Integral Calculus

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1 $\mathbb{R}$-spaces

1.1 Vocabulary

1.1.1

We shall call a pair $\mathfrak{X} = (X, \mathscr{O})$, where X is a set and $\mathscr{O} \subseteq \mathbb{R}^X$ is a unital $\mathbb{R}$-algebra of real valued functions on a set X, an $\mathbb{R}$-space. Here X will be called the *support* of $\mathfrak{X}$ and will be denoted $|\mathfrak{X}|$, while $\mathscr{O} = \mathscr{O}(\mathfrak{X})$ will be called the *structural algebra* of $\mathfrak{X}$.

In the category of $\mathbb{R}$-spaces morphisms from $\mathfrak{X}$ to $\mathfrak{Y}$ are maps $\phi : |\mathfrak{X}| \to |\mathfrak{Y}|$ such that

$$\phi^*\mathscr{O}(\mathfrak{Y}) := \{f \circ \phi \mid f \in \mathscr{O}(\mathfrak{Y})\} \subseteq \mathscr{O}(\mathfrak{X}).$$

We shall denote the category of $\mathbb{R}$-spaces by $\mathbb{R}$-**Spc**.

1.1.2 An $\mathbb{R}$-space associated with a topological space

With any topological space $(X, \mathscr{T})$ one can naturally associate an $\mathbb{R}$-space $(X, C(X))$ where $C(X) = C(X, \mathscr{T})$ denotes the algebra of functions $X \to \mathbb{R}$ continuous with respect to topology $\mathscr{T}$. This defines a functor from the category of topological spaces **Top** to $\mathbb{R}$-**Spc**

$$C : \mathbf{Top} \rightsquigarrow \mathbb{R}\text{-}\mathbf{Spc}, \qquad (X, \mathscr{T}) \longmapsto (X, C(X, \mathscr{T})). \tag{1}$$

1.1.3 $\mathbb{R}$-spaces associated with a subset of $\mathbb{R}^n$

We shall say that a function $f : D \to \mathbb{R}$ on a subset $D \subseteq \mathbb{R}^n$ is of class C^r, $0 \leq r \leq \infty$, (analytic, polynomial), if f is the restriction to D of a function of class C^r (respectively, analytic, polynomial) defined on an open subset containing D.

Functions of class C^r (respectively, analytic, polynomial) on D form an algebra denoted below $C^r(D)$ (respectively, $\mathcal{O}^{\mathrm{an}}(D)$, $\mathcal{O}^{\mathrm{pol}}(D)$). With each of the above algebras there is associated a corresponding $\mathbb{R}$-space: $\mathbf{D}^r = (D, C^r(D))$, $\mathbf{D}^{\mathrm{an}} = (D, \mathcal{O}^{\mathrm{an}}(D))$, and $\mathbf{D}^{\mathrm{an}} = (D, \mathcal{O}^{\mathrm{pol}}(D))$, respectively.

1.1.4 The canonical topology

The topology $\mathscr{T} = \mathscr{T}(\mathfrak{X})$ generated by the family of preimages of open subsets of $\mathbb{R}$ by members of $\mathcal{O} = \mathcal{O}(\mathfrak{X})$,

$$\mathscr{B} = \left\{ f^{-1}(V) \mid f \in \mathcal{O},\ V \subseteq \mathbb{R} \text{ open} \right\},$$

is the weakest topology on $X = |\mathfrak{X}|$ in which all $f \in \mathcal{O}$ are continuous as functions $X \to \mathbb{R}$. We shall call it the *canonical topology* of an $\mathbb{R}$-space $\mathfrak{X}$.

This defines a functor

$$T : \mathbb{R}\text{-}\mathbf{Spc} \rightsquigarrow \mathbf{Top}, \qquad \mathfrak{X} = (X, \mathcal{O}) \longmapsto (X, \mathscr{T}(\mathfrak{X})). \tag{2}$$

Exercise 1 *Show that the canonical topology is generated by the family of preimages of intervals* $(-\varepsilon, \varepsilon)$*:*

$$\mathscr{B}_0 = \left\{ f^{-1}(-\varepsilon, \varepsilon) \mid f \in \mathcal{O},\ \varepsilon > 0 \right\}.$$

Exercise 2 *Show that the canonical topology is* completely regular*, i.e., for any closed subset $Z \subseteq X$ and a point $p \notin Z$, there exists a function $f : X \to \mathbb{R}$, continuous in canonical topology, such that*

$$f(p) = 1 \qquad \textit{and} \qquad f_{|Z} = 1.$$

Exercise 3 *Let $(X, \mathscr{T})$ be a topological space and $\mathfrak{Y} = (Y, \mathcal{O})$ be an $\mathbb{R}$-space. Show that a map $\phi : (X, \mathscr{T}) \to (Y, \mathscr{T}(\mathfrak{Y}))$ is continuous if and only if $\phi^*(\mathcal{O}) \subseteq C(X, \mathscr{T})$.*

Derive from this that the functor **Top** $\rightsquigarrow$ $\mathbb{R}$-**Spc**, *defined in* (1)*, is left adjoint to the functor* $\mathbb{R}$-**Spc** $\rightsquigarrow$ **Top** *defined in* (2)*.*

Exercise 4 *Show that*

$$S \circ T \circ S = S \qquad \textit{and} \qquad T \circ S \circ T = T.$$

1.2 Integral of a differential form over a parametric patch

1.2.1 Regions in Euclidean space

A subset $D \subseteq \mathbb{R}^q$ will be called a region if it is contained in the closure of its interior. We shall mostly deal with open or closed regions.

1.2.2

Denote by Ω^1_D the free $\mathscr{O}(D)$-module of rank q with basis

$$d^c x_1, \dots, d^c x_q.$$

The map

$$d^c : \mathscr{O}(D) \longrightarrow \Omega^1_D, \qquad f \mapsto d^c f = \sum_{i=1}^{q} \frac{\partial f}{\partial x_i} d^c x_i, \tag{3}$$

is an $\mathbb{R}$-linear derivation of algebra $\mathscr{O}(D)$. It is in fact a *universal continuous* derivation with values in a locally convex $\mathscr{O}(D)$-module. The subscript c indicates that and also serves the reader warning not to confuse $d^c f \in \Omega^1_D$ with $df \in \Omega^1_{\mathscr{O}(D)/\mathbb{R}}$.

Derivation (3) induces an $\mathscr{O}(D)$-linear and obviously surjective map

$$\Omega^1_{\mathscr{O}(D)/\mathbb{R}} \longrightarrow \Omega^1_D \qquad df \mapsto d^c f,$$

which in turn induces a surjective map of differential graded $\mathscr{O}(D)$-algebras

$$\Omega^*_{\mathscr{O}(D)/\mathbb{R}} \longrightarrow \Omega^*_D := \textstyle\bigwedge^*_{\mathscr{O}(D)} \Omega^1_D, \qquad \alpha \mapsto \alpha^c. \tag{4}$$

Note that

$$(f_0 df_1 \wedge \cdots \wedge df_p)^c = f_0 d^c f_1 \wedge \cdots \wedge d^c f_p.$$

1.2.3 Volume forms

Since Ω^1_D is free of rank q, its p-th exterior power, Ω^p_D is free of rank $\binom{q}{p}$. In particular, Ω^q_D is a free $\mathscr{O}(D)$-module of rank 1,

$$\Omega^q_D = \mathscr{O}(D)\, d^c x_1 \wedge \cdots \wedge d^c x_q. \tag{5}$$

1.2.4 A parametric "patch"

Let $\mathfrak{X} = (X, \mathscr{O})$ be an $\mathbb{R}$-space. For any region $D \in \mathbb{R}^q$, a morphism $\gamma : (D, C^\infty(D)) \to \mathfrak{X}$ will be a called a *q-patch* (of class C^∞) in $\mathfrak{X}$.

1.2.5

Any such morphism induces a morphism of differential graded $\mathbb{R}$-algebras

$$\Omega^*_{\mathcal{O}(\mathfrak{x})/\mathbb{R}} \longrightarrow \Omega^*_{C^\infty(D)/\mathbb{R}}. \tag{6}$$

Its composition with with epimorphism (4) will be denoted γ^* and called the *pull-back* map (associated with the patch).

1.2.6

For any q-form $\alpha \in \Omega^q_{\mathcal{O}}$ and any q-patch γ, its pullback, $\gamma^*\alpha$ is a volume form on D. In particular,

$$\gamma^*\alpha = f d^c x_1 \wedge \cdots \wedge d^c x_q$$

for a unique function $f \in C^\infty(D)$. This function will be denoted

$$\frac{\gamma^*\alpha}{d^c x_1 \wedge \cdots \wedge d^c x_q}. \tag{7}$$

We define then $\int_\gamma \alpha$ as the q-tuple integral

$$\int_\gamma \alpha := \int_D f = \int_D \frac{\gamma^*\alpha}{d^c x_1 \wedge \cdots \wedge d^c x_q}. \tag{8}$$

Integral in (33) is meant in the sense of Riemann q-dimensional integral if D is bounded. If it is not, then (33) can be understood as an improper integral:

$$\int_\gamma \alpha := \lim_{r\to\infty} \int_{D\cap B_r(0)} \frac{\gamma^*\alpha}{d^c x_1 \wedge \cdots \wedge d^c x_q}.$$

1.2.7

2 Singular homology of an $\mathbb{R}$-space

2.1 Euclidean q-simplices

2.1.1

A standard model of the topological *q-dimensional simplex* Δ^q, where $0 \le q < \infty$, represents it as the following subspace of $\mathbb{R}^{q+1}$:

$$\{(t_0, \ldots, t_q) \in \mathbb{R}^{q+1} \mid t_i \ge 0; t_0 + \cdots + t_q = 1\}. \tag{9}$$

2.1.2 Barycentric coordinates

Restrictions to Δ^q of the $q+1$ projections $\mathbb{R}^{q+1} \to \mathbb{R}$ are called *barycentric coordinates*.

2.1.3 Face maps

If $q > 0$, then the q-simplex has $q+1$ *faces* of dimension $q-1$:

$$\Delta^q_i := \{(t_0, \ldots, t_q) \in \mathbb{R}^{q+1} \mid t_i = 0\} \qquad (0 \le i \le q). \tag{10}$$

Each face is identified with Δ^{q-1} via one of the following $q+1$ *face maps*:

$$d^q_i : \Delta^{q-1} \longrightarrow \Delta^q, \qquad (t_0, \ldots, t_{q-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{q-1}). \tag{11}$$

Note that

$$d^q_0 : (t_0, \ldots, t_{q-1}) \mapsto (0, t_0, \ldots, t_{q-1}) \quad \text{and} \quad d^q_{q+1} : (t_0, \ldots, t_{q-1}) \mapsto (t_0, \ldots, t_{q-1}, 0).$$

Exercise 5 *Show that*

$$d^{q+1}_j d^q_i = d^{q+1}_i d^q_{j-1} \qquad (0 \le j < i \le q). \tag{12}$$

We shall refer to (12) as the *Face Relations*.

2.1.4 $\mathbb{R}$-space structures on the topological simplices

Face maps (11) are as important as spaces Δ^q themselves. When equipping Δ^q with an $\mathbb{R}$-space structure we should do this simultaneously for all q and in a manner compatible with the face maps. In other words, let $\mathcal{O}^q$ be, for each $q \in \mathbb{N}$, a subalgebra of the algebra of all $\mathbb{R}$-valued functions $\mathbb{R}^{\Delta^q}$ such that

$$\left(d^q_i\right)^* \mathcal{O}^q \subseteq \mathcal{O}^{q-1} \qquad (q > 1; 0 \le i \le q).$$

We shall call the resulting family of $\mathbb{R}$-spaces

$$\boldsymbol{\Delta} = \{(\Delta^q, \mathcal{O}^q)\}_{q \in \mathbb{N}}$$

a Δ-*realization*.

There are several natural realizations

Set theoretic realization $\boldsymbol{\Delta}^{\text{set}} = \left\{\left(\Delta^q, \mathbb{R}^{\Delta^q}\right)\right\}$,

Topological realization $\mathbf{\Delta}^{\text{top}} = \{(\Delta^q, C(\Delta^q)\}$,

Realization of class C^r $(0 \le r \le \infty)$ $\mathbf{\Delta}^{(r)} = \{(\Delta^q, C^r(\Delta^q)\}$,

Smooth realization $\mathbf{\Delta}^{\text{sm}} = \{(\Delta^q, C^\infty(\Delta^q)\}$,

Analytic realization $\mathbf{\Delta}^{\text{an}} = \{(\Delta^q, \mathscr{O}^{\text{an}}(\Delta^q)\}$,

Polynomial realization $\mathbf{\Delta}^{\text{pol}} = \{(\Delta^q, \mathscr{O}^{\text{pol}}(\Delta^q)\}$.

Note that the realizations of class C^0 and C^∞ are the same as the topological and, respectively, smooth realizations.

2.2 Singular chain complexes

2.2.1 Singular q-simplices

Fix a realization $\mathbf{\Delta}$. Given an $\mathbb{R}$-space $\mathfrak{X} = (X, \mathscr{O})$, morphisms

$$\gamma : \mathbf{\Delta}^q = (\Delta^q, \mathscr{O}^q) \longrightarrow \mathfrak{X}$$

will be called *singular q-simplices* in $\mathfrak{X}$.

2.2.2 Singular q-chains

Elements of the *free* abelian group generated by singular q-simplices

$$C_q(\mathfrak{X}) := \mathbb{Z}\,\mathrm{Hom}_{\mathbb{R}\text{-}\mathbf{Spc}}(\mathbf{\Delta}^q, \mathfrak{X}) \tag{13}$$

are called *singular q-chains* in $\mathfrak{X}$. It is customary to put

$$C_q(\mathfrak{X}) = 0 \qquad (q < 0)$$

in view of the fact that the sets of singular q-simplices are empty for $q < 0$.

2.2.3 The boundary maps

For any $q \ge 0$, the formula

$$\partial_q := d_0^* - d_1^* + \cdots + (-1)^q d_q^* \tag{14}$$

or, more explicitly,

$$\partial_q(\sigma) := \sigma \circ d_0 - \sigma \circ d_1 + \cdots + (-1)^q \sigma \circ d_q, \tag{15}$$

defines a homomorphism of abelian groups

$$\partial_q : C_q(\mathfrak{X}) \longrightarrow C_{q-1}(\mathfrak{X}). \tag{16}$$

Exercise 6 *Show that*

$$\partial_{q-1} \circ \partial_q = 0 \qquad (q \in \mathbb{Z}). \tag{17}$$

2.3 Chain complex vocabulary

2.3.1 Chain complexes of A-modules

Let A be an algebra. A sequence $C_\bullet$ of (left) A-modules $\{C_q\}_{q\in\mathbb{Z}}$ and of A-module maps $\partial_q : M_q \to M_{q-1}$ is called a *chain complex* of A-modules if maps ∂_q satisfy identity (16).

Maps $\{\partial_q\}_{q\in\mathbb{Z}}$ satisfying (16) are called *boundary maps*.

2.3.2 Cycles

Elements of $\operatorname{Ker} \partial_q$ are called *q-cycles*. They form an A-submodule of C_q which is usually denoted Z_q (“Zyklen” in German).

2.3.3 Boundaries

Elements of $\operatorname{Im} \partial_{q+1}$ are called *q-boundaries*. They form an A-submodule of Z_q which is usually denoted B_q.

2.3.4 Homology groups of a chain complex

Boundaries are considered to be “trivial” cycles. The *homology groups*, which are defined as the quotients

$$H_q(C_\bullet) := \operatorname{Ker} \partial_q / \operatorname{Im} \partial_{q+1}, \tag{18}$$

measure the difference between cycles and boundaries: H_q vanishes precisely when every q-cycle is a boundary. The homology groups of a chain complex of A-modules are A-modules themselves: the terminology “homology groups” is only a lasting tribute to tradition.

2.3.5 The category of chain complexes

Chain complexes of A-modules naturally form a category: morphisms $\varphi : (C_\bullet, \partial_\bullet) \to (C'_\bullet, \partial'_\bullet)$ consist of sequences of A-module maps $\varphi_q : C_q \to C'_q$ such that all the squares in the following diagram

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{\partial_{q-1}} & C_{q-1} & \xleftarrow{\partial_q} & C_q & \xleftarrow{\partial_{q+1}} & C_{q+1} & \xleftarrow{\partial_{q+2}} & \cdots \\
 & & \downarrow{\scriptstyle \varphi_{q-1}} & \circlearrowleft & \downarrow{\scriptstyle \varphi_q} & \circlearrowleft & \downarrow{\scriptstyle \varphi_{q+1}} & & \\
\cdots & \xleftarrow[\partial'_{q-1}]{} & C'_{q-1} & \xleftarrow[\partial'_q]{} & C'_q & \xleftarrow[\partial'_{q+1}]{} & C'_{q+1} & \xleftarrow[\partial'_{q+2}]{} & \cdots
\end{array}$$

commute. We shall denote this category $\mathscr{C}(A)$.

A morphism between complexes induces a sequence of A-module maps between the corresponding homology groups

$$H_q(\varphi) : H_q(C) \longrightarrow H_q(C').$$

Each H_q is thus a functor

$$H_q : \mathscr{C}(A) \rightsquigarrow A\textbf{-mod}. \tag{19}$$

One can also collectively think of $H_\bullet = \{H_q\}_{q \in \mathbb{Z}}$ as a functor from $\mathscr{C}(A)$ into the category of *graded* A-modules.

2.3.6 Null-homotopic morphisms

A morphism $\varphi : C_\bullet \to C'_\bullet$ is said to be *null-homotopic* (or, *homotopic to zero*) if it can be represented as the "supercommutator"

$$\varphi_q = h_{q-1} \circ \partial_q + \partial'_{q+1} \circ h_q \qquad (q \in \mathbb{Z}). \tag{20}$$

of the boundary maps and of a certain map $h : C_\bullet \to C'_\bullet$ of *degree* 1. The latter means that $h = \{h_q\}_{q \in \mathbb{Z}}$ where h_q is an A-module map $C_q \to C'_{q+1}$.

If h satisfies (20), then we call it a *contracting homotopy* for morphism φ.

Exercise 7 *Show that*

$$H_q(\varphi) = 0 \qquad (q \in \mathbb{Z})$$

for any null-morphism.

2.3.7 Homotopy classes of morphisms

We say that two morphisms φ and ψ from $C_\bullet$ to $C'_\bullet$ are *chain homotopic* if $\varphi - \psi$ is null-homotopic.

Chain homotopy is an equivalence relation on the sets of morphisms

$$\mathrm{Hom}_{\mathscr{C}(A)}(C_\bullet, C'_\bullet).$$

Null-homotopic morphisms define an *ideal* in the category of chain complexes of A-modules. The quotient category, which has chain complexes of A-modules as its objects, and homotopy classes of morphisms as its morphisms, is called the *homotopy category of chain complexes of A-modules*.

It follows from Exercise 7 that the homology functors (19) factorize through the homotopy category.

2.3.8 Homotopy equivalences

We say that a morphism $\varphi : C_\bullet \to C'_\bullet$ is a *homotopy equivalence* if it becomes an isomorphism between $C_\bullet$ and $C'_\bullet$ in the homotopy category.

Explicitly, φ is a *homotopy equivalence* if there exists a morphism $\psi : C'_\bullet \to C_\bullet$ such that $\varphi \circ \psi$ is homotopic to $\mathrm{id}_{C'_\bullet}$ and $\psi \circ \varphi$ is homotopic to $\mathrm{id}_{C_\bullet}$.

2.3.9 Contractible complexes

A complex $C_\bullet$ is said to be *contractible* if it is homotopy equivalent to the *zero* complex.

Exercise 8 *Show that $C_\bullet$ is contractible if and only if* $\mathrm{id}_{C_\bullet}$ *is null-homotopic.*

2.3.10

For any A-module M, consider the chain complex $M[0]$

$$M[0]_q := \begin{cases} M & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

The correspondence $M \mapsto M[0]$ defines a canonical embedding of the category of A-modules into the category of chain complexes of A-modules.

2.3.11 Shift functors

For any $j \in \mathbb{Z}$ and any chain complex $C_\bullet$, define $C_\bullet[j]$ as

$$(C[j])_q := C_{q-j} \qquad \text{and} \qquad (\partial[j])_q := (-1)^j \partial_{q-j}. \tag{22}$$

This defines so called *shift* functors $\mathscr{C}(A) \rightsquigarrow \mathscr{C}(A)$.

Note that

$$[i] \circ [j] = [i+j] \qquad \text{and} \qquad [0] = \mathrm{id}_{\mathscr{C}(A)}\,.$$

2.4 Singular homology

2.4.1 The singular chain complexes of an $\mathbb{R}$-space

In view of identities (17), the sequence of abelian groups $\{C_q(\mathfrak{X})\}_{q\in\mathbb{Z}}$ and homomorphisms $\{\partial_q\}_{q\in\mathbb{Z}}$ forms a chain complex of abelian groups (i.e., $\mathbb{Z}$-modules). We shall denote it $C_\bullet(\mathfrak{X})$ and refer to it as the *singular chain complex* of $\mathfrak{X}$.

This complex and its homology depend on the chosen realization $\boldsymbol{\Delta}$. To indicate this dependence we may be also using notation $C_\bullet^{\boldsymbol{\Delta}}(\mathfrak{X})$.

Note that the correspondence $\mathfrak{X} \mapsto C_\bullet^{\boldsymbol{\Delta}}(\mathfrak{X})$ is functorial in $\mathfrak{X}$, in other words, it defines a functor from the category of $\mathbb{R}$-spaces to the category of chain complexes of abelian groups.

2.4.2

In special cases like the ones mentioned in 2.1.4, we shall be speaking of *set-theoretic*, *continuous* (or *topological*), *smooth* (or *class* C^∞), *analytic* and, respectively, *polynomial* singular chains. The corresponding complexes will be denoted $C_\bullet^{\mathrm{set}}(\mathfrak{X})$, $C_\bullet^{\mathrm{top}}(\mathfrak{X})$, $C_\bullet^{\mathrm{sm}}(\mathfrak{X})$, $C_q^{\mathrm{an}}(\mathfrak{X})$ and, respectively, $C_q^{\mathrm{pol}}(\mathfrak{X})$.

2.4.3

Note that $C_\bullet^{\mathrm{set}}(\mathfrak{X})$ depends only on the underlying set $|\mathfrak{X}|$, not on the structural algebra $\mathscr{O}(\mathfrak{X})$.

2.4.4 The singular homology groups of an $\mathbb{R}$-space

The homology groups of $C_\bullet^{\boldsymbol{\Delta}}(\mathfrak{X})$ will be denoted $H_\bullet^{\boldsymbol{\Delta}}(\mathfrak{X})$ and referred to as the *singular homology groups* of $\mathfrak{X}$ (with respect to a given realization $\boldsymbol{\Delta}$).

2.4.5

In special cases mentioned in 2.1.4, we shall be speaking of *set-theoretic*, *continuous* (or *topological*), *smooth* (or *class* C^∞), *analytic* and, respectively, *polynomial* singular homology groups of $\mathfrak{X}$. The corresponding groups will be denoted $H_\bullet^{\text{set}}(\mathfrak{X})$, $H_\bullet^{\text{top}}(\mathfrak{X})$, $H_\bullet^{\text{sm}}(\mathfrak{X})$, $H_q^{\text{an}}(\mathfrak{X})$ and, respectively, $H_q^{\text{pol}}(\mathfrak{X})$.

2.4.6 Example: singular homology of a point

A set consisting of a single element $X = \{*\}$ admits a unique $\mathbb{R}$-space structure: $\mathcal{O} = \mathbb{R}$. For every $q \in \mathbb{Z}$, there is only one singular q-simplex: the unique map $\sigma^q : \Delta^q \to \{*\}$, irrespective of the actual simplicial realization $\mathbf{\Delta}$ we use. It follows that each singular chain group is a free group of rank 1:

$$C_q^\Delta(*) = \mathbb{Z}\sigma^q \qquad (q \in \mathbb{Z}).$$

Exercise 9 *Show that ∂_q in $C_\bullet^{\mathbf{\Delta}}(*)$ is zero for any odd q, and that ∂_q is an isomorphism $C_q \simeq C_{q-1}$ for any even $q \geq 2$.*

Exercise 10 *Show that the inclusion of $\mathbb{Z}[0]$ into $C_\bullet^{\mathbf{\Delta}}(*)$ is a homotopy equivalence.*

2.4.7

We have noted before that the set-theoretic singular complex $C_\bullet^{\text{set}}(\mathfrak{X})$ depends only on the underlying set $X = |\mathfrak{X}|$, not on the structural algebra $\mathcal{O}(\mathfrak{X})$. We shall therefore also denote it by $C_\bullet^{\text{set}}(X)$.

We will now prove that $C_\bullet^{\text{set}}(X)$ is homotopy equivalent to $C_\bullet^{\text{set}}(*)$ which we already know is homotopy equivalent to $\mathbb{Z}[0]$.

Proposition 2.1 *For any two maps ϕ and ψ from a set X to a set Y, the induced morphisms of the set-theoretic singular chain complexes are homotopy equivalent.*

Proof. Since the cardinality of Y^X is less or equal 1 when one of the sets is empty, we can assume that both X and Y are nonempty.

Let $y \in Y$. It suffices to show that, for any map $\phi : X \to Y$, the morphism $\phi_\bullet : C_\bullet^{\text{set}}(X) \to C_\bullet^{\text{set}}(Y)$ is chain homotopic to the morphism $\psi_\bullet$ induced by the map that sends every element of X to y:

$$\psi(x) = y \qquad (x \in X).$$

Define the maps $h_q : C_q^{\text{set}}(X) \to C_{q+1}^{\text{set}}(Y)$ as follows:

$$h(\sigma)\big(t_0, \ldots, t_{q+1}\big) := \begin{cases} \phi\Big(\sigma\Big(\frac{t_1}{1-t_0}, \ldots, \frac{t_{q+1}}{1-t_0}\Big)\Big) & \big(0 \le t_0 < 1\big) \\ y & \big(t_0 = 1\big) \end{cases}. \tag{23}$$

We have

$$\big((\partial h)(\sigma)\big)\big(t_0, \ldots, t_q\big) = \\ h(\sigma)\big(0, t_0, \ldots, t_q\big) - h(\sigma)\big(t_0, 0, t_1, \ldots, t_q\big) + \cdots \\ + (-1)^{q+1} h(\sigma)\big(t_0, \ldots, t_q, 0\big) \tag{24}$$

and

$$\big((h\partial)(\sigma)\big)\big(t_0, \ldots, t_q\big) = \\ h(\sigma)\big(t_0, 0, t_1, \ldots, t_q\big) + \cdots + (-1)^q h(\sigma)\big(t_0, \ldots, t_q, 0\big). \tag{25}$$

By combining (24)–(25) with (23) we obtain

$$\big((\partial h + h\partial)\sigma\big)\big(t_0, \ldots, t_q\big) = h(\sigma)\big(0, t_0, \ldots, t_q\big) = \sigma\big(t_0, \ldots, t_q\big),$$

i.e.,

$$(\partial h + h\partial)\sigma = \sigma \qquad \big(\sigma \in C_q^{\text{set}}(X)\big).$$

□

Corollary 2.2 *Any map between nonempty sets $\phi : X \to Y$ induces a homotopy equivalence between $C_\bullet^{\text{set}}(X)$ and $C_\bullet^{\text{set}}(Y)$.*

Indeed, for any map $\psi : Y \to X$, morphism $\phi_\bullet \circ \psi_\bullet = (\phi \circ \psi)_\bullet$ is chain homotopic to $\mathrm{id}_{C_\bullet^{\text{set}}(Y)}$ and $\psi_\bullet \circ \phi_\bullet = (\psi \circ \phi)_\bullet$ is chain homotopic to $\mathrm{id}_{C_\bullet^{\text{set}}(X)}$ in view of just proven Proposition 2.1.

Corollary 2.3 *For any nonempty set X, the set-theoretic singular chain complex $C_\bullet^{\text{set}}(X)$ is homotopy equivalent to $\mathbb{Z}[0]$. In particular,*

$$H_q^{\text{set}}(X) = \begin{cases} \mathbb{Z} & (q = 0) \\ 0 & (q > 0) \end{cases}. \tag{26}$$

2.5 De Rham Theory

2.6 De Rham Pairing

2.6.1

Let $\mathfrak{X} = (X, \mathcal{O})$ be an $\mathbb{R}$-space. For any $q \in \mathbb{N}$, there is an obvious pairing

$$C_q^{\mathrm{sm}}(\mathfrak{X}) \times \Omega^q_{\mathcal{O}/\mathbb{R}} \longrightarrow \mathbb{R}, \qquad (\sigma, \alpha) \mapsto \int_\sigma \alpha \tag{27}$$

where the integration is extended by $\mathbb{Z}$-linearity from singular q-simplices to singular q-chains:

$$\text{if} \quad \sigma = \sum m_\gamma \gamma, \quad \text{then} \quad \int_\sigma \alpha := \sum m_\gamma \int_\gamma \alpha. \tag{28}$$

2.6.2

de Rham pairing is obviously additive (i.e., $\mathbb{Z}$-linear) in left argument and $\mathbb{R}$-linear in right argument.

Theorem 2.4 (Stokes Theorem) *The boundary map $\partial : C_q(\mathfrak{X}) \to C_{q-1}(\mathfrak{X})$ and the de Rham differential $d : \Omega^{q-1}_{\mathcal{O}/\mathbb{R}}$ are adjoint to each other, i.e.,*

$$\int_{\partial\sigma} \beta = \int_\sigma d\beta \qquad (\sigma \in C_q(\mathfrak{X}),\ \beta \in \Omega^{q-1}_{\mathcal{O}/\mathbb{R}}). \tag{29}$$

2.6.3

An equivalent formulation of Stokes' Theorem is obtained by considering the singular *cochain complex* $(C^*_{\mathbf{\Delta}}(\mathfrak{X}; \mathbb{R}), \delta)$ which is defined as the *dual* of the singular chain complex:

$$C^q_{\mathbf{\Delta}} := \mathrm{Hom}_{\mathbb{Z}\text{-}\mathbf{mod}}(C_q^{\mathbf{\Delta}}(\mathfrak{X}), \mathbb{R}) = \mathrm{Map}(\mathrm{Hom}_{\mathbb{R}\text{-}\mathbf{Spc}}(\mathbf{\Delta}^q, \mathfrak{X}), \mathbb{R}) \tag{30}$$

and

$$(\delta^q(\varphi))(\sigma) := \varphi(\partial\sigma) \qquad \left(\varphi \in C^q_{\mathbf{\Delta}}(\mathfrak{X}),\ \sigma \in C_q^{\mathbf{\Delta}}(\mathfrak{X})\right). \tag{31}$$

2.6.4 De Rham Map

De Rham Map is the dual form of the de Rham Pairing introduced in (27)

$$\Omega^q_{\mathscr{O}(\mathfrak{X})/\mathbb{R}} \longrightarrow C^q_{\mathrm{sm}}(\mathfrak{X};\mathbb{R}), \qquad \alpha \mapsto \int \alpha, \tag{32}$$

where $\int \alpha$ is a singular cochain

$$\int \alpha : \sigma \mapsto \int_\sigma \alpha \qquad \left(\sigma \in C_q^{\mathrm{sm}}(\mathfrak{X})\right). \tag{33}$$

Theorem 2.5 (Stokes Theorem (dual form)) *One has*

$$\int d\beta = \delta\left(\int d\beta\right) \qquad \left(\beta \in \Omega^*_{\mathscr{O}/\mathbb{R}}\right),$$

i.e., de Rham Map is a morphism of cochain complexes,

$$\left(\Omega^q_{\mathscr{O}/\mathbb{R}}, d\right) \longrightarrow \left(C^q_{\mathrm{sm}}(\mathfrak{X};\mathbb{R}), \delta\right).$$

2.6.5

It follows that the de Rham Map induces a homomorphism of cohomology groups (which are graded $\mathbb{R}$-vector spaces):

$$H^*_{\mathrm{dR}}(\mathscr{O}(\mathfrak{X})/\mathbb{R}) \longrightarrow H^*_{\mathrm{sm}}(\mathfrak{X};\mathbb{R}).$$