Notes on Uniform Structures

Annex to H104

Mariusz Wodzicki

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1 Vocabulary

1.1 Binary Relations

1.1.1 The power set

Given a set X, we denote the set of all subsets of X by $\mathscr{P}(X)$ and by $\mathscr{P}^*(X)$ — the set of all nonempty subsets. The set of subsets $E \subseteq X$ which contain a given subset A will be denoted $\mathscr{P}_A(X)$. If $A \neq \emptyset$, then $\mathscr{P}_A(X)$ is a *filter*. Note that one has $\mathscr{P}_\emptyset(X) = \mathscr{P}(X)$.

1.1.2

In these notes we identify binary relations between elements of a set X and a set Y with subsets $E \subseteq X \times Y$ of their Cartesian product $X \times Y$. To a given relation $\sim$ corresponds the subset:

$$E_\sim := \{(x, y) \in X \times Y \mid x \sim y\} \tag{1}$$

and, vice-versa, to a given subset $E \subseteq X \times Y$ corresponds the relation:

$$x \sim_E y \qquad \text{if and only if} \qquad (x, y) \in E. \tag{2}$$

1.1.3 The opposite relation

We denote by

$$E^{\mathrm{op}} := \{(y, x) \in Y \times X \mid (x, y) \in E\} \tag{3}$$

the **opposite** relation.

1.1.4

The correspondence

$$E \longmapsto E^{\mathrm{op}} \qquad (E \subseteq X \times X) \tag{4}$$

defines an involution[1] of $\mathscr{P}(X \times X)$. It induces the corresponding involution of $\mathscr{P}(\mathscr{P}(X \times X))$:

$$\mathscr{E} \longmapsto \mathrm{op}_*(\mathscr{E}) := \{E^{\mathrm{op}} \mid E \in \mathscr{E}\} \qquad (\mathscr{E} \subseteq \mathscr{P}(X \times X)). \tag{5}$$

Exercise 1 *Show that* $\mathrm{op}_*(\mathscr{E})$

(a) possesses the Finite Intersection Property, if $\mathscr{E}$ *possesses the Finite Intersection Property;*

(b) is a filter-base, if $\mathscr{E}$ *is a filter-base;*

(c) is a filter, if $\mathscr{E}$ *is a filter.*

1.1.5 The identity relation

For any set X, we shall denote by Δ_X the **identity** relation $\{(x, x') \in X \times X \mid x = x'\}$. We shall often omit subscript X when set X is clear from the context.

Exercise 2 *Let* A *and* B *be subsets of a set* X. *Show that*

$$(A \times B) \cap \Delta = \varnothing \qquad \text{if and only if} \qquad A \cap B = \varnothing, \tag{6}$$

i.e., sets A *and* B *are disjoint.*

1.1.6 The sets of (left) E-relatives

For any subset $B \subseteq Y$ we shall denote by $E \cdot B$ the set of *left* E-relatives of elements of B:

$$E \cdot B := \{x \in X \mid \exists_{y \in B}\, x \sim_E y\}. \tag{7}$$

We shall also denote by Ey the set $E \cdot \{y\}$.

Definition 1.1 *We say that an element* $x \in X$ *is* E***-related****, (or* E***-close****) to an element* $y \in Y$, *and write* $x \sim_E y$, *if* $(x, y) \in E$.

[1] Recall that a mapping $f\colon S \longrightarrow S$ is called an *involution* (of a set S) if $f \circ f = \mathrm{id}_S$.

In particular, $y \sim_{E^{\mathrm{op}}} x$ if and only if $x \sim_E y$.

Exercise 3 *Show that:*

$$E \cdot \emptyset \;=\; \emptyset \tag{8}$$

$$E \cdot B \;=\; \bigcup_{y \in B} Ey \tag{9}$$

$$E \cdot B \subseteq E \cdot B' \qquad \textit{if} \qquad B \subseteq B' \tag{10}$$

$$(E \cdot B) \cup (E \cdot B') \;=\; E \cdot (B \cup B') \tag{11}$$

$$(E \cdot B) \cap (E \cdot B') \;\supseteq\; E \cdot (B \cap B') \tag{12}$$

where B and B' are arbitrary subsets of Y. Give an example demonstrating that

$$(E \cdot B) \cap (E \cdot B') \neq E \cdot (B \cap B')$$

in general.

1.1.7 The sets of right E-relatives

For any subset $A \subseteq X$ we shall denote by $A \cdot E$ the set of *right* E-relatives of elements of A:

$$A \cdot E := \{y \in Y \mid \exists_{x \in A}\, x \sim_E y\}. \tag{13}$$

We shall also denote by xE the set $\{x\} \cdot E$.

Exercise 4 *Show that*

$$A \cdot E \;=\; E^{\mathrm{op}} \cdot A \qquad (A \subseteq X;\; E \subseteq X \times Y).$$

Exercise 5 *Let A and B be subsets of X and Y respectively, and let $E \subseteq X \times Y$. Show that the following conditions are equivalent*

there exist $a \in A$ and $b \in B$ such that $a \sim_E b$, (14a)

$(A \times B) \cap E \neq \emptyset$, (14b)

$A \cap (E \cdot B) \neq \emptyset$, (14c)

$(A \cdot E) \cap B \neq \emptyset$. (14d)

1.2 Composition of binary relations

1.2.1

If $E \subseteq X{\times}Y$ and $F \subseteq Y{\times}Z$, then $E \circ F \subseteq X{\times}Z$ is defined as

$$\begin{aligned} E \circ F &:= \{(x,z) \in X{\times}Z \mid \exists_{y\in Y}\ (x,y) \in E \text{ and } (y,z) \in F\} \\ &= \{(x,z) \in X{\times}Z \mid \exists_{y\in Y}\ x \sim_E y \text{ and } y \sim_F z\}. \end{aligned} \tag{15}$$

1.2.2 Associativity

Composition of binary relations is *associative*:

$$E \circ (F \circ G) = (E \circ F) \circ G \tag{16}$$

where $G \subseteq Z{\times}W$. Note also that

$$\Delta_X \circ E = E = E \circ \Delta_Y \tag{17}$$

Identities (16)–(17) mean that, equipped with $\cdot$, the set of all binary relations on X becomes a monoid.

Exercise 6 *Let $E \subseteq X{\times}Y$, $F \subseteq Y{\times}Z$ and $C \subseteq Z$. Show that, for any $B \subseteq Z$,*

$$(E \circ F) \cdot B \;=\; E \cdot (F \cdot B). \tag{18}$$

Exercise 7 *Let $D \subseteq W{\times}X$, $E \subseteq X{\times}Y$, $F \subseteq Y{\times}Z$, and let $p \in X$ and $q \in Y$. Show that*

$$Dp{\times}qF \;\subseteq\; D \circ E \circ F \tag{19}$$

if $p \sim_E q$.

Exercise 8 *Let A and C be subsets of X and Z respectively, and let $E \subseteq X{\times}Y$ and $F \subseteq Y{\times}Z$. Show that the following conditions are equivalent*

$$\text{there exist } a \in A \text{ and } c \in C \text{ such that } a \sim_{E\circ F} c, \tag{20a}$$

$$(A{\times}C) \cap (E \circ F) \neq \emptyset, \tag{20b}$$

$$A \cap ((E \circ F) \cdot C) \neq \emptyset, \tag{20c}$$

$$(A \cdot E) \cap (F \cdot C) \neq \emptyset, \tag{20d}$$

$$(A \cdot (E \circ F)) \cap C \neq \emptyset. \tag{20e}$$

1.2.3 Monotonicity

Composition of binary relations is *monotonic* in both arguments:

$$\textit{if } E \subseteq E' \textit{ and } F \subseteq F', \textit{ then } E \circ F \subseteq E' \circ F'. \tag{21}$$

1.2.4

The correspondence

$$(E, F) \longmapsto E \circ F \qquad (E \subseteq X \times Y,\ F \subseteq Y \times Z) \tag{22}$$

defines a mapping

$$\mathscr{P}(X \times Y) \times \mathscr{P}(Y \times Z) \longrightarrow \mathscr{P}(X \times Z).$$

This, in turn, induces the mapping

$$\mathscr{P}(\mathscr{P}(X \times Y)) \times \mathscr{P}(\mathscr{P}(Y \times Z)) \longrightarrow \mathscr{P}(\mathscr{P}(X \times Z)) \tag{23}$$

where

$$(\mathscr{E}, \mathscr{F}) \longmapsto \mathscr{E} \circ \mathscr{F} := \{E \circ F \mid E \in \mathscr{E} \text{ and } F \in \mathscr{F}\}. \tag{24}$$

1.2.5

Since $E \circ F$ may be empty while E and F are not empty, the composite, $\mathscr{E} \circ \mathscr{F}$ of two families with the Finite Intersection Property may not have that property. However, monotonicity of the composition of binary relations means that (23) induces the pairings

$$\mathscr{P}(\mathscr{P}_C(X \times Y)) \times \mathscr{P}(\mathscr{P}_D(Y \times Z)) \longrightarrow \mathscr{P}(\mathscr{P}_{C \circ D}(X \times Z)) \tag{25}$$

In particular, if $C \circ D \neq \emptyset$, then, for any $\mathscr{E} \subseteq \mathscr{P}_C((X \times Y)$ and $\mathscr{F} \subseteq \mathscr{P}_D(\mathscr{P}((Y \times Z))$, the family $\mathscr{E} \circ \mathscr{F}$ has the Finite Intersection Property since it is contained in the filter $\mathscr{P}_{C \circ D}(X \times Z)$.

1.2.6

A superset of $E \circ F$ is generally not of the form $E' \circ F'$ for some relations $E' \subseteq X \times Y$ and $F' \subseteq Y \times Z$. In particular, the composite of filters, $\mathscr{E} \circ \mathscr{F}$, when it exists is generally only a filter-base.

In view of this, we pose

$$\mathscr{E} \sharp \mathscr{F} := (\mathscr{E} \circ \mathscr{F})_{\sharp}, \tag{26}$$

where $\mathscr{E}$ and $\mathscr{F}$ are filters such that $\mathscr{E} \circ \mathscr{F}$ has the Finite Intersection Property. We will call (26) the *composite* of filters $\mathscr{E}$ and $\mathscr{F}$.

1.2.7

We note one more identity

$$(E \circ F)^{\mathrm{op}} = F^{\mathrm{op}} \circ E^{\mathrm{op}} \qquad (E \subseteq X{\times}Y;\ F \subseteq Y{\times}Z). \tag{27}$$

1.2.8 Anti-involutions

Definition 1.2 *Given a binary operation on a set S,*

$$S{\times}S \longrightarrow S, \qquad (s,t) \longmapsto s \cdot t, \tag{28}$$

an operation

$$s \longmapsto s^* \qquad (m \in M) \tag{29}$$

is said to be an anti-involution *if it satisfies the identity*

$$(s \cdot t)^* = t^* \cdot s^* \qquad (s,t \in S)$$

and

$$(s^*)^* = s \qquad (s \in S).$$

1.2.9 *-semigroups

If $\cdot$ is associative, then the structure $(S, \cdot, ^*)$ is called a **-semigroup*.

1.2.10 *-monoids

Exercise 9 *Let e be a left (respectively, right) identity for* (28). *Show that e^* is a right (respectively, left identity.*

It follows that if e is a two-sided identity, then e^* is also a two-sided identity. Recalling that a two-sided identity is unique, we infer that

$$e^* = e.$$

A monoid with an anti-involution is referred to as a *-monoid.

1.2.11 A group as an example of a *-monoid

The operation on a *group* $(G, \cdot)$ which sends $g \in G$ to its inverse g^{-1} is an anti-involution. Thus $(G, \cdot, ()^{-1})$ is a *-monoid.

1.2.12

Identities (16), (17) and (27) mean that the set of binary relations on a given set X, $\mathscr{P}(X{\times}X)$, equipped with the operation $\circ$ and the anti-involution ${}^{\mathrm{op}}$, is a *-monoid.

1.2.13 The filter of reflexive relations

A relation $E \subseteq X{\times}X$ is *reflexive* if $\Delta_X \subseteq E$. If both E and F are reflexive, then

$$\Delta_X = \Delta_X \circ \Delta_X \subseteq E \circ F \qquad \text{and} \qquad \Delta_X = (\Delta_X)^{\mathrm{op}} \subseteq E^{\mathrm{op}}.$$

In particular the filter $\mathscr{P}_\Delta(X \times X)$ of reflexive relations on X is a *-submonoid of $\mathscr{P}(X{\times}X)$.

1.3 The induced relations between power sets

1.3.1

A binary relation $E \subseteq X{\times}Y$ naturally induces the following two relations between $\mathscr{P}(X)$ and $\mathscr{P}(Y)$:

$$E^{\vee} \;:=\; \{(A,B) \in \mathscr{P}(X){\times}\mathscr{P}(Y) \mid A{\times}B \cap E \neq \emptyset\} \tag{30}$$

and

$$E^{\wedge} \;:=\; \{(A,B) \in \mathscr{P}(X){\times}\mathscr{P}(Y) \mid A{\times}B \subseteq E\}. \tag{31}$$

Equivalently,

$$A \sim_{E^{\vee}} B \qquad \text{if} \qquad x \sim_E y \text{ for } \textit{some } x \in A \text{ and } y \in B, \tag{32}$$

and

$$A \sim_{E^{\wedge}} B \qquad \text{if} \qquad x \sim_E y \text{ for } \textit{all } x \in A \text{ and } y \in B. \tag{33}$$

Exercise 10 *Let $A \subseteq X$, $B \subseteq Y$ and $C \subseteq Z$. Let $E \subseteq X{\times}Y$ and $F \subseteq Y{\times}Z$. Show that*

$$\textit{if } A \sim_{E^{\wedge}} B \textit{ and } B \sim_{F^{\wedge}} C\textit{, then } A \sim_{(E\circ F)^{\wedge}} C. \tag{34}$$

provided $B \neq \emptyset$. Explain why (34) *does not hold, in general, when $B = \emptyset$.*

1.3.2

In particular, if we restrict the induced ${}^{\wedge}$-relations to nonempty subsets, then

$$E^{\wedge} \circ F^{\wedge} \;\subseteq\; (E \circ F)^{\wedge}. \tag{35}$$

1.3.3

There are no analogs of (34) or (35), for the $^{\vee}$-relations. Generally speaking, one cannot say too much about $E^{\vee} \circ F^{\vee}$ in terms of $(E \circ F)^{\vee}$, and vice-versa.

1.3.4 The iterated induced relations

One can iterate the above procedures to obtain 2^n binary relations between $\mathscr{P}^n(X)$ and $\mathscr{P}^n(Y)$ induced by a single relation E between X and Y.

The case $n = 2$, i.e., the case of the induced relations on sets of families of subsets represents a special interest in Analysis and Topology. A single relation E between X and Y in this case induces four relations between $\mathscr{P}(\mathscr{P}(X))$ and $\mathscr{P}(\mathscr{P}(Y))$,

$$E^{\vee\vee} := (E^{\vee})^{\vee}, \quad E^{\wedge\vee} := (E^{\wedge})^{\vee}, \quad E^{\vee\wedge} := (E^{\vee})^{\wedge} \quad \text{and} \quad E^{\wedge\wedge} := (E^{\wedge})^{\wedge}. \tag{36}$$

Exercise 11 *Given a family $\mathscr{A} \subseteq \mathscr{P}(X)$ and a family $\mathscr{B} \subseteq \mathscr{P}(Y)$, show that*

$$\mathscr{A} \sim_{E^{\wedge\vee}} \mathscr{B} \qquad \textit{if and only if} \qquad A \times B \subseteq E \textit{ for some } A \in \mathscr{A} \textit{ and } B \in \mathscr{B}. \tag{37}$$

Exercise 12 *Let $\mathscr{A} \subseteq \mathscr{P}(X)$, $\mathscr{B} \subseteq \mathscr{P}(Y)$ and $\mathscr{C} \subseteq \mathscr{P}(Z)$. Let $E \subseteq X \times Y$ and $F \subseteq Y \times Z$. Show that*

$$\textit{if } \mathscr{A} \sim_{E^{\wedge\vee}} \mathscr{B} \textit{ and } \mathscr{B} \sim_{F^{\wedge\vee}} \mathscr{C}, \textit{ then } A \sim_{(E\circ F)^{\wedge\vee}} \mathscr{C}. \tag{38}$$

provided $\mathscr{B}$ is a filter base. Explain why (34) *does not hold, in general, when $\mathscr{B}$ is not a filter base.*

1.4 The binary relation associated with a family $\mathscr{A} \subseteq \mathscr{P}(X)$

1.4.1

Any equivalence relation $\sim$ on a set X defines a certain family of subsets $\mathscr{A}_{\sim}$, namely the family $X/_{\sim}$ of equivalence classes of relation $\sim$.

We can recover the equivalence relation from that family of subsets by means of the following general construction.

1.4.2

For any family of subsets $\mathscr{A}$ of a set X, let us consider the binary relation on X:

$$\square_{\mathscr{A}} := \bigcup_{A \in \mathscr{A}} A \times A. \tag{39}$$

Exercise 13 *Show that the union of $A \in \mathscr{A}$ which contain $x \in X$ coincides with the set of points $\square_{\mathscr{A}}$-close to x:*

$$x \square_{\mathscr{A}} \ = \ \square_{\mathscr{A}} x \ = \ \bigcup_{\substack{A \in \mathscr{A} \\ A \ni x}} A. \tag{40}$$

1.4.3

In the special case of the family of all singleton subsets,

$$\mathscr{X} = \big\{ \{x\} \mid x \in X \big\}, \tag{41}$$

we obtain the identity relation

$$\square_{\mathscr{X}} = \Delta_X. \tag{42}$$

1.4.4

The associated relation is automatically symmetric. It is reflexive precisely when $\mathscr{A}$ is a cover of X.

Exercise 14 *Show that relation $\sim_{\square_{\mathscr{A}}}$ is reflexive, i.e.,*

$$\square_{\mathscr{A}} \supseteq \Delta_X$$

if and only if $\mathscr{A}$ covers X.

1.4.5

Vice-versa, for any reflexive and symmetric relation $E \subseteq X \times X$, there exists a cover $\mathscr{A}$ of X such that $E = \square_{\mathscr{A}}$.

Exercise 15 *Given a reflexive symmetric relation $E \subseteq X \times X$, let*

$$\mathscr{A}_E \ := \ \{ A \subseteq X \mid A \times A \subseteq E \}. \tag{43}$$

Show that $E = \square_{\mathscr{A}_E}$.

1.4.6

Let us introduce the following operation on $\mathscr{P}(\mathscr{P}^*(X))$, the set of all families of *nonempty* subsets of X,

$$\mathscr{A} \triangledown \mathscr{B} := \{A \cup B \mid A \in \mathscr{A},\ B \in \mathscr{B}, \text{and } A \cap B \neq \emptyset\}. \tag{44}$$

It is commutative, and $\mathscr{X}$, the family introduced in (41), is its identity.

Exercise 16 *Show that, for any $\mathscr{A} \subseteq \mathscr{P}^*(X)$,*

$$\mathscr{X} \triangledown \mathscr{A} = \mathscr{A} = \mathscr{A} \triangledown \mathscr{X}.$$

Exercise 17 *Show that*

$$(A \times B) \circ (B' \times C) = \begin{cases} A \times C & \textit{if } B \cap B' \neq \emptyset \\ \emptyset & \textit{otherwise} \end{cases}$$

where $A \subseteq X$, $C \subseteq Z$, while B and B' are subsets of Y.

Exercise 18 *Show that*

$$\square_{\mathscr{A}} \circ \square_{\mathscr{B}} \cup \square_{\mathscr{B}} \circ \square_{\mathscr{A}} \ \subseteq\ \square_{\mathscr{A} \triangledown \mathscr{B}} \ \subseteq\ \square_{\mathscr{A}} \circ \square_{\mathscr{A}} \cup \square_{\mathscr{A}} \circ \square_{\mathscr{B}} \cup \square_{\mathscr{B}} \circ \square_{\mathscr{A}} \cup \square_{\mathscr{A}} \circ \square_{\mathscr{B}}. \tag{45}$$

Deduce from (45) *that*

$$\square_{\mathscr{A} \triangledown \mathscr{A}} \ =\ \square_{\mathscr{A}} \circ \square_{\mathscr{A}}. \tag{46}$$

1.4.7

We say that $\mathscr{A}$ is an n-cover of a subset $B \subseteq X$ if for any n elements $x_1, \ldots, x_n \in B$, there exists $A \in \mathscr{A}$ such that $x_1, \ldots, x_n \in A$. General families of subsets are 0-covers, covers are the same as 1-covers.

1.4.8

If $\mathscr{A}$ consists of disjoint subsets and $\mathscr{A}$ covers B, then $\mathscr{A}$ is an n-cover of B for any n.

Exercise 19 *Show that the following conditions are equivalent:*

relation $\sim_{\square_{\mathscr{A}}}$ is transitive, (47a)

for any $A, A' \in \mathscr{A}$ with $A \cap A'$ nonempty, $\mathscr{A}$ is a 2-cover of $A \cup A'$, (47b)

$\mathscr{A} \triangledown \mathscr{A} = \mathscr{A}$. (47c)

1.5 Preorders $⋺$ and $⋲$

1.5.1

We shall say that a family $\mathscr{A}' \subseteq \mathscr{P}(X)$ *refines* a family $\mathscr{A} \subseteq \mathscr{P}(X)$ (or, that it is a *refinement* of $\mathscr{A}$) if

$$\textit{for any } A' \in \mathscr{A}', \textit{ there exists } A \in \mathscr{A} \textit{ such that } A' \subseteq A. \tag{48}$$

We denote it by $\mathscr{A}' ⋺ \mathscr{A}$. This terminology is frequently employed in considerations involving *covers*.

Exercise 20 *Show that*

$$\mathscr{A}' ⋺ \mathscr{A} \qquad \textit{implies} \qquad \square_{\mathscr{A}'} \subseteq \square_{\mathscr{A}}. \tag{49}$$

1.5.2

Dually, we shall say that a family $\mathscr{A}' \subseteq \mathscr{P}(X)$ is *inscribed* into a family $\mathscr{A}$ if

$$\textit{for any } A \in \mathscr{A}, \textit{ there exists } A' \in \mathscr{A}' \textit{ such that } A' \subseteq A. \tag{50}$$

We denote it by $\mathscr{A}' ⋲ \mathscr{A}$.

2 Uniform spaces

2.1 Uniform structures

2.1.1

Definition 2.1 *A filter $\mathfrak{U}$ on $X \times X$ is said to be a **uniform structure** if it satisfies the following conditions*

($\mathbf{U_1}$) $\bigcap \mathfrak{U} \supseteq \Delta$;

($\mathbf{U_2}$) *if $E \in \mathfrak{U}$, then $E^{\mathrm{op}} \in \mathfrak{U}$;*

($\mathbf{U_3}$) *for any $E \in \mathfrak{U}$, there exists $E' \in \mathfrak{U}$ such that $E' \circ E' \subseteq E$.*

2.1.2

The above conditions are equivalent to the following three:

($\mathbf{U_1'}$) $\mathcal{U} \subseteq \mathscr{P}_\Delta(X \times X)$;

($\mathbf{U_2'}$) $\mathcal{U} = \mathrm{op}_*(\mathcal{U})$;

$\left(\mathbf{U_3'}\right)$ $\mathcal{U} = \mathcal{U}^{\sharp 2}$.

where

$$\mathcal{U}^{\sharp n} := \underbrace{\mathcal{U}\sharp \cdots \sharp\mathcal{U}}_{n \text{ times}}. \tag{51}$$

2.1.3

Note that condition (3) alone is equivalent to

$$\mathcal{U}^{\sharp 2} \succcurlyeq \mathcal{U}$$

which, for filters, is to

$$\mathcal{U}^{\sharp 2} \supseteq \mathcal{U}.$$

The reverse containment is, however, automatic for subfilters of the filter of reflexive relations $\mathscr{P}_\Delta(X \times X)$.

Exercise 21 *Show that for any subfilter $\mathscr{F}$ of $\mathscr{P}_\Delta(X \times X)$, one has*

$$\mathscr{F}^{\sharp 2} \subseteq \mathscr{F}.$$

Definition 2.2 *A set X equipped with a uniform structure $\mathcal{U}$ is called a* ***uniform space*** *and the filter $\mathcal{U}$ is often referred to as its* ***uniformity****.*

2.1.4 Entourages

Members of $\mathcal{U}$ are usually referred to as **entourages**. For two points p and q of X we shall say that p is E**-close** to q if $p \sim_E q$. Thus, $E \cdot B$ is the set of points $p \in X$ which are E*-close* to a subset $B \subseteq X$.

2.1.5 Symmetric entourages

Since $\mathcal{U}$ is a filter, and $F = E \cap E^{\mathrm{op}}$ is clearly *symmetric*, i.e., $F = F^{\mathrm{op}}$, symmetric entourages form a base of filter $\mathcal{U}$.

2.1.6 E-small sets

Definition 2.3 *We say that a subset $A \subseteq X$ is E-**small** if $A \times A \subseteq E$, i.e., if any two elements of A are E-close,*

$$\forall_{s,s' \in A}\ s \sim_E s'. \tag{52}$$

Exercise 22 *Show that $E \subseteq E \circ E$ for any entourage $E \in \mathcal{U}$.*

Exercise 23 *Let E be an entourage. Show that, for any $n \geq 2$, there exists $D \in \mathcal{U}$ such that*

$$D^{\circ n} := \underbrace{D \circ \cdots \circ D}_{n \text{ times}} \subseteq E. \tag{53}$$

Exercise 24 *Let E be a symmetric entourage. Show that, if $B \subseteq X$ is E-small, then $E \cdot B$ is $E \circ E \circ E$-small.*

2.1.7 A fundamental system of entourages

A base of the uniformity filter is sometimes referred to as a *fundamental system of entourages* of a uniform space $(X, \mathcal{U})$.

Any member B of a fundamental system of entourages $\mathscr{B}$ satisfies the following triple condition:

$(\mathbf{UB_1})$ $B \supseteq \Delta_X$;

$(\mathbf{UB_2})$ there exists $C \in \mathscr{B}$ such that $B \supseteq C^{\mathrm{op}}$;

$(\mathbf{UB_3})$ there exists $D \in \mathscr{B}$ such that $B \supseteq D^{\circ 2}$.

2.1.8 Uniformity-bases

Vice-versa, if any member $B \in \mathscr{B}$ of some filter-base $\mathscr{B}$ on $X \times X$ satisfies $(\mathbf{UB_1})$–$(\mathbf{UB_3})$, then the generated filter $\mathscr{B}_\sharp$ is a uniformity.

We shall refer to such a filter-base as a *uniformity-base*.

2.1.9

For a given family $\mathscr{B}$ of subsets of $X \times X$, let us consider the family of relations on the the set $\mathrm{FB}(X)$ of all filter-bases on X, which are ${}^{\wedge\vee}$-induced by entourages from $\mathscr{B}$,

$$\{B^{\wedge\vee} \mid B \in \mathscr{B}\} \tag{54}$$

Abusing slightly notation, we shall be denoting (54) by $\mathscr{B}^{\wedge\vee}$.

Exercise 25 *Show that, if $\mathscr{B}$ satisfies* $(\mathbf{UB_2})$, *so does* $\mathscr{B}^{\wedge\vee}$.

Exercise 26 *Show that, if $\mathscr{B}$ satisfies* $(\mathbf{UB_3})$, *so does* $\mathscr{B}^{\wedge\vee}$.

2.1.10

On the contrary, Property $(\mathbf{UB_1})$ is not inherited by $\mathscr{B}^{\wedge\vee}$. This leads to the following definition.

Definition 2.4 *A family $\mathscr{A}$ of subsets of a uniform space $(X, \mathcal{U})$ is said to be a* Cauchy *family if*

$$\mathscr{A} \sim_{E^{\wedge\vee}} \mathscr{A} \quad \text{for any} \quad E \in \mathcal{U}. \tag{55}$$

Equivalently, if for any entourage $E \in \mathcal{U}$, family $\mathscr{A}$ possesses an E-small member set.

2.1.11

In the above definition one can replace $\mathcal{U}$ by any uniformity-base. Observe, that the set of Cauchy families on X is obviously an upset in the partially ordered set $(\mathscr{P}(\mathscr{P}(X)), \subseteq)$ of all families of subsets of X.

2.1.12 The induced uniformity on $\mathbf{FB}_{\mathbf{Cauchy}}(X, \mathcal{U})$

If we denote by $\mathrm{FB}_{\mathrm{Cauchy}}(X, \mathcal{U})$ the set of Cauchy filter-bases on the uniform space $(X, \mathcal{U})$, then the family $\mathcal{U}^{\wedge\vee}$ becomes a uniformity-base on $\mathrm{FB}_{\mathrm{Cauchy}}(X, \mathcal{U})$. In particular, $\mathrm{FB}_{\mathrm{Cauchy}}(X, \mathcal{U})$ equipped with the generated uniformity $\mathcal{U}^{\wedge\vee})_\sharp$ becomes a uniform space in its own right. This space is usually highly non-separable, however.

Exercise 27 *Show that, if $\mathscr{A}$ is a Cauchy family and $\mathscr{B} \Subset \mathscr{A}$, then*

$$\mathscr{A} \sim_{E^{\wedge\vee}} \mathscr{B} \quad \text{for any} \quad E \in \mathcal{U}.$$

2.2 The partially ordered set of uniformities on an arbitrary set

2.2.1

The set of all uniformities on a set X is the subset of the partially ordered set of all filters on $X \times X$,

$$\mathrm{Unif}(X) \subseteq \mathrm{Filt}(X \times X). \tag{56}$$

It possesses the smallest element

$$\mathscr{P}_{X\times X}(X\times X) = \{X\times X\}$$

and the greatest element

$$\mathscr{P}_{\Delta}(X\times X) = \{E \in \mathscr{P}(X\times X) \mid E \supseteq \Delta\},$$

namely the set of all *reflexive* relations on X.

In particular, for any family of uniformities, $\mathfrak{U}$, its infimum and supremum exist in the set of all filters on $X\times X$:

$$\inf_{\mathrm{Filt}(X\times X)}\mathfrak{U} = \bigcap \mathscr{U} = \bigcap_{\mathcal{U}\in\mathfrak{U}} \mathcal{U} \tag{57}$$

and

$$\sup_{\mathrm{Filt}(X\times X)}\mathfrak{U} = \Big(\bigcup \mathfrak{U}\Big)_{\sharp}. \tag{58}$$

Exercise 28 *Show that both* (57) *and* (58) *are uniformities (in case of* (58)*, show that* $\bigcup\mathfrak{U}$ *is a uniformity-base).*

Thus, $\mathrm{Unif}(X)$ is a complete lattice and the inclusion

$$\mathrm{Unif}(X) \hookrightarrow \mathrm{Filt}(X\times X)$$

is both inf- and sup-continuous.

2.2.2

Note that $\mathrm{Filt}(X\times X)$ itself does not possess a greatest element if X has more than one element.

3 Metrization

3.1 The uniform structure associated with a semi-metric

3.1.1

Suppose $\rho\colon X\times X \longrightarrow [0,\infty)$ is a semi-metric on a set X. The sets

$$E_\epsilon := \{(p,q) \in X\times X \mid \rho(p,q) < \epsilon\} \tag{59}$$

form a basis of a filter on $X\times X$.

Note that

$$\bigcap_{\epsilon>0} E_\epsilon \supseteq \Delta, \tag{60}$$

$E_\epsilon = E_\epsilon^{\mathrm{op}}$, and the triangle inequality yields

$$E_\epsilon \circ E_{\epsilon'} \subseteq E_{\epsilon+\epsilon'}.$$

It follows that the filter generated by $\{E_\epsilon \mid \epsilon > 0\}$ satisfies the three conditions of a uniformity, cf. Definition 2.1. We shall denote it $\mathcal{U}_\rho$.

3.1.2

Function ρ *separates* points of X, i.e., is a metric on set X, precisely when

$$\bigcap_{\epsilon>0} E_\epsilon = \Delta_X. \tag{61}$$

3.1.3

Exercise 29 *Let $c > 0$. Show that $\rho' := \rho \wedge c$,*

$$\rho'(p,q) \;:=\; \min(\rho(p,q), c) \qquad (p,q \in X), \tag{62}$$

is a semi-metric and

$$\mathcal{U}_{\rho'} \;=\; \mathcal{U}_\rho.$$

3.1.4

The uniformity $\mathcal{U}_\rho$ associated with ρ possesses a countable base: take for example

$$\left\{ E_{\frac{1}{n}} \mid n = 1, 2, \ldots \right\}.$$

In the next section we will show the reverse: *any uniformity with a countable base is the associated uniformity of some semi-metric on X.*

3.2 The semi-metric associated with a flag of entourages

3.2.1 A flag of entourages

Let us call a nested sequence of entourages

$$\mathscr{E}: \quad X \times X = E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots \tag{63}$$

a *flag of entourages*.

3.2.2 The associated semi-metric

Given a flag (63), define a function

$$f(p) := \begin{cases} \frac{1}{2^n} & \text{if } p \in E_n \setminus E_{n+1} \\ 0 & \text{if } p \in \bigcap_{i=0}^{\infty} E_i \end{cases}, \tag{64}$$

and then produce the corresponding semi-metric by enforcing the Triangle Inequality as described in the *Notes on Topology*:

$$\rho_{\mathscr{E}} = f^t, \quad \text{i.e.,} \quad \rho_{\mathscr{E}}(p,q) = \inf\left\{ \sum_{i=1}^{n} f(x_{i-1}, x_i) \mid x_0 = p, x_n = q \right\} \tag{65}$$

where the infimum is taken over all finite sequences $\{x_i\}_{i\in\{0,\ldots,n\}}$ of elements of X of any length which start at p and terminate at q.

3.2.3

One obviously has the inequality

$$\rho_{\mathscr{E}} \le f.$$

In particular,

$$E_n \subseteq \left\{ (p,q) \in X \times X \mid \rho_{\mathscr{E}}(p,q) < \frac{1}{2^n} \right\}$$

Lemma 3.1 *If the flag satisfies the following condition*

$$E_n \circ E_n \circ E_n \subseteq E_{n-1} \qquad (n = 1, 2, \ldots). \tag{66}$$

then

$$\frac{1}{2} f \le \rho_{\mathscr{E}}. \tag{67}$$

Proof. We shall prove by induction on n that

$$\frac{1}{2} f \le \sum_{i=1}^{l} f(p_{i-1}, p_i) \tag{68}$$

for any sequence

$$p_0 = p, \ldots, p_l = q. \tag{69}$$

There is nothing to prove for $l = 1$.

For a given sequence (69), denote by d the sum

$$\sum_{i=1}^{l} f(p_{i-1}, p_i).$$

If $d = 0$, then $f(p_{i-1}, p_i) = 0$ for each $i \in \{1, \ldots, l\}$ which means that $(p_i, p_{i+1}) \in E_n$ for any n. Hence,

$$(p, q) \in \underbrace{E_n \circ \cdots \circ E_n}_{l \text{ times}} \subseteq E_m$$

for any $m \leq n - \log_3 l$. In particular, $(p, q) \in \bigcap \mathscr{E}$, and thus $f(p, q) = 0$.

Suppose that $d > 0$. Denote by m be the largest index in $\{0, \ldots, l\}$ such that

$$\sum_{i=1}^{m} f(p_{i-1}, p_i) \leq \frac{1}{2} d.$$

Note that $m < l$, and also

$$f(p_m, p_{m+1}) > 0 \qquad \text{and} \qquad \sum_{i=m+2}^{l} f(p_{i-1}, p_i) < \frac{1}{2} d$$

Combined with inductive hypothesis we obtain

$$f(p, p_m) \leq 2 \cdot \frac{1}{2} d = d \qquad \text{and} \qquad f(p_{m+1}, q) \leq 2 \cdot \frac{1}{2} d = d$$

and, obviously, also

$$f(p_m, p_{m+1}) \leq d.$$

The above inequalities mean that if n is the largest integer such that

$$\frac{1}{2^n} \leq d,$$

then (p, p_m), $(p_m, p_{m+1}$, and (p_{m+1}, q) all belong to E_n. In particular,

$$(p, q) \in E_n \circ E_n \circ E_n \subseteq E_{n-1}$$

which means that

$$f(p, q) \leq \frac{1}{2} d.$$

□

Corollary 3.2 *If a flag $\mathscr{E}$ satisfies condition* (66)*, then the uniformity it generates, $\mathscr{E}_\sharp$, is associated with semi-metric $\rho_{\mathscr{E}}$.*

We arrive at the following important result.

Theorem 3.3 (Metrization Theorem) *A uniformity $\mathcal{U}$ is associated with some semi-metric if and only if it possesses a countable base.*

Proof. Existence of a countable base is obviously a necessary condition for $\mathscr{E}_\sharp$ to be associated with a semi-metric.

If $\mathcal{U}$ possesses a countable base, then it possesses a base $\mathscr{E}$ satisfying condition (66). Then,

$$E_n \subseteq \left\{(p,q) \in X{\times}X \mid \rho_{\mathscr{E}}(p,q) < \frac{1}{2^n}\right\} \subseteq E_{n-1}. \qquad (n = 1, 2, \dots).$$

□

3.3 The uniform structure associated with a family of semi-metrics

3.3.1

Let $\varrho = (\rho_i)_{i \in I}$ be a family of semi-metrics on a set X. We shall associate with it the uniformity generated by the family of uniformities $\mathcal{U}_{\rho_i}$, associated with semi-metrics ρ_i,

$$\mathcal{U}_\varrho := \sup_{\mathrm{Filt}(X\times X)}\{\mathcal{U}_{\rho_i} \mid i \in I\}. \tag{70}$$

3.3.2

The argument that was used to construct a semi-metric on a uniform space with a countable uniformity-base, can be used to show that *any* uniformity $\mathcal{U}$ on a set X is of the form (70) for a suitable family of semi-metrics $\varrho = (\rho_i)_{i \in I}$ on X.

3.3.3

If the indexing set I is countable, then the associated uniformity $\mathcal{U}_\varrho$ has a countable base. By the Metrization Theorem then there exists a single semi-metric ρ such that

$$\mathcal{U}_\varrho = \mathcal{U}_\rho. \tag{71}$$

One can show that (71) holds for, for example, the semi-metric

$$\rho := \sum_{n=0}^{\infty} \rho_n \wedge \frac{1}{2^n}$$

where I is identified with $\mathbb{N}$.

4 Uniform topology

With any uniformity $\mathcal{U}$ on X, we shall associate a topology on X. We shall do this, first, by defining the associated family of neighborhood filters, then we shall do this by defining an appropriate closure operation.

4.1 The associated family of neighborhood filters

4.1.1

Definition 4.1 *For any point $p \in X$, we set $\mathscr{N}_p$ to be the filter with the base formed by the sets of points E-close to p, where $E \in \mathcal{U}$,*

$$\mathscr{B}_p := \{Ep \mid E \in \mathcal{U}\}. \tag{72}$$

4.1.2

In the above definition, one can replace $\mathcal{U}$ by *any* uniformity base.

Definition 4.2 *We declare a subset $U \in X$ to be* ***open*** *if, for any $P \in U$, there exists $E \in \mathcal{U}$ such that $Ep \subseteq U$.*

Exercise 30 *Show that*

$$\mathscr{T}^{\mathcal{U}} := \{U \subseteq X \mid U \text{ is open}\} \tag{73}$$

satisfies the axioms of a topology.

4.1.3

The above topology will be referred to as the **uniform topology** and filters $\mathscr{N}_p$, cf. Definition 4.1, are the neighborhood filters of this topology.

4.2 The associated closure operation

4.2.1

Definition 4.3 *Define the closure operation on the $\mathscr{P}(X)$ by*

$$A \longmapsto \bar{A} := \bigcap_{E\in\mathfrak{U}} E\cdot A = \bigcap_{E\in\mathfrak{U}} A\cdot E. \tag{74}$$

4.2.2

Note that the intersection of all $E\cdot A$ coincides with the intersection of all $A\cdot E$ since for every $E\in\mathfrak{U}$ also $E^{\mathrm{op}}\in\mathfrak{U}$, and $A\cdot E^{\mathrm{op}} = E\cdot A$.

Exercise 31 *Show that the operation defined in* (74) *satisfies the axioms of the topological closure operation*

$$S \subseteq \bar{A} \tag{75}$$
$$\bar{\bar{A}} = \bar{A} \tag{76}$$
$$\overline{A\cup B} = \bar{A}\cup\bar{B} \tag{77}$$
$$\overline{\emptyset} = \emptyset \tag{78}$$

where A and B are arbitrary subsets of X.

Definition 4.4 *We declare a subset $Z\subseteq X$ to be **closed** if $Z=\overline{Z}$.*

Proposition 4.5 *A subset $U\subseteq X$ is open if and only if $X\setminus U$ is closed.*

Proof. Let A be a subset of X. Suppose that $p\notin\bar{A}$. Then $p\notin E\cdot A$ for some $E\in\mathfrak{U}$. Let $D\in\mathfrak{U}$ be such that $D\circ D\subseteq E$. Then $p\notin (D\circ D)\cdot A$ and thus $pD\cap D\cdot A=\emptyset$, cf. Exercise 8. It follows that

$$pD\cap\bar{A} = pD\cap\bigcap_{F\in\mathfrak{U}} F\cdot A \subseteq pD\cap(D\cdot A)=\emptyset,$$

i.e., $pD\in X\setminus\bar{A}$. Hence $X\setminus\bar{A}$ is open.

Let U be an open subset of X and $p\in U$. Then there exists $E\in\mathfrak{U}$ such that $pE\cap(X\setminus U)=\emptyset$. The latter is equivalent to

$$\{p\}\cap E\cdot(X\setminus U)=\emptyset,$$

cf. Exercise 5. Thus, $p \notin E \cdot (X \setminus U)$. In particular, $p \notin \overline{X \setminus U}$. It follows that

$$U \subseteq X \setminus \overline{(X \setminus U)}$$

or equivalently,

$$X \setminus U \supseteq \overline{(X \setminus U)}.$$

In view of $X \setminus U \subseteq \overline{(X \setminus U)}$, we infer that $X \setminus U = \overline{(X \setminus U)}$, i.e., $X \setminus U$ is closed. □

4.2.3 Separability properties of uniform spaces

Exercise 32 *Show that X is T_0 in the uniform topology if and only if*

$$\bigcap \mathcal{U} = \Delta_X.$$

Show that if X is T_0 in the uniform topology, then it is automatically T_2.

4.3 Uniform continuity

4.3.1

For a mapping $f\colon X \longrightarrow Y$, let $f \times f\colon X \times X \longrightarrow Y \times Y$ be the mapping

$$(p, q) \longmapsto (f(p), f(q)). \tag{79}$$

Definition 4.6 *We say that a mapping $f\colon X \longrightarrow Y$ between uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$, is* ***uniformly continuous*** *if $(f \times f)^{-1}(E) \in \mathcal{U}$ for any $E \in \mathcal{V}$. In othere words, if*

$$\mathcal{V} \subseteq (f \times f)_\sharp \mathcal{U}.$$

Exercise 33 *Show that $f\colon X \longrightarrow Y$ is uniformly continuous if and only if it satisfies the following condition*

$$\forall_{E \in \mathcal{V}} \, \exists_{D \in \mathcal{U}} \, \forall_{p,q \in X} \; \left(p \sim_D q \;\Rightarrow\; f(p) \sim_E f(q) \right). \tag{80}$$

4.3.2

A uniformly continuous mapping is continuous in respective uniform topologies. The reverse is generally false: continuous mappings are not necessarily uniformly continuous.

Exercise 34 *Prove that the homeomorphism* $f\colon (0,\infty)\longrightarrow(0,\infty)$*,*

$$f(x) = \frac{1}{x}$$

is continuous but not uniformly continuous. Here $(0,\infty)$ *is equipped with the usual* length *metric* $\rho(a,b) = |a-b|$*.*

4.3.3

In view of this, the following result is one of the crucial reasons why compactness plays such an important role in Mathematics, and especially in Mathematical Analysis.

Theorem 4.7 *If* X *is compact in the uniform topology, then any continuous mapping from* X *into a uniform topological space* Y *is uniformly continuous.*

Proof. Let $\mathcal{U}$ denote the uniformity of X and $\mathcal{V}$ denote the uniformity of Y. For a given $E \in \mathcal{V}$, let $E' \in \mathcal{V}$ be a symmetric entourage such that $E' \circ E' \subseteq E$.

If $f\colon X\longrightarrow Y$ is continuous, then for each $p \in X$, there exists an entourage $D'_p \in \mathcal{U}$ such that

$$f\left(D'_p p\right) \subseteq E'(f(p)). \tag{81}$$

Let $D_p \in \mathcal{U}$ be an entourage such that $D_p \circ D_p \subseteq D'_p$. Since each $p \in X$ belongs to $(D_p(p)^\circ)$, the interiors $\{(D_p(p)^\circ)\}_{p\in X}$ form an open cover of X. In view of compactness of X, one has

$$X = D_{p_1}p_1 \cup \cdots \cup D_{p_n}p_n \tag{82}$$

for certain points $p_1, \ldots, p_n \in X$.

Set $D \coloneqq D_{p_1} \cap \cdots \cap D_{p_n}$. The latter is an entourage of X.

Let p and q be arbitrary points of X. Suppose that $p \sim_D q$. In view of (82), one has $q \sim_{D_{p_i}} p_i$ for some p_i. It follows that $p \sim_{D\circ D_{p_i}} p_i$. Since $D_{p_i} \subseteq D'_{p_i}$ and $D \circ D_{p_i} \subseteq D_{p_i} \circ D_{p_i} \subseteq D'_{p_i}$ we obtain

$$p \sim_{D'_{p_i}} p_i \qquad \text{and} \qquad q \sim_{D'_{p_i}} p_i. \tag{83}$$

By combining (83) with (81) we obtain

$$f(p) \sim_{E'} f(p_i) \qquad \text{and} \qquad f(q) \sim_{E'} f(p_i).$$

and, since E' is symmetric, $f(p) \sim_{E'\circ E'} f(q)$. Recalling that $E' \circ E' \subseteq E$, we deduce that $f(p) \sim_E f(q)$. ◻

Corollary 4.8 *A compact topological space* $(X, \mathscr{T})$ *admits no more than one uniformity* $\mathcal{U}$ *such that* $\mathscr{T} = \mathscr{T}^{\mathcal{U}}$.

Proof. We will prove that, if $\mathcal{U}$ and $\mathcal{U}'$ are two uniformities on a set X such that $\mathscr{T}^{\mathcal{U}} = \mathscr{T} = \mathscr{T}^{\mathcal{U}'}$, and $(X, \mathscr{T})$ is compact, then $\mathcal{U} = \mathcal{U}'$. Indeed, the identity maps

$$(X, \mathscr{T}^{\mathcal{U}}) \longrightarrow (X, \mathscr{T}^{\mathcal{U}'}) \qquad \text{and} \qquad (X, \mathscr{T}^{\mathcal{U}'}) \longrightarrow (X, \mathscr{T}^{\mathcal{U}})$$

are continuous, hence uniformly continuous by Theorem 4.7, which means that $\mathcal{U} \succeq \mathcal{U}'$ and $\mathcal{U}' \succeq \mathcal{U}$. Since uniformities are filters, this is equivalent to $\mathcal{U} \supseteq \mathcal{U}'$ and $\mathcal{U}' \supseteq \mathcal{U}$, i.e., $\mathcal{U} = \mathcal{U}'$. $\square$

5 Uniformization

5.1 The neighborhood filter of the diagonal

5.1.1 Uniformizable topological spaces

We say that a topological space $(X, \mathscr{T})$ is *uniformizable* if $\mathscr{T} = \mathscr{T}^{\mathcal{U}}$ for some uniformity $\mathcal{U}$ on X.

5.1.2

Let X be a topological space. Below we shall examine when the neighborhood filter $\mathscr{N}_\Delta$ of the diagonal Δ_X in $X \times X$ is a uniformity. Obviously, $\mathscr{N}_\Delta$ satisfies Axiom $(\mathbf{U_1})$.

Exercise 35 *Show that* $\mathscr{N}_\Delta$ *satisfies Axiom* $(\mathbf{U_2})$.

5.1.3

Exercise 36 *Show that a family* $\mathscr{U} \subseteq \mathscr{P}(X)$ *is an open cover of* X *if and only if* $\square_{\mathscr{U}}$ *is an open neighborhood of the diagonal.*

5.1.4

For any point $p \in X$ and $E \in \mathscr{N}_\Delta$, the set of E-relatives of of p is a neighborhood of p. Indeed, since E is a neighborhood of Δ_X, there exists a pair of open neighborhoods U and V of p such that

$$U \times V \subseteq E.$$

It follows that

$$U = (U \times V)p \subseteq Ep.$$

Thus, filter-base

$$\{Ep \mid E \in \mathscr{N}_\Delta\}$$

generates a filter not finer than $\mathscr{N}_p$.

Proposition 5.1 *If a point $p \in X$ possesses a fundamental system of closed neighborhoods, then any open neighborhood of p is of the form Ep for some $E \in \mathscr{N}_\Delta$.*

In particular,

$$\{Ep \mid E \in \mathscr{N}_\Delta\} \;=\; \{pE \mid E \in \mathscr{N}_\Delta\} \tag{84}$$

is a fundamental system of neighborhoods of point p.

Proof. For an open neighborhood $U \in \mathscr{N}_p$, let $N \in \mathscr{N}_p$ be a closed neighborhood such that $N \subseteq U$. Then

$$\mathscr{U} \;:=\; \{U, N^c\}$$

is an open cover of X, and thus $\square_{\mathscr{U}}$ is an open neighborhood of the diagonal Δ_X. Since $p \in U$ and $p \notin N^c$, we have

$$\square_{\mathscr{U}}\, p \;=\; U,$$

cf. (40). □

5.2 Uniformizable spaces

5.2.1

A topological space $(X, \mathscr{T})$ is said to be **uniformizable**, if there exists a uniform structure $\mathcal{U}$ on X such that $\mathscr{T}$ is the associated uniform topology $\mathscr{T}^{\mathcal{U}}$, cf. (73).

Exercise 37 *Show that a uniformizable space is necessarily regular.*

5.2.2

In view of Proposition 5.1, A regular topological space is uniformizable if and only if the neighborhood filter of the diagonal $\mathscr{N}_\Delta$ satisfies Axiom $(\mathbf{U}_3)$.

Exercise 38 *Show that the family*

$$\mathscr{B} = \{\square_{\mathscr{U}} \mid \mathscr{U} \text{ is an open cover of } X\} \tag{85}$$

is a base of the neighborhood filter $\mathscr{N}_\Delta$.

5.2.3

The family

$$\mathscr{B}^{\circ 2} := \{\square_{\mathscr{U}} \circ \square_{\mathscr{V}} \mid \mathscr{U} \text{ and } \mathscr{V} \text{ are open covers of } X\} \tag{86}$$

is again a filter-base and, in view of

$$\square_{\mathscr{U}} = \square_{\mathscr{U}} \circ \Delta_X \subseteq \square_{\mathscr{U}} \circ \square_{\mathscr{V}},$$

it generates a subfilter of $\mathscr{N}_\Delta$. In particular, the neighborhood filter of the diagonal $\mathscr{N}_\Delta$ satisfies Axiom $(\mathbf{U}_3)$, and thus is a uniformity on X, precisely when (86) generates $\mathscr{N}_\Delta$.

5.2.4

We shall now show that in a regular topological space separable pairs of points (p, q) can be separated from the diagonal Δ_X using neighborhoods from $\mathscr{B}^{\circ 2}$.

Lemma 5.2 *Let p and q be a pair of points in a regular topological space X such that $\mathscr{N}_p \neq \mathscr{N}_q$. Then, there exists an open cover $\mathscr{U}$ of X and a neighborhood W of (p, q), such that*

$$W \cap (\square_{\mathscr{U}} \circ \square_{\mathscr{U}}) = \emptyset. \tag{87}$$

Proof. In a regular space any pair of points p and q with $\mathscr{N}_p \neq \mathscr{N}_q$ can be separated by a pair of open neighborhoods $U \in \mathscr{N}_p$ and $V \in \mathscr{N}_q$:

$$U \cap V = \emptyset.$$

In a regular space, closed neighborhoods of a point form a fundamental system of neighborhoods of that point. Hence, there exist closed neighborhoods $M \in \mathscr{N}_p$ and $N \in \mathscr{N}_q$ such that

$$M \subseteq U \qquad \text{and} \qquad N \subseteq V.$$

In particular,

$$\mathscr{U} := \{U, V, (M \cup N)^c\}$$

is an open cover of X. The final two steps of the proof we leave as exercises.

Exercise 39 *Show that*

$$\mathscr{U} \triangledown \mathscr{U} = \{M^c, N^c\}.$$

In particular,

$$\square_{\mathscr{U}} \circ \square_{\mathscr{U}} = \square_{\mathscr{U} \triangledown \mathscr{U}} = (M^c \times M^c) \cup (N^c \times N^c).$$

Exercise 40 *Describe*

$$X \times X \setminus \big((M^c \times M^c) \cup (N^c \times N^c)\big)$$

and derive from your description that

$$(M \times N) \cap \square_{\mathscr{U} \triangledown \mathscr{U}} = \emptyset.$$

Thus, set $W = M \times N$ is a desired neighborhood of (p, q) in $X \times X$.

□

Theorem 5.3 (Compact Uniformization Theorem) *The neighborhood filter, $\mathscr{N}_\Delta$, of the diagonal $\mathscr{D} \subset X \times X$, is a uniform structure on X if X is a compact regular space.*

Proof. If $\mathscr{N}_\Delta$ does not satisfy Axiom $(\mathbf{U_3})$, then there exists an open neighborhood $E \in \mathscr{N}_\Delta$ such that

$$(D \circ D) \setminus E \neq \emptyset \qquad (D \in \mathscr{N}_\Delta). \tag{88}$$

Thus, the family

$$\mathscr{A} := \{(D \circ D) \setminus E \mid D \in \mathscr{N}_\Delta\} \tag{89}$$

consists of non-empty subsets, and is directed below In other words, (89) is a filter-base on $X \times X \setminus E$.

The latter being a closed subset of the product of two compact spaces is compact by Tichonov's Theorem. It follows that $\mathscr{A}$ adheres to a certain point $(p,q) \in X \times X \setminus E$.

Note that p is not E-close to q, i.e., the neighborhood Eq of q does not contain p and, similarly, the neighborhood pE of p does not contain q.

By Lemmma 5.2, there exists an open neighborhood D of Δ such that $D \circ D$ is disjoint with a certain neighborhood of (p,q). It follows that (p,q) cannot be a cluster point of (89).

The contradiction proves that $\mathscr{N}_\Delta$ satisfies Axiom $(\mathbf{U}_3)$. □

6 Completeness

6.1 Cauchy filters

6.1.1 Convergent filters are Cauchy

Exercise 41 *Show that any filter $\mathscr{F}$ convergent in the uniform topology is Cauchy.*

6.1.2 Minimal Cauchy filters

Exercise 42 *Show that, for any filter-base $\mathscr{B}$ on X, the collection of sets*

$$\{E \cdot B \mid E \in \mathfrak{U} \text{ and } B \in \mathscr{B}\} \tag{90}$$

is a filter-base.

6.1.3

The filter generated by (90) will be denoted $\mathscr{B}^{\mathfrak{U}}$. Since *open* entourages form a base of the uniformity, the family of open subsets of X

$$\{E \cdot B \mid E \in \mathfrak{U} \text{ is an open subset of } X \times X \text{ and } B \in \mathscr{B}\}$$

forms a base of $\mathscr{B}^{\mathfrak{U}}$.

Exercise 43 *Show that $\mathscr{B}^{\mathfrak{U}}$ is Cauchy if $\mathscr{B}$ is Cauchy.*

6.1.4

For any $p \in X$, the family consisting of a single set $\{p\}$ is, of course, a Cauchy family, and $\mathscr{N}_p = \{\{p\}\}^{\mathcal{U}}$ by the definition of $\mathscr{N}_p$.

Exercise 44 *Show that $\mathscr{F}^{\mathcal{U}} \subseteq \mathscr{G}$ if $\mathscr{G} \subseteq \mathscr{F}$ and $\mathscr{G}$ is a Cauchy filter.*

It follows from Exercises 43 and 44 that $\mathscr{F}^{\mathcal{U}}$ is the smallest Cauchy subfilter contained in a given Cauchy filter $\mathscr{F}$. We shall say that $\mathscr{F}$ is a **minimal** Cauchy filter if $\mathscr{F} = \mathscr{F}^{\mathcal{U}}$.

6.1.5

In particular, the neighborhood filters $\mathscr{N}_p$ are minimal Cauchy filters.

Exercise 45 *Show that any adherence point p of a Cauchy filter $\mathscr{F}$ is its point of convergence.*

In particular, a Cauchy filter $\mathscr{F}$ converges if and only if $\mathrm{Adh}(\mathscr{F}) \neq \emptyset$.

6.1.6 The canonical map $\mathscr{N} : X \longrightarrow \mathbf{Filt}^{\mathbf{min}}_{\mathbf{Cauchy}}(X, \mathcal{U})$

Thus, the canonical correspondence

$$\mathscr{N} : p \longmapsto \mathscr{N}_p \qquad (p \in X) \tag{91}$$

defines a map $X \longrightarrow \mathrm{Filt}^{\mathrm{min}}_{\mathrm{Cauchy}}(X, \mathcal{U})$.

6.1.7

In Section 2.1.12 we equipped the set $\mathrm{FB}_{\mathrm{Cauchy}}(X, \mathcal{U})$ of all Cauchy filter-bases with a natural uniformity. We saw, cf. Exercise 27, that the neighborhood filter of any Cauchy filter-base $\mathscr{B}$ coincides with the neighborhood filter of the generated filter $\mathscr{B}_\sharp$, and thus also with the neighborhood filter of $(\mathscr{B}_\sharp)^{\mathcal{U}}$, the unique minimal Cauchy subfilter of $\mathscr{B}_\sharp$.

6.1.8

If we restrict the uniformity generated by $\mathcal{U}^{\wedge\vee}$ to the set $\mathrm{Filt}^{\mathrm{min}}_{\mathrm{Cauchy}}(X, \mathcal{U})$, the latter becomes a T_0- and therefore also a Hausdorff uniform space.

6.1.9

If we restrict further restrict $\mathcal{U}^{\wedge\vee}$ to the image of the $\mathscr{N}$-map,

$$\mathscr{N}(\mathrm{X}) \;=\; \{\mathscr{N}_p \mid p \in X\}, \tag{92}$$

we obtain a uniformity generated by the following filter-base

$$(\mathscr{N} \times \mathscr{N})_\sharp \mathcal{U}. \tag{93}$$

Indeed, if $\mathscr{N}_p \sim_{E^{\wedge\vee}} \mathscr{N}_q$, then there exists $A \in \mathscr{N}_p \cap \mathscr{N}_q$ which is E-small, which means that

$$(p,q) \in A \times A \subseteq E,$$

i.e., $p \sim_E q$.

Vice-versa, given $E \in \mathcal{U}$, let $D \in \mathcal{U}$ be a symmetric entourage satisfying $D \circ D \circ D \subseteq E$. If $p \sim_D q$, then

$$Dp \times qD \subseteq D \circ D \circ D \subseteq E$$

and

$$qD \times Dp \subseteq (D \circ D \circ D)^{\mathrm{op}} = D^{\mathrm{op}} \circ D^{\mathrm{op}} \circ D^{\mathrm{op}} = D \circ D \circ D \subseteq E.$$

Also,

$$Dp \times Dp = Dp \times pD \subseteq D \circ \Delta_X \circ D \subseteq D \circ D \circ D \subseteq E$$

and

$$qD \times qD = Dq \times qD \subseteq D \circ \Delta_X \circ D \subseteq D \circ D \circ D \subseteq E.$$

In all four cases we made use of Exercise 7. Let $A := Dp \cup qD$. It follows that

$$A \times A \subseteq D \circ D \circ D \subseteq E,$$

i.e., A is E-small. Additionally, A belongs to $\mathscr{N}_p$ since $A \supseteq Dp$, and it belongs to $\mathscr{N}_q$ since $A \supseteq qD$. □

6.1.10

We not only proved that

$$\mathscr{N} : (X, \mathcal{U}) \longrightarrow \mathrm{Filt}^{\min}_{\mathrm{Cauchy}}(X, \mathcal{U})$$

is uniformly continuous: we also proved that any uniformly continuous mapping from $(X, \mathcal{U})$ to any *separable* uniform space uniquely factorizes through the subspace $\mathscr{N}(X)$ of the uniform space $\mathrm{Filt}^{\min}_{\mathrm{Cauchy}}(X, \mathcal{U})$ which consists of the neighborhood filters of the topology associated with $\mathcal{U}$.

6.2 Complete uniform spaces

6.2.1

We say that a uniform space is *complete* if every Cauchy filter converges.

6.2.2 Completion of a uniform space

Since the canonical map (91) is uniformly continuous, it sends Cauchy filters to Cauchy filters. If $\mathscr{N}_\sharp(\mathscr{F})$ denotes the direct image of a Cauchy filter $\mathscr{F}$, then $\mathscr{N}_\sharp(\mathscr{F})$ converges in $\mathrm{Filt}^{\mathrm{min}}_{\mathrm{Cauchy}}(X,\mathcal{U})$ to the "point" corresponding to the unique minimal ultrafilter $\mathscr{F}^{\mathcal{U}}$.

Exercise 46 *Show that $\mathscr{N}_\sharp(\mathscr{F})$ converges to $\mathscr{F}^{\mathcal{U}}$.*

6.2.3 The universal property of the map $\mathscr{N}$

In particular, the closure $X^\sim$ of the image of (91) in $\mathrm{Filt}^{\mathrm{min}}_{\mathrm{Cauchy}}(X,\mathcal{U})$ is a *complete* uniform space which, as is easy to see, possesses the following universal property:

any uniformly continuous mapping $f\colon (X,\mathcal{U})\longrightarrow(Y,\mathcal{V})$ into a Hausdorff complete uniform space $(Y,\mathcal{V})$ factorizes $f=\tilde{f}\circ\mathscr{N}$ for a unique uniformly continuous map

$$\tilde{f}\colon (X^\sim,(\mathcal{U}^{\wedge\vee})_\sharp)\longrightarrow(Y,\mathcal{V}) \tag{94}$$

.

6.2.4 $X^\sim = \mathbf{Filt}^{\mathbf{min}}_{\mathbf{Cauchy}}(X,\mathcal{U})$

Let $\mathscr{F}$ be any minimal Cauchy filter on X. It possesses a base consisting of open subsets. In particular, for any entourage $E\in\mathcal{U}$, there exists an E-small open set $F_E\in\mathscr{F}$. That set belongs to $\mathscr{N}_p$ for any $p\in F_E$. In particular, such $\mathscr{N}_p$ is $E^{\wedge\vee}$-close to $\mathscr{F}$. This proves that in any neighborhood of $\mathscr{F}$ in $\mathrm{Filt}^{\mathrm{min}}_{\mathrm{Cauchy}}(X,\mathcal{U})$ there are filters of the form $\mathscr{N}_p$ for some $p\in X$. In other words, the image of X in $\mathrm{Filt}^{\mathrm{min}}_{\mathrm{Cauchy}}(X,\mathcal{U})$ is *dense*, i.e., $X^\sim=\mathrm{Filt}^{\mathrm{min}}_{\mathrm{Cauchy}}(X,\mathcal{U})$.