

Notes on Ordered Sets

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1 Vocabulary

1.1 Definitions

Definition 1.1 A binary relation \preceq on a set S is said to be a **partial order** if it is *reflexive*,

$$x \preceq x,$$

weakly antisymmetric,

$$\text{if } x \preceq y \text{ and } y \preceq x, \text{ then } x = y,$$

and *transitive*,

$$\text{if } x \preceq y \text{ and } y \preceq z, \text{ then } x \preceq z$$

Above x, y, z , are arbitrary elements of S .

Definition 1.2 Let $E \subseteq S$. An element $y \in S$ is said to be an **upper bound** for E if

$$x \preceq y \quad \text{for any } x \in E. \tag{1}$$

By definition, any element of S is declared to be an upper bound for \emptyset , the empty subset.

We shall denote by $U(E)$ the set of all upper bounds for E

$$U(E) := \{y \in S \mid x \preceq y \text{ for any } x \in E\}.$$

Note that $U(\emptyset) = S$.

Definition 1.3 We say that a subset $E \subseteq S$ is **bounded (from) above**, if $U(E) \neq \emptyset$, i.e., when there exists at least one element $y \in S$ satisfying (1).

Definition 1.4 If $y, y' \in U(E) \cap E$, then

$$y \preceq y' \quad \text{and} \quad y' \preceq y.$$

Thus, $y = y'$, and that unique upper bound of E which belongs to E will be denoted $\max E$ and called the **largest element** of E .

It follows that $U(E) \cap E$ is empty when E has no largest element, and consists of a single element, namely $\max E$, when it does.

If we replace \preceq by \succeq everywhere above, we shall obtain the definitions of a **lower bound** for set E , of the set of all lower bounds,

$$L(E) := \{y \in S \mid x \succeq y \text{ for any } x \in E\},$$

and, respectively, of the **smallest element** of E . The latter will be denoted $\min E$.

Exercise 1 Show that, for any subsets E and F , one has

$$L(E) \supseteq F \quad \text{if and only if} \quad E \subseteq U(F).$$

1.1.1 The Principle of Duality

Note that the relation defined by

$$x \preceq^{\text{rev}} y \quad \text{if} \quad x \succeq y$$

is also an order relation on S . We will refer to it as the **reverse ordering**.

Any general statement about partially ordered sets has the corresponding *dual* statement that is obtained by replacing (S, \preceq) with $(S, \preceq^{\text{rev}})$. Under this “duality” *upper* bounds become *lower* bounds, maxima become minima, suprema become infima, and vice-versa.

Therefore most general theorems about partially ordered sets possess the corresponding *dual* theorems. Below, I will be usually formulating those dual statements for the convenience of reference.

Exercise 2 Show that if $E \subseteq S$ is bounded below and nonempty, then $L(E)$ is bounded above and nonempty.

Dually, if E is bounded above and nonempty, then $U(E)$ is bounded below and nonempty.

1.1.2

If $E \subseteq F \subseteq S$, then

$$\max F \in U(E) \quad (2)$$

when $\max F$ exists, and, dually,

$$\min F \in L(E)$$

when $\min F$ exists.

If both $\max E$ and $\max F$ exist, then

$$\max E \preceq \max F.$$

Dually, if both $\min E$ and $\min F$ exist, then

$$\min F \preceq \min E.$$

Exercise 3 (Sandwich Lemma for maxima) Show that if $E'' \subseteq E \subseteq E'$ and both $\max E'$ and $\max E''$ exist and are equal, then $\max E$ exists and

$$\max E'' = \max E = \max E'.$$

Dually, if both $\min E'$ and $\min E''$ exist and are equal, then $\min E$ exists and

$$\min E' = \min E = \min E''.$$

1.1.3 Supremum and infimum

Definition 1.5 When $\min U(E)$ exists it is called the **least upper bound** of E , or the **supremum** of E , and is denoted $\sup E$.

Dually, when $\max L(E)$ exists it is called the **largest lower bound** of E , or the **infimum** of E , and is denoted $\inf E$.

For the supremum of E to exist, subset E must be bounded above. The supremum of E may exist for some bounded above subsets of S and may not exist for others.

1.1.4 Example

Let us consider $S = \mathbb{Q}$, the set of rational numbers, with the usual order. Both the following subset $E_1 \subseteq \mathbb{Q}$,

$$E_1 := \{x \in \mathbb{Q} \mid x^2 < 1\}$$

and the subset $E_2 \subseteq \mathbb{Q}$,

$$E_2 := \{x \in \mathbb{Q} \mid x^2 < 2\},$$

are simultaneously bounded above and below. None of them has either the largest nor the smallest element but

$$\sup E_1 = 1 \quad \text{and} \quad \inf E_1 = -1$$

while neither $\sup E_2$ nor $\inf E_2$ exist in $S = \mathbb{Q}$.

Exercise 4 *Show that*

$$\sup \emptyset = \min S \quad \text{and} \quad \inf \emptyset = \max S.$$

In particular, $\sup \emptyset$ exists if and only if S has the smallest element; similarly, $\inf \emptyset$ exists if and only if S has the largest element.

1.1.5 Down-intervals and up-intervals

Let (S, \preceq) be a partially ordered set. For each $s \in S$, we define the *down-interval*

$$\langle s] := \{t \in S \mid t \preceq s\}.$$

and the *up-interval*

$$[s) := \{t \in S \mid s \preceq t\}.$$

Exercise 5 *Show that, for $E \subseteq S$, one has*

$$L(E) = \langle s] \quad \text{for some } s \in S$$

if and only if $\inf E$ exists. In this case,

$$L(E) = \langle \inf E].$$

Dually,

$$U(E) = [s], \quad \text{for some } s \in S,$$

if and only if $\sup E$ exists. In this case,

$$U(E) = [\sup E].$$

1.1.6 Example

Consider the set of natural numbers,

$$\mathbb{N} := \{0, 1, 2, \dots\},$$

equipped with the ordering given by

$$m \preceq n \quad \text{if} \quad m \mid n$$

(“ m divides n ”).

Exercise 6 Does (\mathbb{N}, \mid) have the maximum? the minimum? If yes, then what are they?

Exercise 7 For a given $n \in \mathbb{N}$, describe intervals $\langle n \rangle$ and $[n \rangle$ in (\mathbb{N}, \mid) .

Exercise 8 For given $m, n \in \mathbb{N}$, is set $\{m, n\}$ bounded below? Does it possess infimum? If yes, then describe $\inf\{m, n\}$.

Exercise 9 For given $m, n \in \mathbb{N}$, is set $\{m, n\}$ bounded above? Does it possess supremum? If yes, then describe $\sup\{m, n\}$.

Exercise 10 Does every subset of \mathbb{N} possess infimum in (\mathbb{N}, \mid) ? Does every subset of \mathbb{N} possess supremum?

1.1.7 Totally ordered sets

Definition 1.6 We say that a partially ordered set (S, \preceq) is **totally**, or **linearly**, ordered if any two elements s and t of S are comparable

$$\text{either } s \preceq t \text{ or } t \preceq s.$$

Totally ordered subsets in any given partially ordered set are called **chains**.

Exercise 11 Let (S, \preceq) be a totally ordered set and $E, F \subseteq S$ be two subsets. Show that

$$\text{either } L(E) \subseteq L(F) \text{ or } L(F) \subseteq L(E).$$

1.2 Observations

1.2.1

For any subset $E \subseteq S$, one has

$$E \subseteq LU(E) := L(U(E)) \quad (3)$$

and

$$E \subseteq UL(E) := U(L(E)). \quad (4)$$

1.2.2

If $E \subseteq F \subseteq S$, then

$$U(E) \supseteq U(F) \quad (5)$$

and

$$L(E) \supseteq L(F). \quad (6)$$

Exercise 12 Show that if $E \subseteq F$ and both $\sup E$ and $\sup F$ exist, then

$$\sup E \preceq \sup F.$$

Dually, if both $\inf E$ and $\inf F$ exist, then

$$\inf F \preceq \inf E. \quad (7)$$

Exercise 13 (Sandwich Lemma for infima) Show that if $E'' \subseteq E \subseteq E'$ and both $\inf E'$ and $\inf E''$ exist and are equal, then $\inf E$ exists and

$$\inf E'' = \inf E = \inf E'.$$

Dually, if both $\sup E'$ and $\sup E''$ exist and are equal, then $\sup E$ exists and

$$\sup E' = \sup E = \sup E''.$$

1.2.3

By applying (5) to the pair of subsets in (3), one obtains

$$U(E) \supseteq ULU(E) := U(L(U(E)))$$

while (4) applied to subset $U(E)$ yields

$$U(E) \subseteq ULU(E).$$

It follows that

$$U(E) = ULU(E). \quad (8)$$

Dually,

$$L(E) = LUL(E). \quad (9)$$

Note that equality (9) is nothing but equality (8) for the *reverse* ordering.

1.2.4

For any subsets $E \subseteq S$ and $F \subseteq S$, one has

$$U(E \cup F) = U(E) \cap U(F)$$

and

$$L(E \cup F) = L(E) \cap L(F).$$

1.2.5

For any $E \subseteq S$, $\max E$ exists if and only if $\sup E$ exists and belongs to E , and they are equal

$$\sup E = \max E. \quad (10)$$

Dually, $\min E$ exists if and only if $\inf E$ exists and belongs to E , and they are equal

$$\inf E = \min E.$$

Indeed, if $\max E$ exists, then it is the least element of $U(E)$, cf. Definition (1.4).

If $\sup E$ exists and is a member of E , then it belongs to $U(E) \cap E$ which as we established, cf. Definition (1.4), consists of the single element $\max E$ when $U(E) \cap E$ is nonempty.

The case of $\min E$ and $\inf E$ follows if we apply the already proven to $(S, \preceq^{\text{rev}})$.

1.2.6

For any $E \subseteq S$, $\inf U(E)$ exists if and only if $\sup E$ exists, and they are equal

$$\max LU(E) = \inf U(E) = \min U(E) = \sup E. \quad (11)$$

Indeed,

$$\inf U(E) := \max LU(E) \in U(E),$$

in view of $E \subseteq LU(E)$, cf., (3), combined with (2) where $F = LU(E)$. Thus,

$$\inf U(E) = \min U(E)$$

by (10).

Dually, $\sup L(E)$ exists if and only if $\inf E$ exists, and they are equal

$$\min UL(E) = \sup L(E) = \max L(E) = \inf E.$$

1.2.7 Example: the power set as a partially ordered set

Let $S = \mathcal{P}(A)$ be the *power set* of a set A :

$$\mathcal{P}(A) := \text{the set of all subsets of } A.$$

Containment \subseteq is a partial order relation on $\mathcal{P}(A)$.

Subsets \mathcal{E} of $\mathcal{P}(A)$ are the same as *families* of subsets of A . Since $S = \mathcal{P}(A)$ contains the largest element, namely A , and the smallest element, namely \emptyset , every subset of $\mathcal{P}(A)$ is bounded above and below.

The union of all members of a family \mathcal{E} ,

$$\bigcup_{E \in \mathcal{E}} E := \{a \in A \mid a \in E \text{ for some } E \in \mathcal{E}\} \quad (12)$$

is the smallest subset of A which *contains every member* of family \mathcal{E} . Hence, $\sup \mathcal{E}$ exists and equals (12).

Dually, the intersection of all members of family \mathcal{E} ,

$$\bigcap_{E \in \mathcal{E}} E := \{a \in A \mid a \in E \text{ for all } E \in \mathcal{E}\} \quad (13)$$

is the largest subset of A which is *contained in every member* of family \mathcal{E} . Hence, $\inf \mathcal{E}$ exists and equals (13).

The power set provides an example of a partially ordered set in which every subset (including the empty set) possesses both supremum and infimum.

1.3 Completeness

1.3.1

Definition 1.7 We say that a partially ordered set (S, \preceq) has the **largest-lower-bound property** if $\inf E$ exists for every subset $E \subseteq S$ which is nonempty and bounded below.

Dually, we say that S has the **least-upper-bound property** if $\sup E$ exists for subset $E \subseteq S$ which is nonempty and bounded above.

1.3.2 An alternative terminology

Partially ordered sets with the largest-lower-bound property are said to be **inf-complete**, and those with the least-upper-bound property are said to be **sup-complete**.

Lemma 1.8 A partially ordered set S has the largest-lower-bound property if and only if it has the least-upper-bound property.

Proof. Suppose that S is inf-complete. If $E \subseteq S$ is bounded above and nonempty, then the set of upper bounds, $U(E)$ is nonempty. Since

$$L(U(E)) \supseteq E \neq \emptyset$$

subset $U(E)$ is also bounded below. Then $\inf U(E)$ exists in view of our assumption about (S, \preceq) . But then it coincides with $\sup E$ in accordance with (11). This shows that S is sup-complete.

The reverse implication,

$$\text{sup-completeness} \Rightarrow \text{inf-completeness}$$

follows by applying the already proven implication

$$\text{inf-completeness} \Rightarrow \text{sup-completeness}$$

to the reverse order on S . □

Since sup- and inf-completeness are equivalent we shall simply call such sets *complete*.

1.3.3 Lattices

Definition 1.9 A partially ordered set (S, \preceq) is called a **pre-lattice** if every nonempty finite subset $E \subseteq S$ has supremum and infimum.

Exercise 14 Show that (S, \preceq) is a **pre-lattice** if and only if, for any $s, t \in S$, both $\sup\{s, t\}$ and $\inf\{s, t\}$ exist.

Definition 1.10 A partially ordered set (S, \preceq) is called a **lattice** if every finite subset $E \subseteq S$, including $\emptyset \subseteq S$, has supremum and infimum.

1.3.4 Complete lattices

Complete partially ordered sets with the largest and the smallest elements are the same as *complete lattices*. Note that in such sets every subset is bounded below and above.

For example, the totally ordered set of rational numbers, (\mathbb{Q}, \leq) , is a pre-lattice but not a lattice, and it is not complete.

The power set of an arbitrary set, $(\mathcal{P}(A), \subseteq)$, is an example of a complete lattice. A less obvious example is the subject of the next section.

1.4 Down-sets, up-sets

1.4.1 Down-sets

A subset L of a partially ordered set (S, \preceq) is called a **down-set** if

$$\text{for any } s \in L \text{ and } s' \preceq s, \text{ also } s' \in L.$$

Exercise 15 Show that the union and the intersection of any family $\mathcal{L} \subseteq \mathcal{P}(A)$ of down-sets is a down-set.

Exercise 16 Show that a down-set L is a down-interval if and only if $\sup L$ exists and belongs to L .

1.4.2 Down-set closure of a subset

The family of down-sets containing a given subset $E \subseteq S$ is nonempty since $E \subseteq S$ and S is a down-set. It follows that the intersection of all down-sets L containing E ,

$$\text{Cl}^\downarrow(E) := \bigcap_{L \supseteq E} L,$$

is the *smallest* down-set containing E .

1.4.3 Down-set interior of a subset

The family of down-sets contained in a given subset $E \subseteq S$ is nonempty since $\emptyset \subseteq E$ and \emptyset is a down-set. It follows that the union of all down-sets L contained in E ,

$$\text{Int}^\downarrow(E) := \bigcup_{L \subseteq E} L,$$

is the *largest* down-set contained in E .

1.4.4

By definition, one has

$$\text{Int}^\downarrow(E) \subseteq E \subseteq \text{Cl}^\downarrow(E)$$

Exercise 17 Show that $E \subseteq S$ is a down-set if and only if

$$\text{Int}^\downarrow(E) = E$$

if and only if

$$E = \text{Cl}^\downarrow(E).$$

Exercise 18 Any subset $E \subseteq S$ is contained in $\bigcup_{s \in E} \langle s \rangle$. Show that E is a down-set if and only if

$$E = \bigcup_{s \in E} \langle s \rangle.$$

1.4.5

Let E and F be two subsets of a partially ordered set (S, \preceq) . We may say that F *dominates* E *from above*, and express this symbolically with $E \rightarrow F$, if

for every $s \in E$ there exists $t \in F$ such that $s \preceq t$.

Exercise 19 Show that $U(E) \supseteq U(F)$ whenever $E \rightarrow F$. In particular

$$\sup E \preceq \sup F$$

when both suprema exist.

Exercise 20 Show that $E \rightarrow F$ if and only if $\text{Cl}^\downarrow(E) \subseteq \text{Cl}^\downarrow(F)$.

1.4.6 Cofinal pairs of subsets

We shall say that subsets E and F are **cofinal** if $E \rightarrow F$ and $F \rightarrow E$. For cofinal subsets $U(E) = U(F)$. In particular, $\sup E$ exists if and only if $\sup F$ exists and they are equal (cf. Ex. 19).

1.4.7

It follows that E and F are *cofinal* if and only if $\text{Cl}^\downarrow(E) = \text{Cl}^\downarrow(F)$.

1.4.8 Up-sets

Up-sets are defined, by duality, as down-sets for the reverse ordering \preceq^{rev} . In particular, a subset E is an up-set if and only if

$$E = \bigcup_{s \in E} [s).$$

One can also define the corresponding notions of the *up-closure*, $\text{Cl}^\uparrow(E)$, and the *up-interior*, $\text{Int}^\uparrow(E)$, of a subset E .

Exercise 21 Define the dual concept

F *dominates* E *from below*

(one can denote this fact by using notation $F \varepsilon E$).

1.4.9

If we replace \preceq by the reverse order, \preceq^{rev} , we obtain another pair of relations between subsets of S :

$$E \preceq^{\text{rev}} F \quad \text{and} \quad F \preceq^{\text{rev}} E.$$

Note that

$$F \preceq E \quad \text{if and only if} \quad E \preceq^{\text{rev}} F \quad (\text{not } E \preceq F!).$$

1.4.10

The dual concept to a cofinal pair of subsets is a **coinitial** pair of subsets.

1.5 Partially ordered subsets

1.5.1

In $S \subseteq T$ is a subset of a partially ordered set (T, \preceq) , then it can be regarded as a partially ordered set in its own right. One has to be cautioned, however, that S with the induced order may have vastly different properties.

1.5.2

For a subset $E \subseteq S$, the sets of upper and lower bounds will generally depend on whether one considers E as a subset of S or T . In particular, E may be not bounded as a subset of S yet be bounded as a subset of T .

When necessary, we shall indicate this by subscript S or T . Thus,

$$L_T(E), \quad U_T(E), \quad \inf_T E, \quad \sup_T E,$$

will denote the set of lower bounds, the set of upper bounds, the infimum, and the supremum, when E is viewed as a subset of T .

Exercise 22 Show that, for $E \subseteq S$, one has

$$\sup_T E \preceq \sup_S E$$

whenever both suprema exist.

1.5.3

Dually, one has

$$\inf_S E \preceq \inf_T E$$

whenever both infima exist.

Exercise 23 For $E \subseteq S$, suppose that $\sup_T E$ exists and belongs to S . Show that $\sup_S E$ exists and

$$\sup_S E = \sup_T E.$$

1.5.4

Dually, if $\inf_T E$ exists and belongs to S , then $\inf_S E$ exists and

$$\inf_S E = \inf_T E. \quad (14)$$

Exercise 24 Let $E \subseteq S$, and suppose that (S, \preceq) is a partially ordered subset of (T, \preceq) . Show that, if $\inf_T E$ exists, then

$$L(E) = \langle \inf_T E \rangle \cap S = \{s \in S \mid s \preceq \inf_T E\}. \quad (15)$$

(Here $L(E) = L_S(E)$.)

1.5.5

Dually, one has

$$U(E) = [\sup_T E) \cap S = \{s \in S \mid \sup_T E \preceq s\}$$

if $\sup_T E$ exists.

Exercise 25 Find examples of pairs $E \subseteq S$ of subsets of \mathbb{Q} such that:

- (a) E is unbounded above in S yet bounded in \mathbb{Q} ;
- (b) E is bounded in S , and $\sup_{\mathbb{Q}} E$ exists but $\sup_S E$ does not;
- (c) E is bounded in S , and $\sup_S E$ exists but $\sup_{\mathbb{Q}} E$ does not;
- (d) both $\sup_S E$ and $\sup_{\mathbb{Q}} E$ exist but $\sup_S E \neq \sup_{\mathbb{Q}} E$.

1.5.6 Sublattices

1.5.7

A subset S of a lattice (T, \preceq) forms a *sublattice* if

$$\inf_S E = \inf_T E \quad \text{and} \quad \sup_S E = \sup_T E$$

for any *finite* $E \subseteq S$.

1.5.8

A subset S of a *complete* lattice (T, \preceq) forms a *sublattice* if

$$\inf_S E = \inf_T E \quad \text{and} \quad \sup_S E = \sup_T E$$

for *any* $E \subseteq S$.

1.5.9

If $A \subseteq B$, then $(\mathcal{P}(A), \subseteq)$ forms naturally a sublattice of $(\mathcal{P}(B), \subseteq)$.

1.5.10

According to Exercise 15, the family of all down-sets $(\mathcal{P}^\downarrow(S), \subseteq)$ in a partially ordered set (S, \preceq) forms a complete sublattice of $(\mathcal{P}(S), \subseteq)$. Similarly, the family of all up-sets $(\mathcal{P}^\uparrow(S), \subseteq)$ forms a complete sublattice of $(\mathcal{P}(S), \subseteq)$.

1.6 Density

1.6.1

Definition 1.11 We say that a subset $S \subseteq T$ is *sup-dense* if every element $t \in T$ equals

$$t = \sup E$$

for some subset $E \subseteq S$.

By replacing \sup with \inf , one obtains the definition of a *inf-dense* subset.

1.6.2 Example

Suppose that a partially ordered set (S, \preceq) is the union of three subsets $S = X \cup Y \cup Z$ such that

$$x \preceq y \quad \text{and} \quad x \preceq z \quad \text{for any } x \in X, y \in Y, \text{ and } z \in Z,$$

and no $y \in Y$ and $z \in Z$ are comparable. Additionally, let us assume that neither Y nor Z possess the minimal elements.

Let us extend the ordering relation to $T = S \cup \{v, \zeta\}$ by setting

$$x \prec v \prec y \quad \text{for any } x \in X, y \in Y,$$

and

$$x \prec \zeta \prec z \quad \text{for any } x \in X, z \in Z,$$

and making v and ζ not comparable. Finally, denote by T' the subset $S \cup \{v\}$ of T .

a) (S, \preceq) as a subset of (T, \preceq) . One has $L_T(Y) = \langle v \rangle$ and $L_T(Z) = \langle \zeta \rangle$. It follows that

$$v = \inf_T Y \quad \text{and} \quad \zeta = \inf_T Z,$$

and therefore S is inf-dense in T . In addition,

$$\langle v \rangle \cap S = X = \langle \zeta \rangle \cap S$$

but $v \neq \zeta$, they are not even comparable.

Exercise 26 Show that neither v nor ζ equals $\sup_T E$ for any $E \subseteq S$. In particular, S is inf-dense in T but not sup-dense.

b) (S, \preceq) as a subset of (T', \preceq) . One has

$$v = \inf_{T'} Y$$

hence S is inf-dense in T' . In addition,

$$L(Y) = \langle v \rangle \cap S = X = L(Z)$$

and $v = \inf_{T'} Y$ while $\inf_{T'} Z$ does not exist.

Exercise 27 Let (S, \preceq) be a partially ordered subset of (T, \preceq) . Show that if

$$t = \inf_T E$$

for some $E \subseteq S$, then

$$t = \inf_T([t] \cap S).$$

Dually, if

$$t = \sup_T F \tag{16}$$

for some $F \subseteq S$, then

$$t = \sup_T(\langle t \rangle \cap S).$$

Exercise 28 (A criterion of equality for infima) Let (S, \preceq) be a partially ordered subset of (T, \preceq) and $E \subseteq S$. Suppose that both $\inf E$ and $t = \inf_T E$ exist. Show that if there exists a subset $F \subseteq S$ such that (16) holds, then

$$\inf_T E = \inf E.$$

Exercise 29 Formulate the dual criterion of equality for suprema.

2 Mappings between partially ordered sets

2.1 Morphisms

2.1.1

Definition 2.1 Given two partially ordered sets (S, \preceq) and (S', \preceq') , a mapping $f: S \rightarrow S'$ which preserves order,

$$\text{if } s \preceq t, \text{ then } f(s) \preceq' f(t) \quad (s, t \in S),$$

is said to be a **morphism** $(S, \preceq) \rightarrow (S', \preceq')$.

2.1.2

Recall that the **image** of a subset $E \subseteq S$ under f is the set

$$f(E) := \{s' \in S' \mid s' = f(a) \text{ for some } a \in E\},$$

and the **preimage** of a subset $E' \subseteq S'$ under f is the set

$$f^{-1}(E') := \{s \in S \mid f(s) \in E'\}.$$

Exercise 30 Show that

$$f \text{ is a morphism } (S, \preceq) \longrightarrow (S', \preceq')$$

if and only if

the preimage, $f^{-1}(L')$, of any down-set $L' \subseteq S'$ is a down-set in S

if and only if

the preimage, $f^{-1}(U')$, of any up-set $U' \subseteq S'$ is an up-set in S .

Exercise 31 Let f be a morphism $(S, \preceq) \longrightarrow (S', \preceq')$. Show that, for any subset $E \subseteq S$, one has

$$\min f(E) = f(\min E) \quad \text{and} \quad \max f(E) = f(\max E)$$

whenever $\min E$ or $\max E$ exists.

Exercise 32 Let f be a morphism $(S, \preceq) \longrightarrow (S', \preceq')$. Show that, for any nonempty subset $E \subseteq S$, one has

$$U(f(E)) \supseteq f(U(E)) \quad \text{and} \quad L(f(E)) \supseteq f(L(E)).$$

Deduce from this the inequalities

$$f(\inf E) \preceq' \inf f(E) \preceq' \sup f(E) \preceq' f(\sup E)$$

whenever the corresponding infima and suprema exist.

2.1.3

Definition 2.2 We say that a morphism $f: (S, \preceq) \longrightarrow (S', \preceq')$ is **sup-continuous** if f preserves the suprema. More precisely, if it has the following property

for any $E \subseteq S$, if $\sup E$ exists, then $\sup f(E)$ exists, and

$$\sup f(E) = f(\sup E).$$

Exercise 33 State the dual definition of an **inf-continuous** morphism.

2.2 Order embeddings

2.2.1

Definition 2.3 A mapping $\iota: S \rightarrow S'$ is said to be an **order embedding**,

$$(S, \preceq) \hookrightarrow (S', \preceq'), \quad (17)$$

if it satisfies a stronger condition

$$s \preceq t \quad \text{if and only if} \quad \iota(s) \preceq' \iota(t) \quad (s, t \in S).$$

Exercise 34 Show that an order embedding is injective.

Definition 2.4 A morphism $f: (S, \preceq) \rightarrow (S', \preceq')$ is said to be an **isomorphism** if it has an inverse, i.e., if there is a morphism $g: (S', \preceq') \rightarrow (S, \preceq)$ such that $f \circ g = \text{id}_{S'}$ and $g \circ f = \text{id}_S$.

Exercise 35 Show that an order embedding, (17), is an isomorphism onto its image, $(\iota(S), \preceq')$.

Exercise 36 For any partially ordered (S, \preceq) , let S^* denote the subset obtained by removing $\max S$ and $\min S$ if they exist. Show that, for any nonempty bounded subset E of S^* ,

$$\sup_{S^*} E = \sup_S E \quad \text{and} \quad \inf_{S^*} E = \inf_S E.$$

Exercise 37 (A criterion of continuity of an embedding) Let (S, \preceq) be a sup-dense subset of (T, \preceq) . Show that the inclusion

$$(S, \preceq) \hookrightarrow (T, \preceq) \quad (18)$$

is inf-continuous.

Dually, if (S, \preceq) is a inf-dense subset of (T, \preceq) , then inclusion (18) is sup-continuous.

2.2.2 Example: the canonical embedding $(S, \preceq) \hookrightarrow (\mathcal{P}(S), \subseteq)$

For any pair of elements s and t in a partially ordered set (S, \preceq) , one has

$$s \preceq t \quad \text{if and only if} \quad \langle s \rangle \subseteq \langle t \rangle.$$

Thus, the correspondence

$$\langle \cdot \rangle : S \longrightarrow \mathcal{P}(S), \quad s \longmapsto \langle s \rangle, \quad (19)$$

is an order embedding of (S, \preceq) into $(\mathcal{P}(S), \subseteq)$. Embedding (19) identifies (S, \preceq) with the set of down-intervals $(\mathcal{P}^\downarrow(S), \subseteq)$.

Exercise 38 Show that, for any $E \subseteq S$, one has

$$\sup \langle E \rangle = \bigcup_{s \in E} \langle s \rangle = \text{Cl}^\downarrow(E)$$

and

$$\inf \langle E \rangle = L(E) \quad (20)$$

in $(\mathcal{P}(S), \subseteq)$.¹

It follows that

$$\sup \langle E \rangle = \langle \sup E \rangle$$

in $(\mathcal{P}(S), \subseteq)$ if and only if $\max E$ exists. In particular, (19) is nearly never *sup*-continuous. Indeed, $\sup \langle E \rangle$ always exists, it coincides with the down-closure of E , and the latter is a *proper* subset of $\langle \sup E \rangle$ if when E has no maximal element.

On the other hand, the canonical embedding is *inf*-continuous.

Exercise 39 Show that the canonical embedding, (19), is *inf*-continuous.

Exercise 40 Explain why $\mathcal{P}^\downarrow(S, \subseteq) = \langle S \rangle$ is, generally, neither *sup*- nor *inf*-dense in $\mathcal{P}(S)$.

¹Here $\langle E \rangle$ denotes the *image* of a subset $E \subseteq S$ under the canonical embedding, (19), *not* the interval, $\langle E \rangle = \{X \in \mathcal{P}(S) \mid X \subseteq E\}$, in partially ordered set $(\mathcal{P}(S), \subseteq)$. The latter coincides with $\mathcal{P}(E)$, the set of all subsets of E .

2.3 Galois connections

2.3.1 Residuated mappings

If f is a morphism, then the preimage, $f^{-1}(\langle s' \rangle)$, of any interval $\langle s' \rangle \subseteq S'$ is a down-set in (S, \preceq) but not necessarily a down-interval. Mappings $f: S \rightarrow S'$ which have the property that the preimage of any down-interval $\langle s' \rangle$ in (S', \preceq') is a down-interval in (S, \preceq) are said to be **residuated**.

Exercise 41 Show that a residuated mapping is a morphism.

Exercise 42 Show that $f_1 \circ f_2$ is residuated if both f_1 and f_2 are residuated.

2.3.2 Residual mappings

Dually, we say that a mapping f is **residual**, if the preimage, $f^{-1}([s'])$, of any up-interval $[s'] \subseteq S'$ is an up-interval in (S, \preceq) .

2.3.3

The following proposition gives a simple but very useful characterization of residuated mappings.

Proposition 2.5 A mapping $f: S \rightarrow S'$ is residuated if and only if it is a morphism with the property that the preimage of any down-interval has supremum in (S, \preceq) and,

$$f\left(\sup f^{-1}(\langle s' \rangle)\right) \preceq' s' \quad (s' \in S'). \quad (21)$$

We shall prove each implication separately.

Necessity. If f is residuated, then $f^{-1}(\langle s' \rangle) = \langle t \rangle$, for some $t \in S$, and therefore $\sup f^{-1}(\langle s' \rangle)$, which coincides with t , exists.

Noting that $\langle t \rangle \subseteq f^{-1}(\langle s' \rangle)$ is equivalent to $f(\langle t \rangle) \subseteq \langle s' \rangle$, we infer that $f(t) \preceq' s'$. This is inequality (21).

Sufficiency. Denote $\sup f^{-1}(\langle s' \rangle)$ by t and note that

$$f^{-1}(\langle s' \rangle) \subseteq \langle t \rangle$$

is the obvious containment $E \subseteq \langle \sup E \rangle$ for $E = \sup f^{-1}(\langle s' \rangle)$.

The preimage $f^{-1}(\langle s' \rangle)$ is a down-set since f is assumed to be a morphism, while inequality (21) means that $t \in f^{-1}(\langle s' \rangle)$. Hence

$$f^{-1}(\langle s' \rangle) \supseteq \langle t \rangle.$$

□

Exercise 43 Formulate the dual of Proposition 2.5.

2.3.4 The residual of a residuated mapping

A residuated mapping $f: S \rightarrow S'$ defines a mapping $g: S' \rightarrow S$ by setting

$$g(s') := t \quad \text{where} \quad f^{-1}(\langle s' \rangle) = \langle t \rangle.$$

Exercise 44 Show that

$$f(s) \preceq' s' \quad \text{if and only if} \quad s \preceq g(s') \quad (s \in S, s' \in S') \quad (22)$$

and deduce that g is a residual mapping.

Exercise 45 Show that, for a given mapping $g: S' \rightarrow S$, there exists a residuated mapping $f: S \rightarrow S'$ such that g is the residual of f if and only if $g^{-1}([s])$ is an up-interval in (S', \preceq') for any $s \in S$. Show that in this case

$$g^{-1}([s]) = [f(s)] \quad (s \in S).$$

2.3.5 Galois connections

2.3.6

A pair of mappings

$$f: S \rightarrow S' \quad \text{and} \quad g: S' \rightarrow S$$

satisfying condition (22) is called a **Galois connection** between partially ordered sets (S, \preceq) and (S', \preceq') .

2.3.7 Terminology

If (f, g) forms a Galois connection, f is referred to as the *lower* (or, *left*) *adjoint* of g , and g is called the *upper* (or, *right*) *adjoint* of f .

Exercise 46 Let (f, g) be a Galois connection between (S, \preceq) and (S', \preceq') , and (h, k) be a Galois connection between (S', \preceq') and (S'', \preceq'') . Show that $(h \circ f, g \circ k)$ is a Galois connection between (S, \preceq) and (S'', \preceq'') .

2.3.8 Duality between residuated and residual mappings

A pair (f, g) is a Galois connection between (S, \preceq) and (S', \preceq') if and only if (g, f) is a Galois connection between $(S', (\preceq')^{\text{rev}})$ and $(S, \preceq^{\text{rev}})$. Reversing simultaneously the orderings on S and S' exchanges the roles of the residual and the residuated mapping.

Exercise 47 Show that

$$\text{id}_S \preceq g \circ f \quad \text{and} \quad f \circ g \preceq' \text{id}_{S'}$$

Exercise 48 Show that if f is an isomorphism between (S, \preceq) and (S', \preceq') , then (f, f^{-1}) is a Galois connection between these sets.

Exercise 49 Show that if (f, g) is a Galois connection between (S, \preceq) and (S', \preceq') , and (g, f) is a Galois connection between (S', \preceq') and (S, \preceq) , then $g = f^{-1}$. In particular, f and g are isomorphisms of partially ordered sets.

Exercise 50 Show that if (f, g) is a Galois connection, then mapping f is residuated and g is the residual of f .

2.3.9

By combining Exercises 44 and 50, we obtain the following proposition.

Proposition 2.6 A mapping $f: S \rightarrow S'$ is residuated if and only if there exists $g: S' \rightarrow S$ such that (f, g) is a Galois connection. In particular, there is a natural correspondence

$$\left\{ \begin{array}{l} \text{residuated mappings} \\ f: (S, \preceq) \rightarrow (S', \preceq') \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Galois connections } (f, g) \\ \text{between } (S, \preceq) \text{ and } (S', \preceq') \end{array} \right\}.$$

2.3.10

Dually, there is a natural correspondence

$$\left\{ \begin{array}{l} \text{Galois connections } (f, g) \\ \text{between } (S, \preceq) \text{ and } (S', \preceq') \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{residual mappings} \\ g: (S', \preceq') \rightarrow (S, \preceq) \end{array} \right\}.$$

Proposition 2.7 Let (f, g) be a Galois connection between partially ordered sets (S, \preceq) and (S', \preceq') . Then, for any $E' \subseteq S'$,

$$\text{sets } g(L(E')) \text{ and } L(g(E')) \text{ are cofinal} \quad (23)$$

and, for any $E \subseteq S$,

$$\text{sets } f(U(E)) \text{ and } U(f(E)) \text{ are coinital.} \quad (24)$$

Proof. Let $s \in L(g(E'))$. Since g is a morphism from (S', \preceq') to (S, \preceq) , one has $f(s) \in L(E')$. Noting that $g(f(s)) \in L(g(E'))$ and combining this observation with inequality $s \preceq g(f(s))$ shows that the set $g(L(E'))$ dominates the set $L(g(E'))$ from above. Since the latter set contains the former, the two sets are cofinal.

Statement (24) is statement (23) for the reverse ordering. \square

Corollary 2.8 Let (f, g) be a Galois connection between partially ordered sets (S, \preceq) and (S', \preceq') . Then, f is sup-continuous and g is inf-continuous.

Equivalently, any residuated mapping is sup-continuous and any residual mapping is inf-continuous. \square

2.3.11

Corollary 2.8 provides necessary conditions for a given mapping to be the left or the right component of a Galois connection. When (S, \preceq) is a complete lattice, those conditions are also sufficient as we will now demonstrate.

Corollary 2.9 Suppose that (S, \preceq) is a complete lattice, cf. 1.3.4, i.e., every subset of S has supremum.² Then a mapping f from S to S' is residuated if and only if it is a sup-continuous morphism $(S, \preceq) \rightarrow (S', \preceq')$.

Proof. In view of Corollary 2.8 it suffices to show that a sup-continuous morphism is residuated. We will demonstrate that f satisfies the equivalent condition of Proposition 2.5.

For any $s' \in S'$, supremum $\sup f^{-1}(\langle s' \rangle)$ of course exists and

$$f\left(\sup f^{-1}(\langle s' \rangle)\right) = \sup f\left(f^{-1}(\langle s' \rangle)\right),$$

by sup-continuity of f , and

$$\sup f\left(f^{-1}(\langle s' \rangle)\right) \preceq' \sup \langle s' \rangle = s',$$

since $f\left(f^{-1}(\langle s' \rangle)\right) \subseteq \langle s' \rangle$. \square

²Note that in that case every subset has also infimum.

2.3.12

Expressed succinctly,

$$\left\{ \begin{array}{c} \text{residuated} \\ \text{mappings} \end{array} \right\} = \left\{ \begin{array}{c} \text{sup-continuous} \\ \text{morphisms} \end{array} \right\}$$

and, by duality,

$$\left\{ \begin{array}{c} \text{residual} \\ \text{mappings} \end{array} \right\} = \left\{ \begin{array}{c} \text{inf-continuous} \\ \text{morphisms} \end{array} \right\}$$

whenever their domain is a complete lattice.

2.4 Galois connections between power sets

2.4.1 sup-continuous morphisms $(\mathcal{P}(X), \subseteq) \longrightarrow (\mathcal{P}(Y), \subseteq)$

Every subset $A \subseteq X$ can be represented as the union of the family of single-element subsets $\mathcal{A} := \{\{x\} \mid x \in A\}$,

$$A = \bigcup_{x \in A} \{x\} = \sup_{\mathcal{P}(X)} \mathcal{A}.$$

Note that \mathcal{A} is a subset of $\mathcal{P}(X)$.

If F is a sup-continuous morphism from $(\mathcal{P}(X), \subseteq)$ to $(\mathcal{P}(Y), \subseteq)$, then

$$F(A) = F(\sup \mathcal{A}) = \sup_{\mathcal{P}(Y)} \{F(\{x\}) \mid x \in A\} = \bigcup_{x \in A} F(\{x\}). \quad (25)$$

This shows that F is uniquely determined by its restriction to single-element subsets of X .

Vice-versa, one can use identity (25) to extend an arbitrary mapping $\varphi: X \longrightarrow \mathcal{P}(Y)$ to a mapping $F: \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$.

Exercise 51 For an arbitrary mapping $\varphi: X \longrightarrow \mathcal{P}(Y)$, let $F: \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ be defined by the formula

$$F(A) := \bigcup_{x \in A} \varphi(x).$$

Show that F is a morphism from $(\mathcal{P}(X), \subseteq)$ to $(\mathcal{P}(Y), \subseteq)$ and is sup-continuous.

2.4.2

We obtain the following natural correspondence

$$\left\{ \begin{array}{c} \text{sup-continuous morphisms} \\ (\mathcal{P}(X), \subseteq) \longrightarrow (\mathcal{P}(Y), \subseteq) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{mappings} \\ X \longrightarrow \mathcal{P}(Y) \end{array} \right\}. \quad (26)$$

2.4.3

Any mapping $\varphi: X \longrightarrow \mathcal{P}(Y)$ defines a subset of $X \times Y$,

$$\Gamma_\varphi := \bigcup_{x \in X} (\{x\} \times \varphi(x)).$$

We shall refer to Γ_φ as the **graph** of φ .

Vice-versa, any subset Γ of $X \times Y$ is of this form for a unique $\varphi: X \longrightarrow \mathcal{P}(Y)$, namely

$$\varphi(x) := \{y \in Y \mid y \in \varphi(x)\}.$$

Recall that subsets of $X \times Y$ correspond to binary relations between elements of X and elements of Y ,

$$\left\{ \begin{array}{c} \text{binary relations between} \\ \text{elements of } X \text{ and elements of } Y \end{array} \right\} \longleftrightarrow \mathcal{P}(X \times Y),$$

via the identification

$$\text{a relation } \sim \longleftrightarrow \{(x, y) \in X \times Y \mid x \sim y\}.$$

In terms of the corresponding relation the associated mapping $F: \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ can be expressed as

$$F(A) = \{y \in Y \mid \text{there exists } x \in A \text{ such that } x \sim y\}. \quad (27)$$

Let us denote this mapping by F_\sim . The correspondence

$$\text{a relation } \sim \longleftrightarrow F_\sim$$

thus provides a natural correspondence between binary relations and sup-continuous mappings of the power sets involved

$$\left\{ \begin{array}{c} \text{binary relations between} \\ \text{elements of } X \text{ and elements of } Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{sup-continuous morphisms} \\ (\mathcal{P}(X), \subseteq) \longrightarrow (\mathcal{P}(Y), \subseteq) \end{array} \right\}.$$

2.4.4 inf-continuous morphisms $(\mathcal{P}(Y), \subseteq) \longrightarrow (\mathcal{P}(X), \subseteq)$

Every subset $E' \subseteq Y$ can be obtained from Y by removing all elements $y \notin B$. In other words, B is the intersection of the family $\mathcal{B}' := \{Y \setminus \{y\} \mid y \notin B\}$ of *complements* of single-element subsets formed by elements of Y *not belonging* to B ,

$$B = \bigcap_{y \notin B} (Y \setminus \{y\}) = \bigcap_{y \notin B} \{y\}^c = \inf_{\mathcal{P}(X)} \mathcal{B}'.$$

If G is an inf-continuous morphism from $(\mathcal{P}(Y), \subseteq)$ to $(\mathcal{P}(X), \subseteq)$, then

$$G(B) = G(\inf \mathcal{B}') = \inf_{\mathcal{P}(X)} \{G(\{y\}^c) \mid y \notin B\} = \bigcap_{y \notin B} G(\{y\}^c). \quad (28)$$

Like before, an inf-continuous mapping G is uniquely determined by its restriction to the *complements* of single-element subsets of X . And vice-versa, one can use identity (28) to extend an arbitrary mapping $\psi: Y \longrightarrow \mathcal{P}(X)$ to a mapping $G: \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ where set Y is identified with a subset of $\mathcal{P}(Y)$ via the order-reversing embedding

$$y \longmapsto \{y\}^c = Y \setminus \{y\}.$$

Exercise 52 For an arbitrary mapping $\psi: Y \longrightarrow \mathcal{P}(X)$, let $G: \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ be defined by the formula

$$G(B) := \bigcap_{y \notin B} \psi(y).$$

Show that G is a morphism from $(\mathcal{P}(Y), \subseteq)$ to $(\mathcal{P}(X), \subseteq)$ and is inf-continuous.

2.4.5

We obtain the following natural correspondence

$$\left\{ \begin{array}{l} \text{inf-continuous morphisms} \\ (\mathcal{P}(Y), \subseteq) \longrightarrow (\mathcal{P}(X), \subseteq) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{mappings} \\ Y \longrightarrow \mathcal{P}(X) \end{array} \right\}.$$

The $G \leftrightarrow \psi$ correspondence is *dual* to the $F \leftrightarrow \varphi$ correspondence, cf. (26). In the latter X is identified with the subset of all single-element subsets of X while in the former Y is identified with the set of *complements* of single-element subsets of Y .

2.4.6 The relation associated with an inf-continuous mapping

Given an inf-continuous mapping G from $(\mathcal{P}(Y), \subseteq)$ to $(\mathcal{P}(X), \subseteq)$, define the binary relation between elements of X and Y by

$$x \sim y \quad \text{if} \quad x \notin \psi(y) \quad (x \in X, y \in Y), \quad (29)$$

where $\psi(y) = G(\{y\}^c)$. In terms of relation (29), mapping $G: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ can be expressed as

$$G(B) = \{x \in X \mid \text{for every } y \notin B \text{ one has } x \approx y\} \quad (30)$$

where \approx denotes the *negation* of relation \sim ,

$$x \approx y \quad \text{if and only if} \quad \neg(x \sim y).$$

Let us denote this mapping by G^\sim . The correspondence

$$\text{a relation } \sim \quad \longleftrightarrow \quad G^\sim$$

thus provides a natural correspondence between binary relations and inf-continuous mappings of the power sets involved

$$\left\{ \begin{array}{l} \text{binary relations between} \\ \text{elements of } X \text{ and elements of } Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{inf-continuous morphisms} \\ (\mathcal{P}(Y), \subseteq) \rightarrow (\mathcal{P}(X), \subseteq) \end{array} \right\}.$$

2.4.7 Galois connections

According to Corollary 2.9, residuated mappings between power sets are precisely the sup-continuous mappings and residual mappings are precisely inf-continuous mappings. We described the sup- and inf-continuous mappings above, and we will use these descriptions to describe Galois connections between power sets.

Exercise 53 Let \sim be a binary relation between elements of a set X and elements of a set Y . Show that (F, G) is a Galois connection between $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$ where F is defined by equality (27) and G is defined by equality (30).

In view of the already established fact that every residuated mapping F between $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$ is of the form (27) for a unique binary relation \sim , we obtain the following

Proposition 2.10 *There is a natural correspondence*

$$\left\{ \begin{array}{l} \text{Galois connections } (F, G) \text{ between} \\ (\mathcal{P}(X), \subseteq) \text{ and } (\mathcal{P}(Y), \subseteq) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{binary relations } \sim \text{ between} \\ \text{elements of } X \text{ and } Y \end{array} \right\}$$

where the pair of mappings, (F, G) ,

$$A \xrightarrow{F} \{y \in Y \mid \exists_{x \in A} x \sim y\} \quad (31)$$

$$\{x \in X \mid \forall_{y \notin B} x \not\sim y\} \xleftarrow{G} B$$

corresponds to a relation \sim .³

2.4.8 The Galois connection associated with a mapping

Exercise 54 Let $f: X \rightarrow Y$ be an arbitrary mapping between sets X and Y . Show that the pair of mappings

$$F: \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \quad A \mapsto f(A) \quad (A \subseteq X),$$

and

$$G: \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad B \mapsto f^{-1}(B) \quad (B \subseteq Y),$$

forms a Galois connection between $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$. It corresponds to the graph-of-a-function relation

$$x \sim y \quad \text{if and only if} \quad y = f(x) \quad (x \in X, y \in Y).$$

2.4.9 The (Conjunction, Implication) Galois connection

Exercise 55 Given a subset B of an arbitrary set X , let

$$F: A \mapsto A \cap B \quad \text{and} \quad G: C \mapsto B \Rightarrow C \quad (A, C \subseteq X)$$

where

$$B \Rightarrow C := B^c \cup C = (X \setminus B) \cup C.$$

Show that (F, G) is a Galois connection on $(\mathcal{P}(X), \subseteq)$. It corresponds to the relation

$$x \sim y \quad \text{if and only if} \quad x = y \in B \quad (x \in X, y \in Y).$$

³To make notation more compact we used symbolic notation: \exists ("there exists") and \forall ("for all").

2.5 Order reversing Galois connections between power sets

2.5.1

A Galois connection (Φ, Ψ) between $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \supseteq)$ is a pair of mappings $\Phi: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ and $\Psi: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ such that

$$\Phi(A) \subseteq B \quad \text{if and only if} \quad \Psi(B) \subseteq A \quad (A \subseteq X, B \subseteq Y)$$

(note the symmetry between Φ and Ψ).

Since *passing-to-the-complement* induces an isomorphism between partially ordered sets $(\mathcal{P}(Y), \subseteq)$ and $(\mathcal{P}(Y), \supseteq)$, whose inverse is given by the same operation, by taking into account that composition of Galois connections is a Galois connection, cf. Exercise 46, we obtain from (Φ, Ψ) the Galois connection (F, G) ,

$$F := (\)^c \circ \Phi \quad \text{and} \quad G := \Psi \circ (\)^c,$$

between $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$.

All such connections were described in Proposition 2.10. Noting that

$$\Phi = (\)^c \circ F \quad \text{and} \quad \Psi = G \circ (\)^c,$$

we arrive at the complete description of arbitrary order reversing Galois connections between power sets.

Corollary 2.11 *There is a natural correspondence*

$$\left\{ \begin{array}{l} \text{Galois connections } (\Phi, \Psi) \text{ between} \\ (\mathcal{P}(X), \subseteq) \text{ and } (\mathcal{P}(Y), \supseteq) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{binary relations } \sim \text{ between} \\ \text{elements of } X \text{ and } Y \end{array} \right\} \quad (32)$$

where the pair of mappings, (Φ, Ψ) ,

$$A \xrightarrow{\Phi} \{y \in Y \mid \forall_{x \in A} x \sim y\} \quad (33)$$

$$\{x \in X \mid \forall_{y \in B} x \sim y\} \xleftarrow{\Psi} B$$

corresponds to a relation \sim .

Proof. If $F = (\)^c \circ \Phi$ and $G = \Psi \circ (\)^c$ are given by (31), then

$$\Phi(A) = \{y \in Y \mid \forall_{x \in A} x \approx y\} \quad \text{and} \quad \Psi(B) = \{x \in X \mid \forall_{y \in B} x \approx y\}$$

and every Galois connection between $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \supseteq)$ is of this form for a unique binary relation \sim . By composing this correspondence between Galois connections and relations with the involution on the set of relations

$$\sim \longleftrightarrow \approx,$$

we obtain correspondence (32)–(33). \square

2.5.2 The canonical Galois connection associated with a partial order

Note that, for any pair of subsets A and B of a partially ordered set (S, \preceq) , one obviously has

$$E \subseteq U(B) \quad \text{if and only if} \quad L(A) \supseteq B.$$

This means that (U, L) forms a Galois connection between $(\mathcal{P}(S), \subseteq)$ and $(\mathcal{P}(S), \supseteq)$. We shall refer to it as the *canonical* Galois connection of (S, \preceq) . Fittingly, Galois connection (U, L) corresponds, under (32)–(33), to relation \preceq .

2.6 Closure and interior operations

2.6.1 The floor mapping

Let (S, \preceq) be a partially ordered set and E be a subset of S . It follows from Proposition 2.5 that the canonical inclusion $\iota_E: E \hookrightarrow S$ is a residuated mapping $(E, \preceq) \rightarrow (S, \preceq)$ if and only if, for every $s \in S$,

$$\langle s \rangle \cap E \text{ is a down-interval in } (E, \preceq),$$

i.e., $\langle s \rangle \cap E$ has a maximum element. In this case we shall say that E is a *floor* subset of (S, \preceq) , and the right adjoint

$$s \longmapsto \lfloor s \rfloor_E := \max \langle s \rangle \cap E,$$

will be called the *floor mapping* associated with subset E of (S, \preceq) .

2.6.2 The ceiling mapping

Similarly, the inclusion map, $E \hookrightarrow S$, is residual if and only if

$$[s] \cap E \text{ is an up-interval in } (E, \preceq),$$

i.e., $[s) \cap E$ has a minimum element. In this case we shall say that E is a *ceiling* subset of (S, \preceq) , and the left adjoint

$$s \mapsto [s]^E := \min [s) \cap E,$$

will be called the *ceiling mapping* associated with subset E of (S, \preceq) .

2.6.3 Closure operations on a partially ordered set

Definition 2.12 Let (S, \preceq) be a partially ordered set. A self-mapping

$$s \mapsto \bar{s}, \quad S \longrightarrow S,$$

is said to be a **closure operation** if it enjoys the following three properties

$$s \preceq \bar{s}, \tag{34}$$

$$\text{if } s \preceq t, \text{ then } \bar{s} \preceq \bar{t}, \tag{35}$$

and

$$\bar{\bar{s}} = \bar{s} \tag{36}$$

where s and t denote arbitrary elements of S .

In this case, we say that an element $s \in S$ is **closed** if $\bar{s} = s$.

2.6.4

Denote by $Z \subseteq S$ the set of closed elements. If $z \in [s) \cap Z$, then $s \preceq z$, and hence

$$\bar{s} \preceq \bar{z} = z,$$

in view of equality (36). But $\bar{s} \in [s)$, in view of inequality (34), and $\bar{s} \in Z$, in view of equality (36) again. Thus

$$\bar{s} = \min([s) \cap Z)$$

which means that Z is a ceiling subset of (S, \preceq) and the closure operation is the associated ceiling mapping

$$\bar{s} = [s]^Z \quad (s \in S).$$

Exercise 56 Let Z be a ceiling subset of a partially ordered set (S, \preceq) . Show that

$$s \mapsto \lceil s \rceil^Z \quad (s \in S)$$

satisfies conditions (34)–(36), i.e. it defines a closure operation on (S, \preceq) .

We obtain

Proposition 2.13 *There is a natural correspondence between closure operations on a partially ordered set (S, \preceq) and ceiling subsets of (S, \preceq) ,*

$$\left\{ \begin{array}{l} \text{ceiling subsets} \\ Z \subseteq S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{closure operations} \\ \text{on } (S, \preceq) \end{array} \right\},$$

where

$$Z \longleftrightarrow \lceil \rceil^Z.$$

2.6.5

Let $Z' \subseteq Z$ be any subset of the set of closed elements, and suppose that $\zeta = \inf Z'$ exists. Then, for any $z' \in Z'$, the inequality $\zeta \preceq z'$ implies that

$$\bar{\zeta} \preceq z' = z',$$

i.e., $\bar{\zeta}$ is a lower bound of set Z' . In particular, $\bar{\zeta} \preceq \zeta$. By combining this with inequality (34), we obtain $\bar{\zeta} = \zeta$, i.e., $\inf Z'$ is a closed element of S . Note that this argument applies also for $Z' = \emptyset$ since in this case $\inf Z'$ is the greatest element of (S, \preceq) and the greatest element is closed.

Exercise 57 Show that the greatest element of (S, \preceq) is closed when it exists.

We established the following interesting fact

Proposition 2.14 *For any subset $Z' \subseteq Z$ of the set of closed elements of a closure operation, if $\inf Z'$ exists in (S, \preceq) , then it belongs to Z . \square*

2.6.6 Interior operations on a partially ordered set

Closure operations for the reversed order are sometimes called **interior operations**. In this case, notation $s \mapsto \mathring{s}$ or s° is used and elements $s \in S$ such that $\mathring{s} = s$ are said to be **open**.

Exercise 58 State the dual version of Proposition 2.14 for an interior operation and the associated set T of open elements.

2.6.7 Example: closure and interior operations associated with a Galois connection

Let (f, g) be a Galois connection between partially ordered sets (S, \preceq) and (S', \preceq') .

Exercise 59 Show that

$$f \circ g \circ f = f \quad \text{and} \quad g \circ f \circ g = g.$$

Exercise 60 Show that

$$s \mapsto (g \circ f)(s) \quad (s \in S)$$

is a closure operation on (S, \preceq) and

$$s' \mapsto (f \circ g)(s') \quad (s' \in S')$$

is an interior operation on (S', \preceq') .

2.6.8

It follows that a Galois connection (f, g) between partially ordered sets (S, \preceq) and (S', \preceq') induces a closure operation on (S, \preceq) and an interior operation on (S', \preceq') . Denote by $Z \subseteq S$ and by $T' \subseteq S'$ the corresponding sets of closed and, respectively, open elements.

Exercise 61 Show that an element $s \in S$ is closed if and only if s is in the image of g , i.e., $s = g(s')$ for some $s' \in S'$. Likewise, show that an element $s' \in S'$ is open if and only if s' is in the image of f , i.e., $s' = f(s)$ for some $s \in S$.

Exercise 62 Show that the restriction of f to Z and the restriction of g to T' are mutually inverse. In particular, pair (f, g) induces an isomorphism of partially ordered sets (Z, \preceq) and (T', \preceq') .

2.6.9

Note that if we denote the induced isomorphism from (Z, \preceq) to (T', \preceq') by ϕ , then

$$f = \phi \circ \lceil \rceil^Z \quad \text{and} \quad g = \phi^{-1} \circ \lfloor \rfloor_{T'}.$$

Indeed, $g \circ f = \lceil \rceil^Z$ and $f \circ g = \lfloor \rfloor_{T'}$, while $\phi = f|_{g(S')}$ and $\phi^{-1} = g|_{f(S)}$. Hence

$$\phi \left(\lceil s \rceil^Z \right) = f(g(f(s))) = f(s) \quad \text{and} \quad \phi^{-1} \left(\lfloor s' \rfloor_{T'} \right) = g(f(g(s'))) = g(s').$$

Exercise 63 Let Z be an arbitrary ceiling subset of a partially ordered set (S, \preceq) , let T' be an arbitrary floor subset of a partially ordered set (S', \preceq') , and let ϕ be an arbitrary isomorphism between partially ordered sets (Z, \preceq) and (T', \preceq') . Show that the pair of mappings

$$f := \phi \circ \lceil \rceil^Z \quad \text{and} \quad g := \phi^{-1} \circ \lfloor \rfloor_{T'} \quad (37)$$

forms a Galois connection between (S, \preceq) and (S', \preceq') .

By combining Exercise 63 with the previous argument, cf. 2.6.9, we obtain

Proposition 2.15 *There is a natural correspondence between Galois connections and triples (Z, T', ϕ)*

$$\left\{ \begin{array}{l} \text{Galois connections } (f, g) \\ \text{between } (S, \preceq) \text{ and } (S', \preceq') \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{triples} \\ (Z, T', \phi) \end{array} \right\}, \quad (38)$$

where $Z \subseteq S$ denotes any ceiling subset, $T' \subseteq S'$ denotes any floor subset, and ϕ is an arbitrary isomorphism between partially ordered sets (Z, \preceq) and (T', \preceq') . Correspondence (38) associates with a Galois connection (f, g) the triple $(g(S'), f(S), f|_{g(S')})$. The inverse correspondence associates with a triple (Z, T', ϕ) the pair (f, g) defined in (37).

2.6.10 Example: a structure of a topological space on a set

To equip a set X with a structure of a topological space is the same as to equip $(\mathcal{P}(X), \subseteq)$ with the closure operation which is *inf-continuous*,

for any family $\mathcal{E} \subseteq \mathcal{P}(X)$, one has

$$\overline{\inf \mathcal{E}} = \inf \bar{\mathcal{E}}$$

where $\bar{\mathcal{E}}$ is the family of the closures of subsets belonging to \mathcal{E} .

and *finitely sup-continuous*,

for any *finite* family $\mathcal{E} \subseteq \mathcal{P}(X)$, one has

$$\overline{\sup \mathcal{E}} = \sup \bar{\mathcal{E}}. \quad (39)$$

Exercise 64 Show that for any closure operation on $(\mathcal{P}(X), \subseteq)$ which satisfies (39), one has

$$\overline{\emptyset} = \emptyset.$$

Hint. Consider the empty family of subsets.

2.6.11

An equivalent way to equip a set X with a structure of a topological space is to equip $(\mathcal{P}(X), \subseteq)$ with an interior operation which is *sup-continuous*,

for any family $\mathcal{E} \subseteq \mathcal{P}(X)$, one has

$$\sup \mathcal{E}^\circ = \sup \mathring{\mathcal{E}}$$

where $\mathring{\mathcal{E}}$ is the family of the interiors of subsets belonging to \mathcal{E} .

and *finitely inf-continuous*,

for any *finite* family $\mathcal{E} \subseteq \mathcal{P}(X)$, one has

$$\inf \mathcal{E}^\circ = \inf \mathring{\mathcal{E}}.$$

2.7 Closure and interior operations on the power set of a partially ordered set

2.7.1 Down- and up-closure operations

Both

$$E \mapsto \text{CI}^\downarrow(E) \quad \text{and} \quad E \mapsto \text{CI}^\uparrow(E)$$

are closure operations. The first one is *sup-continuous*, the second one is *inf-continuous*.

2.7.2 LU- and UL-closure operations

A partial ordering (S, \preceq) induces two closure operations on $(\mathcal{P}(S), \subseteq)$, the *LU-closure*,

$$E \mapsto \bar{E}^{LU} := LU(E),$$

and the *UL-closure*,

$$E \mapsto \bar{E}^{UL} := UL(E).$$

Indeed, for the *LU-closure*, (3) is property (34), property (35) follows from the combination of (5) and (6), while (9) implies property (36).

Note that, according to (9), a subset F of S is *LU-closed* if and only if it is of the form $F = L(E)$ for some $E \subseteq S$.

Note that

$$LU(\emptyset) = \begin{cases} \emptyset & \text{if } S \text{ has no least element} \\ \{\min S\} & \text{otherwise.} \end{cases}$$

Dually,

$$UL(\emptyset) = \begin{cases} \emptyset & \text{if } S \text{ has no largest element} \\ \{\max S\} & \text{otherwise.} \end{cases}$$

It follows that \emptyset is *LU-closed* (respectively, *UL-closed*) if and only if S has no least element (respectively, no largest element).

On the other hand, S is always both *LU*- and *UL*-closed.

3 Dedekind-MacNeille Completion of a Partially Ordered Set

3.1 The set of *LU*-closed subsets as a partially ordered set

3.1.1

Let (S, \preceq) be a partially ordered set. Let $\mathcal{L}(S) \subseteq \mathcal{P}(S)$ be the subset of the power set which consists of all *LU*-closed subsets of S . Recall that $F \subseteq S$ is *LU*-closed if and only if it has the form

$$F = L(E)$$

for some $E \subseteq S$. In fact, one can take $E = U(F)$.

Exercise 65 Show that, for any family $\mathcal{E} \subseteq \mathcal{L}(S)$ of *LU*-closed subsets of S , the supremum of \mathcal{E} in $\mathcal{L}(S)$ exists and equals

$$\sup \mathcal{E} = LU\left(\bigcup_{E \in \mathcal{E}} E\right),$$

i.e., $\sup_{\mathcal{Z}(S)} \mathcal{E}$ is the LU-closure of $\sup_{\mathcal{P}(S)} \mathcal{E}$,

$$\sup_{\mathcal{Z}(S)} \mathcal{E} = \overline{\sup_{\mathcal{P}(S)} \mathcal{E}}^{LU}.$$

Exercise 66 Show that

$$\min \mathcal{Z}(S) = \begin{cases} \emptyset & \text{if } S \text{ has no least element} \\ \{\min S\} & \text{otherwise} \end{cases}.$$

Thus, every family $\mathcal{E} \subseteq \mathcal{Z}(S)$ is bounded below and, obviously, the infimum of the empty family exists and is equal to

$$\inf \emptyset = \max \mathcal{Z}(S) = S.$$

By invoking Lemma 1.8 we deduce that the infimum of every family $\mathcal{E} \subseteq \mathcal{Z}(S)$ exists in $\mathcal{Z}(S)$, and we established the following fact.

Proposition 3.1 For any partially ordered set (S, \preceq) , the set of LU-closed subsets, $(\mathcal{Z}(S), \subseteq)$ is a complete lattice. \square

Exercise 67 Show that in $\mathcal{Z}(S)$ one has

$$\sup \langle E \rangle = LU(E)$$

and

$$\inf \langle E \rangle = L(E).$$

3.1.2

Since intervals $\langle s \rangle$ are LU-closed,

$$\langle s \rangle = L(\{s\}),$$

the image of set S under the canonical embedding, (19), is contained in $\mathcal{Z}(S)$.

Exercise 68 Show that $\langle S \rangle$ is both sup- and inf-dense in $\mathcal{Z}(S)$.

Exercise 69 Show that the embedding

$$\langle \cdot \rangle: (S, \preceq) \hookrightarrow (\mathcal{Z}(S), \subseteq) \quad (40)$$

is both sup- and inf-continuous.

In other words, show that in $\mathcal{Z}(S)$ one has

$$\sup_{\mathcal{Z}(S)} \langle E \rangle = \langle \sup E \rangle$$

whenever $\sup E$ exists in (S, \preceq) , and

$$\inf_{\mathcal{Z}(S)} \langle E \rangle = \langle \inf E \rangle$$

whenever $\inf E$ exists in (S, \preceq) .

By combining the results of Exercises 68 and 69 with Proposition 3.1, we establish the following important fact.

Theorem 3.2 Any partially ordered set (S, \preceq) admits an embedding onto a dense subset of a complete lattice, and this embedding preserves suprema and infima (i.e., is sup- and inf-continuous). \square

The lattice $\mathcal{Z}(S), \subseteq$ is called the **Dedekind-Macneille envelope** of a partially ordered set (S, \preceq) , or the **Dedekind-Macneille completion** of (S, \preceq) . Strictly speaking, the latter should apply to the embedding, (40), of (S, \preceq) into $\mathcal{Z}(S), \subseteq$.

It is worth noticing that for any subset $E \subseteq S$ which is not bounded above in (S, \preceq) , one has

$$\sup \langle E \rangle = \max \mathcal{Z}(S) = S$$

and, for a subset not bounded below

$$\inf \langle E \rangle = \min \mathcal{Z}(S) = \emptyset$$

(note that $\min \mathcal{Z}(S) = \emptyset$ in this case).

3.2 An important variant

3.2.1

Dedekind-MacNeille completion adds to any partially ordered set all the “missing” suprema and infima, including the suprema and infima for unbounded sets.

A simple modification provides a variant of the above construction allows to “add” only the suprema of nonempty subsets that are *bounded above*, and the infima of nonempty subsets which are *bounded below* while leaving unbounded sets not bounded.

Let (S, \preceq) be a partially ordered set. Let us consider the subset $\mathcal{L}^*(S) \subseteq \mathcal{L}(S)$ which consists of the *LU*-closures of nonempty subsets of S which are bounded above.

Exercise 70 Show that $\emptyset \notin \mathcal{L}^*(S)$.

Exercise 71 Show that $S \in \mathcal{L}^*(S)$ if and only if S has the largest element.

It follows that

$$\mathcal{L}(S) = \begin{cases} \mathcal{L}^*(S) & \text{if both } \min S \text{ and } \max S \text{ exist} \\ \mathcal{L}^*(S) \cup \{S\} & \text{if } \min S \text{ exists and } \max S \text{ does not} \\ \mathcal{L}^*(S) \cup \{\emptyset\} & \text{if } \max S \text{ exists and } \min S \text{ does not} \\ \mathcal{L}^*(S) \cup \{\emptyset, S\} & \text{if neither } \min S \text{ nor } \max S \text{ exist} \end{cases}$$

Thus, $\mathcal{L}^*(S)$ has the same elements as $\mathcal{L}(S)$ except possibly for the two extremal ones: \emptyset and S .

Exercise 72 Show that $\langle S \rangle \subseteq \mathcal{L}^*(S)$.

By Exercise 36, suprema and infima in $\mathcal{L}^*(S)$ coincide for nonempty bounded subsets of $\mathcal{L}^*(S)$ with those in $\mathcal{L}(S)$, hence $(\mathcal{L}^*(S), \subseteq)$ is a complete partially ordered set.

In particular, the embedding

$$\langle \rangle: (S, \preceq) \longrightarrow (\mathcal{L}^*(S), \subseteq)$$

is both sup- and inf-continuous.

Finally, since $\langle S \rangle$ is sup- and inf-dense in $\mathcal{L}(S)$ it is dense also in $\mathcal{L}^*(S)$.

We established thus the following variant of Theorem 3.2.

Theorem 3.3 Any partially ordered set (S, \preceq) admits an embedding onto a dense subset of a complete partially ordered set, and this embedding preserves suprema and infima (i.e., is sup- and inf-continuous), and also unboundedness of subsets:

if $E \subseteq S$ is not bounded above (below) in S , then E is not bounded above (resp., below) in the completion.

□

3.2.2

It follows from Exercise 11 that $(\mathcal{L}(S), \subseteq)$, and therefore also $(\mathcal{L}^*(S), \subseteq)$, are totally ordered if (S, \preceq) is totally ordered. Thus one can strengthen the statements of Theorems 3.2 and 3.3 by adding that the corresponding completions of totally ordered sets are themselves totally ordered.

3.2.3 \mathbb{R} and $\overline{\mathbb{R}} = [-\infty, \infty]$

In the case of (\mathbb{Q}, \leq) we obtain two totally ordered completions

$(\mathcal{L}^*(\mathbb{Q}), \subseteq)$ is a model for the set of real numbers,

while

$(\mathcal{L}(\mathbb{Q}), \subseteq)$ is a model for the set of extended real numbers

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty].$$

4 Universal Properties of the Dedekind-MacNeille Completion

In this section (S, \preceq) denotes a subset of (T, \preceq) .

4.1 Extending the Canonical Embedding

4.1.1

The correspondence

$$t \mapsto \langle t \rangle \cap S = \{s \in S \mid s \preceq t\} \quad (41)$$

defines a morphism $(T, \preceq) \rightarrow (\mathcal{P}(S), \subseteq)$ which extends the canonical embedding, $(S, \preceq) \hookrightarrow (\mathcal{P}(S), \subseteq)$, cf. (19), to (T, \preceq) .

Exercise 73 Show that (41) is an order embedding if (S, \preceq) is sup-dense in (T, \preceq) .

In particular,

morphism (41) is injective if (S, \preceq) is sup-dense in (T, \preceq) .

If (S, \preceq) is *not* sup-dense, then (41) is generally not injective.

Thus,

$$\langle v \rangle \cap S = \langle \zeta \rangle \cap S$$

in Example 1.6.2.a but $v \neq \zeta$.

4.1.2

If $t = \inf_T E$ for some $E \subseteq S$, then

$$\langle t \rangle \cap S = L(E) \in \mathcal{L}(S),$$

according to (15). It follows that

*the image of morphism (41) is contained in $\mathcal{L}(S)$
if (S, \preceq) is inf-dense in (T, \preceq) .*

The following example demonstrates that the image of morphism (41) may be contained in $\mathcal{L}(S)$ even though *no* element of $T \setminus S$ may be of the form $\inf_T E$ for some $E \subseteq S$.

4.1.3 Example

Let $T := \mathbb{Q}\{0, 1\}$ equipped with the usual order. Let

$$S := \{x \in \mathbb{Q} \mid x < 0 \text{ or } x > 1\} = \mathbb{Q}_{<0} \cup \mathbb{Q}_{>1}.$$

Then, all $t \in T \setminus S = \{x \in \mathbb{Q} \mid 0 < x < 1\}$ are being sent by morphism (41) to one and the same *LU*-closed subset of S :

$$\langle t \rangle \cap S = \mathbb{Q}_{<0} = L(\mathbb{Q}_{>1}).$$

Exercise 74 *Show that every $t \in T \setminus S$ is neither of the form $\inf_T E$ nor of the form $\sup_T E$ for some $E \subseteq S$.*

4.1.4

When (S, \preceq) is inf-dense in (T, \preceq) , then one can easily describe the image of T under morphism (41):

a subset F of $\mathcal{Z}(S)$ is in the image if and only if it is of the form

$$\langle \inf_T E \rangle \cap S = L_T(E) \cap S = L(E) \quad (42)$$

for some $E \subseteq S$ which possesses infimum in T .

This is so since every element of T equals $\inf_T E$ for a suitable subset $E \subseteq S$ and then $\langle t \rangle = L_T(E)$.

Note, however, that $L(E)$, for a particular subset E , may belong to the image of morphism (41) while $\inf_T E$ may not exist. We only know that there must be another subset $E' \subseteq S$ such that

$$L(E) = L(E') \quad \text{and} \quad \inf_T E' \text{ exists.}$$

In Example 1.6.2.b, one has

$$L(Z) = \langle v \rangle \cap S = L(Y)$$

where Y possesses infimum in the larger set, which is denoted there T' , while Z does not.

4.1.5

By combining (42) with the assertion of Exercise 73, we deduce the following important universal property of the Dedekind-MacNeille completion.

Theorem 4.1 *If (S, \preceq) is a subset of (T, \preceq) which is both inf- and sup-dense, then morphism (41) embeds (T, \preceq) isomorphically onto the subset of $\mathcal{Z}(S)$:*

$$\{F \in \mathcal{Z}(S) \mid F = L(E) \text{ for some } E \subseteq S \text{ which possesses infimum in } T\}.$$

In particular, if every subset of S possesses infimum in T , then (41) establishes a canonical isomorphism between (T, \preceq) and the Dedekind-MacNeille completion, $(\mathcal{Z}(S), \subseteq)$ which extends the embedding of (S, \preceq) into $(\mathcal{Z}(S), \subseteq)$. \square

4.2 Completing a subset in a bigger partially ordered set

4.2.1

In general, there is no largest subset $T' \subseteq T$ such that (S, \preceq) would be sup-dense, inf-dense, or both sup- and inf-dense in (T', \preceq) .

4.2.2 Example

Let $S = \mathbb{Q} \setminus \{0\}$ be equipped with usual order and $T = S \cup \{a, b\}$ with

$$x < a < y \quad \text{and} \quad x < b < y$$

for any $x \in \mathbb{Q}_{<0}$ and $y \in \mathbb{Q}_{>0}$. In this case, S is both sup- and inf-dense in $T_1 = S \cup \{a\}$ and $T_2 = S \cup \{b\}$ but is neither sup-dense nor inf-dense in $T = T_1 \cup T_2$.

Exercise 75 Show that neither a nor b is of the form $\inf_T E$ or $\sup_T E$ for any subset $E \subseteq S$.

Thus, T_1 and T_2 are two distinct *maximal* subsets of T in which S is dense in any of the three spelled out senses.

4.2.3

For any partially ordered set (T, \preceq) containing (S, \preceq) , there is, however, a subset that is a perfect analog of the Dedekind-MacNeille completion:

$$\widehat{S}_T := \{t \in T \mid t = \inf_T U(E) \text{ for some } E \subseteq S\}.$$

If we identify S with its isomorphic image $\langle S \rangle$ in $\mathcal{P}(S)$, then

$$\widehat{S}_{\mathcal{P}(S)} = \mathcal{L}(S)$$

since every $F \in \mathcal{L}(S)$ is of the form

$$F = LU(E) = \inf_{\mathcal{P}(S)} U(E),$$

cf. equalities (9) and (20).

4.2.4

If $t = \inf_T U(E)$, then

$$E \subseteq LU(E) = L_T(U(E)) \cap S \subseteq L_T(U(E)) = \langle t \rangle$$

This shows that $t \in U_T(E)$.

If another element of $\widehat{S_T}$, say $t' = \inf_T U(E')$, belongs to $U_T(E)$, then

$$x \preceq t' \preceq y$$

for any $x \in E$ and $y \in U(E')$. It follows that

$$U(E') \subseteq U(E)$$

and hence

$$t = \inf_T E \preceq \inf_T E' = t',$$

cf. (7). Thus, t is the smallest element of $U_T(E) \cap \widehat{S_T} = U_{\widehat{S_T}}(E)$, i.e., $\sup_{\widehat{S_T}} E$ exists and equals t :

$$\sup_{\widehat{S_T}} E = \inf_T U(E).$$

Note that, in view of (14), the infimum of $U(E)$ in $\widehat{S_T}$ exists and coincides with $\inf_T U(E)$,

$$\inf_{\widehat{S_T}} U(E) = \inf_T U(E).$$

In particular,

$$\sup_{\widehat{S_T}} E = \inf_{\widehat{S_T}} U(E).$$

We established the following fact.

Proposition 4.2 *Any subset S of a partially ordered set (T, \preceq) is both sup- and inf-dense in $(\widehat{S_T}, \preceq)$. \square*

4.2.5

By the Universal Property of the Dedekind-MacNeille completion cf. Theorem 4.1, the order embedding of (S, \preceq) into $\mathcal{Z}(S), \subseteq$ then extends to an order embedding

$$(\widehat{S_T}, \preceq) \hookrightarrow (\mathcal{Z}(S), \subseteq). \quad (43)$$

4.2.6

If (T, \preceq) is a complete lattice, then embedding (43) is *surjective* since, for every $E \subseteq S$, one has

$$LU(E) = \langle \inf_T U(E) \rangle \cap S,$$

cf. Exercise 24.

In particular, (S_T^\wedge, \preceq) is isomorphic to the Dedekind-MacNeille completion.

This way we arrive at our final result.

Theorem 4.3 *Any order embedding $(S, \preceq) \hookrightarrow (T, \preceq)$ into a complete lattice induces an order embedding of the Dedekind-MacNeille completion,*

$$(\mathcal{L}(S), \subseteq) \hookrightarrow (T, \preceq)$$

which identifies $(\mathcal{L}(S), \subseteq)$ with (S_T^\wedge, \preceq) . In particular, (S_T^\wedge, \preceq) is itself a complete lattice. \square