Notes on Ordered Sets

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1 Vocabulary

1.1 Definitions

Definition 1.1 *A binary relation $\preceq$ on a set S is said to be a **partial order** if it is **reflexive**,*

$$x \preceq x,$$

***weakly antisymmetric**,*

$$\text{if } x \preceq y \text{ and } y \preceq x \text{, then } x = y,$$

*and **transitive**,*

$$\text{if } x \preceq y \text{ and } y \preceq z \text{, then } x \preceq z$$

Above x, y, z, are arbitrary elements of S.

Definition 1.2 *Let $E \subseteq S$. An element $y \in S$ is said to be an **upper bound** for E if*

$$x \preceq y \qquad \text{for any } x \in E. \tag{1}$$

By definition, any element of S is declared to be an upper bound for $\emptyset$, the empty subset.

We shall denote by $U(E)$ the set of all upper bounds for E

$$U(E) := \{y \in S \mid x \preceq y \text{ for any } x \in E\}.$$

Note that $U(\emptyset) = S$.

Definition 1.3 *We say that a subset* $E \subseteq S$ *is* ***bounded (from) above****, if* $U(E) \neq \emptyset$*, i.e., when there exists at least one element* $y \in S$ *satisfying* (1).

Definition 1.4 *If* $y, y' \in U(E) \cap E$*, then*

$$y \preceq y' \quad \textit{and} \quad y' \preceq y.$$

Thus, $y = y'$*, and that unique upper bound of* E *which belongs to* E *will be denoted* $\max E$ *and called the* ***largest element*** *of* E*.*

It follows that $U(E) \cap E$ is empty when E has no largest element, and consists of a single element, namely $\max E$, when it does.

If we replace $\preceq$ by $\succeq$ everywhere above, we shall obtain the definitions of a **lower bound** for set E, of the set of all lower bounds,

$$L(E) := \{y \in S \mid x \succeq y \text{ for any } x \in E\},$$

and, respectively, of the **smallest element** of E. The latter will be denoted $\min E$.

Exercise 1 *Show that, for any subsets* E *and* F*, one has*

$$L(E) \supseteq F \qquad \textit{if and only if} \qquad E \subseteq U(F).$$

1.1.1 The Principle of Duality

Note that the relation defined by

$$x \preceq^{\text{rev}} y \quad \text{if} \quad x \succeq y$$

is also an order relation on S. We will refer to it as the **reverse ordering**.

Any general statement about partially ordered sets has the corresponding *dual* statement that is obtained by replacing $(S, \preceq)$ with $(S, \preceq^{\text{rev}})$. Under this "duality" *upper* bounds become *lower* bounds, maxima becoma minima, suprema become infima, and vice-versa.

Therefore most general theorems about partially ordered sets possess the corresponding *dual* theorems. Below, I will be usually formulating those dual statements for the convenience of reference.

Exercise 2 *Show that if* $E \subseteq S$ *is bounded below and nonempty, then* $L(E)$ *is bounded above and nonempty.*

Dually, if E is bounded above and nonempty, then $U(E)$ is bounded below and nonempty.

1.1.2

If $E \subseteq F \subseteq S$, then
$$\max F \in U(E) \tag{2}$$
when $\max F$ exists, and, dually,
$$\min F \in L(E)$$
when $\min F$ exists.

If both $\max E$ and $\max F$ exist, then
$$\max E \preceq \max F.$$

Dually, if both $\min E$ and $\min F$ exist, then
$$\min F \preceq \min E.$$

Exercise 3 (Sandwich Lemma for maxima) *Show that if $E'' \subseteq E \subseteq E'$ and both $\max E'$ and $\max E''$ exist and are equal, then $\max E$ exists and*
$$\max E'' = \max E = \max E'.$$

Dually, if both $\min E'$ and $\min E''$ exist and are equal, then $\min E$ exists and
$$\min E' = \min E = \min E''.$$

1.1.3 Supremum and infimum

Definition 1.5 *When $\min U(E)$ exists it is called the **least upper bound** of E, or the **supremum** of E, and is denoted $\sup E$.*

*Dually, when $\max L(E)$ exists it is called the **largest lower bound** of E, or the **infimum** of E, and is denoted $\inf E$.*

For the supremum of E to exist, subset E must be bounded above. The supremum of E may exist for some bounded above subsets of S and may not exist for others.

1.1.4 Example

Let us consider $S = \mathbb{Q}$, the set of rational numbers, with the usual order. Both the following subset $E_1 \subseteq \mathbb{Q}$,

$$E_1 := \{x \in \mathbb{Q} \mid x^2 < 1\}$$

and the subset $E_2 \subseteq \mathbb{Q}$,

$$E_2 := \{x \in \mathbb{Q} \mid x^2 < 2\},$$

are simultaneously bounded above and below. None of them has either the largest nor the smallest element but

$$\sup E_1 = 1 \qquad \text{and} \qquad \inf E_1 = -1$$

while neither $\sup E_2$ nor $\inf E_2$ exist in $S = \mathbb{Q}$.

Exercise 4 *Show that*

$$\sup \emptyset = \min S \qquad \textit{and} \qquad \inf \emptyset = \max S.$$

In particular, $\sup \emptyset$ exists if and only if S has the smallest element; similarly, $\inf \emptyset$ exists if and only if S has the largest element.

1.1.5 Down-intervals and up-intervals

Let $(S, \preceq)$ be a partially ordered set. For each $s \in S$, we define the *down-interval*

$$\langle s] := \{t \in S \mid t \preceq s\}.$$

and the *up-interval*

$$[s\rangle := \{t \in S \mid s \preceq t\}.$$

Exercise 5 *Show that, for* $E \subseteq S$, *one has*

$$L(E) = \langle s] \qquad \textit{for some } s \in S$$

if and only if $\inf E$ *exists. In this case,*

$$L(E) = \langle \inf E].$$

Dually,

$$U(E) = [s\rangle, \qquad \text{for some } s \in S,$$

if and only if $\sup E$ exists. In this case,

$$U(E) = [\sup E\rangle.$$

1.1.6 Example

Consider the set of natural numbers,

$$\mathbb{N} := \{0, 1, 2, \dots\},$$

equipped with the ordering given by

$$m \preceq n \qquad \text{if} \qquad m \mid n$$

("*m divides n*").

Exercise 6 *Does $(\mathbb{N}, |\,)$ have the maximum? the minimum? If yes, then what are they?*

Exercise 7 *For a given $n \in \mathbb{N}$, describe intervals $\langle n]$ and $[n\rangle$ in $(\mathbb{N}, |\,)$.*

Exercise 8 *For given $m, n \in \mathbb{N}$, is set $\{m, n\}$ bounded below? Does it possess infimum? If yes, then describe* $\inf\{m, n\}$.

Exercise 9 *For given $m, n \in \mathbb{N}$, is set $\{m, n\}$ bounded above? Does it possess supremum? If yes, then describe* $\sup\{m, n\}$.

Exercise 10 *Does every subset of $\mathbb{N}$ possess infimum in $(\mathbb{N}, |\,)$? Does every subset of $\mathbb{N}$ possess supremum?*

1.1.7 Totally ordered sets

Definition 1.6 *We say that a partially ordered set $(S, \preceq)$ is **totally**, or **linearly**, ordered if any two elements s and t of S are* comparable

$$\textit{either} \quad s \preceq t \quad \textit{or} \quad t \preceq s.$$

Totally ordered subsets in any given partially ordered set are called **chains**.

Exercise 11 *Let* $(S, \preceq)$ *be a totally ordered set and* $E, F \subseteq S$ *be two subsets. Show that*

$$\text{either } L(E) \subseteq L(F) \text{ or } L(F) \subseteq L(E).$$

1.2 Observations

1.2.1

For any subset $E \subseteq S$, one has

$$E \subseteq LU(E) := L(U(E)) \tag{3}$$

and

$$E \subseteq UL(E) := U(L(E)). \tag{4}$$

1.2.2

If $E \subseteq F \subseteq S$, then

$$U(E) \supseteq U(F) \tag{5}$$

and

$$L(E) \supseteq L(F). \tag{6}$$

Exercise 12 *Show that if* $E \subseteq F$ *and both* $\sup E$ *and* $\sup F$ *exist, then*

$$\sup E \preceq \sup F.$$

Dually, if both $\inf E$ and $\inf F$ exist, then

$$\inf F \preceq \inf E. \tag{7}$$

Exercise 13 (Sandwich Lemma for infima) *Show that if* $E'' \subseteq E \subseteq E'$ *and both* $\inf E'$ *and* $\inf E''$ *exist and are equal, then* $\inf E$ *exists and*

$$\inf E'' = \inf E = \inf E'.$$

Dually, if both $\sup E'$ and $\sup E''$ exist and are equal, then $\sup E$ exists and

$$\sup E' = \sup E = \sup E''.$$

1.2.3

By applying (5) to the pair of subsets in (3), one obtains

$$U(E) \supseteq ULU(E) := U(L(U(E)))$$

while (4) applied to subset $U(E)$ yields

$$U(E) \subseteq ULU(E).$$

It follows that

$$U(E) = ULU(E). \tag{8}$$

Dually,

$$L(E) = LUL(E). \tag{9}$$

Note that equality (9) is nothing but equality (8) for the *reverse* ordering.

1.2.4

For any subsets $E \subseteq S$ and $F \subseteq S$, one has

$$U(E \cup F) = U(E) \cap U(F)$$

and

$$L(E \cup F) = L(E) \cap L(F).$$

1.2.5

For any $E \subseteq S$, $\max E$ *exists if and only if* $\sup E$ *exists and belongs to* E, *and they are equal*

$$\sup E = \max E. \tag{10}$$

Dually, $\min E$ *exists if and only if* $\inf E$ *exists and belongs to* E, *and they are equal*

$$\inf E = \min E.$$

Indeed, if $\max E$ exists, then it is the least element of $U(E)$, cf. Definition (1.4).

If $\sup E$ exists and is a member of E, then it belongs to $U(E) \cap E$ which as we established, cf. Definition (1.4), consists of the single element $\max E$ when $U(E) \cap E$ is nonempty.

The case of $\min E$ and $\inf E$ follows if we apply the already proven to $(S, \preceq^{\text{rev}})$.

1.2.6

For any $E \subseteq S$, $\inf U(E)$ *exists if and only if* $\sup E$ *exists, and they are equal*

$$\max LU(E) = \inf U(E) = \min U(E) = \sup E. \tag{11}$$

Indeed,

$$\inf U(E) := \max LU(E) \in U(E),$$

in view of $E \subseteq LU(E)$, cf., (3), combined with (2) where $F = LU(E)$. Thus,

$$\inf U(E) = \min U(E)$$

by (10).

Dually, $\sup L(E)$ *exists if and only if* $\inf E$ *exists, and they are equal*

$$\min UL(E) = \sup L(E) = \max L(E) = \inf E.$$

1.2.7 Example: the power set as a partially ordered set

Let $S = \mathscr{P}(A)$ be the *power set* of a set A:

$$\mathscr{P}(A) := \text{the set of all subsets of } A.$$

Containment $\subseteq$ is a partial order relation on $\mathscr{P}(A)$.

Subsets $\mathscr{E}$ of $\mathscr{P}(A)$ are the same as *families* of subsets of A. Since $S = \mathscr{P}(A)$ contains the largest element, namely A, and the smallest element, namely $\emptyset$, every subset of $\mathscr{P}(A)$ is bounded above and below.

The union of all members of a family $\mathscr{E}$,

$$\bigcup_{E \in \mathscr{E}} E := \{a \in A \mid a \in E \text{ for some } E \in \mathscr{E}\} \tag{12}$$

is the smallest subset of A which *contains every member* of family $\mathscr{E}$. Hence, $\sup \mathscr{E}$ exists and equals (12).

Dually, the intersection of all members of family $\mathscr{E}$,

$$\bigcap_{E \in \mathscr{E}} E := \{a \in A \mid a \in E \text{ for all } E \in \mathscr{E}\} \tag{13}$$

is the largest subset of A which is *contained in every member* of family $\mathscr{E}$. Hence, $\inf \mathscr{E}$ exists and equals (13).

The power set provides an example of a partially ordered set in which every subset (including the empty set) possesses both suppremum and infimum.

1.3 Completeness

1.3.1

Definition 1.7 *We say that a partially ordered set $(S, \preceq)$ has the **largest-lower-bound property** if* $\inf E$ *exists for every subset $E \subseteq S$ which is nonempty and bounded below.*

*Dually, we say that S has the **least-upper-bound property** if* $\sup E$ *exists for subset $E \subseteq S$ which is nonempty and bounded above.*

1.3.2 An alternative terminology

Partially ordered sets with the largest-lower-bound property are said to be inf**-complete**, and those with the least-upper-bound property are said to be sup**-complete**.

Lemma 1.8 *A partially ordered set S has the largest-lower-bound property if and only if it has the least-upper-bound property.*

Proof. Suppose that S is inf-complete. If $E \subseteq S$ is bounded above and nonempty, then the set of upper bounds, $U(E)$ is nonempty. Since

$$L(U(E)) \supseteq E \neq \emptyset$$

subset $U(E)$ is also bounded below. Then $\inf U(E)$ exists in view of our assumption about $(S, \preceq)$. But then it coincides with $\sup E$ in accordance with (11). This shows that S is sup-complete.

The reverse implication,

$$\text{sup-completeness} \Rightarrow \text{inf-completeness}$$

follows by applying the already proven implication

$$\text{inf-completeness} \Rightarrow \text{sup-completeness}$$

to the reverse order on S. □

Since sup- and inf-completeness are equivalent we shall simply call such sets *complete*.

1.3.3 Lattices

Definition 1.9 *A partially ordered set $(S, \preceq)$ is called a **pre-lattice** if every nonempty finite subset $E \subseteq S$ has supremum and infimum.*

Exercise 14 *Show that $(S, \preceq)$ is a **pre-lattice** if and only if , for any $s, t \in S$, both $\sup\{s, t\}$ and $\inf\{s, t\}$ exist.*

Definition 1.10 *A partially ordered set $(S, \preceq)$ is called a **lattice** if every finite subset $E \subseteq S$, including $\emptyset \subseteq S$, has supremum and infimum.*

1.3.4 Complete lattices

Complete partially ordered sets with the largest and the smallest elements are the same as *complete lattices*. Note that in such sets every subset is bounded below and above.

For example, the totally ordered set of rational numbers, $(\mathbb{Q}, \leq)$, is a pre-lattice but not a lattice, and it is not complete.

The power set of an arbitrary set, $(\mathscr{P}(A), \subseteq)$, is an example of a complete lattice. A less obvious example is the subject of the next section.

1.4 Down-sets, up-sets

1.4.1 Down-sets

A subset L of a partially ordered set $(S, \preceq)$ is called a **down-set** if

$$\textit{for any } s \in L \textit{ and } s' \preceq s, \textit{ also } s' \in L.$$

Exercise 15 *Show that the union and the intersection of any family* $\mathscr{L} \subseteq \mathscr{P}(A)$ *of down-sets is a down-set.*

Exercise 16 *Show that a down-set* L *is a down-interval if and only if* $\sup L$ *exists and belongs to* L.

1.4.2 Down-set closure of a subset

The family of down-sets containing a given subset $E \subseteq S$ is nonempty since $E \subseteq S$ and S is a down-set. It follows that the intersection of all down-sets L containing E,

$$\mathrm{Cl}^{\downarrow}(E) := \bigcap_{L \supseteq E} L,$$

is the *smallest* down-set containing E.

1.4.3 Down-set interior of a subset

The family of down-sets contained in a given subset $E \subseteq S$ is nonempty since $\emptyset \subseteq E$ and $\emptyset$ is a down-set. It follows that the union of all down-sets L contained in E,

$$\mathrm{Int}^{\downarrow}(E) := \bigcup_{L \subseteq E} L,$$

is the *largest* down-set contained in E.

1.4.4

By definition, one has

$$\mathrm{Int}^{\downarrow}(E) \subseteq E \subseteq \mathrm{Cl}^{\downarrow}(E)$$

Exercise 17 *Show that* $E \subseteq S$ *is a down-set if and only if*

$$\mathrm{Int}^{\downarrow}(E) = E$$

if and only if

$$E = \mathrm{Cl}^{\downarrow}(E).$$

Exercise 18 *Any subset* $E \subseteq S$ *is contained in* $\bigcup_{s \in E} \langle s]$. *Show that* E *is a down-set if and only if*

$$E = \bigcup_{s \in E} \langle s].$$

1.4.5

Let E and F be two subsets of a partially ordered set $(S, \preceq)$. We may say that F *dominates* E *above*, and express this symbolically with $E \dashv F$, if

$$\textit{for every } s \in E \textit{ there exists } t \in F \textit{ such that } s \preceq t.$$

Exercise 19 *Show that* $U(E) \supseteq U(F)$ *whenever* $E \dashv F$*. In particular*

$$\sup E \preceq \sup F$$

when both suprema exist.

Exercise 20 *Show that* $E \dashv F$ *if and only if* $\mathrm{Cl}^{\downarrow}(E) \subseteq \mathrm{Cl}^{\downarrow}(F)$*.*

1.4.6 Cofinal pairs of subsets

We shall say that subsets E and F are **cofinal** if $E \dashv F$ and $F \dashv E$. For cofinal subsets $U(E) = U(F)$. In particular, $\sup E$ exists if and only if $\sup F$ exists and they are equal (cf. Ex. 19.

1.4.7

It follows that E and F are *cofinal* if and only if $\mathrm{Cl}^{\downarrow}(E) = \mathrm{Cl}^{\downarrow}(F)$.

1.4.8 Up-sets

Up-sets are defined, by duality, as down-sets for the reverse ordering $\preceq^{\mathrm{rev}}$. In particular, a subset E is an up-set if and only if

$$E = \bigcup_{s \in E} [s\rangle.$$

One can also define the corresponding notions of the *up-closure*, $\mathrm{Cl}^{\uparrow}(E)$, and the *up-interior*, $\mathrm{Int}^{\uparrow}(E)$, of a subset E.

Exercise 21 *Define the dual concept*

$$F \textit{ dominates } E \textit{ below}$$

(*one can denote this fact by using notation* $F \vDash E$).

1.4.9

If we replace $\preceq$ by the reverse order, $\preceq^{\mathrm{rev}}$, we obtain another pair of relations between subsets of S:

$$E \mathrel{-\!\!3}^{\mathrm{rev}} F \qquad \text{and} \qquad F \mathrel{\varepsilon\!\!-}^{\mathrm{rev}} E.$$

Note that

$$F \mathrel{\varepsilon\!\!-} E \quad \text{if and only if} \quad E \mathrel{-\!\!3}^{\mathrm{rev}} F \quad (\text{not } E \mathrel{-\!\!3} F!).$$

1.4.10

The dual concept to a cofinal pair of subsets is a **coinitial** pair of subsets.

1.5 Partially ordered subsets

1.5.1

In $S \subseteq T$ is a subset of a partially ordered set $(T, \preceq)$, then it can be regarded as a partially ordered set in its own right. One has to be cautioned, however, that S with the induced order may have vastly different properties.

1.5.2

For a subset $E \subseteq S$, the sets of upper and lower bounds will generally depend on whether one considers E as a subset of S or T. In particular, E may be not bounded as a subset of S yet be bounded as a subset of T.

When necessary, we shall indicate this by subscript S or T. Thus,

$$L_T(E), \quad U_T(E), \quad \inf_T E, \quad \sup_T E,$$

will denote the set of lower bounds, the set of upper bounds, the infimum, and the supremum, when E is viewed as a subset of T.

Exercise 22 *Show that, for $E \subseteq S$, one has*

$$\sup_T E \preceq \sup_S E$$

whenever both suprema exist.

1.5.3

Dually, one has

$$\inf_S E \preceq \inf_T E$$

whenever both infima exist.

Exercise 23 *For $E \subseteq S$, suppose that $\sup_T E$ exists and belongs to S. Show that $\sup_S E$ exists and*

$$\sup_S E = \sup_T E.$$

1.5.4

Dually, if $\inf_T E$ exists and belongs to S, then $\inf_S E$ exists and

$$\inf_S E = \inf_T E. \tag{14}$$

Exercise 24 *Let $E \subseteq S$, and suppose that $(S, \preceq)$ is a partially ordered subset of $(T, \preceq)$. Show that, if $\inf_T E$ exists, then*

$$L(E) = \langle \inf_T E] \cap S = \{s \in S \mid s \preceq \inf_T E\}. \tag{15}$$

(Here $L(E) = L_S(E)$.)

1.5.5

Dually, one has

$$U(E) = [\sup_T E\rangle \cap S = \{s \in S \mid \sup_T E \preceq s\}$$

if $\sup_T E$ exists.

Exercise 25 *Find examples of pairs $E \subseteq S$ of subsets of $\mathbb{Q}$ such that:*

(a) E is unbounded above in S yet bounded in $\mathbb{Q}$;

(b) E is bounded in S, and $\sup_{\mathbb{Q}} E$ exists but $\sup_S E$ does not;

(c) E is bounded in S, and $\sup_S E$ exists but $\sup_{\mathbb{Q}} E$ does not;

(d) both $\sup_S E$ and $\sup_{\mathbb{Q}} E$ exist but $\sup_S E \neq \sup_{\mathbb{Q}} E$.

1.5.6 Sublattices

1.5.7

A subset S of a lattice $(T, \preceq)$ forms a *sublattice* if

$$\inf_S E = \inf_T E \qquad \text{and} \qquad \sup_S E = \sup_T E$$

for any *finite* $E \subseteq S$.

1.5.8

A subset S of a *complete* lattice $(T, \preceq)$ forms a *sublattice* if

$$\inf_S E = \inf_T E \qquad \text{and} \qquad \sup_S E = \sup_T E$$

for *any* $E \subseteq S$.

1.5.9

If $A \subseteq B$, then $(\mathscr{P}(A), \subseteq)$ forms naturally a sublattice of $(\mathscr{P}(B), \subseteq)$.

1.5.10

According to Exercise 15, the family of all down-sets $(\mathscr{P}^{\downarrow}(S), \subseteq)$ in a partially ordered set $(S, \preceq)$ forms a complete sublattice of $(\mathscr{P}(S), \subseteq)$. Similarly, the family of all up-sets $(\mathscr{P}^{\uparrow}(S), \subseteq)$ forms a complete sublattice of $(\mathscr{P}(S), \subseteq)$.

1.6 Density

1.6.1

Definition 1.11 *We say that a subset $S \subseteq T$ is* sup***-dense*** *if every element $t \in T$ equals*

$$t = \sup E$$

for some subset $E \subseteq S$.

By replacing sup *with* inf*, one obtains the definition of a* inf***-dense*** *subset.*

1.6.2 Example

Suppose that a partially ordered set $(S, \preceq)$ is the union of three subsets $S = X \cup Y \cup Z$ such that

$$x \preceq y \quad \text{and} \quad x \preceq z \qquad \text{for any } x \in X,\ y \in Y, \text{ and } z \in Z,$$

and no $y \in Y$ and $z \in Z$ are comparable. Additionally, let us assume that neither Y nor Z possess the minimal elements.

Let us extend the ordering relation to $T = S \cup \{v, \zeta\}$ by setting

$$x \prec v \prec y \qquad \text{for any } x \in X,\ y \in Y,$$

and

$$x \prec \zeta \prec z \qquad \text{for any } x \in X,\ z \in Z,$$

and making v and ζ not comparable. Finally, denote by T' the subset $S \cup \{v\}$ of T.

a) $(S, \preceq)$ **as a subset of** $(T, \preceq)$**.** One has $L_T(Y) = \langle v]$ and $L_T(Z) = \langle \zeta]$. It follows that

$$v = \inf_T Y \qquad \text{and} \qquad \zeta = \inf_T Z,$$

and therefore S is inf-dense in T. In addition,

$$\langle v] \cap S = X = \langle \zeta] \cap S$$

but $v \neq \zeta$, they are not even comparable.

Exercise 26 *Show that neither v nor ζ equals* $\sup_T E$ *for any $E \subseteq S$. In particular, S is* inf*-dense in T but not* sup*-dense.*

b) $(S, \preceq)$ **as a subset of** $(T', \preceq)$**.** One has

$$v = \inf_{T'} Y$$

hence S is inf-dense in T'. In addition,

$$L(Y) = \langle v] \cap S = X = L(Z)$$

and $v = \inf_{T'} Y$ while $\inf_{T'} Z$ does not exist.

Exercise 27 *Let* $(S, \preceq)$ *be a partially ordered subset of* $(T, \preceq)$. *Show that if*

$$t = \inf_T E$$

for some $E \subseteq S$, *then*

$$t = \inf_T([t\rangle \cap S).$$

Dually, if

$$t = \sup_T F \tag{16}$$

for some $F \subseteq S$, then

$$t = \sup_T(\langle t] \cap S).$$

Exercise 28 (A criterion of equality for infima) *Let* $(S, \preceq)$ *be a partially ordered subset of* $(T, \preceq)$ *and* $E \subseteq S$. *Suppose that both* $\inf E$ *and* $t = \inf_T E$ *exist. Show that if there exists a subset* $F \subseteq S$ *such that* (16) *holds, then*

$$\inf_T E = \inf E.$$

Exercise 29 *Formulate the dual criterion of equality for suprema.*

2 Mappings between partially ordered sets

2.1 Morphisms

2.1.1

Definition 2.1 *Given two partially ordered sets* $(S, \preceq)$ *and* $(S', \preceq')$, *a mapping* $f\colon S \longrightarrow S'$ *which preserves order,*

$$\text{if } s \preceq t, \text{ then } f(s) \preceq' f(t) \qquad (s, t \in S),$$

is said to be a ***morphism*** $(S, \preceq) \longrightarrow (S', \preceq')$.

2.1.2

Recall that the **image** of a subset $E \subseteq S$ under f is the set

$$f(E) := \{s' \in S' \mid s' = f(a) \text{ for some } a \in E\},$$

and the **preimage** of a subset $E' \subseteq S'$ under f is the set

$$f^{-1}(E') := \{s \in S \mid f(s) \in E'\}.$$

Exercise 30 *Show that*

$$f \text{ is a morphism } (S, \preceq) \longrightarrow (S', \preceq')$$

if and only if

$$\text{the preimage, } f^{-1}(L'), \text{ of any down-set } L' \subseteq S' \text{ is a down-set in } S$$

if and only if

$$\text{the preimage, } f^{-1}(U'), \text{ of any up-set } U' \subseteq S' \text{ is an up-set in } S.$$

Exercise 31 *Let f be a morphism $(S, \preceq) \longrightarrow (S', \preceq')$. Show that, for any subset $E \subseteq S$, one has*

$$\min f(E) = f(\min E) \qquad \text{and} \qquad \max f(E) = f(\max E)$$

whenever $\min E$ or $\max E$ exists.

Exercise 32 *Let f be a morphism $(S, \preceq) \longrightarrow (S', \preceq')$. Show that, for any nonempty subset $E \subseteq S$, one has*

$$U(f(E)) \supseteq f(U(E)) \qquad \text{and} \qquad L(f(E)) \supseteq f(L(E)).$$

Deduce from this the inequalities

$$f(\inf E) \preceq' \inf f(E) \preceq' \sup f(E) \preceq' f(\sup E)$$

whenever the corresponding infima and suprema exist.

2.1.3

Definition 2.2 *We say that a morphism $f\colon (S, \preceq) \longrightarrow (S', \preceq')$ is* $\sup$***-continuous*** *if f preserves the suprema. More precisely, if it has the following property*

$$\text{for any } E \subseteq S, \text{ if } \sup E \text{ exists, then } \sup f(E) \text{ exists, and}$$

$$\sup f(E) = f(\sup E).$$

Exercise 33 *State the dual definition of an* $\inf$***-continuous*** *morphism.*

2.2 Order embeddings

2.2.1

Definition 2.3 *A mapping* $\iota\colon S \longrightarrow S'$ *is said to be an* ***order embedding****,*

$$(S, \preceq) \hookrightarrow (S', \preceq'), \tag{17}$$

if it satisfies a stronger condition

$$s \preceq t \quad \textit{if and only if} \quad \iota(s) \preceq' \iota(t) \qquad (s, t \in S).$$

Exercise 34 *Show that an order embedding is injective.*

Definition 2.4 *A morphism* $f\colon (S, \preceq) \longrightarrow (S', \preceq')$ *is said to be an* ***isomorphism*** *if it has an inverse, i.e., if there is a morphism* $g\colon (S', \preceq') \longrightarrow (S, \preceq)$ *such that* $f \circ g = \mathrm{id}_{S'}$ *and* $g \circ f = \mathrm{id}_S$*.*

Exercise 35 *Show that an order embedding,* (17)*, is an isomorphism onto its image,* $(\iota(S), \preceq')$*.*

Exercise 36 *For any partially ordered* $(S, \preceq)$*, let* S^* *denote the subset obtained by removing* $\max S$ *and* $\min S$ *if they exist. Show that, for any nonempty bounded subset* E *of* S^**,*

$$\sup_{S^*} E = \sup_S E \qquad \textit{and} \qquad \inf_{S^*} E = \inf_S E.$$

Exercise 37 (A criterion of continuity of an embedding) *Let* $(S, \preceq)$ *be a* sup*-dense subset of* $(T, \preceq)$*. Show that the inclusion*

$$(S, \preceq) \hookrightarrow (T, \preceq) \tag{18}$$

is inf*-continuous.*

Dually, if $(S, \preceq)$ is a inf-dense subset of $(T, \preceq)$, then inclusion (18) is sup-continuous.

2.2.2 Example: the canonical embedding $(S, \preceq) \hookrightarrow (\mathscr{P}(S), \subseteq)$

For any pair of elements s and t in a partially ordered set $(S, \preceq)$, one has

$$s \preceq t \qquad \text{if and only if} \qquad \langle s] \subseteq \langle t].$$

Thus, the correspondence

$$\langle]: S \longrightarrow \mathscr{P}(S), \qquad s \longmapsto \langle s], \tag{19}$$

is an order embedding of $(S, \preceq)$ into $(\mathscr{P}(S), \subseteq)$. Embedding (19) identifies $(S, \preceq)$ with the set of down-intervals $(\mathscr{I}^{\downarrow}(S), \subseteq)$.

Exercise 38 *Show that, for any $E \subseteq S$, one has*

$$\sup\langle E] = \bigcup_{s \in E} \langle s] = \mathrm{Cl}^{\downarrow}(E)$$

and

$$\inf\langle E] = L(E) \tag{20}$$

in $(\mathscr{P}(S), \subseteq)$.[1]

It follows that

$$\sup\langle E] = \langle \sup E]$$

in $(\mathscr{P}(S), \subseteq)$ if and only if $\max E$ exists. In particular, (19) is nearly never sup-continuous. Indeed, $\sup\langle E]$ always exists, it coincides with the down-closure of E, and the latter is a *proper* subset of $\langle \sup E]$ if when E has no maximal element.

On the other hand, the canonical embedding is $\inf$-continuous.

Exercise 39 *Show that the canonical embedding,* (19)*, is* $\inf$*-continuous.*

Exercise 40 *Explain why $\mathscr{I}^{\downarrow}(S), \subseteq) = \langle S]$ is, generally, neither* $\sup$- *nor* $\inf$-*dense in $\mathscr{P}(S)$.*

[1] Here $\langle E]$ denotes the *image* of a subset $E \subseteq S$ under the canonical embedding, (19), *not* the interval, $\langle E] = \{X \in \mathscr{P}(S) \mid X \subseteq E\}$, in partially ordered set $(\mathscr{P}(S), \subseteq)$. The latter coincides with $\mathscr{P}(E)$, the set of all subsets of E.

2.3 Galois connections

2.3.1 Residuated mappings

If f is a morphism, then the preimage, $f^{-1}(\langle s'])$, of any interval $\langle s'] \subseteq S'$ is a down-set in $(S, \preceq)$ but not necessarily a down-interval. Mappings $f\colon S \longrightarrow S'$ which have the property that the preimage of any down-interval $\langle s']$ in $(S', \preceq')$ is a down-interval in $(S, \preceq)$ are said to be **residuated**.

Exercise 41 *Show that a residuated mapping is a morphism.*

Exercise 42 *Show that $f_1 \circ f_2$ is residuated if both f_1 and f_2 are residuated.*

2.3.2 Residual mappings

Dually, we say that a mapping f is **residual**, if the preimage, $f^{-1}([s'\rangle)$, of any up-interval $[s'\rangle \subseteq S'$ is an up-interval in $(S, \preceq)$.

2.3.3

The following proposition gives a simple but very useful characterization of residuated mappings.

Proposition 2.5 *A mapping $f\colon S \longrightarrow S'$ is residuated if and only if it is a morphism with the property that the preimage of any down-interval has supremum in $(S, \preceq)$ and,*

$$f\left(\sup f^{-1}(\langle s'])\right) \preceq' s' \qquad (s' \in S'). \tag{21}$$

We shall prove each implication separately.

Necessity. If f is residuated, then $f^{-1}(\langle s']) = \langle t]$, for some $t \in S$, and therefore $\sup f^{-1}(\langle s'])$, which coincides with t, exists.

Noting that $\langle t] \subseteq f^{-1}(\langle s'])$ is equivalent to $f(\langle t]) \subseteq \langle s']$, we infer that $f(t) \preceq' s'$. This is inequality (21).

Sufficiency. Denote $\sup f^{-1}(\langle s'])$ by t and note that

$$f^{-1}(\langle s']) \subseteq \langle t]$$

is the obvious containment $E \subseteq \langle \sup E]$ for $E = \sup f^{-1}(\langle s'])$.

The preimage $f^{-1}(\langle s'])$ is a down-set since f is assumed to be a morphism, while inequality (21) means that $t \in f^{-1}(\langle s'])$. Hence

$$f^{-1}(\langle s']) \supseteq \langle t].$$

$\square$

Exercise 43 *Formulate the dual of Proposition 2.5.*

2.3.4 The residual of a residuated mapping

A residuated mapping $f\colon S \longrightarrow S'$ defines a mapping $g\colon S' \longrightarrow S$ by setting

$$g(s') := t \qquad \text{where} \qquad f^{-1}\left(\langle s']\right) = \langle t].$$

Exercise 44 *Show that*

$$f(s) \preceq' s' \quad \textit{if and only if} \quad s \preceq g(s') \qquad (s \in S,\ s' \in S') \tag{22}$$

and deduce that g *is a residual mapping.*

Exercise 45 *Show that, for a given mapping* $g\colon S' \longrightarrow S$*, there exists a residuated mapping* $f\colon S \longrightarrow S'$ *such that* g *is the residual of* f *if and only if* $g^{-1}\left([s\rangle\right)$ *is an up-interval in* $(S', \preceq)$ *for any* $s \in S$*. Show that in this case*

$$g^{-1}\left([s\rangle\right) = [f(s)\rangle \qquad (s \in S).$$

2.3.5 Galois connections

2.3.6

A pair of mappings

$$f\colon S \longrightarrow S' \qquad \text{and} \qquad g\colon S' \longrightarrow S$$

satisfying condition (22) is called a **Galois connection** between partially ordered sets $(S, \preceq)$ and $(S', \preceq')$.

2.3.7 Terminology

If (f, g) forms a Galois connection, f is referred to as the *lower* (or, *left*) *adjoint* of g, and g is called the *upper* (or, *right*) *adjoint* of f.

Exercise 46 *Let* (f, g) *be a Galois connection between* $(S, \preceq)$ *and* $(S', \preceq')$*, and* (h, k) *be a Galois connection between* $(S', \preceq')$ *and* $(S'', \preceq'')$*. Show that* $(h \circ f, g \circ k)$ *is a Galois connection between* $(S, \preceq)$ *and* $(S'', \preceq'')$*.*

2.3.8 Duality between residuated and residual mappings

A pair (f, g) is a Galois connection between $(S, \preceq)$ and $(S', \preceq')$ if and only if (g, f) is a Galois connection between $(S', (\preceq')^{\text{rev}})$ and $(S, \preceq^{\text{rev}})$. Reversing simultaneously the orderings on S and S' exchanges the roles of the residual and the residuated mapping.

Exercise 47 *Show that*

$$\mathrm{id}_S \preceq g \circ f \qquad \textit{and} \qquad f \circ g \preceq' \mathrm{id}_{S'}$$

Exercise 48 *Show that if f is an isomorphism between $(S, \preceq)$ and $(S', \preceq')$, then (f, f^{-1}) is a Galois connection between these sets.*

Exercise 49 *Show that if (f, g) is a Galois connection between $(S, \preceq)$ and $(S', \preceq')$, and (g, f) is a Galois connection between $(S', \preceq')$ and $(S, \preceq)$, then $g = f^{-1}$. In particular, f and g are isomorphisms of partially ordered sets.*

Exercise 50 *Show that if (f, g) is a Galois connection, then mapping f is residuated and g is the residual of f.*

2.3.9

By combining Exercises 44 and 50, we obtain the following proposition.

Proposition 2.6 *A mapping $f\colon S \longrightarrow S'$ is residuated if and only if there exists $g\colon S' \longrightarrow S$ such that (f, g) is a Galois connection. In particular, there is a natural correspondence*

$$\left\{ \begin{matrix} \textit{residuated mappings} \\ f\colon (S, \preceq) \longrightarrow (S', \preceq') \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{Galois connections } (f, g) \\ \textit{between } (S, \preceq) \textit{ and } (S', \preceq') \end{matrix} \right\}.$$

2.3.10

Dually, there is a natural correspondence

$$\left\{ \begin{matrix} \textit{Galois connections } (f, g) \\ \textit{between } (S, \preceq) \textit{ and } (S', \preceq') \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{residual mappings} \\ g\colon (S', \preceq') \longrightarrow (S, \preceq) \end{matrix} \right\}.$$

Proposition 2.7 *Let (f, g) be a Galois connection between partially ordered sets $(S, \preceq)$ and $(S', \preceq')$. Then, for any $E' \subseteq S'$,*

$$\text{sets } \; g(L(E')) \; \text{ and } \; L(g(E')) \text{ are cofinal} \tag{23}$$

and, for any $E \subseteq S$,

$$\text{sets } \; f(U(E)) \text{ and } \; U(f(E)) \text{ are coinitial.} \tag{24}$$

Proof. Let $s \in L(g(E'))$. Since g is a morphism from $(S', \preceq')$ to $(S, \preceq)$, one one has $f(s) \in L(E')$. Noting that $g(f(s)) \in L(g(E'))$ and combining this observation with inequality $s \preceq g(f(s))$ shows that the set $g(L(E'))$ dominates the set $L(g(E'))$ from above. Since the latter set contains the former, the two sets are cofinal.

Statement (24) is statement (23) for the reverse ordering. □

Corollary 2.8 *Let (f, g) be a Galois connection between partially ordered sets $(S, \preceq)$ and $(S', \preceq')$. Then, f is* sup*-continuous and g is* inf*-continuous.*

Equivalently, any residuated mapping is sup*-continuous and any residual mapping is* inf*-continuous.* □

2.3.11

Corollary 2.8 provides necessary conditions for a given mapping to be the left or the right component of a Galois connection. When $(S, \preceq)$ is a complete lattice, those conditions are also sufficient as we will now demonstrate.

Corollary 2.9 *Suppose that $(S, \preceq)$ is a complete lattice, cf. 1.3.4, i.e., every subset of S has supremum.*[2] *Then a mapping f from S to S' is residuated if and only if it is a* sup*-continuous morphism $(S, \preceq) \longrightarrow (S', \preceq')$.*

Proof. In view of Corollary 2.8 it suffices to show that a sup-continuous morphism is residuated. We will demonstrate that f satisfies the equivalent condition of Proposition 2.5.

For any $s' \in S'$, supremum $\sup f^{-1}(\langle s']) $ of course exists and

$$f\left(\sup f^{-1}(\langle s'])\right) = \sup f\left(f^{-1}(\langle s'])\right),$$

by sup-continuity of f, and

$$\sup f\left(f^{-1}(\langle s'])\right) \preceq' \sup \langle s'] = s',$$

since $f\left(f^{-1}(\langle s'])\right) \subseteq \langle s']$. □

[2]Note that in that case every subset has also infimum.

2.3.12

Expressed succinctly,

$$\left\{\begin{matrix}\text{residuated}\\ \text{mappings}\end{matrix}\right\} = \left\{\begin{matrix}\text{sup-continuous}\\ \text{morphisms}\end{matrix}\right\}$$

and, by duality,

$$\left\{\begin{matrix}\text{residual}\\ \text{mappings}\end{matrix}\right\} = \left\{\begin{matrix}\text{inf-continuous}\\ \text{morphisms}\end{matrix}\right\}$$

whenever their domain is a complete lattice.

2.4 Galois connections between power sets

2.4.1 sup-continuous morphisms $(\mathscr{P}(X), \subseteq) \longrightarrow (\mathscr{P}(Y), \subseteq)$

Every subset $A \subseteq X$ can be represented as the union of the family of single-element subsets $\mathscr{A} := \{\{x\} \mid x \in A\}$,

$$A = \bigcup_{x \in A} \{a\} = \sup_{\mathscr{P}(X)} \mathscr{A}.$$

Note that $\mathscr{A}$ is a subset of $\mathscr{P}(X)$.

If F is a sup-continuous morphism from $(\mathscr{P}(X), \subseteq)$ to $(\mathscr{P}(Y), \subseteq)$, then

$$F(A) = F(\sup \mathscr{A}) = \sup_{\mathscr{P}(Y)} \{F(\{x\}) \mid x \in A\} = \bigcup_{x \in A} F(\{x\}). \qquad (25)$$

This shows that F is uniquely determined by its restriction to single-element subsets of X.

Vice-versa, one can use identity (25) to extend an arbitrary mapping $\varphi\colon X \longrightarrow \mathscr{P}(Y)$ to a mapping $F\colon \mathscr{P}(X) \longrightarrow \mathscr{P}(Y)$.

Exercise 51 *For an arbitrary mapping $\varphi\colon X \longrightarrow \mathscr{P}(Y)$, let $F\colon \mathscr{P}(X) \longrightarrow \mathscr{P}(Y)$ be defined by the formula*

$$F(A) := \bigcup_{x \in A} \varphi(x).$$

Show that F is a morphism from $(\mathscr{P}(X), \subseteq)$ to $(\mathscr{P}(Y), \subseteq)$ and is sup*-continuous.*

2.4.2

We obtain the following natural correspondence

$$\left\{\begin{matrix}\text{sup-continuous morphisms}\\ (\mathscr{P}(X),\subseteq)\longrightarrow(\mathscr{P}(Y),\subseteq)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{mappings}\\ X\longrightarrow\mathscr{P}(Y)\end{matrix}\right\}. \qquad (26)$$

2.4.3

Any mapping $\varphi\colon X\longrightarrow\mathscr{P}(Y)$ defines a subset of $X\times Y$,

$$\Gamma_\varphi := \bigcup_{x\in X}\big(\{x\}\times\varphi(x)\big).$$

We shall refer to Γ_φ as the **graph** of φ.

Vice-versa, any subset Γ of $X\times Y$ is of this form for a unique $\varphi\colon X\longrightarrow\mathscr{P}(Y)$, namely

$$\varphi(x) := \{y\in Y \mid y\in\varphi(x)\}.$$

Recall that subsets of $X\times Y$ correspond to binary relations between elements of X and elements of Y,

$$\left\{\begin{matrix}\text{binary relations between}\\ \text{elements of } X \text{ and elements of } Y\end{matrix}\right\} \longleftrightarrow \mathscr{P}(X\times Y),$$

via the identification

$$\text{a relation } \sim \quad\longleftrightarrow\quad \{(x,y)\in X\times Y \mid x\sim y\}.$$

In terms of the corresponding relation the associated mapping $F\colon\mathscr{P}(X)\longrightarrow\mathscr{P}(Y)$ can be expressed as

$$F(A) = \{y\in Y \mid \text{there exists } x\in A \text{ such that } x\sim y\}. \qquad (27)$$

Let us denote this mapping by $F_\sim$. The correspondence

$$\text{a relation } \sim \quad\longleftrightarrow\quad F_\sim$$

thus provides a natural correspondence between binary relations and sup-continuous mappings of the power sets involved

$$\left\{\begin{matrix}\text{binary relations between}\\ \text{elements of } X \text{ and elements of } Y\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{sup-continuous morphisms}\\ (\mathscr{P}(X),\subseteq)\longrightarrow(\mathscr{P}(Y),\subseteq)\end{matrix}\right\}.$$

2.4.4 inf**-continuous morphisms** $(\mathscr{P}(Y), \subseteq) \longrightarrow (\mathscr{P}(X), \subseteq)$

Every subset $E' \subseteq Y$ can be obtained from Y by removing all elements $y \notin B$. In other words, B is the intersection of the family $\mathscr{B}' := \{Y \setminus \{y\} \mid y \notin B\}$ of *complements* of single-element subsets formed by elements of Y *not belonging* to B,

$$B = \bigcap_{y \notin B} (Y \setminus \{y\}) = \bigcap_{y \notin B} \{y\}^c = \inf_{\mathscr{P}(X)} \mathscr{B}'.$$

If G is an inf-continuous morphism from $(\mathscr{P}(Y), \subseteq)$ to $(\mathscr{P}(X), \subseteq)$, then

$$G(B) = G(\inf \mathscr{B}') = \inf_{\mathscr{P}(X)} \{G(\{y\}^c) \mid y \notin B\} = \bigcap_{y \notin B} G(\{y\}^c). \qquad (28)$$

Like before, an inf-continuous mapping G is uniquely determined by its restriction to the *complements* of single-element subsets of X. And vice-versa, one can use identity (28) to extend an arbitrary mapping $\psi \colon Y \longrightarrow \mathscr{P}(X)$ to a mapping $G \colon \mathscr{P}(Y) \longrightarrow \mathscr{P}(X)$ where set Y is identified with a subset of $\mathscr{P}(Y)$ via the order-reversing embedding

$$y \longmapsto \{y\}^c = Y \setminus \{y\}.$$

Exercise 52 *For an arbitrary mapping* $\psi \colon Y \longrightarrow \mathscr{P}(X)$*, let* $G \colon \mathscr{P}(Y) \longrightarrow \mathscr{P}(X)$ *be defined by the formula*

$$G(B) := \bigcap_{y \notin B} \psi(y).$$

Show that G *is a morphism from* $(\mathscr{P}(Y), \subseteq)$ *to* $(\mathscr{P}(X), \subseteq)$ *and is* inf*-continuous.*

2.4.5

We obtain the following natural correspondence

$$\left\{ \begin{matrix} \text{inf-continuous morphisms} \\ (\mathscr{P}(Y), \subseteq) \longrightarrow (\mathscr{P}(X), \subseteq) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{mappings} \\ Y \longrightarrow \mathscr{P}(X) \end{matrix} \right\}.$$

The $G \leftrightarrow \psi$ correspondence is *dual* to the $F \leftrightarrow \varphi$ correspondence, cf. (26). In the latter X is identified with the subset of all single-element subsets of X while in the former Y is identified with the set of *complements* of single-element subsets of Y.

2.4.6 The relation associated with an $\inf$-continuous mapping

Given an inf-continuous mapping G from $(\mathscr{P}(Y), \subseteq)$ to $(\mathscr{P}(X), \subseteq)$, define the binary relation between elements of X and Y by

$$x \sim y \qquad \text{if} \qquad x \notin \psi(y) \qquad (x \in X,\ y \in Y), \tag{29}$$

where $\psi(y) = G(\{y\}^c)$. In terms of relation (29), mapping $G\colon \mathscr{P}(Y) \longrightarrow \mathscr{P}(X)$ can be expressed as

$$G(B) = \{x \in X \mid \text{for every } y \notin B \text{ one has } x \nsim y\} \tag{30}$$

where $\nsim$ denotes the *negation* of relation $\sim$,

$$x \nsim y \qquad \text{if and only if} \qquad \neg(x \sim y).$$

Let us denote this maping by $G^{\sim}$. The correspondence

$$\text{a relation } \sim \quad \longleftrightarrow \quad G^{\sim}$$

thus provides a natural correspondence between binary relations and inf-continuous mappings of the power sets involved

$$\left\{\begin{matrix}\text{binary relations between} \\ \text{elements of } X \text{ and elements of } Y\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{inf-continuous morphisms} \\ (\mathscr{P}(Y), \subseteq) \longrightarrow (\mathscr{P}(X), \subseteq)\end{matrix}\right\}.$$

2.4.7 Galois connections

According to Corollary 2.9, residuated mappings between power sets are precisely the sup-continuous mappings and residual mappings are precisely inf-continuous mappings. We described the sup- and inf-continuous mappings above, and we will use these descriptions to describe Galois connections between power sets.

Exercise 53 *Let $\sim$ be a binary relation between elements of a set X and elements of a set Y. Show that (F, G) is a Galois connection between $(\mathscr{P}(X), \subseteq)$ and $(\mathscr{P}(Y), \subseteq)$ where F is defined by equality* (27) *and G is defined by equality* (30).

In view of the already established fact that every residuated mapping F between $(\mathscr{P}(X), \subseteq)$ and $(\mathscr{P}(Y), \subseteq)$ is of the form (27) for a unique binary relation $\sim$, we obtain the following

Proposition 2.10 *There is a natural correspondence*

$$\left\{\begin{matrix}\text{Galois connections } (F,G) \text{ between}\\ (\mathscr{P}(X),\subseteq) \text{ and } (\mathscr{P}(Y),\subseteq)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{binary relations } \sim \text{ between}\\ \text{elements of } X \text{ and } Y\end{matrix}\right\}$$

where the pair of mappings, (F,G)*,*

$$A \overset{F}{\longmapsto} \{y \in Y \mid \exists_{x\in A}\, x \sim y\} \tag{31}$$

$$\{x \in X \mid \forall_{y\notin B}\, x \nsim y\} \overset{G}{\longleftarrow\!\!\!\shortmid} B$$

corresponds to a relation $\sim$.[3]

2.4.8 The Galois connection associated with a mapping

Exercise 54 *Let* $f\colon X \longrightarrow Y$ *be an arbitrary mapping between sets* X *and* Y*. Show that the pair of mappings*

$$F\colon \mathscr{P}(X)\longrightarrow\mathscr{P}(Y), \qquad A \longmapsto f(A) \qquad (A \subseteq X),$$

and

$$G\colon \mathscr{P}(Y)\longrightarrow\mathscr{P}(X), \qquad B \longmapsto f^{-1}(B) \qquad (E' \subseteq Y),$$

forms a Galois connection between $(\mathscr{P}(X),\subseteq)$ *and* $(\mathscr{P}(Y),\subseteq)$*. It corresponds to the* graph-of-a-function *relation*

$$x \sim y \qquad \text{if and only if} \qquad y = f(x) \qquad (x \in X,\ y \in Y).$$

2.4.9 The (*Conjunction, Implication*) Galois connection

Exercise 55 *Given a subset* B *of an arbitrary set* X*, let*

$$F\colon A \longmapsto A \cap B \qquad \text{and} \qquad G\colon C \longmapsto B{\Rightarrow}C \qquad (A, C \subseteq X)$$

where

$$B{\Rightarrow}C := B^c \cup C = (X \setminus B) \cup C.$$

Show that (F,G) *is a Galois connection on* $(\mathscr{P}(X),\subseteq)$*. It corresponds to the relation*

$$x \sim y \qquad \text{if and only if} \qquad x = y \in B \qquad (x \in X,\ y \in Y).$$

[3] To make notation more compact we used symbolic notation: $\exists$ ("there exists") and $\forall$ ("for all").

2.5 Order reversing Galois connections between power sets

2.5.1

A Galois connection (Φ, Ψ) between $(\mathscr{P}(X), \subseteq)$ and $(\mathscr{P}(Y), \supseteq)$ is a pair of mappings $\Phi\colon \mathscr{P}(X) \longrightarrow \mathscr{P}(Y)$ and $\Psi\colon \mathscr{P}(Y) \longrightarrow \mathscr{P}(X)$ such that

$$\Phi(A) \subseteq B \qquad \text{if and only if} \qquad \Psi(B) \subseteq A \qquad (A \subseteq X,\ E' \subseteq Y)$$

(note the symmetry between Φ and Ψ).

Since *passing-to-the-complement* induces an isomorphism between partially ordered sets $(\mathscr{P}(Y), \subseteq)$ and $(\mathscr{P}(Y), \supseteq)$, whose inverse is given by the same operation, by taking into account that composition of Galois connections is a Galois connection, cf. Exercise 46, we obtain from (Φ, Ψ) the Galois connection (F, G),

$$F := (\ )^c \circ \Phi \qquad \text{and} \qquad G := \Psi \circ (\ )^c,$$

between $(\mathscr{P}(X), \subseteq)$ and $(\mathscr{P}(Y), \subseteq)$.

All such connections were described in Proposition 2.10. Noting that

$$\Phi = (\ )^c \circ F \qquad \text{and} \qquad \Psi = G \circ (\ )^c,$$

we arrive at the complete description of arbitrary order reversing Galois connections between power sets.

Corollary 2.11 *There is a natural correspondence*

$$\left\{ \begin{matrix} \textit{Galois connections } (\Phi, \Psi) \textit{ between} \\ (\mathscr{P}(X), \subseteq) \textit{ and } (\mathscr{P}(Y), \supseteq) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{binary relations } \sim \textit{ between} \\ \textit{elements of } X \textit{ and } Y \end{matrix} \right\} \tag{32}$$

where the pair of mappings, (Φ, Ψ),

$$A \xmapsto{\ \Phi\ } \{y \in Y \mid \forall_{x \in A}\ x \sim y\} \tag{33}$$

$$\{x \in X \mid \forall_{y \in B}\ x \sim y\} \xleftarrow{\ \Psi\ } B$$

corresponds to a relation $\sim$.

Proof. If $F = (\ )^c \circ \Phi$ and $G = \Psi \circ (\ )^c$ are given by (31), then

$$\Phi(A) = \{y \in Y \mid \forall_{x \in A}\ x \nsim y\} \qquad \text{and} \qquad \Psi(B) = \{x \in X \mid \forall_{y \in B}\ x \nsim y\}$$

and every Galois connection between between $(\mathscr{P}(X), \subseteq)$ and $(\mathscr{P}(Y), \supseteq)$ is of this form for a unique binary relation $\sim$. By composing this correspondence between Galois connections and relations with the involution on the set of relations

$$\sim \quad \longleftrightarrow \quad \nsim,$$

we obtain correspondence (32)–(33). $\square$

2.5.2 The canonical Galois connection associated with a partial order

Note that, for any pair of subsets A and B of a partially ordered set $(S, \preceq)$, one obviously has

$$E \subseteq U(B) \qquad \text{if and only if} \qquad L(A) \supseteq B.$$

This means that (U, L) forms a Galois connection between $(\mathscr{P}(S), \subseteq)$ and $(\mathscr{P}(S), \supseteq)$. We shall refer to it as the *canonical* Galois connection of $(S, \preceq)$. Fittingly, Galois connection (U, L) corresponds, under (32)–(33), to relation $\preceq$.

2.6 Closure and interior operations

2.6.1 The floor mapping

Let $(S, \preceq)$ be a partially ordered set and E be a subset of S. It follows from Proposition 2.5 that the canonical inclusion $\iota_E \colon E \hookrightarrow S$ is a residuated mapping $(E, \preceq) \longrightarrow (S, \preceq)$ if and only if , for every $s \in S$,

$$\langle s] \cap E \quad \text{is a down-interval in} \quad (E, \preceq),$$

i.e., $\langle s] \cap E$ has a maximum element. In this case we shall say that E is a *floor* subset of $(S, \preceq)$, and the right adjoint

$$s \longmapsto \lfloor s \rfloor_E := \max \langle s] \cap E,$$

will be called the *floor mapping* associated with subset E of $(S, \preceq)$.

2.6.2 The ceiling mapping

Similarly, the inclusion map, $E \hookrightarrow S$, is residual if and only if

$$[s\rangle \cap E \quad \text{is an up-interval in} \quad (E, \preceq),$$

i.e., $[s\rangle \cap E$ has a minimum element. In this case we shall say that E is a *ceiling* subset of $(S, \preceq)$, and the left adjoint

$$s \longmapsto \lceil s \rceil^E := \min [s\rangle \cap E,$$

will be called the *ceiling mapping* associated with subset E of $(S, \preceq)$.

2.6.3 Closure operations on a partially ordered set

Definition 2.12 *Let $(S, \preceq)$ be a partially ordered set. A self-mapping*

$$s \longmapsto \bar{s}, \qquad S \longrightarrow S,$$

*is said to be a **closure operation** if it enjoys the following three properties*

$$s \preceq \bar{s}, \tag{34}$$

$$\textit{if } \ s \preceq t\,, \ \textit{then } \ \bar{s} \preceq \bar{t}, \tag{35}$$

and

$$\bar{\bar{s}} = \bar{s} \tag{36}$$

where s and t denote arbitrary elements of S.

*In this case, we say that an element $s \in S$ is **closed** if $\bar{s} = s$.*

2.6.4

Denote by $Z \subseteq S$ the set of closed elements. If $z \in [s\rangle \cap Z$, then $s \preceq z$, and hence

$$\bar{s} \preceq \bar{z} = z,$$

in view of equality (36). But $\bar{s} \in [s\rangle$, in view of inequality (34), and $\bar{s} \in Z$, in view of equality (36) again. Thus

$$\bar{s} = \min([s\rangle \cap Z)$$

which means that Z is a ceiling subset of $(S, \preceq)$ and the closure operation is the associated ceiling mapping

$$\bar{s} = \lceil s \rceil^Z \qquad (s \in S).$$

Exercise 56 *Let Z be a ceiling subset of a partially ordered set $(S, \preceq)$. Show that*

$$s \longmapsto \lceil s \rceil^Z \qquad (s \in S)$$

satisfies conditions (34)–(36), *i.e. it defines a closure operation on $(S, \preceq)$.*

We obtain

Proposition 2.13 *There is a natural correspondence between closure operations on a partially ordered set $(S, \preceq)$ and ceiling subsets of $(S, \preceq)$,*

$$\left\{ \begin{matrix} \textit{ceiling subsets} \\ Z \subseteq S \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{closure operations} \\ \textit{on } (S, \preceq) \end{matrix} \right\},$$

where

$$Z \qquad \longleftrightarrow \qquad \lceil\ \rceil^Z .$$

2.6.5

Let $Z' \subseteq Z$ be any subset of the set of closed elements, and suppose that $\zeta = \inf Z'$ exists. Then, for any $z' \in Z'$, the inequality $\zeta \preceq z'$ implies that

$$\bar{\zeta} \preceq \bar{z}' = z',$$

i.e., $\bar{\zeta}$ is a lower bound of set Z'. In particular, $\bar{\zeta} \preceq \zeta$. By combining this with inequality (34), we obtain $\bar{\zeta} = \zeta$, i.e., $\inf Z'$ is a closed element of S. Note that this argument applies also for $Z' = \emptyset$ since in this case $\inf Z'$ is the greatest element of $(S, \preceq)$ and the greatest element is closed.

Exercise 57 *Show that the greatest element of $(S, \preceq)$ is closed when it exists.*

We established the following interesting fact

Proposition 2.14 *For any subset $Z' \subseteq Z$ of the set of closed elements of a closure operation, if $\inf Z'$ exists in $(S, \preceq)$, then it belongs to Z.* □

2.6.6 Interior operations on a partially ordered set

Closure operations for the reversed order are sometimes called **interior operations**. In this case, notation $s \longmapsto \mathring{s}$ or s° is used and elements $s \in S$ such that $\mathring{s} = s$ are said to be **open**.

Exercise 58 *State the dual version of Proposition 2.14 for an interior operation and the associated set T of open elements.*

2.6.7 Example: closure and interior operations associated with a Galois connection

Let (f, g) be a Galois connection between partially ordered sets $(S, \preceq)$ and $(S', \preceq')$.

Exercise 59 *Show that*

$$f \circ g \circ f = f \qquad \text{and} \qquad g \circ f \circ g = g.$$

Exercise 60 *Show that*

$$s \longmapsto (g \circ f)(s) \qquad (s \in S)$$

is a closure operation on $(S, \preceq)$ *and*

$$s' \longmapsto (f \circ g)(s') \qquad (s' \in S')$$

is an interior operation on $(S', \preceq')$.

2.6.8

It follows that a Galois connection (f, g) between partially ordered sets $(S, \preceq)$ and $(S', \preceq')$ induces a closure operation on $(S, \preceq)$ and an interior operation on $(S', \preceq')$. Denote by $Z \subseteq S$ and by $T' \subseteq S'$ the corresponding sets of closed and, respectively, open elements.

Exercise 61 *Show that an element* $s \in S$ *is closed if and only if* s *is in the image of* g*, i.e.,* $s = g(s')$ *for some* $s' \in S'$*. Likewise, show that an element* $s' \in S'$ *is open if and only if* s' *is in the image of* f*, i.e.,* $s' = f(s)$ *for some* $s \in S$.

Exercise 62 *Show that the restriction of* f *to* Z *and the restriction of* g *to* T *are mutually inverse. In particular, pair* (f, g) *induces an isomorphism of partially ordered sets* $(Z, \preceq)$ *and* $(T', \preceq')$.

2.6.9

Note that if we denote the induced isomorphism from $(Z, \preceq)$ to $(T', \preceq')$ by ϕ, then

$$f = \phi \circ \lceil \ \rceil^{Z} \qquad \text{and} \qquad g = \phi^{-1} \circ \lfloor \ \rfloor_{T'} .$$

Indeed, $g \circ f = \lceil\ \rceil^Z$ and $f \circ g = \lfloor\ \rfloor_{T'}$, while $\phi = f_{|g(S')}$ and $\phi^{-1} = g_{|f(S)}$. Hence

$$\phi\left(\lceil s\rceil^Z\right) = f(g(f(s)) = f(s) \quad \text{and} \quad \phi^{-1}\left(\lfloor s'\rfloor_{T'}\right) = g(f(g(s')) = g(s').$$

Exercise 63 *Let Z be an arbitrary ceiling subset of a partially ordered set $(S, \preceq)$, let T' be an arbitrary floor subset of a partially ordered set $(S', \preceq')$, and let ϕ be an arbitrary isomorphism between partially ordered sets $(Z, \preceq)$ and $(T', \preceq')$. Show that the pair of mappings*

$$f := \phi \circ \lceil\ \rceil^Z \qquad \textit{and} \qquad g := \phi^{-1} \circ \lfloor\ \rfloor_{T'} \tag{37}$$

forms a Galois connection between $(S, \preceq)$ and $(S', \preceq')$.

By combining Exercise 63 with the previous argument, cf. 2.6.9, we obtain

Proposition 2.15 *There is a natural correspondence between Galois connections and triples (Z, T', ϕ)*

$$\left\{\begin{matrix}\textit{Galois connections } (f,g) \\ \textit{between } (S,\preceq) \textit{ and } (S',\preceq')\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{triples} \\ (Z, T', \phi)\end{matrix}\right\}, \tag{38}$$

where $Z \subseteq S$ denotes any ceiling subset, $T' \subseteq S'$ denotes any floor subset, and ϕ is an arbitrary isomorphism between partially ordered sets $(Z, \preceq)$ and $(T', \preceq')$. Correspondence (38) *associates with a Galois connection (f, g) the triple $(g(S'), f(S), f_{|g(S')})$. The inverse correspondence associates with a triple (Z, T', ϕ) the pair (f, g) defined in* (37).

2.6.10 Example: a structure of a topological space on a set

To equip a set X with a structure of a topological space is the same as to equip $(\mathscr{P}(X), \subseteq)$ with the closure operation which is inf-*continuous*,

> for any family $\mathscr{E} \subseteq \mathscr{P}(X)$, one has
>
> $$\overline{\inf \mathscr{E}} = \inf \bar{\mathscr{E}}$$
>
> where $\bar{\mathscr{E}}$ is the family of the closures of subsets belonging to $\mathscr{E}$.

and *finitely* sup-*continuous*,

> for any *finite* family $\mathscr{E} \subseteq \mathscr{P}(X)$, one has
>
> $$\overline{\sup \mathscr{E}} = \sup \bar{\mathscr{E}}. \tag{39}$$

Exercise 64 *Show that for any closure operation on* $(\mathscr{P}(X), \subseteq)$ *which satisfies* (39)*, one has*

$$\overline{\emptyset} = \emptyset.$$

Hint. Consider the empty family of subsets.

2.6.11

An equivalent way to equip a set X with a structure of a topological space is to equip $(\mathscr{P}(X), \subseteq)$ with an interior operation which is sup*-continuous*,

> for any family $\mathscr{E} \subseteq \mathscr{P}(X)$, one has
>
> $$\sup \mathscr{E}^{\circ} = \sup \mathring{\mathscr{E}}$$
>
> where $\mathring{\mathscr{E}}$ is the family of the interiors of subsets belonging to $\mathscr{E}$.

and *finitely* inf*-continuous*,

> for any *finite* family $\mathscr{E} \subseteq \mathscr{P}(X)$, one has
>
> $$\inf \mathscr{E}^{\circ} = \inf \mathring{\mathscr{E}}.$$

2.7 Closure and interior operations on the power set of a partially ordered set

2.7.1 Down- and up-closure operations

Both

$$E \longmapsto \mathrm{Cl}^{\downarrow}(E) \qquad \text{and} \qquad E \longmapsto \mathrm{Cl}^{\uparrow}(E)$$

are closure operations. The first one is sup-continuous, the second one is inf-continuous.

2.7.2 LU- and UL-closure operations

A partial ordering $(S, \preceq)$ induces two closure operations on $(\mathscr{P}(S), \subseteq)$, the LU*-closure*,

$$E \longmapsto \bar{E}^{LU} := LU(E),$$

and the *UL-closure*,
$$E \longmapsto \bar{E}^{UL} := UL(E).$$

Indeed, for the *LU-closure*, (3) is property (34), property (35) follows from the combination of (5) and (6), while (9) implies property (36).

Note that, according to (9), a subset F of S is LU-closed if and only if it is of the form $F = L(E)$ for some $E \subseteq S$.

Note that

$$LU(\emptyset) = \begin{cases} \emptyset & \text{if } S \text{ has no least element} \\ \{\min S\} & \text{otherwise.} \end{cases}$$

Dually,
$$UL(\emptyset) = \begin{cases} \emptyset & \text{if } S \text{ has no largest element} \\ \{\max S\} & \text{otherwise.} \end{cases}$$

It follows that $\emptyset$ is LU-closed (respectively, UL-closed) if and only if S has no least element (respectively, no largest element).

On the other hand, S is always both LU- and UL-closed.

3 Dedekind-MacNeille Completion of a Partially Ordered Set

3.1 The set of LU-closed subsets as a partially ordered set

3.1.1

Let $(S, \preceq)$ be a partially ordered set. Let $\mathscr{X}(S) \subseteq \mathscr{P}(S)$ be the subset of the power set which consists of all LU-closed subsets of S. Recall that $F \subseteq S$ is LU-closed if and only if it has the form

$$F = L(E)$$

for some $E \subseteq S$. In fact, one can take $E = U(F)$.

Exercise 65 *Show that, for any family $\mathscr{E} \subseteq \mathscr{X}(S)$ of LU-closed subsets of S, the supremum of $\mathscr{E}$ in $\mathscr{X}(S)$ exists and equals*

$$\sup \mathscr{E} = LU\Big(\bigcup_{E \in \mathscr{E}} E\Big),$$

i.e., $\sup_{\mathscr{L}(S)}\mathscr{E}$ *is the* LU*-closure of* $\sup_{\mathscr{P}(S)}\mathscr{E}$*,*

$$\sup_{\mathscr{L}(S)}\mathscr{E} = \overline{\sup_{\mathscr{P}(S)}\mathscr{E}}^{LU}.$$

Exercise 66 *Show that*

$$\min \mathscr{L}(S) = \begin{cases} \emptyset & \text{if } S \text{ has no least element} \\ \{\min S\} & \text{otherwise} \end{cases}.$$

Thus, every family $\mathscr{E} \subseteq \mathscr{L}(S)$ is bounded below and, obviously, the infimum of the empty family exists and is equal to

$$\inf \emptyset = \max \mathscr{L}(S) = S.$$

By invoking Lemma 1.8 we deduce that the infimum of every family $\mathscr{E} \subseteq \mathscr{L}(S)$ exists in $\mathscr{L}(S)$, and we established the following fact.

Proposition 3.1 *For any partially ordered set* $(S, \preceq)$*, the set of* LU*-closed subsets,* $(\mathscr{L}(S), \subseteq)$ *is a complete lattice.* $\square$

Exercise 67 *Show that in* $\mathscr{L}(S)$ *one has*

$$\sup\langle E] = LU(E)$$

and

$$\inf\langle E] = L(E).$$

3.1.2

Since intervals $\langle s]$ are LU-closed,

$$\langle s] = L(\{s\}),$$

the image of set S under the canonical embedding, (19), is contained in $\mathscr{L}(S)$.

Exercise 68 *Show that* $\langle S]$ *is both* sup- *and* inf-*dense in* $\mathscr{L}(S)$.

Exercise 69 *Show that the embedding*

$$\langle\,]\colon (S,\preceq) \hookrightarrow (\mathscr{X}(S),\subseteq) \tag{40}$$

is both sup- *and* inf-*continuous.*

In other words, show that in $\mathscr{X}(S)$ *one has*

$$\sup_{\mathscr{X}(S)}\langle E] = \langle \sup E]$$

whenever $\sup E$ *exists in* $(S,\preceq)$, *and*

$$\inf_{\mathscr{X}(S)}\langle E] = \langle \inf E]$$

whenever $\inf E$ *exists in* $(S,\preceq)$.

By combining the results of Exercises 68 and 69 with Proposition 3.1, we establish the following important fact.

Theorem 3.2 *Any partially ordered set* $(S,\preceq)$ *admits an embedding onto a dense subset of a complete lattice, and this embedding preserves suprema and infima (i.e., is* sup- *and* inf-*continuous).* □

The lattice $\mathscr{X}(S),\subseteq)$ is called the **Dedekind-Macneille envelope** of a partially ordered set $(S,\preceq)$, or the **Dedekind-Macneille completion** of $(S,\preceq)$. Strictly speaking, the latter should apply to the embedding, (40), of $(S,\preceq)$ into $\mathscr{X}(S),\subseteq)$.

It is worth noticing that for any subset $E\subseteq S$ which is not bounded above in $(S,\preceq)$, one has

$$\sup\langle E] = \max \mathscr{X}(S) = S$$

and, for a subset not bounded below

$$\inf\langle E] = \min \mathscr{X}(S) = \emptyset$$

(note that $\min \mathscr{X}(S) = \emptyset$ in this case).

3.2 An important variant

3.2.1

Dedekind-MacNeille completion adds to any partially ordered set all the "missing" suprema and infima, including the suprema and infima for unbouded sets.

A simple modification provides a variant of the above construction allows to "add" only the suprema of nonempty subsets that are *bounded above*, and the infima of nonempty subsets which are *bounded below* while leaving unbounded sets not bounded.

Let $(S, \preceq)$ be a partially ordered set. Let us consider the subset $\mathscr{L}^*(S) \subseteq \mathscr{L}(S)$ which consists of the LU-closures of nonempty subsets of S which are bounded above.

Exercise 70 *Show that* $\emptyset \notin \mathscr{L}^*(S)$.

Exercise 71 *Show that* $S \in \mathscr{L}^*(S)$ *if and only if* S *has the largest element.*

It follows that

$$\mathscr{L}(S) = \begin{cases} \mathscr{L}^*(S) & \text{if both } \min S \text{ and } \max S \text{ exist} \\ \mathscr{L}^*(S) \cup \{S\} & \text{if } \min S \text{ exists and } \max S \text{ does not} \\ \mathscr{L}^*(S) \cup \{\emptyset\} & \text{if } \max S \text{ exists and } \min S \text{ does not} \\ \mathscr{L}^*(S) \cup \{\emptyset, S\} & \text{if neither } \min S \text{ nor } \max S \text{ exist} \end{cases}.$$

Thus, $\mathscr{L}^*(S)$ has the same elements as $\mathscr{L}(S)$ except possibly for the two extremal ones: $\emptyset$ and S.

Exercise 72 *Show that* $\langle S] \subseteq \mathscr{L}^*(S)$.

By Exercise 36, suprema and infima in $\mathscr{L}^*(S)$ coincide for nonempty bounded subsets of $\mathscr{L}^*(S)$ with those in $\mathscr{L}(S)$, hence $(\mathscr{L}^*(S), \subseteq)$ is a complete partially ordered set.

In particular, the embedding

$$\langle\,]\colon (S, \preceq) \longrightarrow (\mathscr{L}^*(S), \subseteq)$$

is both sup- and inf-continuous.

Finally, since $\langle S]$ is sup- and inf-dense in $\mathscr{L}(S)$ it is dense also in $\mathscr{L}^*(S)$.

We established thus the following variant of Theorem 3.2.

Theorem 3.3 *Any partially ordered set* $(S, \preceq)$ *admits an embedding onto a dense subset of a complete partially ordered set, and this embedding preserves suprema and infima (i.e., is* sup*- and* inf*-continuous), and also unboundedness of subsets:*

> *if* $E \subseteq S$ *is not bounded above (below) in* S, *then* E *is not bounded above (resp., below) in the completion.*

□

3.2.2

It follows from Exercise 11 that $(\mathscr{Z}(S), \subseteq)$, and therefore also $(\mathscr{Z}^*(S), \subseteq)$, are totally ordered if $(S, \preceq)$ is totally ordered. Thus one can strengthen the statements of Theorems 3.2 and 3.3 by adding that the corresponding completions of totally ordered sets are themselves totally ordered.

3.2.3 $\mathbb{R}$ and $\overline{\mathbb{R}} = [-\infty, \infty]$

In the case of $(\mathbb{Q}, \leq)$ we obtain two totally ordered completions

> $(\mathscr{Z}^*(\mathbb{Q}), \subseteq)$ *is a model for the set of real numbers,*

while

> $(\mathscr{Z}(\mathbb{Q}), \subseteq)$ *is a model for the set of* extended *real numbers*

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty].$$

4 Universal Properties of the Dedekind-MacNeille Completion

In this section $(S, \preceq)$ denotes a subset of $(T, \preceq)$.

4.1 Extending the Canonical Embedding

4.1.1

The correspondence

$$t \longmapsto \langle t] \cap S = \{s \in S \mid s \preceq t\} \tag{41}$$

defines a morphism $(T, \preceq) \longrightarrow (\mathscr{P}(S), \subseteq)$ which extends the canonical embedding, $(S, \preceq) \hookrightarrow (\mathscr{P}(S), \subseteq)$, cf. (19), to $(T, \preceq)$.

Exercise 73 *Show that* (41) *is an order embedding if* $(S, \preceq)$ *is* sup*-dense in* $(T, \preceq)$.

In particular,

morphism (41) *is injective if* $(S, \preceq)$ *is* sup*-dense in* $(T, \preceq)$.

If $(S, \preceq)$ is *not* sup-dense, then (41) is generally not injective. Thus,

$$\langle v] \cap S = \langle \zeta] \cap S$$

in Example 1.6.2.a but $v \neq \zeta$.

4.1.2

If $t = \inf_T E$ for some $E \subseteq S$, then

$$\langle t] \cap S = L(E) \in \mathscr{L}(S),$$

according to (15). It follows that

the image of morphism (41) *is contained in* $\mathscr{L}(S)$
if $(S, \preceq)$ *is* inf*-dense in* $(T, \preceq)$.

The following example demonstrates that the image of morphism (41) may be contained in $\mathscr{L}(S)$ even though *no* element of $T \setminus S$ may be of the form $\inf_T E$ for some $E \subseteq S$.

4.1.3 Example

Let $T := \mathbb{Q}\{0, 1\}$ equipped with the usual order. Let

$$S := \{x \in \mathbb{Q} \mid x < 0 \text{ or } x > 1\} = \mathbb{Q}_{<0} \cup \mathbb{Q}_{>1}.$$

Then, all $t \in T \setminus S = \{x \in \mathbb{Q} \mid 0 < x < 1\}$ are being sent by morphism (41) to one and the same LU-closed subset of S:

$$\langle t] \cap S = \mathbb{Q}_{<0} = L(\mathbb{Q}_{>1}).$$

Exercise 74 *Show that every* $t \in T \setminus S$ *is neither of the form* $\inf_T E$ *nor of the form* $\sup_T E$ *for some* $E \subseteq S$.

4.1.4

When $(S, \preceq)$ is inf-dense in $(T, \preceq)$, then one can easily describe the image of T under morphism (41):

> *a subset F of $\mathscr{L}(S)$ is in the image* if and only if *it is of the form*
>
> $$\langle \mathrm{inf}_T E] \cap S = L_T(E) \cap S = L(E) \tag{42}$$
>
> *for some $E \subseteq S$ which possesses infimum in T.*

This is so since every element of T equals $\mathrm{inf}_T E$ for a suitable subset $E \subseteq S$ and then $\langle t] = L_T(E)$.

Note, however, that $L(E)$, *for a particular* subset E, may belong to the image of morphism (41) while $\mathrm{inf}_T E$ may not exist. We only know that there must be another subset $E' \subseteq S$ such that

$$L(E) = L(E') \qquad \text{and} \qquad \mathrm{inf}_T E' \text{ exists.}$$

In Example 1.6.2.b, one has

$$L(Z) = \langle v] \cap S = L(Y)$$

where Y possesses infimum in the larger set, which is denoted there T', while Z does not.

4.1.5

By combining (42) with the assertion of Exercise 73, we deduce the following important universal property of the Dedekind-MacNeille completion.

Theorem 4.1 *If $(S, \preceq)$ is a subset of $(T, \preceq)$ which is both* inf- *and* sup*-dense, then morphism* (41) *embeds $(T, \preceq)$ isomorphically onto the subset of $\mathscr{L}(S)$:*

$$\{F \in \mathscr{L}(S) \mid F = L(E) \text{ for some } E \subseteq S \text{ which possesses infimum in } T\}.$$

In particular, if every *subset of S possesses infimum in T, then* (41) *establishes a canonical isomorphism between $(T, \preceq)$ and the Dedekind-MacNeille completion, $(\mathscr{L}(S), \subseteq)$ which extends the embedding of $(S, \preceq)$ into $(\mathscr{L}(S), \subseteq)$.* □

4.2 Completing a subset in a bigger partially ordered set

4.2.1

In general, there is no largest subset $T' \subseteq T$ such that $(S, \preceq)$ would be sup-dense, inf-dense, or both sup- and inf-dense in $(T', \preceq)$.

4.2.2 Example

Let $S = \mathbb{Q} \setminus \{0\}$ be equipped with usual order and $T = S \cup \{a, b\}$ with

$$x < a < y \qquad \text{and} \qquad x < b < y$$

for any $x \in \mathbb{Q}_{<0}$ and $y \in \mathbb{Q}_{>0}$. In this case, S is both sup- and inf-dense in $T_1 = S \cup \{a\}$ and $T_2 = S \cup \{b\}$ but is neither sup-dense nor inf-dense in $T = T_1 \cup T_2$.

Exercise 75 *Show that neither a nor b is of the form* $\inf_T E$ *or* $\sup_T E$ *for any subset* $E \subseteq S$.

Thus, T_1 and T_2 are two distinct *maximal* subsets of T in which S is dense in any of the three spelled out senses.

4.2.3

For any partially ordered set $(T, \preceq)$ containing $(S, \preceq)$, there is, however, a subset that is a perfect analog of the Dedekind-MacNeille completion:

$$\widehat{S_T} := \{t \in T \mid t = \inf_T U(E) \text{ for some } E \subseteq S\}.$$

If we identify S with its isomorphic image $\langle S]$ in $\mathscr{P}(S)$, then

$$\widehat{S_{\mathscr{P}(S)}} = \mathscr{L}(S)$$

since every $F \in \mathscr{L}(S)$ is of the form

$$F = LU(E) = \inf_{\mathscr{P}(S)} U(E),$$

cf. equalities (9) and (20).

4.2.4

If $t = \inf_T U(E)$, then

$$E \subseteq LU(E) = L_T(U(E)) \cap S \subseteq L_T(U(E)) = \langle t]$$

This shows that $t \in U_T(E)$.

If another element of $\widehat{S_T}$, say $t' = \inf_T U(E')$, belongs to $U_T(E)$, then

$$x \preceq t' \preceq y$$

for any $x \in E$ and $y \in U(E')$. It follows that

$$U(E') \subseteq U(E)$$

and hence

$$t = \inf_T E \preceq \inf_T E' = t',$$

cf. (7). Thus, t is the smallest element of $U_T(E) \cap \widehat{S_T} = U_{\widehat{S_T}}(E)$, i.e., $\sup_{\widehat{S_T}} E$ exists and equals t:

$$\sup_{\widehat{S_T}} E = \inf_T U(E).$$

Note that, in view of (14), the infimum of $U(E)$ in $\widehat{S_T}$ exists and coincides with $\inf_T U(E)$,

$$\inf_{\widehat{S_T}} U(E) = \inf_T U(E).$$

In particular,

$$\sup_{\widehat{S_T}} E = \inf_{\widehat{S_T}} U(E).$$

We established the following fact.

Proposition 4.2 *Any subset S of a partially oredered set $(T, \preceq)$ is both* sup- *and* inf-*dense in $(\widehat{S_T}, \preceq)$.* □

4.2.5

By the Universal Property of the Dedekind-MacNeille completion cf. Theorem 4.1, the order embedding of $(S, \preceq)$ into $\mathscr{L}(S), \subseteq)$ then extends to an order embedding

$$(\widehat{S_T}, \preceq) \hookrightarrow (\mathscr{L}(S), \subseteq). \quad (43)$$

4.2.6

If $(T, \preceq)$ is a complete lattice, then embedding (43) is *surjective* since, for every $E \subseteq S$, one has

$$LU(E) = \langle \mathrm{inf}_T U(E)] \cap S,$$

cf. Exercise 24.

In particular, $(\widehat{S_T}, \preceq)$ is isomorphic to the Dedekind-MacNeille completion.

This way we arrive at our final result.

Theorem 4.3 *Any order embedding $(S, \preceq) \hookrightarrow (T, \preceq)$ into a complete lattice induces an order embedding of the Dedekind-MacNeille completion,*

$$(\mathscr{L}(S), \subseteq) \hookrightarrow (T, \preceq)$$

which identifies $(\mathscr{L}(S), \subseteq)$ with $(\widehat{S_T}, \preceq)$. In particular, $(\widehat{S_T}, \preceq)$ is itself a complete lattice. □