1 \( \mathbb{R} \)-spaces

1.1 Vocabulary

1.1.1

We shall call a pair \( \mathcal{X} = (X, \mathcal{O}) \), where \( X \) is a set and \( \mathcal{O} \subseteq \mathbb{R}^X \) is a unital \( \mathbb{R} \)-algebra of real valued functions on a set \( X \), an \( \mathbb{R} \)-space. Here \( X \) will be called the support of \( \mathcal{X} \) and will be denoted \( |\mathcal{X}| \), while \( \mathcal{O} = \mathcal{O}(\mathcal{X}) \) will be called the structural algebra of \( \mathcal{X} \).

In the category of \( \mathbb{R} \)-spaces morphisms from \( \mathcal{X} \) to \( \mathcal{Y} \) are maps \( \phi : |\mathcal{X}| \to |\mathcal{Y}| \) such that

\[
\phi^* \mathcal{O}(\mathcal{Y}) := \{ f \circ \phi \mid f \in \mathcal{O}(\mathcal{Y}) \} \subseteq \mathcal{O}(\mathcal{X}).
\]

We shall denote the category of \( \mathbb{R} \)-spaces by \( \mathbb{R} \)-Spc.

1.1.2 An \( \mathbb{R} \)-space associated with a topological space

With any topological space \( (X, \mathcal{T}) \) one can naturally associate an \( \mathbb{R} \)-space \( (X, C(X)) \) where \( C(X) = C(X, \mathcal{T}) \) denotes the algebra of functions \( X \to \mathbb{R} \) continuous with respect to topology \( \mathcal{T} \). This defines a functor from the category of topological spaces \( \text{Top} \) to \( \mathbb{R} \)-Spc

\[
C : \text{Top} \to \mathbb{R} \text{-Spc}, \quad (X, \mathcal{T}) \mapsto (X, C(X, \mathcal{T})). \tag{1}
\]

1.1.3 \( \mathbb{R} \)-spaces associated with a subset of \( \mathbb{R}^n \)

We shall say that a function \( f : D \to \mathbb{R} \) on a subset \( D \subseteq \mathbb{R}^n \) is of class \( C^r \), \( 0 \leq r \leq \infty \), (analytic, polynomial), if \( f \) is the restriction to \( D \) of a function of
class $C^r$ (respectively, analytic, polynomial) defined on an open subset containing $D$.

Functions of class $C^r$ (respectively, analytic, polynomial) on $D$ form an algebra denoted below $C^r(D)$ (respectively, $\mathcal{O}^{an}(D)$, $\mathcal{O}^{pol}(D)$). With each of the above algebras there is associated a corresponding $\mathbb{R}$-space: $D^r = (D, C^r(D))$, $D^{an} = (D, \mathcal{O}^{an}(D))$, and $D^{an} = (D, \mathcal{O}^{pol}(D))$, respectively.

### 1.1.4 The canonical topology

The topology $\mathcal{T} = \mathcal{T}(\mathfrak{X})$ generated by the family of preimages of open subsets of $\mathbb{R}$ by members of $\mathcal{O} = \mathcal{O}(\mathfrak{X})$,

$$\mathcal{B} = \{ f^{-1}(V) \mid f \in \mathcal{O}, V \subseteq \mathbb{R} \text{ open} \},$$

is the weakest topology on $X = |\mathfrak{X}|$ in which all $f \in \mathcal{O}$ are continuous as functions $X \to \mathbb{R}$. We shall call it the canonical topology of an $\mathbb{R}$-space $\mathfrak{X}$.

This defines a functor

$$T : \mathbb{R}\text{-Spc} \to \text{Top}, \quad \mathfrak{X} = (X, \mathcal{O}) \longmapsto (X, \mathcal{T}(\mathfrak{X})). \quad (2)$$

**Exercise 1** Show that the canonical topology is generated by the family of preimages of intervals $(-\epsilon, \epsilon)$:

$$\mathcal{B}_0 = \{ f^{-1}(-\epsilon, \epsilon) \mid f \in \mathcal{O}, \epsilon > 0 \}.$$

**Exercise 2** Show that the canonical topology is completely regular, i.e., for any closed subset $Z \subseteq X$ and a point $p \notin Z$, there exists a function $f : X \to \mathbb{R}$, continuous in canonical topology, such that

$$f(p) = 1 \quad \text{and} \quad f|_Z = 1.$$

**Exercise 3** Let $(X, \mathcal{T})$ be a topological space and $\mathfrak{Y} = (Y, \mathcal{O})$ be an $\mathbb{R}$-space. Show that a map $\phi : (X, \mathcal{T}) \to (Y, \mathcal{T}(\mathfrak{Y}))$ is continuous if and only if $\phi^*(\mathcal{O}) \subseteq C(X, \mathcal{T})$.

Derive from this that the functor $\text{Top} \to \mathbb{R}\text{-Spc}$, defined in (1), is left adjoint to the functor $\mathbb{R}\text{-Spc} \to \text{Top}$ defined in (2).

**Exercise 4** Show that

$$S \circ T \circ S = S \quad \text{and} \quad T \circ S \circ T = T.$$
1.2 Integral of a differential form over a parametric patch

1.2.1 Regions in Euclidean space

A subset $D \subseteq \mathbb{R}^q$ will be called a region if it is contained in the closure of its interior. We shall mostly deal with open or closed regions.

1.2.2 Denote by $\Omega^1_D$ the free $\mathcal{O}(D)$-module of rank $q$ with basis $dx_1, \ldots, dx_q$.

The map

$$d^c : \mathcal{O}(D) \longrightarrow \Omega^1_D, \quad f \mapsto d^c f = \sum_{i=1}^{q} \frac{\partial f}{\partial x_i} d^c x_i,$$

(3)

is an $\mathbb{R}$-linear derivation of algebra $\mathcal{O}(D)$. It is in fact a *universal continuous* derivation with values in a locally convex $\mathcal{O}(D)$-module. The subscript $c$ indicates that and also serves the reader warning not to confuse $d^c f \in \Omega^1_D$ with $df \in \Omega^1_{\mathcal{O}(D)/\mathbb{R}}$.

Derivation (3) induces an $\mathcal{O}(D)$-linear and obviously surjective map

$$\Omega^1_{\mathcal{O}(D)/\mathbb{R}} \longrightarrow \Omega^1_D \quad df \mapsto d^c f,$$

which in turn induces a surjective map of differential graded $\mathcal{O}(D)$-algebras

$$\Omega^*_{\mathcal{O}(D)/\mathbb{R}} \longrightarrow \Omega^*_D : = \bigwedge^*_{\mathcal{O}(D)} \Omega^1_D, \quad \alpha \mapsto \alpha^c.$$

(4)

Note that

$$(f_0 df_1 \wedge \cdots \wedge df_p)^c = f_0 d^c f_1 \wedge \cdots \wedge d^c f_p.$$

1.2.3 Volume forms

Since $\Omega^1_D$ is free of rank $q$, its $p$-th exterior power, $\Omega^p_D$ is free of rank $\binom{q}{p}$. In particular, $\Omega^1_D$ is a free $\mathcal{O}(D)$-module of rank 1,

$$\Omega^1_D = \mathcal{O}(D) dx_1 \wedge \cdots \wedge dx_q.$$

(5)

1.2.4 A parametric "patch"

Let $\mathfrak{X} = (X, \mathcal{O})$ be an $\mathbb{R}$-space. For any region $D \in \mathbb{R}^q$, a morphism $\gamma : (D, C^\infty(D)) \rightarrow \mathfrak{X}$ will be called a $q$-patch (of class $C^\infty$) in $\mathfrak{X}$.
1.2.5

Any such morphism induces a morphism of differential graded \(\mathbb{R}\)-algebras

\[
\Omega^*_{\mathcal{O}(\mathfrak{v})/\mathbb{R}} \to \Omega^*_{C^\infty(D)/\mathbb{R}}.
\]

(6)

Its composition with with epimorphism (4) will be denoted \(\gamma^*\) and called the pull-back map (associated with the patch).

1.2.6

For any \(q\)-form \(\alpha \in \Omega^q_{\mathcal{O}}\) and any \(q\)-patch \(\gamma\), its pullback, \(\gamma^*\alpha\) is a volume form on \(D\). In particular,

\[
\gamma^*\alpha = f d^c x_1 \wedge \cdots \wedge d^c x_q
\]

for a unique function \(f \in C^\infty(D)\). This function will be denoted

\[
\frac{\gamma^*\alpha}{d^c x_1 \wedge \cdots \wedge d^c x_q}.
\]

(7)

We define then \(\int_\gamma \alpha\) as the \(q\)-tuple integral

\[
\int_\gamma \alpha := \int_D f = \int_D \frac{\gamma^*\alpha}{d^c x_1 \wedge \cdots \wedge d^c x_q}.
\]

(8)

Integral in (33) is meant in the sense of Riemann \(q\)-dimensional integral if \(D\) is bounded. If it is not, then (33) can be understood as an improper integral:

\[
\int_\gamma \alpha := \lim_{r \to \infty} \int_{D \cap B_r(0)} \frac{\gamma^*\alpha}{d^c x_1 \wedge \cdots \wedge d^c x_q}.
\]

1.2.7

2 Singular homology of an \(\mathbb{R}\)-space

2.1 Euclidean \(q\)-simplices

2.1.1

A standard model of the topological \(q\)-dimensional simplex \(\Delta^q\), where \(0 \leq q < \infty\), represents it as the following subspace of \(\mathbb{R}^{q+1}\):

\[
\{(t_0, \ldots, t_q) \in \mathbb{R}^{q+1} \mid t_i \geq 0; t_0 + \cdots + t_q = 1\}.
\]

(9)
2.1.2 Barycentric coordinates

Restrictions to $\Delta^q$ of the $q+1$ projections $\mathbb{R}^{q+1} \to \mathbb{R}$ are called barycentric coordinates.

2.1.3 Face maps

If $q > 0$, then the $q$-simplex has $q+1$ faces of dimension $q-1$:

$$\Delta^q_i := \{(t_0, \ldots, t_q) \in \mathbb{R}^{q+1} \mid t_i = 0\} \quad (0 \leq i \leq q).$$

(10)

Each face is identified with $\Delta^{q-1}$ via one of the following $q+1$ face maps:

$$d^q_i : \Delta^{q-1} \longrightarrow \Delta^q, \quad (t_0, \ldots, t_{q-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{q-1}).$$

(11)

Note that $d^q_0 : (t_0, \ldots, t_{q-1}) \mapsto (0, t_0, \ldots, t_{q-1})$ and $d^q_{q+1} : (t_0, \ldots, t_{q-1}) \mapsto (t_0, \ldots, t_{q-1}, 0)$.

Exercise 5 Show that

$$d^{q+1}_j d^q_i = d^{q+1}_i d^{q-1}_j \quad (0 \leq j < i \leq q).$$

(12)

We shall refer to (12) as the Face Relations.

2.1.4 $\mathbb{R}$-space structures on the topological simplices

Face maps (11) are as important as spaces $\Delta^q$ themselves. When equipping $\Delta^q$ with an $\mathbb{R}$-space structure we should do this simultaneously for all $q$ and in a manner compatible with the face maps. In other words, let $\mathcal{O}^q$ be, for each $q \in \mathbb{N}$, a subalgebra of the algebra of all $\mathbb{R}$-valued functions $\mathbb{R}^{\Delta^q}$ such that

$$\left(d^q_i\right)^* \mathcal{O}^q \subseteq \mathcal{O}^{q-1} \quad (q > 1; 0 \leq i \leq q).$$

We shall call the resulting family of $\mathbb{R}$-spaces

$$\Delta = \{(\Delta^q, \mathcal{O}^q)\}_{q \in \mathbb{N}}$$

a $\Delta$-realization.

There are several natural realizations

Set theoretic realization $\Delta^{\text{set}} = \left\{(\Delta^q, \mathbb{R}^{\Delta^q})\right\}$,
Topological realization \( \Delta^{\text{top}} = \{(\Delta^q, C(\Delta^q))\} \),

Realization of class \( C^r \) \((0 \leq r \leq \infty)\) \( \Delta^{(r)} = \{(\Delta^q, C^r(\Delta^q))\} \),

Smooth realization \( \Delta^{\text{sm}} = \{(\Delta^q, C^\infty(\Delta^q))\} \),

Analytic realization \( \Delta^{\text{an}} = \{(\Delta^q, \mathcal{O}^{\text{an}}(\Delta^q))\} \),

Polynomial realization \( \Delta^{\text{pol}} = \{(\Delta^q, \mathcal{O}^{\text{pol}}(\Delta^q))\} \).

Note that the realizations of class \( C^0 \) and \( C^\infty \) are the same as the topological and, respectively, smooth realizations.

2.2 Singular chain complexes

2.2.1 Singular \( q \)-simplices

Fix a realization \( \Delta \). Given an \( \mathbb{R} \)-space \( \mathcal{X} = (X, \mathcal{O}) \), morphisms

\[ \gamma : \Delta^q = (\Delta^q, \mathcal{O}^q) \rightarrow \mathcal{X} \]

will be called singular \( q \)-simplices in \( \mathcal{X} \).

2.2.2 Singular \( q \)-chains

Elements of the free abelian group generated by singular \( q \)-simplices

\[ C_q(\mathcal{X}) := \mathbb{Z} \text{Hom}_{\mathbb{R}\text{-Spc}}(\Delta^q, \mathcal{X}) \]  

are called singular \( q \)-chains in \( \mathcal{X} \). It is customary to put

\[ C_q(\mathcal{X}) = 0 \quad (q < 0) \]

in view of the fact that the sets of singular \( q \)-simplices are empty for \( q < 0 \).

2.2.3 The boundary maps

For any \( q \geq 0 \), the formula

\[ \partial_q := d_0^* - d_1^* + \cdots + (-1)^q d_q^* \]  

(14)
or, more explicitly,
\[
\partial_q(\sigma) := \sigma \circ d_0 - \sigma \circ d_1 + \cdots + (-1)^q \sigma \circ d_q,
\]
(15)
defines a homomorphism of abelian groups
\[
\partial_q : C_q(\mathcal{X}) \longrightarrow C_{q-1}(\mathcal{X}).
\]
(16)

**Exercise 6** Show that
\[
\partial_{q-1} \circ \partial_q = 0 \quad (q \in \mathbb{Z}).
\]
(17)

### 2.3 Chain complex vocabulary

#### 2.3.1 Chain complexes of \(A\)-modules

Let \(A\) be an algebra. A sequence \(C\), of (left) \(A\)-modules \(\{C_q\}_{q \in \mathbb{Z}}\) and of \(A\)-module maps \(\partial_q : M_q \rightarrow M_{q-1}\) is called a **chain complex** of \(A\)-modules if maps \(\partial_q\) satisfy identity (16).

Maps \(\{\partial_q\}_{q \in \mathbb{Z}}\) satisfying (16) are called **boundary maps**.

#### 2.3.2 Cycles

Elements of \(\text{Ker} \ \partial_q\) are called **\(q\)-cycles**. They form an \(A\)-submodule of \(C_q\) which is usually denoted \(Z_q\) (“Zyklen” in German).

#### 2.3.3 Boundaries

Elements of \(\text{Im} \ \partial_{q+1}\) are called **\(q\)-boundaries**. They form an \(A\)-submodule of \(Z_q\) which is usually denoted \(B_q\).

#### 2.3.4 Homology groups of a chain complex

Boundaries are considered to be “trivial” cycles. The **homology groups**, which are defined as the quotients
\[
H_q(C) := \text{Ker} \partial_q / \text{Im} \partial_{q+1},
\]
(18)
measure the difference between cycles and boundaries: \(H_q\) vanishes precisely when every \(q\)-cycle is a boundary. The homology groups of a chain complex of \(A\)-modules are \(A\)-modules themselves: the terminology “homology groups” is only a lasting tribute to tradition.
2.3.5 The category of chain complexes

Chain complexes of $A$-modules naturally form a category: morphisms $\varphi : (C, \partial) \to (C', \partial')$ consist of sequences of $A$-module maps $\varphi_q : C_q \to C'_q$ such that all the squares in the following diagram commute. We shall denote this category $\mathcal{C}(A)$.

A morphism between complexes induces a sequence of $A$-module maps between the corresponding homology groups

$$H_q(\varphi) : H_q(C) \longrightarrow H_q(C').$$

Each $H_q$ is thus a functor

$$H_q : \mathcal{C}(A) \to A\text{-mod}. \quad (19)$$

One can also collectively think of $H_* = \{H_q\}_{q \in \mathbb{Z}}$ as a functor from $\mathcal{C}(A)$ into the category of graded $A$-modules.

2.3.6 Null-homotopic morphisms

A morphism $\varphi : C \to C'$ is said to be null-homotopic (or, homotopic to zero) if it can be represented as the “supercommutator”

$$\varphi_q = h_{q-1} \circ \partial_q + \partial'_{q+1} \circ h_q \quad (q \in \mathbb{Z}). \quad (20)$$

of the boundary maps and of a certain map $h : C \rightarrow C'$ of degree 1. The latter means that $h = \{h_q\}_{q \in \mathbb{Z}}$ where $h_q$ is an $A$-module map $C_q \rightarrow C'_{q+1}$.

If $h$ satisfies (20), then we call it a contracting homotopy for morphism $\varphi$.

**Exercise 7** Show that

$$H_q(\varphi) = 0 \quad (q \in \mathbb{Z})$$

for any null-morphism.
2.3.7 Homotopy classes of morphisms

We say that two morphisms \( \varphi \) and \( \psi \) from \( C_* \) to \( C'_* \) are chain homotopic if \( \varphi - \psi \) is null-homotopic.

Chain homotopy is an equivalence relation on the sets of morphisms

\[
\text{Hom}_{\text{Gr}(A)}(C_*, C'_*). \]

Null-homotopic morphisms define an ideal in the category of chain complexes of \( A \)-modules. The quotient category, which has chain complexes of \( A \)-modules as its objects, and homotopy classes of morphisms as its morphisms, is called the homotopy category of chain complexes of \( A \)-modules.

It follows from Exercise 7 that the homology functors (19) factorize through the homotopy category.

2.3.8 Homotopy equivalences

We say that a morphism \( \varphi : C_* \to C'_* \) is a homotopy equivalence if it becomes an isomorphism between \( C_* \) and \( C'_* \) in the homotopy category.

Explicitly, \( \varphi \) is a homotopy equivalence if there exists a morphism \( \psi : C'_* \to C_* \) such that \( \varphi \circ \psi \) is homotopic to \( \text{id}_{C'_*} \) and \( \psi \circ \varphi \) is homotopic to \( \text{id}_{C_*} \).

2.3.9 Contractible complexes

A complex \( C_* \) is said to be contractible if it is homotopy equivalent to the zero complex.

Exercise 8 Show that \( C_* \) is contractible if and only if \( \text{id}_{C_*} \) is null-homotopic.

2.3.10

For any \( A \)-module \( M \), consider the chain complex \( M[0] \)

\[
M[0]_q := \begin{cases} M & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}
\]

The correspondence \( M \mapsto M[0] \) defines a canonical embedding of the category of \( A \)-modules into the category of chain complexes of \( A \)-modules.
2.3.11 Shift functors

For any \( j \in \mathbb{Z} \) and any chain complex \( C_\ast \), define \( C_\ast[j] \) as

\[
(C[j])_q := C_{q-j} \quad \text{and} \quad (\partial[j])_q := (-1)^j \partial_{q-j}.
\]

(22)

This defines so called shift functors \( \mathcal{C}(A) \to \mathcal{C}(A) \).

Note that \([i] \circ [j] = [i + j]\) and \([0] = \text{id}_{\mathcal{C}(A)}\).

2.4 Singular homology

2.4.1 The singular chain complexes of an \( \mathbb{R} \)-space

In view of identities (17), the sequence of abelian groups \( \{C_q(\mathcal{X})\}_{q \in \mathbb{Z}} \) and homomorphisms \( \{\partial_q\}_{q \in \mathbb{Z}} \) forms a chain complex of abelian groups (i.e., \( \mathbb{Z} \)-modules). We shall denote it \( C_\ast(\mathcal{X}) \) and refer to it as the singular chain complex of \( \mathcal{X} \).

This complex and its homology depend on the chosen realization \( \Delta \). To indicate this dependence we may be also using notation \( C^\Delta_\ast(\mathcal{X}) \).

Note that the correspondence \( \mathcal{X} \mapsto C^\Delta_\ast(\mathcal{X}) \) is functorial in \( \mathcal{X} \), in other words, it defines a functor from the category of \( \mathbb{R} \)-spaces to the category of chain complexes of abelian groups.

2.4.2

In special cases like the ones mentioned in 2.1.4, we shall be speaking of set-theoretic, continuous (or topological), smooth (or class \( C^\infty \)), analytic and, respectively, polynomial singular chains. The corresponding complexes will be denoted \( C^{\text{set}}_\ast(\mathcal{X}), C^{\text{top}}_\ast(\mathcal{X}), C^{\text{sm}}_\ast(\mathcal{X}), C^{\text{an}}_\ast(\mathcal{X}) \) and, respectively, \( C^{\text{pol}}_q(\mathcal{X}) \).

2.4.3

Note that \( C^{\text{set}}_\ast(\mathcal{X}) \) depends only on the underlying set \( |\mathcal{X}| \), not on the structural algebra \( \mathcal{O}(\mathcal{X}) \).

2.4.4 The singular homology groups of an \( \mathbb{R} \)-space

The homology groups of \( C^\Delta_\ast(\mathcal{X}) \) will be denoted \( H^\Delta_\ast(\mathcal{X}) \) and referred to as the singular homology groups of \( \mathcal{X} \) (with respect to a given realization \( \Delta \)).
2.4.5

In special cases mentioned in 2.1.4, we shall be speaking of set-theoretic, continuous (or topological), smooth (or class $C^\infty$), analytic and, respectively, polynomial singular homology groups of $\mathcal{X}$. The corresponding groups will be denoted $H_\text{set}^\ast(\mathcal{X})$, $H_\text{top}^\ast(\mathcal{X})$, $H_\text{sm}^\ast(\mathcal{X})$, $H_\text{an}^\ast(\mathcal{X})$ and, respectively, $H_\text{pol}^\ast(\mathcal{X})$.

2.4.6 Example: singular homology of a point

A set consisting of a single element $X = \{\ast\}$ admits a unique $\mathbb{R}$-space structure: $\emptyset = \mathbb{R}$. For every $q \in \mathbb{Z}$, there is only one singular $q$-simplex: the unique map $\sigma^q : \Delta^q \to \{\ast\}$, irrespective of the actual simplicial realization $\Delta$ we use. It follows that each singular chain group is a free group of rank 1:

$$C_q^\Delta(\ast) = \mathbb{Z}\sigma^q \quad (q \in \mathbb{Z}).$$

Exercise 9 Show that $\partial_q$ in $C_q^\Delta(\ast)$ is zero for any odd $q$, and that $\partial_q$ is an isomorphism $C_q \simeq C_{q-1}$ for any even $q \geq 2$.

Exercise 10 Show that the inclusion of $\mathbb{Z}[0]$ into $C_q^\Delta(\ast)$ is a homotopy equivalence.

2.4.7

We have noted before that the set-theoretic singular complex $C_\text{set}^\ast(\mathcal{X})$ depends only on the underlying set $X = |\mathcal{X}|$, not on the structural algebra $\emptyset(\mathcal{X})$. We shall therefore also denote it by $C_\text{set}^\ast(X)$.

We will now prove that $C_\text{set}^\ast(X)$ is homotopy equivalent to $C_\text{set}^\ast(\ast)$ which we already know is homotopy equivalent to $\mathbb{Z}[0]$.

Proposition 2.1 For any two maps $\phi$ and $\psi$ from a set $X$ to a set $Y$, the induced morphisms of the set-theoretic singular chain complexes are homotopy equivalent.

Proof. Since the cardinality of $Y^X$ is less or equal 1 when one of the sets is empty, we can assume that both $X$ and $Y$ are nonempty.

Let $y \in Y$. It suffices to show that, for any map $\phi : X \to Y$, the morphism $\phi_* : C_\text{set}^\ast(X) \to C_\text{set}^\ast(Y)$ is chain homotopic to the morphism $\psi_*$ induced by the map that sends every element of $X$ to $y$:

$$\psi(x) = y \quad (x \in X).$$
Define the maps $h_q : C^\text{set}_q(X) \to C^\text{set}_{q+1}(Y)$ as follows:

$$h(\sigma)(t_0, \ldots, t_{q+1}) := \begin{cases} \phi(\sigma \left( \frac{t_1}{1-t_0}, \ldots, \frac{t_{q+1}}{1-t_0} \right)) & (0 \leq t_0 < 1) \\ y & (t_0 = 1) \end{cases}.$$ (23)

We have

$$((\partial h)(\sigma))(t_0, \ldots, t_q) = h(\sigma)(0, t_0, \ldots, t_q) - h(\sigma)(t_0, 0, t_1, \ldots, t_q) + \cdots + (-1)^{q+1} h(\sigma)(t_0, \ldots, t_q, 0)$$ (24)

and

$$((h\partial)(\sigma))(t_0, \ldots, t_q) = h(\sigma)(t_0, 0, t_1, \ldots, t_q) + \cdots + (-1)^q h(\sigma)(t_0, \ldots, t_q, 0).$$ (25)

By combining (24)–(25) with (23) we obtain

$$((\partial h + h\partial)\sigma)(t_0, \ldots, t_q) = h(\sigma)(0, t_0, \ldots, t_q) = \sigma(t_0, \ldots, t_q),$$

i.e.,

$$\partial h + h\partial \sigma = \sigma \quad (\sigma \in C^\text{set}_q(X)).$$

Corollary 2.2 Any map between nonempty sets $\phi : X \to Y$ induces a homotopy equivalence between $C^\cdot_{\text{set}}(X)$ and $C^\cdot_{\text{set}}(Y)$.

Indeed, for any map $\psi : Y \to X$, morphism $\phi \circ \psi = (\phi \circ \psi)$, is chain homotopic to $\text{id}_{C^\text{set}_q(Y)}$ and $\psi \circ \phi = (\psi \circ \phi)$, is chain homotopic to $\text{id}_{C^\text{set}_q(X)}$ in view of just proven Proposition 2.1.

Corollary 2.3 For any nonempty set $X$, the set-theoretic singular chain complex $C^\cdot_{\text{set}}(X)$ is homotopy equivalent to $\mathbb{Z}[0]$. In particular,

$$H^\text{set}_q(X) = \begin{cases} \mathbb{Z} & (q = 0) \\ 0 & (q > 0) \end{cases}.$$ (26)
2.5 De Rham Theory

2.6 De Rham Pairing

2.6.1

Let \( \mathfrak{X} = (X, \emptyset) \) be an \( \mathbb{R} \)-space. For any \( q \in \mathbb{N} \), there is an obvious pairing

\[
C^\mathrm{sm}_q(\mathfrak{X}) \times \Omega^q_{\mathbb{C}/\mathbb{R}} \longrightarrow \mathbb{R}, \quad (\sigma, \alpha) \mapsto \int_{\sigma} \alpha
\]

(27)

where the integration is extended by \( \mathbb{Z} \)-linearity from singular \( q \)-simplices to singular \( q \)-chains:

\[
\text{if } \sigma = \sum m_\gamma \gamma, \text{ then } \int_{\sigma} \alpha := \sum m_\gamma \int_{\gamma} \alpha.
\]

(28)

2.6.2

de Rham pairing is obviously additive (i.e., \( \mathbb{Z} \)-linear) in left argument and \( \mathbb{R} \)-linear in right argument.

**Theorem 2.4 (Stokes Theorem)** The boundary map \( \partial : C_q(\mathfrak{X}) \rightarrow C_{q-1}(\mathfrak{X}) \) and the de Rham differential \( d : \Omega^q_{\mathbb{C}/\mathbb{R}} \) are adjoint to each other, i.e.,

\[
\int_{\partial \sigma} \beta = \int_{\sigma} d\beta \quad (\sigma \in C_q(\mathfrak{X}), \beta \in \Omega^q_{\mathbb{C}/\mathbb{R}}).
\]

(29)

2.6.3

An equivalent formulation of Stokes’ Theorem is obtained by considering the singular cochain complex \( (C_q^\Delta(\mathfrak{X}; \mathbb{R}), \delta) \) which is defined as the dual of the singular chain complex:

\[
C^q_{\Delta} := \text{Hom}_{\mathbb{Z}\text{-mod}}(C^q(\mathfrak{X}), \mathbb{R}) = \text{Map}(\text{Hom}_{\text{re-Spc}}(\Delta^q, \mathfrak{X}), \mathbb{R})
\]

(30)

and

\[
(\delta^q(\varphi))(\sigma) := \varphi(\partial \sigma) \quad (\varphi \in C^q_{\Delta}(\mathfrak{X}), \sigma \in C^q_{\Delta}(\mathfrak{X})).
\]

(31)
2.6.4 De Rham Map

De Rham Map is the dual form of the de Rham Pairing introduced in (27)

\[ \Omega^q_{\mathcal{O}(\mathfrak{X})/\mathbb{R}} \longrightarrow C^q_{\mathrm{sm}}(\mathfrak{X}; \mathbb{R}), \quad \alpha \mapsto \int \alpha, \quad (32) \]

where \( \int \alpha \) is a singular cochain

\[ \int \alpha : \sigma \mapsto \int_\sigma \alpha \quad (\sigma \in C^q_{\mathrm{sm}}(\mathfrak{X})). \quad (33) \]

**Theorem 2.5 (Stokes Theorem (dual form))** One has

\[ \int d\beta = \delta \left( \int d\beta \right) \quad (\beta \in \Omega^*_{\mathcal{O}/\mathbb{R}}), \]

i.e., de Rham Map is a morphism of cochain complexes,

\[ (\Omega^q_{\mathcal{O}/\mathbb{R}}, d) \longrightarrow (C^q_{\mathrm{sm}}(\mathfrak{X}; \mathbb{R}), \delta). \]

2.6.5

It follows that the de Rham Map induces a homomorphism of cohomology groups (which are graded \( \mathbb{R} \)-vector spaces):

\[ H^*_{dR}(\mathcal{O}(\mathfrak{X})/\mathbb{R}) \longrightarrow H^*_{\mathrm{sm}}(\mathfrak{X}; \mathbb{R}). \]