A mathematician's vocabulary

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Chapter 1

1.1 Introduction

1.1.1 Structures on a set

1.1.1.1

A very general class of mathematical structures is obtained by equipping a set X with one or more subsets $\Gamma \subseteq F(X)$ where $F(X)$ is a set *naturally* associated with set X. 'Naturally' here means that any map $f\colon X \to Y$ induces a map

$$f_*\colon F(X) \to F(Y) \tag{1.1}$$

or a map

$$f^*\colon F(Y) \to F(X) \tag{1.2}$$

1.1.1.2

In the first case we expect that

$$(f \circ g)_* = f_* \circ g_*, \tag{1.3}$$

and we speak of *covariant* dependence on X, in the second case we require that

$$(f \circ g)^* = g^* \circ f^*, \tag{1.4}$$

and we speak of *contravariant* dependence on X.

1.1.1.3

In modern Mathematics, such associations are called *covariant* and *contravariant functors* from the category of sets to the category of sets.

1.1.2 A few examples of such functors

1.1.2.1 Cartesian powers

Given a set I, consider the correspondence that associates with a set X its I-th Cartesian power

$$X \rightsquigarrow X^I := \{(x_i)_{i\in I} \mid x_i \in X\}. \tag{1.5}$$

The Cartesian power is a covariant functor, a map $f\colon X \to Y$ induces the map

$$f_*\colon X^I \to Y^I, \qquad f_*\left((x_i)_{i\in I}\right) := (f(x_i))_{i\in I}. \tag{1.6}$$

1.1.2.2 Exponents

Given a set A, consider the correspondence that associates with a set X the set of maps from X to A

$$X \rightsquigarrow A^X := \{\phi\colon X \to A\}. \tag{1.7}$$

This functor is contravariant:

$$f^*\colon A^Y \to A^X, \qquad f^*(\phi) := \phi \circ f. \tag{1.8}$$

1.1.2.3 The power set as a covariant functor

This is the functor that associates with a set X the set $\mathscr{P}(X)$ of all of its subsets and, with a map $f\colon X \to Y$, the map $f_*\colon \mathscr{P}(X) \to \mathscr{P}(Y)$ that sends a subset $A \subseteq X$ to its *image* under f,

$$f(A) := \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$

1.1.2.4 The power set as a contravariant functor

This functor associates with a set X, the same set $\mathscr{P}(X)$, and with $f\colon X \to Y$, the map $f^*\colon \mathscr{P}(Y) \to \mathscr{P}(X)$ that sends a subset $B \subseteq Y$ to its *preimage* under f,

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

1.1.2.5

For any $A \subseteq X$ and $B \subseteq Y$, one has

$$f(A) \subseteq B \qquad \text{if and only if} \qquad A \subseteq f^{-1}(B). \tag{1.9}$$

This means that the pair of maps (f_*, f^*) forms a *Galois connection* between partially ordered sets $(\mathscr{P}(X), \subseteq)$ and $(\mathscr{P}(Y), \subseteq)$ (cf. *Notes on Partially Ordered Sets*).

1.1.2.6

For any set X, there exists a *natural* bijection[1]

$$\chi^X \colon \mathscr{P}(X) \to 2^X, \qquad A \mapsto \chi^X_A, \tag{1.10}$$

where

$$\chi^X_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \tag{1.11}$$

is the *characteristic* function of a subset $A \subseteq X$. In the interest of simplifying notation when possible, the superscript X is dropped when X is clear from the context.

1.1.2.7

'Naturality' of (1.10) means that, given a map $f \colon X \to Y$, the following diagram commutes,

$$\begin{array}{ccc} \mathscr{P}(X) & \xleftarrow{\;f^*\;} & \mathscr{P}(Y) \\ \Big\downarrow{\scriptstyle \chi^X} & & \Big\downarrow{\scriptstyle \chi^Y} \\ 2^X & \xleftarrow{\;f^*\;} & 2^Y \end{array} , \tag{1.12}$$

i.e., the composition of arrows either way produces the same result

$$\chi^X \circ f^* = f^* \circ \chi^Y.$$

In categorical language, we could say that χ is a *natural transformation* of the contravariant power-set functor $\mathscr{P}(\;)$ into the exponent functor $2^{(\;)}$ (in this case an *isomorphism* of functors, since all the maps χ^X are isomorphisms in the category of sets, i.e., they are invertible maps).

[1] In the language of sets, $0 = \emptyset$ and $n = \{0, \dots, n-1\}$.

1.1.2.8

Besides the category of sets there are other categories of interest in Mathematics, and there exist several interesting functors between them. Categorical language allows one to see various 'natural' constructions in a clear light, and it facilitates noticing connections between seemingly distant concepts and subjects. For this reason, it became very popular in modern Mathematics to the point of being indispensible, and a 'must-learn' for a beginner. We shall use it too.

1.1.2.9

You are encouraged to familiarize yourself with the language of categories and functors as soon as possible and, after mastering the basics of categorical grammar, to learn also at least the concepts of an equivalence of categories and of a pair of adjoint functors, and study numerous fundamentally important examples these two concepts. To facilitate this, I include the most besic definitions below.

Like with any language, acquiring proficiency requires constant use, so you, after learning the basic concepts, should be constantly observing these concepts at work in various branches of Mathematics.

1.2 First terms in the vocabulary

1.2.1 Families

1.2.1.1 Families of sets

The term a *family of sets* is used in two meanings: as a subset $\mathscr{X} \subseteq \mathscr{P}(U)$ of the power set of some set U or, as a map

$$I \to \mathscr{P}(U), \qquad i \mapsto X_i,$$

which assigns a set X_i to every element $i \in I$ of certain set I. In the latter case we speak of a family of subsets of U *indexed* by set I. The indexing set can be arbitrary and it may come equipped with additional structure like ordering.

1.2.1.2 Notation

It is customary to denote indexed families by $(X_i)_{i \in I}$.

1.2.1.3

A family of subsets of U viewed as a subset of $\mathscr{P}(U)$ is conceptually simpler, as its definition does not rely on the notion of a map yet it can be viewed as a special case of an indexed family, namely as a family indexed by itself:

$$\mathscr{X} \to \mathscr{P}(U), \qquad X \mapsto X.$$

1.2.1.4 Families of elements of a set

A family of elements of a set X will be always used in the sense of a family indexed by some set I. By definition it is a map

$$I \to X, \qquad i \mapsto x_i.$$

Conceptually, there is no difference between a *family of elements* of X and a *map* $I \to X$. The difference is exclusively in notation and in the points of emphasis.

In the language of *families of elements* the focus is on X and its elements. The nature of the indexing set is secondary and generally not very important.

In the language of *maps*, the source and the target of a map are on equal footing, and the map itself is usually sufficiently important to merit its own symbol in notation.

1.2.1.5 Natural numbers

We shall frequently identify natural numbers with the sets:

$$0 := \emptyset, \quad 1 := \{0\}, \quad 2 := \{0, 1\}, \quad \ldots, \quad n := \{0, \ldots, n\text{-}1\}, \quad \ldots \tag{1.13}$$

Exercise 1 *Show that* $2 \neq 3$.

1.2.1.6 Sequences

Families indexed by subsets of the set of natural numbers or, more generally, by ordered countable sets, are called *sequences*.

1.2.1.7 n-tuples

Families indexed by $I = \{1, \dots, n\}$ are called *ordered n-tuples* of elements of X, and notation

$$(x_1, \dots, x_n) \qquad \text{instead of} \qquad (x_i)_{i \in \{1,\dots,n\}}$$

is generally used. Ordered 2-, 3-, 4-tuples are respectively called ordered *pairs*, *triples*, *quadruples*.

1.2.2 Rings of sets and algebras of subsets

1.2.2.1 Rings of sets

A nonempty family of sets $\mathscr{R}$, i.e., a nonempty set whose elements are sets, is said to be a *ring of sets* if the union, $R \cup R'$, and the difference, $R \setminus R'$, belongs to $\mathscr{R}$ for any $R, R' \in \mathscr{R}$.

Exercise 2 *Show that in every ring of sets $\mathscr{R}$ one has*

$$R \cap R' \in \mathscr{R} \qquad \textit{for any} \qquad R, R' \in \mathscr{R}.$$

1.2.2.2 Algebras of subsets

A nonempty family $\mathscr{A} \subseteq \mathscr{P}(X)$ of subsets of a set X is said to be an *algebra of subsets of* X if the intersection, $A \cap A'$, and the complement, $A^c := X \setminus A$, belongs to $\mathscr{A}$ for any $A, A' \in \mathscr{A}$.

Exercise 3 *Show that $\mathscr{A} \subseteq \mathscr{P}(X)$ is an algebra of subsets of a set X* if and only if *$\mathscr{A}$ is a ring of sets which contains X.*

1.2.2.3

The family of all *finite* subsets $\mathscr{P}_{\text{fin}}(X)$ of a set X is a ring of sets which is an algebra of subsets of X if and only if X is finite.

1.2.3 Operations involving families of sets

1.2.3.1 Union

The union of a family $\mathscr{X} \subseteq \mathscr{P}(U)$ is the set

$$\{u \in U \mid u \in X \text{ for some } X \in \mathscr{X}\}. \tag{1.14}$$

This set is denoted

$$\bigcup \mathscr{X} \qquad \text{or} \qquad \bigcup_{X \in \mathscr{X}} X. \tag{1.15}$$

1.2.3.2

The union of an indexed family $(X_i)_{i\in I}$ is defined similarly

$$\bigcup_{i\in I} X_i := \{u \in U \mid u \in X_i \text{ for some } i \in I\}. \tag{1.16}$$

1.2.3.3 Intersection

The intersection a family $\mathscr{X} \subseteq \mathscr{P}(U)$ is the set

$$\{u \in U \mid u \in X \text{ for every } X \in \mathscr{X}\}. \tag{1.17}$$

This set is denoted

$$\bigcap \mathscr{X} \qquad \text{or} \qquad \bigcap_{X\in\mathscr{X}} X. \tag{1.18}$$

1.2.3.4

The intersection of an indexed family $(X_i)_{i\in I}$ is defined similarly

$$\bigcap_{i\in I} X_i := \{u \in U \mid u \in X_i \text{ for every } i \in I\}. \tag{1.19}$$

Exercise 4 *Show that the intersection*

$$\bigcap_{i\in I} \mathscr{R}_i$$

of any family of rings of sets $(\mathscr{R}_i)_{i\in I}$ is a ring of sets. Likewise, show that the intersection

$$\bigcap_{i\in I} \mathscr{A}_i$$

of any family of algebras of subsets $(\mathscr{A}_i)_{i\in I}$ of a given set X is an algebra of subsets of X.

1.2.3.5 Cartesian product

The Cartesian product of an indexed family $(X_i)_{i\in I}$ is the set of all families $\xi = (x_i)_{i\in I}$ of elements of $\bigcup_{i\in I} X_i$ such that $x_i \in X_i$:

$$\prod_{i\in I} X_i := \big\{(x_i)_{i\in I} \mid x_i \in X_i\big\}. \tag{1.20}$$

Equivalently,

$$\prod_{i\in I} X_i := \left\{\xi\colon I \to \bigcup_{i\in I} X_i \mid \xi(i) \in X_i\right\}. \tag{1.21}$$

1.2.3.6 Notation

The Cartesian product of a finite family $(X_1, \ldots, X_n)$ is usually denoted

$$X_1 \times \cdots \times X_n. \tag{1.22}$$

1.2.3.7 Comment

It is important to observe that one can replace $\bigcup_{i \in I} X_i$ in the definition of the Cartesian product by *any* set that contains all X_i. The corresponding 'products' will be essentially identical sets. This is due to the observation that there exists a canonical identification between maps $A \to B$ whose image is contained in a subset $B' \subseteq B$, and maps $A \to B'$.

1.2.3.8 Canonical projections

The Cartesian product comes equipped with the family of surjective maps,

$$\pi_i \colon \prod_{j \in I} X_j \to X_i \qquad \xi \mapsto x_i \qquad (i \in I), \tag{1.23}$$

which send a map $\xi \colon I \to \bigcup_{i \in I} X_i$ to its value at each i. When $I = \{1, \ldots, n\}$, then π_i is the i-th coordinate map

$$\pi_i \colon (x_1, \ldots, x_n) \mapsto x_i \qquad (i = 1, \ldots, n).$$

1.2.3.9 A universal property of the Cartesian product

Given any set Y and a family $(f_i)_{i \in I}$ of maps $f_i \colon Y \to X_i$, there exists a unique map $\tilde{f} \colon Y \to \prod_{i \in I} X_i$ such that

$$f_i = \pi_i \circ \tilde{f} \qquad (i \in I). \tag{1.24}$$

Exercise 5 *Verify that the map*

$$\tilde{f} \colon y \mapsto (f_i(y))_{i \in I} \qquad (y \in Y) \tag{1.25}$$

satisfies (1.24)*, and that any map $g \colon Y \to \prod_{i \in I} X_i$ which satisfies* (1.24) *coincides with $\tilde{f}$.*

1.2.3.10 Disjoint unions of sets

The disjoint union of an indexed family $(X_i)_{i\in I}$ should be thought of as the union of all sets X_i except that we keep as many distinct 'copies' of an element $x \in \bigcup_{i\in I} X_i$ as there are sets X_i which contain x. We achieve this by 'tagging' every element in $\bigcup_{i\in I} X_i$ by the index of the set it belongs to:

$$\coprod_{i\in I} X_i := \big\{(i,x) \in I \times \bigcup_{i\in I} X_i \mid x \in X_i\big\}. \tag{1.26}$$

1.2.3.11 Notation

The disjoint union of a finite family $(X_1,\dots,X_n)$ is usually denoted

$$X_1 \sqcup \cdots \sqcup X_n. \tag{1.27}$$

Exercise 6 *Denote by p the composition of the inclusion map and the canonical projection*

$$\coprod_{i\in I} X_i \hookrightarrow I \times \bigcup_{i\in I} X_i \to \bigcup_{i\in I} X_i. \tag{1.28}$$

Show that p is surjective. Show that the fiber $p^{-1}(x)$ at $x \in \bigcup_{i\in I} X_i$ is

$$p^{-1}(x) = \{(i,x) \mid x \in X_i\}.$$

In particular, $p^{-1}(x)$ is in on-to-one correspondence with the set

$$\{i \in I \mid x \in X_i\}.$$

1.2.3.12

It follows that the disjoint union of a family of sets $(X_i)_{i\in I}$ is canonically identified with their union if and only if sets X_i are disjoint for distinct $i \in I$:

$$X_i \cap X_j = \emptyset \qquad (i \neq j).$$

1.2.3.13 Canonical inclusions

The disjoint union comes equipped with the family of injective maps,

$$\iota_i\colon X_i \to \coprod_{j\in I} X_j \qquad x \mapsto (i,x) \qquad (i \in I). \tag{1.29}$$

1.2.3.14 A universal property of the disjoint union

Given any set Y and a family $(f_i)_{i \in I}$ of maps $f_i\colon X_i \to Y$, there exists a unique map $\tilde{f}\colon \coprod_{i \in I} X_i \to Y$ such that

$$f_i = \tilde{f} \circ \iota_i \qquad (i \in I). \tag{1.30}$$

Exercise 7 *Verify that the map*

$$\tilde{f}\colon (i, x) \mapsto f_i(x) \qquad (i \in I; x \in X_i) \tag{1.31}$$

satisfies (1.30)*, and that any map $g\colon \coprod_{i \in I} X_i \to Y$ which satisfies* (1.30) *coincides with $\tilde{f}$.*

1.2.3.15

Map p defined in (1.28) is precisely such universal map $\tilde{f}$ for the family of inclusion maps

$$f_i\colon X_i \hookrightarrow \bigcup_{j \in I} X_j \qquad (i \in I).$$

1.2.3.16

Note that the properties of the Cartesian product and of the disjoint union of a family of sets are *dual* to each other. We shall explain this concept of *duality* later.

1.3 Associativity properties of operations on families of sets

1.3.1 Associativity of union

1.3.1.1 The indexed families case

Suppose we have two families of sets

$$(X_i)_{i \in I} \qquad \text{and} \qquad (X_k)_{k \in K}.$$

The iterated union

$$\bigcup_{i \in I} X_i \;\cup\; \bigcup_{k \in K} X_k$$

and the union

$$\bigcup_{l \in I \sqcup K} X_l$$

are *equal* as sets. In the case of a pair of finite families $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_n)$, this equality acquires the form

$$(A_1 \cup \cdots \cup A_m) \cup (B_1 \cup \cdots \cup B_n) = A_1 \cup \cdots \cup A_m \cup B_1 \cup \cdots \cup B_n.$$

1.3.1.2 The total family

In general, given any family of families of sets

$$\left((X_{i_j})_{i_j \in I_j} \right)_{j \in J}, \tag{1.32}$$

the universal property of the disjoint union allows us to form the *total* family

$$(X_l)_{l \in L} \qquad \text{where} \qquad L = \coprod_{j \in J} I_j. \tag{1.33}$$

Indeed, regarding all sets to be subsets of a common set U, family of families of (1.32) is the same as a family of maps $I_j \to \mathscr{P}(U)\}_{j \in J}$ and, by the universal property of disjoint union, there exists a unique map $L \to \mathscr{P}(U)$ whose 'restrictions' to I_j are the component-families

$$(I_j \to \mathscr{P}(U))_{j \in J}.$$

We shall refer to $L \to \mathscr{P}(U)$ as the *total* family.

1.3.1.3

Now we are ready to make an observation about iterated unions of families. The following sets are equal

$$\bigcup_{j \in J} \bigcup_{i_j \in I_j} X_{i_j} = \bigcup_{l \in L} X_l.$$

Exercise 8 *Formulate the corresponding associativity laws for intersection of families.*

1.3.1.4 The nonindexed families case

There are two natural maps

$$\bigcup : \mathscr{P}(\mathscr{P}(\mathscr{P}(U))) \longrightarrow \mathscr{P}(\mathscr{P}(U)) \tag{1.34}$$

and

$$\bigcup_* : \mathscr{P}(\mathscr{P}(\mathscr{P}(U))) \longrightarrow \mathscr{P}(\mathscr{P}(U)). \tag{1.35}$$

The first one is the familiar *union-of-a-family* map applied to $\mathscr{P}(U)$ instead of U. It sends $\mathfrak{X} \in \mathscr{P}(\mathscr{P}(\mathscr{P}(U)))$, i.e., a family of families of subsets of U, to the family of subsets of U which belong to at least one member family $\mathscr{X} \in \mathfrak{X}$

$$\bigcup \mathfrak{X} = \bigcup_{\mathscr{X} \in \mathfrak{X}} \mathscr{X}.$$

The other one is induced by the map $\bigcup : \mathscr{P}(\mathscr{P}(U)) \to \mathscr{P}(U)$. It is formed by the *unions* of member families $\mathscr{X} \in \mathfrak{X}$,

$$\bigcup_* (\mathfrak{X}) := \left\{ Y \subseteq U \mid Y = \bigcup \mathscr{X} = \bigcup_{X \in \mathscr{X}} X \text{ for some } \mathscr{X} \in \mathfrak{X} \right\}.$$

Exercise 9 *Show that the following diagram commutes*

$$\begin{array}{ccc} \mathscr{P}(\mathscr{P}(\mathscr{P}(U))) & \xrightarrow{\bigcup_*} & \mathscr{P}(\mathscr{P}(U))) \\ {\scriptstyle\bigcup}\big\downarrow & & \big\downarrow{\scriptstyle\bigcup} \\ \mathscr{P}(\mathscr{P}(U))) & \xrightarrow{\bigcup} & \mathscr{P}(U) \end{array}, \tag{1.36}$$

i.e., show that the following two subsets of U*,*

$$\bigcup \left(\bigcup \mathfrak{X} \right) = \bigcup_{X \in \bigcup \mathfrak{X}} X$$

and

$$\bigcup \bigcup_* (\mathfrak{X}) = \bigcup_{\mathscr{X} \in \mathfrak{X}} \left(\bigcup \mathscr{X} \right) = \bigcup_{\mathscr{X} \in \mathfrak{X}} \left(\bigcup_{X \in \mathscr{X}} X \right),$$

are equal for any family of families $\mathfrak{X} \subseteq \mathscr{P}(\mathscr{P}(U))$.

1.3.1.5 Terminology

If one is going to deal with "families of families of subsets of a set U," et caetera, on an extended basis, then one perhaps should use a less cumbersome terminology. One could, for example, call subsets of the n times iterated power set

$$\mathscr{P}^n(U) = \underbrace{\mathscr{P}(\cdots \mathscr{P}}_{n \text{ times}}(U) \cdots) \qquad (n \geq 0)$$

n-families in a set U. In particular, subsets of $U = \mathscr{P}^0(U)$ are 0-families, families of subsets of U are 1-families, families of families of subsets of U are 2-families, etc.

1.3.2 Cartesian product

1.3.2.1 Associativity of Cartesian product

Instead of equality, we have only a canonical identification between the iterated Cartesian product of a family of families of sets and the Cartesian product of the total family.

Let us consider first the case of a pair of families of sets

$$(X_i)_{i \in I} \qquad \text{and} \qquad (X_k)_{k \in K}.$$

The natural correspondence

$$((x_i)_{i \in I}, (x_k)_{k \in K}) \;\leftrightarrow\; (x_l)_{l \in I \sqcup K}$$

identifies the iterated Cartesian product

$$\prod_{i \in I} X_i \;\times\; \prod_{k \in K} X_k$$

with

$$\prod_{l \in I \sqcup K} X_l.$$

In the case of a pair of finite families $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_n)$, this identification acquires the form

$$\big((a_1, \ldots, a_m), (b_1, \ldots, b_n)\big) \leftrightarrow (a_1, \ldots, a_m, b_1, \ldots, b_n).$$

1.3.2.2 Iterated Cartesian product

In general, given any family of families of sets (1.32), the iterated Cartesian product and the Cartesian product of the total family are naturally identified

$$\prod_{j\in J}\prod_{i_j\in I_j} X_{i_j} \longleftrightarrow \prod_{l\in L} X_l \qquad \text{where} \qquad L = \coprod_{j\in J} I_j. \tag{1.37}$$

Indeed, elements of $\prod_{i_j\in I_j} X_{i_j}$ are maps

$$\xi_j : I_j \to \bigcup_{i_j\in I_j} X_{i_j}$$

such that $\xi_j(i_j) \in X_{i_j}$. By composing maps ξ_j with the inclusions

$$\bigcup_{i_j\in I_j} X_{i_j} \hookrightarrow U := \bigcup_{l\in L} X_l,$$

we can consider all ξ_j as being maps with the common target U. Thus, elements of

$$\prod_{j\in J}\prod_{i_j\in I_j} X_{i_j}$$

become families $(\xi_j)_{j\in J}$ of maps $\xi_j\colon I_j \to U$. By the universal property of the disjoint union, there exists a unique map $\tilde{\xi}\colon L \to U$ whose 'restrictions' to I_j are families maps $\xi_j\colon I_j \to U$.

This map $\tilde{\xi}$ is an element of $\prod_{l\in L} X_l$. Since the correspondence between families $(\xi_j)_{j\in J}$ and maps $\tilde{\xi}$ is bijective, the correspondence in (1.37) is bijective.

Exercise 10 *Formulate and prove the corresponding associativity laws for disjoint union.*

1.3.2.3 Calculus of Cartesian powers of a set

For any sets A, B, and C, one has natural identifications

$$A^B \times A^C \longleftrightarrow A^{B\sqcup C} \tag{1.38}$$

and, more generally,

$$\prod_{j\in J} A^{B_j} \longleftrightarrow A^{\coprod_{j\in J} B_j} \tag{1.39}$$

which are special cases of identifications (1.37).

1.3.2.4

One has also the following natural identification

$$\left(A^B\right)^C \longleftrightarrow A^{B\times C} \tag{1.40}$$

given by the following pair of mutually inverse correspondences

$$\left(A^B\right)^C \ni \ f \mapsto \ \phi \in A^{B\times C}, \qquad \text{where} \qquad \phi(b,c) := \big(f(c)\big)(b)$$

and

$$A^{B\times C} \ni \phi \ \mapsto \ f \in \left(A^B\right)^C, \qquad \text{where} \qquad f(c) := \phi(\,\cdot\,, c).$$

1.3.2.5

Using the *families-of-elements* notation instead of *maps* notation, we can describe identification (1.40) also in this form

$$\left(X^I\right)^J \longleftrightarrow X^{I\times J}, \qquad \big((x_{ij})_{i\in I}\big)_{j\in J} \ \leftrightarrow \ (x_{ij})_{(i,j)\in I\times J}. \tag{1.41}$$

1.4 The language of categories and functors

1.4.1 Oriented graphs

1.4.1.1

An *oriented graph* $\mathcal{C}$ consists of two classes, $\mathcal{C}_0$ (the class of *vertices*) and $\mathcal{C}_1$ (the class of *arrows*) which are related by a pair of correspondences:

$$\begin{array}{ccc} & \mathcal{C}_1 & \\ {}^{t}\swarrow & & \searrow^{s} \\ \mathcal{C}_0 & & \mathcal{C}_0 \end{array} \quad . \tag{1.42}$$

1.4.1.2

For any arrow α we shall refer to $s(\alpha)$ as its *source*, and to $t(\alpha)$ as its *target*.

1.4.1.3

Note that we are saying *classes*—not *sets*. Basic concepts of Category Theory impose on the foundations on which the edifice of Mathematics rests, that one is allowed to talk about classes that are not sets, like the class of all sets, the class of all singleton sets, the class of all vector spaces over a given field of coefficients, etc, and one is likewise allowed to talk about correspondences between classes as if they were mappings between sets.

We henceforth will be cautiously extending to classes certain terminology and notation usually associated with sets. For example, we may indicate that a is a vertex of a graph $\mathcal{C}$ by writing either $a \in \mathcal{C}_0$ or $a \in \operatorname{Vert}\mathcal{C}$. Similarly, we may indicate that α is an arrow of a graph $\mathcal{C}$ by writing either $\alpha \in \mathcal{C}_1$ or $\alpha \in \operatorname{Arr}\mathcal{C}$.

1.4.2 Categories

1.4.2.1 The class of composable arrows

Consider the class $\mathcal{C}_2$ of pairs (α_0, α_1) of arrows such that the source of α_1 is the target of α_0. This class fits naturally into the diagram

$$\begin{array}{ccccccc} & & & \mathcal{C}_2 & & & \\ & & \swarrow^{p_0} & & \searrow^{p_1} & & \\ & \mathcal{C}_1 & & \circlearrowleft & & \mathcal{C}_1 & \\ \swarrow^{t} & & \searrow_{s} & & \swarrow_{t} & & \searrow^{s} \\ \mathcal{C}_0 & & & \mathcal{C}_0 & & & \mathcal{C}_0 \end{array} \tag{1.43}$$

where p_i sends (α_0, α_1) to α_i.

1.4.2.2

A graph equipped with a correspondence

$$m \colon \mathcal{C}_2 \leftrightarrow \mathcal{C}_1 \tag{1.44}$$

is said to be a *category* if (1.44) is *associative*, i.e.,

$$(\alpha_0 \circ \alpha_1) \circ \alpha_2 = \alpha_0 \circ (\alpha_1 \circ \alpha_2) \tag{1.45}$$

for any *composable* triple of arrows. The latter means that

$$s(\alpha_0) = t(\alpha_1) \qquad \text{and} \qquad s(\alpha_1) = t(\alpha_2). \tag{1.46}$$

1.4.2.3 Objects and morphisms

1.4.2.4 $\mathbf{Hom}_{\mathcal{C}}(a, b)$

It was observed early that if one requires in the definition of a category that, for any pair of objects $a, b \in \mathcal{C}_0$, morphisms with a as their source and with b as their target form a *set* and not just a class, then one can avoid essentially all the potential dangers arising from presence of classes in foundations of Category Theory.

This set is usually denoted $\mathrm{Hom}_{\mathcal{C}}(a, b)$ and its elements are referred as morphisms from a to b.

1.4.2.5 The class of composable pairs of morphisms

We say that a pair (α, β) of morphisms is *composable* if $s(\alpha) = t(\beta)$. Denote by $\mathcal{C}_2$ the class of composable pairs of morphisms. We assume that a correspondence

$$m\colon \mathcal{C}_2 \to \mathcal{C}_1, \qquad (\alpha, \beta) \mapsto \alpha \circ \beta, \tag{1.47}$$

is given. It is referred to as *composition* of morphisms, and is possibly the single most important element of the structure of a category.

1.4.2.6 The class of composable triples of morphisms

We say that a triple (α, β, γ) of morphisms is *composable* if $s(\alpha) = t(\beta)$ and $s(\beta) = t(\gamma)$. As can be expected, we denote the class of composable triples of morphisms by $\mathcal{C}_3$. (Binary) composition (1.47) induces two correspondences $\mathcal{C}_3 \to \mathcal{C}_2$

$$m_1\colon (\alpha, \beta, \gamma) \mapsto (\alpha \circ \beta, \gamma) \qquad \text{and} \qquad m_2\colon (\alpha, \beta, \gamma) \mapsto (\alpha, \beta \circ \gamma). \tag{1.48}$$

By applying correspondence (1.47), we obtain two correspondences $\mathcal{C}_3 \to \mathcal{C}_1$. We require them to be equal which means that

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma) \tag{1.49}$$

for any composable triple of morphisms. This condition is called *associativity* of the composition of morphisms.

1.4.2.7

Associativity identity (1.49) can be expressed as commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{C}_3 & \xrightarrow{m_1} & \mathcal{C}_2 \\ {\scriptstyle m_2}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ \mathcal{C}_2 & \xrightarrow{m} & \mathcal{C}_1 \end{array} \tag{1.50}$$

1.4.2.8 The identity morphisms

We could stop here and call the defined structures *categories*. The classical and still a 'default' definition of a category additionally requires presence of a correspondence

$$i\colon \mathcal{C}_0 \to \mathcal{C}_1, \qquad a \mapsto \mathrm{id}_a \in \mathrm{Hom}_{\mathcal{C}}(a,a), \tag{1.51}$$

such that

$$\alpha \circ \mathrm{id}_a = \alpha \qquad \text{and} \qquad \mathrm{id}_b \circ \alpha = \alpha \tag{1.52}$$

for any $\alpha \in \mathrm{Hom}_{\mathcal{C}}(a,b)$. Morphism id_a is referred to as the *identity* morphism of object a.

1.4.2.9

Each of the identities in (1.52) can be expressed as commutativity of a diagram of correspondences:

$$\begin{array}{ccc} \mathcal{C}_1 \times \mathcal{C}_0 & \xrightarrow{(\mathrm{id}_{\mathcal{C}_1}, i)} & \mathcal{C}_2 \\ {\scriptstyle (\mathrm{id}_{\mathcal{C}_1}, s)}\big\uparrow & & \big\downarrow{\scriptstyle m} \\ \mathcal{C}_1 & \overset{\mathrm{id}_{\mathcal{C}_1}}{=\!=\!=} & \mathcal{C}_1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{C}_0 \times \mathcal{C}_1 & \xrightarrow{(i, \mathrm{id}_{\mathcal{C}_1})} & \mathcal{C}_2 \\ {\scriptstyle (t, \mathrm{id}_{\mathcal{C}_1})}\big\uparrow & & \big\downarrow{\scriptstyle m} \\ \mathcal{C}_1 & \overset{\mathrm{id}_{\mathcal{C}_1}}{=\!=\!=} & \mathcal{C}_1 \end{array} \tag{1.53}$$

1.4.2.10

There are very good reasons not to require presence of the identity morphisms in general, and to call the categories that possess such morphisms—*unital* categories.

1.4.2.11 Isomorphisms

We say that a morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ is an *isomorphism* if there exists $\beta \in \mathrm{Hom}_{\mathcal{C}}(b, a)$ such that

$$\alpha \circ \beta = \mathrm{id}_b \qquad \text{and} \qquad \beta \circ \alpha = \mathrm{id}_a\,. \tag{1.54}$$

Exercise 11 *Show that if there exist morphisms* $\beta, \gamma \in \mathrm{Hom}_{\mathcal{C}}(b, a)$ *such that*

$$\alpha \circ \beta = \mathrm{id}_b \qquad \textit{and} \qquad \gamma \circ \alpha = \mathrm{id}_a\,.$$

then $\beta = \gamma$*.*

1.4.2.12

In view of the above exercise, if there exists at least one *right* inverse and at least one *left* inverse for a morphism α, then they are equal, which implies that the two-sided inverse, (1.54), is unique when it exists. It is denoted α^{-1}.

1.4.2.13 Endomorphisms of an object

Morphisms $\alpha\colon a \to a$ are called *endomorphisms* of object a. The set $\mathrm{Hom}_{\mathcal{C}}(a, a)$ is often denoted $\mathrm{End}_{\mathcal{C}}(a)$.

1.4.2.14 Automorphisms of an object

Isomorphisms $\alpha\colon a \to a$ are called *automorphisms* of object a. The set of automorphisms is denoted $\mathrm{Aut}_{\mathcal{C}}(a)$.

1.4.2.15 Symmetries

Before categorical language was proposed and developed as means to describe and study underlying structure of numerous areas of Mathematics, automorphisms of various objects: geometric, physical systems, etc—were often called *symmetries*.

1.4.2.16 Subcategories

For a category $\mathcal{C}$, suppose that, a pair of subclasses $\mathcal{C}'_0 \subseteq \mathcal{C}_0$ and $\mathcal{C}'_1 \subseteq \mathcal{C}_1$ is given such that the source and the target of any morphism in $\mathcal{C}'_1$ is a member of $\mathcal{C}'_0$ and the composition of any two such morphisms is a member of $\mathcal{C}'_1$.

If we equip the pair of classes $(\mathcal{C}_0, \mathcal{C}_1)$ with the source, target, and multiplication correspondences induced from category $\mathcal{C}$, we obtain a category on its own. Denote it $\mathcal{C}'$.

This situation arises frequently. We say that $\mathcal{C}'$ is a *subcategory* of $\mathcal{C}$.

1.4.2.17 Full subcategories

If

$$\mathrm{Hom}_{\mathcal{C}'}(a,b) = \mathrm{Hom}_{\mathcal{C}}(a,b) \qquad (a,b \in \mathcal{C}'_0),$$

then we say that $\mathcal{C}'$ is a *full* subcategory of category $\mathcal{C}$.

1.4.3 Natural definitions of a morphism between sets

1.4.3.1 Set

The category of sets usually takes pride of being presented as the first example of a category. The objects of this category are sets. There are, however, several natural candidates for the morphisms. The standard choice for morphisms $X \to Y$ is to take maps $f\colon X \to Y$:

$$\mathrm{Hom}_{\mathsf{Set}}(X,Y) = Y^X.$$

This category will be denoted Set and referred to as *the* category of sets.

Note that isomorphisms in the category of sets coincide with the class of bijections.

1.4.3.2 Multivalued maps

A *multivalued* map, $\phi\colon X \multimap Y$, from a set X to a set Y, is a map $\phi\colon X \to \mathscr{P}(Y)$. Multivalued maps will be also called *multimaps*.

1.4.3.3 Maps versus multimaps

Every map $f\colon X \to Y$ defines the multimap

$$x \mapsto \phi_f(x) := \{f(x)\} \qquad (x \in X).$$

The correspondence $f \mapsto \phi_f$ identifies maps $f\colon X \to Y$ with multimaps $\phi\colon X \multimap Y$ satisfying the property

$$|\phi(x)| = 1 \qquad (x \in X). \tag{1.55}$$

1.4.3.4 The image map for a multimap

Every multimap $\phi\colon X \multimap Y$ naturally extends to a map $\mathscr{P}(X) \to \mathscr{P}(Y)$,

$$A \longmapsto \phi(A) := \bigcup_{x \in A} \phi(x) \qquad (A \subseteq X). \tag{1.56}$$

We will continue to denote it ϕ and will call it the *image map* associated with multimap ϕ.

1.4.3.5 The reverse of a multimap

Every multimap $\phi\colon X \multimap Y$ also defines a multimap $Y \multimap X$

$$\phi^{\text{rev}}(y) := \{x \in X \mid \phi(x) \ni y\}. \tag{1.57}$$

We shall refer to it as the *reverse* of ϕ. When ϕ is a map $f\colon X \to Y$, then $\phi^{\text{rev}}(x) = \{x \in X \mid f(x) = y\}$ is called the *fiber* of f at $y \in Y$.

1.4.3.6 The preimage map for a multimap

The image map for the reverse multimap, ϕ^{rev}, will be called the *preimage map* for ϕ.

Exercise 12 *Show that*

$$\phi^{\text{rev}}(B) = \{x \in X \mid \phi(x) \cap B \neq \emptyset\} \qquad (B \subseteq Y). \tag{1.58}$$

1.4.3.7 Composition of multimaps

Given multimaps $\phi\colon X \multimap Y$ and $\chi\colon Y \multimap Z$, their *composition*,

$$\chi \circ \phi\colon x \longmapsto \chi(\phi(x)) \qquad (x \in X), \tag{1.59}$$

is a multimap $X \multimap Z$.

Exercise 13 *Given maps $f\colon X \to Y$ and $g\colon Y \to Z$, show that*

$$\phi_g \circ \phi_f = \phi_{g \circ f}. \tag{1.60}$$

Exercise 14 *Show that composition of multimaps is associative, i.e.,*

$$(\chi \circ \phi) \circ v = \chi \circ (\phi \circ v),$$

for any $v\colon W \multimap X$, $\phi\colon X \multimap Y$, and $\chi\colon Y \multimap Z$.

1.4.3.8 $\mathrm{Set}_{\mathrm{mult}}$

Thus, the class of sets equipped with multimaps as morphisms forms a category. We shall denote it $\mathrm{Set}_{\mathrm{mult}}$.

Exercise 15 *Show that the canonical embedding* $\iota_X\colon X \hookrightarrow \mathscr{P}(X)$,

$$\iota_X\colon x \longmapsto \{x\} \qquad (x \in X)$$

is the identity endomorphism of set X *in* $\mathrm{Set}_{\mathrm{mult}}$.

1.4.3.9 Submaps

Let us call a multimap $\phi\colon X \to Y$ satisfying the condition

$$|\phi(x)| \leq 1 \qquad (x \in X), \tag{1.61}$$

a *submap* (compare it with (1.55)).

If multimaps satisfying (1.55) corespond to maps $F\colon X \to Y$, then submaps correspond to *partially defined* maps from X to Y, i.e., to maps $f\colon X' \to Y$ whose domain is a subset of X.

Exercise 16 *Show that* $\chi \circ \phi$ *is a submap if both* ϕ *and* χ *are submaps.*

1.4.3.10 $\mathrm{Set}_{\mathrm{sub}}$

The class of sets with submaps as morphisms defines another category whose objects are sets. We shall denote it $\mathrm{Set}_{\mathrm{sub}}$.

1.4.3.11 $\mathrm{Set}_{\mathrm{fin}}$

More generally, we shall say that $\phi\colon X \multimap Y$ is a *finitely-valued map*, if

$$|\phi(x)| < \infty \qquad (x \in X). \tag{1.62}$$

Exercise 17 *Show that* $\chi \circ \phi$ *is finitely-valued if both* ϕ *and* χ *are finitely-valued.*

In particular, sets with finitely-valued maps as morphisms form a category. We shall denote it $\mathrm{Set}_{\mathrm{fin}}$.

1.4.3.12 $\mathbf{Set_{count}}$

Another possibility is to consider *countably-valued maps* as morphisms,

$$\phi(x) \ \text{ countable for all } x \in X. \tag{1.63}$$

Let us denote denote the corresponding category by $\mathrm{Set}_{\mathrm{count}}$.

1.4.3.13

The above categories form an increasing chain of unital subcategories of the category of sets and multimaps

$$\mathrm{Set} \subseteq \mathrm{Set}_{\mathrm{sub}} \subseteq \mathrm{Set}_{\mathrm{fin}} \subseteq \mathrm{Set}_{\mathrm{count}} \subseteq \mathrm{Set}_{\mathrm{mult}}.$$

Note that they share the same class of objects. They differ only in their morphisms.

1.4.3.14 Composition of binary relations

A different approach to defining morphisms from a set X to a set Y is to consider binary relations $R \subseteq X \times Y$. For $R \subseteq X \times Y$ and $S \subseteq Y \times Z$,

$$R \circ S := \{(x,z) \in X \times Z \mid \text{there exists } y \in Y \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\}. \tag{1.64}$$

is a binary relation between elements of X and Z. If we use notation $x \sim_R y$ (*"element $x \in X$ is in relation R with element $y \in Y$"*) to express the fact that $(x,y) \in R$, then we can rewrite Definition (1.64) as follows

$$R \circ S := \{(x,z) \in X \times Z \mid \text{there exists } y \in Y \text{ such that } x \sim_R y \text{ and } y \sim_S z\}. \tag{1.65}$$

Exercise 18 *Show that composition of binary relations is associative, i.e.,*

$$(Q \circ R) \circ S = Q \circ (R \circ S)$$

for any $Q \subseteq W \times X$, $R \subseteq X \times Y$, and $S \subseteq Y \times Z$.

1.4.3.15 The identity relation

For any set X we shall call the binary relation

$$\Delta_X := \{(x,x') \in X \times X \mid x = x'\} \tag{1.66}$$

the *identity* relation on X.

Exercise 19 *Show that*

$$\Delta_X \circ R = R = R \circ \Delta_Y$$

for any $R \subseteq X \times Y$.

1.4.3.16

Denote the category whose objects are sets and relations $R \subseteq X \times Y$ are morphisms $X \to Y$ by $\mathrm{Set}_{\mathrm{rel}}$.

1.4.4 Discrete categories

1.4.4.1

There are much simpler categories than the categories of sets. The simplest, are perhaps the categories with the *empty* class of morphisms. Such categories are referred to as *discrete*.

1.4.4.2 Discrete unital categories

Every unital category is supposed to have at least the identity morphisms for each object. For this reason, in the context of unital categories *discrete* means: *no morphisms besides the identity morphisms*.

1.4.5 Small categories

1.4.5.1

If the class of objects forms a set, such a category is called a *small* category. In this case, the class of morphisms is a set too. Indeed, it is the union

$$\mathcal{C}_1 = \bigcup_{(a,b)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Hom}_{\mathcal{C}}(a,b)$$

of the family of $\mathrm{Hom}_{\mathcal{C}}(a,b)$ which is indexed by the Cartesian square of the set of objects.

1.4.5.2

Several fundamentally important structures in Mathematics can be interpreted as small categories. We give here just one yet very important example of such structures: a *preordered* set. Other examples will appear later.

1.4.5.3 Preordered sets

We say that a binary relation $\preceq$ on a set X is a *preorder* (the term *quasiorder* is used too), if it is *reflexive*,

$$x \preceq x \qquad (x \in X), \tag{1.67}$$

and *transitive*

$$\textit{if} \;\; x \preceq y \;\; \textit{and} \;\; y \preceq z\,, \;\; \textit{then} \;\; x \preceq z \qquad (x, y, z \in X). \tag{1.68}$$

Of these two properties transitivity is far more important.

A *preordered* set. i.e., a set equipped with a preorder gives rise to the category whose objects are elements of X, and $\mathrm{Hom}(x, y)$ consists of a single element, if $x \preceq y$, and is empty otherwise. Since $\mathrm{Hom}(x, y)$ has at most one element, it does not matter how does one denote it. One may use, for example, symbol $\preceq$ or, to indicate its source and target, $x \preceq y$.

Note that in the associated category, objects x and y are isomorphic if and only if $x \preceq y$ and $y \preceq x$.

1.4.5.4

Vice-versa, any small category $\mathcal{C}$ with the property that, for any $a, b \in \mathcal{C}_0$,

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \; \textit{has at most one element}, \tag{1.69}$$

is obtained this way.

Exercise 20 *For a small category that satisfies* (1.69), *show that*

$$x \preceq y \qquad \textit{if} \qquad \mathrm{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$$

defines a preorder relation on $X := \mathcal{C}_o$.

1.4.5.5 Partially ordered sets

A *partial order* on a set X is a preorder which is *weakly antisymmetric*

$$\textit{if} \;\; x \preceq y \;\; \textit{and} \;\; y \preceq x\,, \;\; \textit{then} \;\; x = y. \tag{1.70}$$

1.4.5.6

Small discrete categories correspond to discrete partially ordered sets, i.e., the sets equipped with the smallest order relation—the *identity* relation:

$$x \preceq_{\mathrm{discr}} y \qquad \text{if} \qquad x = y.$$

1.4.6 Functors

1.4.6.1

A *functor* $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of two correspondences: between the classes of objects and between the classes of morphisms

$$F_0\colon \mathcal{C}_0 \to \mathcal{D}_0 \qquad \text{and} \qquad F_1\colon \mathcal{C}_1 \to \mathcal{D}_1$$

which are compatible with all the elements of the category structure. The latter means that the following diagrams of correspondences

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{F_0} & \mathcal{D}_0 \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s} \\ \mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{D}_1 \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ \mathcal{C}_0 & \xrightarrow{F_0} & \mathcal{D}_0 \end{array} \tag{1.71}$$

and

$$\begin{array}{ccc} \mathcal{C}_2 & \xrightarrow{F_2} & \mathcal{D}_2 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{D}_1 \end{array} \tag{1.72}$$

are commutative. Here, F_2 denotes the correspondence induced by F_1 on the classes of composable pairs:

$$F_2\colon \mathcal{C}_2 \to \mathcal{D}_2, \qquad (\alpha, \beta) \mapsto (F_1(\alpha), F_1(\beta)). \tag{1.73}$$

1.4.6.2 Unital functors

When the corresponding categories are *unital*, i.e., possess identity morphisms, then it is customary to require that a functor $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ is compatible also with the identities. This means that the diagram

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{F_0} & \mathcal{D}_0 \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{D}_1 \end{array} \tag{1.74}$$

is supposed to commute. We shall call such functors *unital*.

1.4.6.3

In the interest of keeping notation as transparent as possible it is customary to omit subscript indices and denote the correspondences between the objects, morphisms, composable pairs of morphisms, etc., using the same symbol F.

1.4.6.4

Commutativity of the two squares in diagram (1.71) then can be expressed as

$$s(F(\alpha)) = F(s(\alpha)) \qquad \text{and} \qquad t(F(\alpha)) = F(t(\alpha)) \qquad (\alpha \in \mathcal{C}_1), \tag{1.75}$$

while commutativity of diagram (1.72) expresses the fact that

$$F(\alpha) \circ F(\beta) = F(\alpha \circ \beta) \tag{1.76}$$

for any pair of composable morphisms α and β in $\mathcal{C}$.

Finally, commutativity of diagram (1.74) means that

$$\mathrm{id}_{F(a)} = F\left(\mathrm{id}_a\right) \qquad (a \in \mathcal{C}_0). \tag{1.77}$$

1.4.6.5 Contravariant functors

The functors we defined above are also called *covariant* functors. The *contravariant* variety is obtained if one requires instead

$$s(F(\alpha)) = F(t(\alpha)) \qquad \text{and} \qquad t(F(\alpha)) = F(s(\alpha)) \qquad (\alpha \in \mathcal{C}_1), \tag{1.78}$$

and

$$F(\beta) \circ F(\alpha) = F(\alpha \circ \beta) \tag{1.79}$$

for any pair of composable morphisms α and β in $\mathcal{C}$.

Exercise 21 *Express requirements* (1.78) *and* (1.79) *with help of diagrams analogous to* (1.71) *and* (1.72).

1.4.6.6 An example: the graph functor

For a multimap $\phi\colon X \multimap Y$, the set

$$\Gamma_\phi := \{(x,y) \in X \times Y \mid \phi(x) \ni y\} \tag{1.80}$$

will be called the *graph* of ϕ. It can be naturally identified with the set

$$\bigcup_{x \in X} \{x\} \times \phi(x).$$

Exercise 22 *Verify that* $\Gamma_{\mathrm{id}_X} = \Delta_X$ *and, for any* $\phi\colon X \multimap Y$ *and* $\chi\colon Y \multimap Z$, *one has*

$$\Gamma_{\chi\circ\phi} = \Gamma_\phi \circ \Gamma_\chi. \tag{1.81}$$

Thus, the double correspondence

$$X \mapsto X, \qquad \phi \mapsto \Gamma_\phi \qquad (X \in \mathrm{Ob}_{\mathsf{Set}},\ \phi \in \mathrm{Arr}_{\mathsf{Set}_{\mathrm{mult}}}), \tag{1.82}$$

defines a cotravariant functor $\Gamma\colon \mathsf{Set}_{\mathrm{mult}} \rightsquigarrow \mathsf{Set}_{\mathrm{rel}}$. When ϕ satisfies condition (1.55), Γ_ϕ cincides with the graph of the corresponding map $f\colon X \to Y$.

1.4.6.7

Note that the correspondence

$$\mathrm{Hom}_{\mathsf{Set}_{\mathrm{mult}}}(X,Y) \longrightarrow \mathscr{P}(X \times Y), \qquad \phi \longmapsto \Gamma_\phi,$$

is bijective: for any $R \subseteq X \times Y$, one has $R = \Gamma_{\phi_R}$ where $\phi_R\colon X \multimap Y$ is the multimap

$$\phi_R(x) := \{y \in Y \mid (x,y) \in R\}.$$

1.4.6.8

Functors very often encode natural constructions in Mathematics. We have already encountered a few functors in Section 1.1.2 of the Introduction, all being functors $\mathsf{Set} \rightsquigarrow \mathsf{Set}$, the first and the third being covariant, the second and the fourth being contravariant.

1.4.6.9 The canonical inclusion functors

Given a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$, the natural inclusion correspondences $\iota_0\colon \mathcal{C}'_0 \to \mathcal{C}_0$ and $\iota_1\colon \mathcal{C}'_1 \to \mathcal{C}_1$ define the *inclusion* functor $\iota\colon \mathcal{C}' \rightsquigarrow \mathcal{C}$.

1.4.6.10 The category of small categories

The category whose objects are small categories and morphisms are *covariant* functors between small categories is itself a category. It is denoted Cat and is called the category of (small nonunital) categories.

1.4.6.11 The category of small unital categories

If we consider only unital small categories and unital functors, then we obtain the category of small unital categories. We shall denote it here Cat_1. The reader should be warned that since categories are usually assumed to possess identity morphisms, the category of small unital categories is often denoted Cat.

1.4.6.12 The category of sets viewed as a subcategory of the category of small categories

Let us identify sets X with small discrete categories $\mathcal{X}$,

$$\mathcal{X}_0 = X, \qquad \mathcal{X}_1 = \varnothing.$$

Any map $f\colon X \to Y$ defines a functor $F\colon \mathcal{X} \rightsquigarrow \mathcal{Y}$,

$$F_0 = f, \qquad F_1 = \mathrm{id}_{\varnothing},$$

and every functor $F\colon \mathcal{X} \rightsquigarrow \mathcal{Y}$ is necessarily of this form since $\mathrm{id}_{\varnothing}$ is the only map from $\varnothing$ to $\varnothing$.

In particular, the category of sets can be viewed as a full subcategory of the category of small categories.

1.4.6.13 Set viewed as a subactory of Cat

In the unital case, we associate with any set X the category $\mathcal{X}'$,

$$\mathcal{X}'_0 = X, \qquad \mathcal{X}'_1 = X$$

with all the structural correspondences being id_X (note that $\mathcal{X}'_2 = \{(x, x) \mid x \in X\}$ is here naturally identified with set X).

Any map $f\colon X \to Y$ defines a functor $F\colon \mathcal{X}' \rightsquigarrow \mathcal{Y}'$,

$$F_0 = f, \qquad F_1 = f, \tag{1.83}$$

Exercise 23 *Show that any* unital *functor* $F\colon \mathcal{X}' \to \mathcal{Y}'$ *is of the form* (1.83).

It follows that Set, the unital category of sets, is a full subcategory of Cat, the category of small unital categories.

1.4.6.14

Since functors between unital categories do not necessarily respect the identity morphisms (an example will be given below), Cat_1 is a subcategory of Cat yet not a full subcategory.

1.4.6.15 Natural transformations of functors

Given two (covariant) functors F and G from a category $\mathcal{C}$ to a category $\mathcal{D}$, a natural transformation between them, denoted $\phi\colon F \Rightarrow G$, consists of a single correspondence $\phi\colon \mathcal{C}_0 \to \mathcal{D}_1$ which is compatible with all the present structures. The latter means that

$$\phi(a) \in \mathrm{Hom}_{\mathcal{D}}(F(a), G(a)) \qquad (a \in \mathcal{C}_0), \tag{1.84}$$

and, for any morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(a, b)$, the following square commutes

$$\begin{array}{ccc} F(a) & \xrightarrow{\phi(a)} & G(a) \\ {\scriptstyle F(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle G(\alpha)} \\ F(b) & \xrightarrow{\phi(b)} & G(b) \end{array} \tag{1.85}$$

1.4.6.16

In the language of correspondences, conditions (1.84) translates into commutativity of the following diagram

$$\begin{array}{ccc} & & \mathcal{D}_0 \\ & \overset{F_0}{\nearrow} & \big\uparrow{\scriptstyle s} \\ \mathcal{C}_0 & \xrightarrow{\phi} & \mathcal{D}_1 \\ & \underset{G_0}{\searrow} & \big\downarrow{\scriptstyle t} \\ & & \mathcal{D}_0 \end{array} \tag{1.86}$$

while conditions (1.85) expresses commutativity of the diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{(\phi\circ t, F_1)} & \mathcal{D}_2 \\ {\scriptstyle (G_1,\phi\circ s)}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{D}_2 & \xrightarrow{m} & \mathcal{D}_1 \end{array} \tag{1.87}$$

Exercise 24 *Formulate the definition of a natural transformation of* contravariant *functors analogous to* (1.84)-(1.85).

Exercise 25 *Formulate the definition of a natural transformation of* contravariant *functors analogous to diagrams* (1.86)-(1.87).

1.4.6.17

We have already encountered a natural transformation of contravariant functors $\chi\colon \mathscr{P}(\ ) \Rightarrow 2^{(\ )}$ in Section 1.1.2.6.

1.4.6.18

Many properties normally expressed as identities involving objects, morphisms, sets, maps, elements of various sets, etc, can be often expressed as commutativity of certain diagrams. This leads to proliferation of what some call 'diagrammatic thinking' in modern Mathematics. Employing diagrams often can significantly clarify the picture.

On some occasions information conveyed by diagrams may be more difficult to understand than the same information expressed differently. I would say that it is probably easier to understand the meaning of conditions (1.85) than the meaning of the commutativity of diagram (1.87). That is probably due to the fact that the conditions (1.85) are themselves expressed in terms of commutativity of some easy-to-understand diagrams.

1.4.7 The opposite category

1.4.7.1

Note that if one retains the clases of objects and arrows, $\mathcal{C}_0$ and $\mathcal{C}_1$, but exchanges the source and the target correspondences, $s\colon \mathcal{C}_1 \to \mathcal{C}_0$ and $t\colon \mathcal{C}_1 \to \mathcal{C}_0$, then one obtains a category again. This is the *opposite* category $\mathcal{C}^{\mathrm{op}}$.

1.4.7.2

More precisely,

$$\mathcal{C}_0^{\mathrm{op}} = \mathcal{C}_0, \qquad \mathcal{C}_1^{\mathrm{op}} = \mathcal{C}_1, \qquad s^{\mathrm{op}} = t, \qquad \text{and} \qquad t^{\mathrm{op}} = s. \tag{1.88}$$

If an object a of $\mathcal{C}$ is considered as an object of $\mathcal{C}^{\mathrm{op}}$, then it should be denoted a^{op}. Similarly for morphisms: if $\alpha\colon a \to b$ is a morphism in $\mathcal{C}$, then α considered as a morphism of the opposite category is a morphism $b^{\mathrm{op}} \to a^{\mathrm{op}}$ and it should be denoted α^{op}.

1.4.7.3

The correspondences

$$a \mapsto a^{\mathrm{op}} \qquad \text{and} \qquad \alpha \mapsto \alpha^{\mathrm{op}} \qquad (a \in \mathcal{C}_0; \alpha \in \mathcal{C}_1),$$

define a *contravariant* functor

$$(\ )^{\mathrm{op}}_{\mathcal{C}}\colon \mathcal{C} \rightsquigarrow \mathcal{C}^{\mathrm{op}}.$$

1.4.7.4

Note that

$$(\ )^{\mathrm{op}}_{\mathcal{C}} \circ (\ )^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}} = \mathrm{id}_{\mathcal{C}^{\mathrm{op}}} \qquad \text{and} \qquad (\ )^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}} \circ (\ )^{\mathrm{op}}_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}.$$

1.4.7.5 An example: a partially ordered set

If $\mathcal{C}$ is the category that corresponds to a partially ordered set $(X, \preceq)$, cf. Section 1.4.5.6, then $\mathcal{C}^{\mathrm{op}}$ corresponds to set X equipped with the *reverse* order, $\preceq^{\mathrm{rev}}$.

1.4.7.6

One of the uses of the concept of the opposite category is that it allows to consider any *contravariant* functor $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ as a *covariant* functor either $\mathcal{C} \rightsquigarrow \mathcal{D}^{\mathrm{op}}$ or $\mathcal{C}^{\mathrm{op}} \rightsquigarrow \mathcal{D}$. Formally speaking, this is done by composing F with $(\ )^{\mathrm{op}}_{\mathcal{D}}$ or $(\ )^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}$,

$$(\ )^{\mathrm{op}}_{\mathcal{D}} \circ F\colon \mathcal{C} \to \mathcal{D}^{\mathrm{op}} \qquad \text{or} \qquad F \circ (\ )^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}\colon \mathcal{C}^{\mathrm{op}} \rightsquigarrow \mathcal{D}.$$

1.4.7.7

Any functor $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$, induces also a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}^{\mathrm{op}}$

$$F^{\mathrm{op}} := (\ )^{\mathrm{op}}_{\mathcal{D}} \circ F \circ (\ )^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}. \tag{1.89}$$

Note that F^{op} is covariant (respectively, contravariant) when F is covariant (respectively, contravariant).

1.4.7.8

Assigning to any category $\mathcal{C}$ its opposite category $\mathcal{C}^{\mathrm{op}}$ is *natural* in $\mathcal{C}$, so one can expect that it gives rise to a functor on the category of (small) categories. This is so indeed, the correspondences

$$\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}} \qquad \text{and} \qquad F \mapsto F^{\mathrm{op}} \qquad (\mathcal{C} \in \mathrm{Cat}_0;\ F \in \mathrm{Cat}_1), \tag{1.90}$$

defined by (1.88) and (1.89), yield a functor $(\ )^{\mathrm{op}}\colon \mathrm{Cat} \rightsquigarrow \mathrm{Cat}$.

Exercise 26 *Is functor* (1.90) *covariant or contravariant?*

1.4.7.9 $\mathbf{Set}_{\mathbf{mult}} \simeq (\mathbf{Set}_{\mathbf{rel}})^{\mathbf{op}}$

The graph functor, $\Gamma\colon \mathrm{Set}_{\mathrm{mult}} \rightsquigarrow \mathrm{Set}_{\mathrm{rel}}$, which was defined in Section 1.4.6.6, identifies the category of sets with multimaps as morphisms with the category opposite to the category of sets with binary relations as morphisms. In other words, $\mathrm{Set}_{\mathrm{mult}}$ is isomorphic to $(\mathrm{Set}_{\mathrm{rel}})^{\mathrm{op}}$.

Isomorphisms between categories are, generally speaking, a rare occurrence.

1.4.7.10 Importance of the opposite category concept

Any diagram in a category $\mathcal{C}$ can be interpreted as the same diagram—but with the *direction of all arrows reversed*—in the opposite category.

An immediate corollary of this simple observation yields the following *Duality Principle*:

> For any categorical concept or construction involving one or more diagrams, there is a *dual* concept or construction.

1.4.8 Categories of arrows

1.4.8.1

For any category there are several naturally associated categories whose objects are morphisms. We shall mention here three.

1.4.8.2 The category of arrows

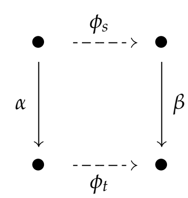
For a category $\mathcal{C}$, let $\mathcal{C}^{\rightarrow}$ be the category whose objects are morphisms of $\mathcal{C}$,

$$(\mathcal{C}^{\rightarrow})_0 := \mathcal{C}_1, \tag{1.91}$$

and morphisms $\phi\colon \alpha \to \beta$ are pairs of morphisms $\phi = (\phi_s, \phi_t)$ in $\mathcal{C}$,

$$\phi_s\colon s(\alpha) \to s(\beta), \qquad \phi_t\colon t(\alpha) \to t(\beta), \tag{1.92}$$

such that the following diagram commutes



(1.93)

1.4.8.3

Category of arrows $\mathcal{C}^{\rightarrow}$ is sometimes also denoted $\operatorname{Arr}\mathcal{C}$. One should be advised however, that $\operatorname{Arr}\mathcal{C}$ may also be used to denote the *class* of morphisms in $\mathcal{C}$.

1.4.8.4 Two comma categories

For any object a in a category $\mathcal{C}$, one can consider two categories: one, $\mathcal{C}^{a\rightarrow}$, whose objects are morphisms in $\mathcal{C}$ with source a,

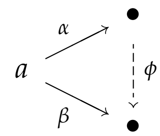
$$(\mathcal{C}^{a\rightarrow})_0 := \{\alpha \in \mathcal{C}_1 \mid s(\alpha) = a\}, \tag{1.94}$$

and another one, $\mathcal{C}^{\rightarrow a}$, whose objects are morphisms with target a,

$$(\mathcal{C}^{a\rightarrow})_0 := \{\alpha \in \mathcal{C}_1 \mid t(\alpha) = a\}. \tag{1.95}$$

1.4.8.5

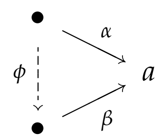
Morphisms $\phi \colon \alpha \to \beta$ in $\mathcal{C}^{a\rightarrow}$ are morphisms $\phi \colon t(\alpha) \to t(\beta)$ such that the following diagram commutes



(1.96)

1.4.8.6

Morphisms $\phi \colon \alpha \to \beta$ in $\mathcal{C}^{\rightarrow a}$ are morphisms $\phi \colon s(\alpha) \to s(\beta)$ such that the following diagram commutes



(1.97)

1.4.9 Categories of diagrams

1.4.9.1

(Covariant) functors from a *small* category Γ to an arbitrary category $\mathcal{C}$ form a category, denoted $\mathcal{C}^{\Gamma}$, with morphisms $\phi \colon F \to G$ being natural transformations of functors.

1.4.9.2 Diagrams as functors

Such functors are often called *diagrams in* $\mathcal{C}$ and the reason will become clear when we look at a series of simple examples.

1.4.9.3 $\mathcal{C}$

Consider the category with a single object, 0, with empty class of morphisms. Denote this category by $\mathbf{1}$. Functors from $\mathbf{1}$ to $\mathcal{C}$ correspond to single objects in $\mathcal{C}$, and $\mathcal{C}^{\mathbf{1}}$ becomes naturally identified with category $\mathcal{C}$ itself.

1.4.9.4 $\mathcal{C}^{\rightarrow}$

Consider the category with two objects, 0 and 1, and a single morphism

$$0 \rightarrow 1.$$

Denote this category by **2**. Functors from **2** to $\mathcal{C}$ correspond to single morphisms in $\mathcal{C}$, and $\mathcal{C}^{\mathbf{2}}$ becomes naturally identified with the category of arrows, $\mathcal{C}^{\rightarrow}$.

1.4.9.5 The category of composable pairs of arrows

Consider the category with three objects, 0, 1 and 2, and just three morphisms, the following two

$$0 \rightarrow 1 \rightarrow 2,$$

and their composition. Denote this category by **3**. Functors from **3** to $\mathcal{C}$ correspond to composable pairs of morphisms in $\mathcal{C}$, and $\mathcal{C}^{\mathbf{3}}$ becomes naturally identified with the category of composable pairs of arrows in $\mathcal{C}$.

Exercise 27 *The category of composable pairs of arrows in $\mathcal{C}$ has class $\mathcal{C}_2$ as its class of objects. Knowing that morphisms $\phi\colon (\alpha_0, \alpha_1) \rightarrow (\beta_0, \beta_1)$ are defined in a natural manner, give the definition of morphisms.*

1.4.9.6

Categories **1**, **2** and **3** correspond to the linearly ordered sets $\{0\}$, $\{0,1\}$, $\{0,1,2\}$. Let **n** be the category with n objects

$$0, 1, \ldots, \text{n-1}$$

which corresponds to the linearly ordered set $\{0, \ldots, n-1\}$.

Exercise 28 *Find the number of morphisms in* **n**.

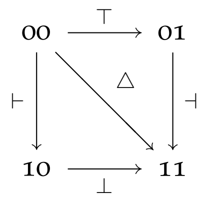
Exercise 29 *Provide a description of $\mathcal{C}^{\mathbf{n}}$ which generalizes to arbitrary n the descriptions given above for $n = 1, 2, 3$.*

1.4.9.7 The category of commuting squares

Consider the category with four objects

$$00, \quad 01, \quad 10, \quad \text{and} \quad 11,$$

and just five morphisms



Denote this category by $\Box$. Objects of $\mathcal{C}^{\Box}$ are commuting squares in $\mathcal{C}$.

Exercise 30 *Describe morphisms in* $\mathcal{C}^{\Box}$.

1.4.9.8 The category of families of objects

Let $\mathbf{I}$ be the category with a set I as its class of objects and empty class of morphisms. Objects of $\mathcal{C}^{\mathbf{I}}$ are families $(a_i)_{i \in I}$ of objects of category $\mathcal{C}$ indexed by set I.

Exercise 31 *Describe morphisms* $\phi\colon (a_i)_{i \in I} \to (b_i)_{i \in I}$.