}$$

with the morphisms $\varprojlim D \to D_n$ being the products over all primes of the canonical quotient maps

$$\mathbb{Z}_p \longrightarrow \mathbb{Z}/p^{\omega_p(n)}\mathbb{Z},$$

where $\omega_p(n)$ denotes the p-order of $n \in \mathbb{Z}_+$. Hint: we use here the canonical decomposition of a cyclic group into the product of the corresponding p-groups of order $p^{\omega_p(n)}$:

$$Z/n\mathbb{Z} \simeq \prod_{p \text{ prime}} \mathbb{Z}/p^{\omega_p(n)}\mathbb{Z}.$$

2.2.6 The formal completion of $\mathbb{Z}$

The inverse limit $\varprojlim D$ in Exercise 5 is called the *formal completion* of the ring of integers. It is denoted $\widehat{\mathbb{Z}}$. The abelianization[2] of the Galois group

$$\mathrm{Gal}(\mathbb{Q}) = \mathrm{Gal}\big(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q}\big)$$

is canonically isomorphic to the additive group of $\widehat{\mathbb{Z}}$. This is a deep result of Algebraic Number Theory.

Exercise 6 *Show that $\widehat{\mathbb{Z}}$ is canonically isomorphic to the ring of endomorphisms*

$$\mathrm{End}_{\mathbf{Ab}}(\mu_\infty(\mathbb{C}))$$

of the group of complex roots of identity

$$\mu_\infty(\mathbb{C}) = \{\zeta \in \mathbb{C} \mid \zeta^n = 1 \textit{ for some } n \in \mathbb{N}\}.$$

[2]The *abelianization* G^{ab} of a group G is the maximal abelian quotient of G; it coincides with the quotient $G/[G,G]$ of G by its commuattor subgroup.

2.2.7 Equalizers and coequalizers

The inverse limit of a parallel pair (12) is called the *equalizer* of f and g. The direct limit is called the *coequalizer* of f and g.

2.2.8 Cartesian squares

A commutative square

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd} \tag{16}$$

in a category $\mathcal{C}$ is said to be *Cartesian* if

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \\ c & \end{tikzcd} \tag{17}$$

is the inverse limit of

$$\begin{tikzcd} & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd} \tag{18}$$

2.2.9 Cocartesian squares

A commutative square (16) is said to be *Cocartesian* if, vice-versa, (18) is the direct limit of (17).

2.2.10

For a given Γ and a general category $\mathcal{C}$, certain Γ-diagrams in $\mathcal{C}$ possess either the direct or the inverse limit while others do not.

A category $\mathcal{C}$ is said to be *complete* if $\varprojlim D$ exists for any small category Γ and any Γ-diagram in $\mathcal{C}$.

If $\varinjlim D$ exists for any Γ-diagram, we say that category $\mathcal{C}$ is *cocomplete*.

Exercise 7 *Show that the category of sets is complete.*

Hint: consider the subset of the product of all D_g,

$$\prod_{g \in \mathrm{Ob}\,\Gamma} D_g,$$

consisting of the tuples

$$\left(x_g\right)_{g \in \mathrm{Ob}\,\Gamma}$$

which satisfy the following relations

$$x_{t(\gamma)} = D(\gamma)\left(x_{s(\gamma)}\right) \qquad (\gamma \in \mathrm{Arr}\,\Gamma).$$

Exercise 8 *Show that the category of sets is cocomplete.*

Hint: consider the quotient of the disjoint union of all D_g,

$$\coprod_{g \in \mathrm{Ob}\,\Gamma} D_g,$$

by the equivalence relation generated by the following relation:

$$x'' \sim x'$$

if

$$x'' = D(\gamma)\left(x'\right), \qquad x' \in D_{s(\gamma)} \qquad \text{and} \qquad x'' \in D_{t(\gamma)},$$

for some $\gamma \in \mathrm{Arr}\,\Gamma$.

Exercise 9 *Show that the category $\mathcal{S}$ associated with a partially ordered set $(S, \leq)$ is complete if and only if $(S, \leq)$ is* inf-*complete, i.e., every subset of S has infimum. Similarly, show that $\mathcal{S}$ is cocomplete if and only if $(S, \leq)$ is* sup-*complete, i.e., every subset of S has supremum.*

2.2.11 Functorial direct limits

It is not infrequent that *every* Γ-diagram in a given category may have the corresponding limits, and that those limits depend functorially on the diagram.

We say that a category $\mathcal{C}$ possesses *functorial* direct limits (for Γ-diagrams), if there exists a functor

$$\varinjlim : \mathcal{C}^{\Gamma} \rightsquigarrow \mathcal{C} \tag{19}$$

such that, for any object $c \in \mathrm{Ob}\,\mathcal{C}$ and Γ-diagram $D \in \mathrm{Ob}\,\mathcal{C}^{\Gamma}$, there exists a *natural* in c and D bijection

$$\mathrm{Hom}_{\mathcal{C}}\left(\varinjlim D, c\right) \leftrightsquigarrow \mathrm{Hom}_{\mathcal{C}^{\Gamma}}(D, \Delta_c). \tag{20}$$

where Δ denotes the diagonal embedding functor introduced in 2.2.2.

2.2.12 Functorial inverse limits

We say that a category $\mathcal{C}$ possesses *functorial* inverse limits (for Γ-diagrams), if there exists a functor

$$\varprojlim : \mathcal{C}^{\Gamma} \rightsquigarrow \mathcal{C} \tag{21}$$

such that, for any object $c \in \operatorname{Ob}\mathcal{C}$ and Γ-diagram $D \in \operatorname{Ob}\mathcal{C}^{\Gamma}$, there exists a *natural* in c and D bijection

$$\operatorname{Hom}_{\mathcal{C}}\left(c, \varprojlim D\right) \leftrightsquigarrow \operatorname{Hom}_{\mathcal{C}^{\Gamma}}(\Delta_c, D). \tag{22}$$

where Δ denotes the diagonal embedding functor introduced in 2.2.2.

2.3 Adjoint functors

2.3.1

In (20) and (22) we encounter a fundamentally important concept: a pair of adjoint functors.

Suppose that, for a pair of functors,

$$\begin{tikzcd} \mathcal{C} \arrow[r, squiggly, shift right, "G"'] & \mathcal{D} \arrow[l, squiggly, shift right, "F"'] \end{tikzcd}, \tag{23}$$

be given. If, for any $c \in \operatorname{Ob}\mathcal{C}$ and $d \in \operatorname{Ob}\mathcal{D}$, there exists a bijective correspondence

$$\operatorname{Hom}_{\mathcal{C}}(Fd, c) \overset{\phi_{dc}}{\leftrightsquigarrow} \operatorname{Hom}_{\mathcal{D}}(d, Gc) \tag{24}$$

which is natural both in c and d, then we say that F is *left adjoint* to G, and G is *right adjoint* to F.

2.3.2

If two functors, F and F', are left adjoint to G, then they are isomorphic.

2.3.3 Example: functorial direct and inverse limits

A category $\mathcal{C}$ possesses functorial direct limits if the diagonal embedding $\Delta : \mathcal{C} \rightsquigarrow \mathcal{C}^{\Gamma}$ admits a left adjoint. Similarly, if Δ admits a *right* adjoint, then $\mathcal{C}$ possesses functorial inverse limits.

Exercise 10 *Let G be a monoid. Show that the functor,*

$$(\,)^G : \mathbf{Set}^G \rightsquigarrow \mathbf{Set}, \qquad X \mapsto X^G := \mathrm{Fix}_G(X) = \{x \in X \mid gx = x \textit{ for any } g \in G\},$$

*which associates with a G-set X the **fixed-point set**, $X^G = \mathrm{Fix}_G(X)$, is right adjoint to the diagonal embedding functor $\Delta : \mathbf{Set} \rightsquigarrow \mathbf{Set}^G$.*

In particular,

$$\varprojlim X = X^G = \mathrm{Fix}_G(X)$$

for any G-set X.

Exercise 11 *Let G be a monoid. Show that the functor,*

$$(\,)_G : \mathbf{Set}^G \rightsquigarrow \mathbf{Set}, \qquad X \mapsto X_G := X_{/G},$$

*which associates with a G-set X the **set of orbits** of G on X, i.e., the set of equivalence classes of the equivalence relation*

$$x' \sim x'' \quad \textit{if} \quad x' = g'x \ \textit{ and } \ x'' = g''x \ \textit{ for some } \ x \in X \ \textit{ and } \ g', g'' \in G.$$

is left adjoint to the diagonal embedding functor $\Delta : \mathbf{Set} \rightsquigarrow \mathbf{Set}^G$.

In particular,

$$\varinjlim X = X_G = X_{/G}$$

for any G-set X.

2.3.4 The sets of G-invariants and G-coinvariants

Notation X^G and X_G is standard in representation theory where X^G is called the set of G-*invariants*, and X_G is called the set of G-*coinvariants* of G-*representation X*.

Notation $\mathrm{Fix}_G(X)$ and $X_{/G}$, and the corresponding terminology: the set of *fixed-point set* and the *set of orbits*, are primarily used for "non-linear" actions, i.e., for G-actions on sets, topological spaces, algebraic varieties.