Notes on the concepts of "space" and continuity

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1 Three natural ways to form the category of sets

1.1 The category of sets and mappings

1.1.1

The objects of the standard category of sets, which will be denoted Set, is formed by the class of all sets and the set of morphisms from a set X to a set Y is the set of all mappings from X to Y,

$$\mathrm{Hom}_{\mathrm{Set}}(X,Y) := Y^X = \{\text{mappings } f\colon X \to Y\}.$$

1.1.2

The word "mapping" is often abbreviated to "map".

1.2 The category of sets and multivalued maps

1.2.1 Multivalued maps

A *multivalued* map $\phi\colon X \multimap Y$ from a set X to a set Y, is a map

$$\phi\colon X \longrightarrow \mathscr{P}(Y).$$

Multivalued maps will be also called *multimaps*.

1.2.2 Maps versus multimaps

Every map $f\colon X \to Y$ defines the multimap

$$x \mapsto \phi_f(x) := \{f(x)\} \qquad (x \in X).$$

The correspondence $f \mapsto \phi_f$ identifies maps $f\colon X \to Y$ with multimaps $\phi\colon X \multimap Y$ satisfying the property

$$|\phi(x)| = 1 \qquad (x \in X). \tag{1}$$

1.2.3 Composition of multimaps

Given multimaps $\phi\colon X \multimap Y$ and $\psi\colon W \multimap X$, we denote by $\phi \circ \psi$ the multimap

$$w \longmapsto (\phi \circ \psi)(w) := \bigcup_{x \in \psi(w)} \phi(x), \qquad (w \in W). \tag{2}$$

1.2.4 Associativity of the composition of multimaps

Exercise 1 *Gven $\upsilon \colon W \multimap X$, $\phi \colon X \multimap Y$ and $\chi \colon Y \multimap Z$, show that*

$$(\chi \circ \phi) \circ \upsilon = \chi \circ (\phi \circ \upsilon).$$

1.2.5 $\mathbf{Set}_{\mathbf{mult}}$

The objects of $\mathrm{Set}_{\mathrm{mult}}$ are sets and

$$\mathrm{Hom}_{\mathrm{Set}_{\mathrm{mult}}}(X, Y) := \{\text{multimaps } \phi \colon X \multimap Y\}.$$

Exercise 2 *Show that the canonical embedding $X \hookrightarrow \mathscr{P}(X)$,*

$$\iota_X : x \longmapsto \{x\} \qquad (x \in X)$$

is the identity morphism of X in $\mathrm{Set}_{\mathrm{mult}}$.

1.3 The category of sets and binary relations

1.3.1

Exercise 3 *For $A \subseteq X$ and $B \subseteq Y$, show that the map*

$$A \times B \longrightarrow X \times Y, \qquad (a, b) \longmapsto (a, b),$$

identifies $A \times B$ with the set

$$p_X^{-1}(A) \cap p_Y^{-1}(B) \tag{3}$$

where p_X and p_Y are the canonical projections

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd} . \tag{4}$$

1.3.2 Multiplication of binary relations

Given $D \subseteq X \times Y$ and $E \subseteq Y \times Z$, we define

$$D \cdot E := \{(x, z) \in X \times Z \mid \text{there exists } y \in Y \text{ such that } (x, y) \in D \text{ and } (y, z) \in E\}. \tag{5}$$

Exercise 4 *Show that*

$$D \cdot E = p_{\hat{Y}} \left(p_{\hat{Z}}^{-1}(D) \cap p_{\hat{X}}^{-1}(E) \right) \tag{6}$$

where p with the corresponding index denotes one of the 6 canonical projections in the diagram

$$\begin{tikzcd}
 & X \times Y & & X & \\
Y & & X \times Y \times Z \arrow[ul, two heads, "p_{\hat{Z}}"'] \arrow[ur, two heads, "p_X"] \arrow[ll, two heads, "p_Y"'] \arrow[rr, two heads, "p_{\hat{Y}}"] \arrow[dl, two heads, "p_{\hat{X}}"'] \arrow[dr, two heads, "p_Z"] & & X \times Z \ . \\
 & Y \times Z & & Z &
\end{tikzcd} \tag{7}$$

1.3.3 Associativity of the multiplication of relations

Exercise 5 *Given $C \subseteq W \times X$, $D \subseteq X \times Y$ and $E \subseteq Y \times Z$, show that*

$$(C \cdot D) \cdot E = C \cdot (D \cdot E).$$

1.3.4 $\mathbf{Set}_{\mathbf{rel}}$

The objects of $\mathrm{Set}_{\mathrm{rel}}$ are sets and

$$\mathrm{Hom}_{\mathrm{Set}_{\mathrm{rel}}}(X, Y) := \mathscr{P}(X \times Y).$$

Note that the composition $E \circ D$ of morphisms $D \in \mathrm{Hom}_{\mathrm{Set}_{\mathrm{rel}}}(X, Y)$ and $E \in \mathrm{Hom}_{\mathrm{Set}_{\mathrm{rel}}}(Y, Z)$ is given by

$$E \circ D := D \cdot E.$$

Exercise 6 *Show that the diagonal*

$$\Delta := \{(x_0, x_1) \in X \times X \mid x_0 = x_1\}$$

is the identity morphism of X in $\mathrm{Set}_{\mathrm{rel}}$.

2 A few functors associated with various categories of sets

2.1 The canonical embedding functor $\iota\colon \mathbf{Set} \rightsquigarrow \mathbf{Set_{mult}}$

2.1.1

Assigning to a map $f\colon X \to Y$ the multimap

$$\phi_f\colon x \longmapsto \{f(x)\}, \qquad (x \in X) \tag{8}$$

defines a functor from the standard category of sets to the category of sets and multimaps. This functor is the identity on the class of objects and one-to-one on the class of morphisms. It allows to view Set as a subcategory of $\mathrm{Set_{mult}}$.

2.2 The graph functor $\Gamma\colon \mathbf{Set_{mult}} \rightsquigarrow \mathbf{Set_{rel}}$

2.2.1

The *graph* of a multimap $\phi\colon X \multimap Y$ is the relation

$$\Gamma_\phi \;:=\; \{(x,y) \in X \times Y \mid \phi(x) \ni y\} = \bigcup_{x \in X} \{x\} \times \phi(x). \tag{9}$$

2.2.2

Exercise 7 *Show that*

$$\Gamma_{\phi \circ \psi} = \Gamma_\psi \cdot \Gamma_\phi. \tag{10}$$

Instead of the fact that Γ *reverses* the order in which the multimaps are composed, Γ is a *covariant* functor from $\mathrm{Set_{mult}}$ to $\mathrm{Set_{rel}}$. This is due to how the composition of morphisms is defined in $\mathrm{Set_{rel}}$.

2.2.3

When $\phi = \phi_f$ is the multimap associated with a map f, we shall denote Γ_ϕ by Γ_f and refer to Γ_f as the *graph of* f.

2.3 The power functors

2.3.1

The correspondence between a set and its set of subsets,

$$X \longmapsto \mathscr{P}(X),$$

gives rise to two functors $\mathrm{Set}_{\mathrm{rel}} \rightsquigarrow \mathrm{Set}$, one covariant, the other one contravariant.

2.3.2 The covariant power functor $\mathbf{Set}_{\mathbf{rel}} \rightsquigarrow \mathbf{Set}$

The covariant power functor sends $D \in \mathscr{P}(X \times Y)$ to the map

$$\cdot D\colon \mathscr{P}(X) \longrightarrow \mathscr{P}(Y), \quad A \longmapsto A \cdot D := \{y \in Y \mid (x,y) \in D \text{ for some } x \in A\}. \tag{11}$$

2.3.3 The contravariant power functor $\mathbf{Set}_{\mathbf{rel}} \rightsquigarrow \mathbf{Set}$

The contravariant power functor sends $D \in \mathscr{P}(X \times Y)$ to the map

$$D\cdot\colon \mathscr{P}(Y) \longrightarrow \mathscr{P}(X), \quad B \longmapsto D \cdot B := \{x \in X \mid (x,y) \in D \text{ for some } y \in B\}. \tag{12}$$

Exercise 8 *Show that*

$$A \cdot D \;=\; p_Y\left(p_X^{-1}(A) \cap D\right) \qquad \textit{and} \qquad D \cdot B \;=\; p_X\left(D \cap p_Y^{-1}(B)\right)$$

where p_X and p_Y are the projections in diagram (4).

2.3.4

The set $A \cdot D$ consists of *right relatives* of elements of $A \subseteq X$ with respect to relation D while $D \cdot B$ consists of *left relatives* of elements of $B \subseteq Y$ with respect to the same relation.

If one identifies X with $\mathbf{1} \times X$ and Y with $Y \times \mathbf{1}$, where

$$\mathbf{1} \;:=\; \{\mathbf{0}\} \qquad \text{and} \qquad \mathbf{0} \;:=\; \emptyset, \tag{13}$$

then we may view $A \subseteq X$ and $B \subseteq Y$ as subsets of $\mathbf{1} \times X$ and $Y \times \mathbf{1}$, respectively, and (11)–(12) then describe the maps of right and, respectively, left multiplication by D.

Exercise 9 *Show that the correspondence* $D \mapsto \phi_D$, *where*

$$\phi_D(x)\colon X \multimap Y, \qquad x \longmapsto \phi_D(x) := \{x\} \cdot D, \tag{14}$$

defines a functor $\mathrm{Set}_{\mathrm{rel}} \rightsquigarrow \mathrm{Set}_{\mathrm{mult}}$. *Show that it is the inverse to the graph functor* Γ.

2.3.5

The category of sets and multimaps and the category of sets and relations are therefore isomorphic and henceforth we shall not be distinguishing between the two. Situations like this are surprisingly rare: much more common is when the two categories are *equivalent*.

2.3.6

From now on we shall be refering to either of them as the category of sets *in the extended sense* while for Set we shall reserve the terminology *the category of sets* or, for emphasis, the category of sets *in the narrow sense*.

2.3.7

When we need to distinguish between the two power functors we shall denote one by $\mathscr{P}^{\mathrm{cov}}$ and the other one by $\mathscr{P}^{\mathrm{ctr}}$.

Exercise 10 *Show that the composite covariant functor* $\mathscr{P}^{\mathrm{cov}} \circ \Gamma \circ \iota\colon \mathrm{Set} \rightsquigarrow \mathrm{Set}$ *sends* $f\colon X \to Y$ *to the "image of* f*" map*

$$f_*\colon \mathscr{P}(X) \longrightarrow \mathscr{P}(Y), \quad A \longmapsto f(A) := \{y \in Y \mid y = f(x) \textit{ for some } x \in A\} \tag{15}$$

while the composite contravariant functor $\mathscr{P}^{\mathrm{ctr}} \circ \Gamma \circ \iota\colon \mathrm{Set} \rightsquigarrow \mathrm{Set}$ *sends* f *to the "inverse image of* f*" map*

$$f^*\colon \mathscr{P}(Y) \longrightarrow \mathscr{P}(X), \quad B \longmapsto f^{-1}(B) := \{x \in X \mid f(x) \in B\}. \tag{16}$$

In other words, show that

$$f(A) = A \cdot \Gamma_f \qquad \textit{and} \qquad f^{-1}(B) = \Gamma_f \cdot B \tag{17}$$

for $A \subseteq X$ *and* $B \subseteq Y$.

3 Natural ways to form a category of *spaces*

3.1 "Spaces"

3.1.1

Below we shall refer to pairs $(X, \mathscr{A})$ where X is a set and $\mathscr{A} \subseteq \mathscr{P}(\mathscr{P}(X))$ is a family of subsets of X as "spaces".

3.1.2

Morphisms from a space $(X, \mathscr{A})$ to a space $(Y, \mathscr{B})$ should be the morphisms between the underlying sets $X \to Y$ (in the narrow or extended sense) which are "compatible" with the corresponding families of subsets. Compatibility will be interpreted in terms of either of the two canonical relations on the iterated power sets $\mathscr{P}(\mathscr{P}(X))$. These are: $\subseteq$ (containment) and $\mathrel{\varepsilon\!-}$ (domination below).[1] Recall that

$$\mathscr{A} \mathrel{\varepsilon\!-} \mathscr{A}' \quad \text{if} \quad \text{for any } A' \in \mathscr{A}' \text{ there exists } A \in \mathscr{A} \text{ such that } A \subseteq A'. \tag{18}$$

The defining condition of (18) expressed using the quantifier notation:

$$\forall_{A' \in \mathscr{A}'} \exists_{A \in \mathscr{A}} \; A \subseteq A'.$$

3.1.3

Containment $\subseteq$ is, so to speak, the "external" ordering while domination below is the "internal" ordering (in fact, $\mathrel{\varepsilon\!-}$ is only a *preorder*; it is transitive and reflexive but not weakly antisymmetric).

3.1.4

Given $D \in \mathscr{P}(X \times Y)$ and a family $\mathscr{A} \subseteq \mathscr{P}(X)$ we have two ways of producing a family on Y:

$$(\cdot D)_*(\mathscr{A}) \qquad \text{or} \qquad (D \cdot)^*(\mathscr{A}).$$

We can then *compare* either of these two families with a family $\mathscr{B} \subseteq Y$, and we can do that using either one of the (pre)-ordering relations $\subseteq$, $\mathrel{\varepsilon\!-}$ or

[1] One can also consider the opposite relation $\supseteq$ and the associated with it relation $\mathrel{-\!\ni}$ of domination above.

$\mathrel{-\!\!\!\ni}$, or their opposites. This yields 12 possibilities for D to be considered "compatible" with $\mathscr{A}$ and $\mathscr{B}$. If we similarly "compare" the families

$$(D\cdot)_*(\mathscr{B}) \qquad \text{or} \qquad (\cdot D)^*(\mathscr{B})$$

with $\mathscr{A}$, this will yield additional 12 possibilities (!).

Exercise 11 *Show that*

$$(\cdot D)_*(\mathscr{A}) = \{B \subseteq Y \mid B = A \cdot D \text{ for some } A \in \mathscr{A}\}$$

and

$$(D\cdot)^*(\mathscr{A}) = \{B \subseteq Y \mid D \cdot B \in \mathscr{A}\}.$$

Write down similar representations for $(D\cdot)_*(\mathscr{B})$ nd $(\cdot D)^*(\mathscr{B})$.

3.1.5

Among these 24 possibilities, some, in fact, are distiguished. Indeed, $((\cdot D)_*, (\cdot D)^*)$ and $((D\cdot)_*, (D\cdot)^*)$ form *Galois connections* for the $\subseteq$ order.

If D is the graph of a map $f\colon X \to Y$, then $((\cdot D)_*, (D\cdot)_*)$ form a Galois connection for the $\mathrel{\in\!\!\!-}$ preorder as the following exrecise shows.

Exercise 12 *Show that*

$$(f_*)_*(\mathscr{A}) \mathrel{\in\!\!\!-} \mathscr{B} \qquad \text{if and only if} \qquad \mathscr{A} \mathrel{\in\!\!\!-} (f^*)_*(\mathscr{B}). \tag{19}$$

Exercise 13 *Formulate the analog of* (19) *for* $\mathrel{-\!\!\!\ni}$.

3.1.6

We shall see soon that two of scenarios are particularly important:

$$(D\cdot)_*(\mathscr{B}) \subseteq \mathscr{A}, \tag{20}$$

which is equivalent to

$$\mathscr{B} \subseteq (D\cdot)^*(\mathscr{A}), \tag{21}$$

and

$$(\cdot D)_*(\mathscr{A}) \mathrel{\in\!\!\!-} \mathscr{B}. \tag{22}$$