Notes on Uniform Structures

Annex to H104

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1 Vocabulary

1.1 Binary Relations

1.1.1

In these notes we identify binary relations between elements of a set X and a set Y with subsets $E \subseteq X \times Y$ of their Cartesian product $X \times Y$. To a given relation $\sim$ corresponds the subset:

$$E_{\sim} := \{(x, y) \in X \times Y \mid x \sim y\} \tag{1}$$

and, vice-versa, to a given subset $E \subseteq X \times Y$ corresponds the relation:

$$x \sim_E y \qquad \text{if and only if} \qquad (x, y) \in E. \tag{2}$$

1.1.2 The inverse relation

We denote by

$$E^{-1} := \{(y, x) \in Y \times X \mid (x, y) \in E\} \tag{3}$$

the **inverse** relation.

1.1.3 The identity relation

For any set X, we shall denote by Δ_X the **identity** relation $\{(x, x') \in X \times X \mid x = x'\}$. We shall often omit subscript X when set X is clear from the context.

Exercise 1 *Let A and B be subsets of a set X. Show that*

$$(A\times B)\cap\Delta=\emptyset \qquad \textit{if and only if} \qquad A\cap B=\emptyset, \tag{4}$$

i.e., sets A and B are disjoint.

1.1.4 Sets of E-relatives

For any subset $A\subseteq Y$ we shall denote by $E(A)$ the set of *left* E-relatives of elements of Y:

$$E(A):=\{x\in X\mid \exists_{s\in A}\, x\sim_E s\}. \tag{5}$$

Definition 1.1 *We say that an element $x\in X$ is E**-related** to an element $y\in Y$, and write $x\sim_E y$, if $(x,y)\in E$.*

In particular, $x\sim_E y$ if and only if $y\sim_{E^{-1}} x$.
We shall also denote by $E(y)$ the set $E(\{y\})$.

Exercise 2 *Show that:*

$$E(\emptyset)=\emptyset \tag{6}$$

$$E(A)=\bigcup_{s\in A}E(s) \tag{7}$$

$$E(A)\subseteq E(B) \quad \textit{if} \quad A\subseteq B \tag{8}$$

$$E(A)\cup E(B)=E(A\cup B) \tag{9}$$

$$E(A)\cap E(B)\supseteq E(A\cap B) \tag{10}$$

where A and B are arbitrary subsets of Y. Give an example demonstrating that

$$E(A)\cap E(B)\neq E(A\cap B)$$

in general.

Exercise 3 *Let A and B be subsets of X and Y respectively, and let $E\subseteq X\times Y$. Show that the following conditions are equivalent*

there exist $a\in A$ and $b\in B$ such that $a\sim_E b$, (11a)

$(A\times B)\cap E\neq\emptyset,$ (11b)

$A\cap E(B)\neq\emptyset,$ (11c)

$E^{-1}(A)\cap B\neq\emptyset.$ (11d)

1.2 Composition of binary relations

1.2.1

If $E \subseteq X \times Y$ and $F \subseteq Y \times Z$, then $E \circ F \subseteq X \times Z$ is defined as

$$\begin{aligned} E \circ F &:= \{(x,z) \in X \times Z \mid \exists_{y \in Y}\ (x,y) \in E \text{ and } (y,z) \in F\} \\ &= \{(x,z) \in X \times Z \mid \exists_{y \in Y}\ x \sim_E y \text{ and } y \sim_F z\}. \end{aligned} \tag{12}$$

1.2.2 Associativity

Composition of binary relations is *associative*:

$$E \circ (F \circ G) = (E \circ F) \circ G \tag{13}$$

where $G \subseteq Z \times W$. Note also that

$$\Delta_X \circ E = E = E \circ \Delta_Y \tag{14}$$

and

$$(E \circ F)^{-1} = F^{-1} \circ E^{-1}. \tag{15}$$

In particular, the set of binary relations on a given set X, $\mathscr{P}(X \times X)$, equipped with the operation $\circ$, is a *monoid*.

Exercise 4 *Let $E \subseteq X \times Y$, $F \subseteq Y \times Z$ and $T \subseteq Z$. Show that, for any $B \subseteq Z$,*

$$(E \circ F)(B) = E(F(B)). \tag{16}$$

Exercise 5 *Let A and B be subsets of X and Y respectively, and let $E \subseteq X \times Y$ and $F \subseteq Y \times Z$. Show that the following conditions are equivalent*

$$\text{there exist } a \in A \text{ and } b \in B \text{ such that } a \sim_{E \circ F} b, \tag{17a}$$

$$(A \times B) \cap (E \circ F) \neq \emptyset, \tag{17b}$$

$$A \cap (E \circ F)(B) \neq \emptyset, \tag{17c}$$

$$E^{-1}(A) \cap F(B) \neq \emptyset, \tag{17d}$$

$$(E \circ F)^{-1}(A) \cap B \neq \emptyset. \tag{17e}$$

1.2.3 Monotonicity

Composition of binary relations is *monotonic* in both arguments:

$$\textit{if } E \subseteq E' \textit{ and } F \subseteq F', \textit{ then } E \circ F \subseteq E' \circ F'. \tag{18}$$

1.3 The binary relation associated with a family $\mathscr{C} \subseteq \mathscr{P}(X)$

1.3.1

Any equivalence relation $\sim$ on a set X defines a certain family of subsets $\mathscr{C}_\sim$, namely the family of equivalence classes of relation $\sim$.

We can recover the equivalence relation from that family of subsets by means of the following general construction.

1.3.2

For any family of subsets $\mathscr{C}$ of a set X, let us consider the binary relation on X:

$$\Delta_{\mathscr{C}} := \bigcup_{C \in \mathscr{C}} C \times C. \tag{19}$$

Exercise 6 *Show that the union of* $C \in \mathscr{C}$ *which contain* $x \in X$ *coincides with the set of points* $\Delta_{\mathscr{C}}$*-close to* x*:*

$$\Delta_{\mathscr{C}}(x) = \bigcup_{C \in \mathscr{C} \textit{ such that } x \in \mathscr{C}} C. \tag{20}$$

1.3.3

In the special case of the family of all singleton subsets,

$$\mathscr{X} = \{\{x\} \mid x \in X\}, \tag{21}$$

we obtain the identity relation

$$\Delta_{\mathscr{X}} = \Delta.$$

1.3.4

The associated relation is automatically symmetric. It is reflexive precisely when $\mathscr{C}$ is a cover of X.

Exercise 7 *Show that relation $\sim_{\Delta_{\mathscr{C}}}$ is reflexive, i.e.,*

$$\Delta_{\mathscr{C}} \supseteq \Delta$$

if and only if $\mathscr{C}$ covers X.

1.3.5

Let us introduce the following operation on $\mathscr{P}(\mathscr{P}(X))$, the set of all families of subsets of X,

$$\mathscr{C} \diamond \mathscr{D} := \{C \cup D \mid C \in \mathscr{C},\ D \in \mathscr{D}, \text{and } C \cap D \neq \emptyset\}. \tag{22}$$

It is associative, commutative, and family $\mathscr{X}$, cf. (21), is its identity.

Exercise 8 *Show that, for any $\mathscr{C} \subseteq \mathscr{P}(X)$,*

$$\mathscr{X} \diamond \mathscr{C} = \mathscr{C} = \mathscr{C} \diamond \mathscr{X}.$$

Exercise 9 *Show that*

$$\Delta_{\mathscr{C}} \circ \Delta_{\mathscr{C}} \cup \Delta_{\mathscr{C}} \circ \Delta_{\mathscr{D}} \cup \Delta_{\mathscr{D}} \circ \Delta_{\mathscr{C}} \cup \Delta_{\mathscr{C}} \circ \Delta_{\mathscr{D}} = \Delta_{\mathscr{C} \diamond \mathscr{D}}. \tag{23}$$

Deduce from (23) *that*

$$\Delta_{\mathscr{C}} \circ \Delta_{\mathscr{C}} = \Delta_{\mathscr{C} \diamond \mathscr{C}}. \tag{24}$$

Exercise 10 *Show that the following conditions are equivalent*

$$\text{relation } \sim_{\Delta_{\mathscr{C}}} \text{ is transitive,} \tag{25a}$$

$$\mathscr{C} \text{ consists of disjoint subsets,} \tag{25b}$$

$$\mathscr{C} \diamond \mathscr{C} = \mathscr{C}. \tag{25c}$$

1.3.6 Refinement

We shall say that a family $\mathscr{C}' \subseteq \mathscr{P}(X)$ *refines* a family $\mathscr{C} \subseteq \mathscr{P}(X)$ (or, that it is a *refinement* of $\mathscr{C}$) if

$$\text{for any } C' \in \mathscr{C}', \text{ there exists } C \in \mathscr{C} \text{ such that } C' \subseteq C. \tag{26}$$

We denote it by $\mathscr{C}' \prec \mathscr{C}$.

1.3.7

Refinement is a reflexive and transitive relation on $\mathscr{P}(\mathscr{P}((X))$. This notion plays an important role in the theory of covers. It should not be confused, however, with 'refinement' in *the sense of filters*.

Exercise 11 *Show that*

$$\mathscr{C}' \dashv \mathscr{C} \qquad \textit{implies} \qquad \Delta_{\mathscr{C}'} \subseteq \Delta_{\mathscr{C}}. \tag{27}$$

2 Uniform spaces

2.1 Uniform structures

2.1.1

Definition 2.1 *A filter $\mathcal{U}$ on $X \times X$ is said to be a **uniform strucure** if it satisfies the following conditions*

$(\mathbf{U_1})$ $\bigcap \mathcal{U} \supseteq \Delta$;

$(\mathbf{U_2})$ *if $E \in \mathcal{U}$, then $E^{-1} \in \mathcal{U}$;*

$(\mathbf{U_3})$ *for any $E \in \mathcal{U}$, there exists $E' \in \mathcal{U}$ such that $E' \circ E' \subseteq E$.*

Definition 2.2 *A set X equipped with a uniform structure $\mathcal{U}$ is called a **uniform space** and the filter $\mathcal{U}$ is often referred to as its **uniformity**.*

2.1.2 Entourages

Members of $\mathcal{U}$ are usually referred to as **entourages**. For two points p and q of X we shall say that they are E**-closed** if $p \sim_E q$. Thus, $E(A)$ is the set of points $p \in X$ which are emph$E$-close to a subset $A \subseteq X$.

2.1.3 Symmetric entourages

Since $\mathcal{U}$ is a filter, and $F = E \cap E^{-1}$ is clearly *symmetric*, i.e., $F = F^{-1}$, symmetric entourages form a base of filter $\mathcal{U}$.

2.1.4

Definition 2.3 *We say that a subset $A \subseteq X$ is E**-small** if $A \times A \subseteq E$, i.e., if any two elements of A are E-close,*

$$\forall_{s,s' \in A}\ s \sim_E s'. \tag{28}$$

Exercise 12 *Show that $E \subseteq E \circ E$ for any entourage $E \in \mathfrak{U}$.*

Exercise 13 *Let E be an entourage. Show that, for any $n \geq 2$, there exists $D \in \mathfrak{U}$ such that*

$$D^{\circ n} := \underbrace{D \circ \cdots \circ D}_{n} \subseteq E. \tag{29}$$

Exercise 14 *Let E be a symmetric entourage. Show that, if $A \subseteq X$ is E-small, then $E(A)$ is $E \circ E \circ E$-small.*

3 Metrization

3.1 The uniform structure associated with a semi-metric

3.1.1

Suppose $\rho\colon X \times X \longrightarrow [0, \infty)$ is a semi-metric on a set X. The sets

$$E_\epsilon := \{(p,q) \in X \times X \mid d_p, q) < \epsilon\} \tag{30}$$

form a basis of a filter on $X \times X$.

Note that

$$\bigcap_{\epsilon > 0} E_\epsilon \supseteq \Delta, \tag{31}$$

$E_\epsilon = E_\epsilon^{-1}$, and the triangle inequality yields

$$E_\epsilon \circ E_{\epsilon'} \subseteq E_{\epsilon + \epsilon'}.$$

It follows that the filter generated by $\{E_\epsilon \mid \epsilon > 0\}$ satisfies the three conditions of a uniformity, cf. Definition 2.1.

3.1.2

Function ρ separates points of X, i.e., is a metric on set X, precisely when

$$\bigcap_{\epsilon>0} E_\epsilon = \Delta. \tag{32}$$

3.1.3

The uniformity associated with a semi-metric possesses a countable base: take for example

$$\left\{ E_{\frac{1}{n}} \mid n = 1, 2, \ldots \right\}.$$

In the next section we will show that any uniformity with a countable base is the associated uniformity of some semi-metric on X.

3.2 A semi-metric associated with a flag of entourages

3.2.1 A flag of entourages

Let us call a nested sequence of entourages

$$\mathscr{E}: \quad X \times X = E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots \tag{33}$$

a *flag of entourages*.

3.2.2 An associated semi-metric

Given a flag (33), define a function

$$f(p) := \begin{cases} \frac{1}{2^n} & \text{if } p \in E_n \setminus E_{n+1} \\ 0 & \text{if } p \in \bigcap_{i=0}^{\infty} E_i \end{cases}, \tag{34}$$

and then produce the corresponding semi-metric by enforcing the Triangle Inequality as described in the *Notes on Topology*:

$$\rho_{\mathscr{E}} = f^t, \quad \text{i.e.,} \quad \rho_{\mathscr{E}}(p,q) = \inf\left\{ \sum_{i=1}^{n} f(x_{i-1}, x_i) \mid x_0 = p, x_n = q \right\} \tag{35}$$

where the infimum is taken over all finite sequences $\{x_i\}_{i\in\{0,\ldots,n\}}$ of elements of X of any length which start at p and terminate at q.

3.2.3

One obviously has the inequality

$$\rho_{\mathscr{E}} \leq f.$$

In particular,

$$E_n \subseteq \left\{ (p,q) \in X \times X \mid \rho_{\mathscr{E}}(p,q) < \frac{1}{2^n} \right\}$$

Lemma 3.1 *If the flag satisfies the following condition*

$$E_n \circ E_n \circ E_n \subseteq E_{n-1} \qquad (n = 1, 2, \dots). \tag{36}$$

then

$$\frac{1}{2} f \leq \rho_{\mathscr{E}}. \tag{37}$$

Proof. We shall prove by induction on n that

$$\frac{1}{2} f \leq \sum_{i=1}^{l} f(p_{i-1}, p_i) \tag{38}$$

for any sequence

$$p_0 = p, \dots, p_l = q. \tag{39}$$

There is nothing to prove for $l = 1$.

For a given sequence (39), denote by d the sum

$$\sum_{i=1}^{l} f(p_{i-1}, p_i).$$

If $d = 0$, then $f(p_{i-1}, p_i) = 0$ for each $i \in \{1, \dots, l\}$ which means that $(p_i, p_{i+1}) \in E_n$ for any n. Hence,

$$(p,q) \in \underbrace{E_n \circ \cdots \circ E_n}_{l \text{ times}} \subseteq E_m$$

for any $m \leq n - \log_3 l$. In particular, $(p,q) \in \bigcap \mathscr{E}$, and thus $f(p,q) = 0$.

Suppose that $d > 0$. Denote by m be the largest index in $\{0, \ldots, l\}$ such that

$$\sum_{i=1}^{m} f(p_{i-1}, p_i) \leq \frac{1}{2}d.$$

Note that $m < l$, and also

$$f(p_m, p_{m+1}) > 0 \qquad \text{and} \qquad \sum_{i=m+2}^{l} f(p_{i-1}, p_i) < \frac{1}{2}d$$

Combined with inductive hypothesis we obtain

$$f(p, p_m) \leq 2 \cdot \frac{1}{2}d = d \qquad \text{and} \qquad f(p_{m+1}, q) \leq 2 \cdot \frac{1}{2}d = d$$

and, obviously, also

$$f(p_m, p_{m+1}) \leq d.$$

The above inequalities mean that if n is the largest integer such that

$$\frac{1}{2^n} \leq d,$$

then (p, p_m), $(p_m, p_{m+1}$, and (p_{m+1}, q) all belong to E_n. In particular,

$$(p, q) \in E_n \circ E_n \circ E_n \subseteq E_{n-1}$$

which means that

$$f(p, q) \leq \frac{1}{2}d.$$

□

Corollary 3.2 *If a flag* $\mathscr{E}$ *satisfies condition* (36)*, then the uniformity it generates,* $\mathscr{E}_*$*, is associated with semi-metric* $\rho_{\mathscr{E}}$*.*

We arrive at the following important result.

Theorem 3.3 (Metrization Theorem) *A uniformity* $\mathcal{U}$ *is associated with some semi-metric if and only if it possesses a countable base.*

Proof. Existence of a countable base is obviously a necessary condition for $\mathscr{E}_*$ to be associated with a semi-metric.

If $\mathcal{U}$ possesses a countable base, then it possesses a base $\mathscr{E}$ satisfying condition (36). Then,

$$E_n \subseteq \left\{ (p, q) \in X \times X \mid \rho_{\mathscr{E}}(p, q) < \frac{1}{2^n} \right\} \subseteq E_{n-1}. \qquad (n = 1, 2, \ldots).$$

□

4 Uniform topology

For any uniform space $(X, \mathfrak{U})$, we shall define an associated topology on X. This can be done by defining either the neighborhood filters or the closure operation.

4.1 The neighborhood filters

4.1.1

Definition 4.1 *For any point $p \in X$, we set $\mathscr{N}_p$ to be the filter with the base*

$$\mathscr{B}_p := \{E(p) \mid E \in \mathfrak{U}\}. \tag{40}$$

Definition 4.2 *We declare a subset $U \in X$ to be **open** if, for any $P \in U$, there exists $E \in \mathfrak{U}$ such that $E(p) \subseteq U$.*

Exercise 15 *Show that*

$$\mathscr{T}^{\mathfrak{U}} := \{U \subseteq X \mid U \text{ is open}\} \tag{41}$$

satisfies the axioms of a topology.

4.1.2

The above topology will be referred to as the **uniform topology** and filters $\mathscr{N}_p$, cf. Definition 4.1, are the neighborhood filters of this topology.

4.2 The closure operation

4.2.1

Definition 4.3 *Define the closure operation on the set, $\mathscr{P}(X)$, of all subsets of X by*

$$A \longmapsto \overline{A} := \bigcap_{E \in \mathfrak{U}} E(A). \tag{42}$$

Exercise 16 *Show that the operation defined in* (42) *satisfies the axioms of the topological closure operation*

$$S \subseteq \overline{A} \tag{43}$$

$$\overline{\overline{A}} = \overline{A} \tag{44}$$

$$\overline{A \cup B} = \overline{A} \cup \overline{B} \tag{45}$$

$$\overline{\varnothing} = \varnothing \tag{46}$$

where A and B are arbitrary subsets of X.

Definition 4.4 *We declare a subset $Z \subseteq X$ to be* ***closed*** *if $Z = \overline{Z}$.*

Proposition 4.5 *A subset $U \subseteq X$ is open if and only if $X \setminus U$ is closed.*

Proof. Let A be a subset of X. Suppose that $p \notin \overline{A}$. Then $p \notin E(A)$ for some $E \in \mathcal{U}$. Let $D \in \mathcal{U}$ be such that $D \circ D \subseteq E$. Then $p \notin (D \circ D)(A)$ and thus $D^{-1}(p) \cap D(A) = \varnothing$, cf. Exercise **??**. It follows that

$$D^{-1}(p) \cap \overline{A} = D^{-1}(p) \cap \bigcap_{F \in \mathcal{U}} F(A) \subseteq D^{-1}(p) \cap D(A) = \varnothing,$$

i.e., $D^{-1}(p) \in X \setminus \overline{A}$. Hence $X \setminus \overline{A}$ is open.

Let U be an open subset of X and $p \in U$. Then there exists $E \in \mathcal{U}$ such that $E(p) \cap (X \setminus U) = \varnothing$. The latter is equivalent to

$$\{p\} \cap E^{-1}(X \setminus U) = \varnothing,$$

cf. Exercise 3. Thus, $p \notin E^{-1}(X \setminus U)$. In particular, $p \notin \bar{(}X \setminus U)$. It follows that

$$U \subseteq X \setminus \overline{(X \setminus U)}$$

or equivalently,

$$X \setminus U \supseteq \overline{(X \setminus U)}.$$

In view of $X \setminus U \subseteq \overline{(X \setminus U)}$, we infer that $X \setminus U = \overline{(X \setminus U)}$, i.e., $X \setminus U$ is closed. ∎

4.3 Uniform continuity

4.3.1

For a function $f\colon X\longrightarrow Y$ let $f\times f\colon X\times X\longrightarrow Y\times Y$ be the function

$$(x,x')\longmapsto (f(x),f(x')). \tag{47}$$

Definition 4.6 *We say that a function $f\colon X\longrightarrow Y$ between uniform spaces $(X,\mathcal{U})$ and $(Y,\mathscr{V})$, is **uniformly continuous** if $(f\times f)^{-1}(E)\in\mathcal{U}$ for any $E\in\mathscr{V}$.*

Exercise 17 *Show that $f\colon X\longrightarrow Y$ is uniformly continuous if and only if it satisfies the following condition*

$$\forall_{E\in\mathscr{V}}\exists_{D\in\mathcal{U}}\forall_{x,x'\in X}\ \left(x\sim_D x' \ \Rightarrow\ f(x)\sim_E f(x')\right). \tag{48}$$

4.3.2

A uniformly continuous function is continuous in respective uniform topologies. The reverse is generally false.

Exercise 18 *Prove that the function $f\colon (0,\infty)\longrightarrow\mathbb{R}$,*

$$f(x)=\frac{1}{x}$$

is continuous but not uniformly continuous. Here $\mathbb{R}$ and its subset $(0,\infty)$ are equipped with the usual metric $d(x,x')=|x-x'|$.

Theorem 4.7 *If X is compact in the uniform topology, then any continuous function from X into a uniform topological space Y is uniformly continuous.*

Proof. Let $\mathcal{U}$ denote the uniformity of X and $\mathscr{V}$ denote the uniformity of Y. For a given $E\in\mathscr{V}$, let $E'\in\mathscr{V}$ be a symmetric entourage such that $E'\circ E'\subseteq E$.

If $f\colon X\longrightarrow Y$ is continuous, then for each $p\in X$, there exists an entourage $D'_p\in\mathcal{U}$ such that

$$f\left(D_p(p)\right)\subseteq E'(f(p)). \tag{49}$$

Let $D_p \in \mathcal{U}$ be an entourage such that $D_p \circ D_p \subseteq D'_p$. Since each $p \in X$ belongs to $(D_p(p)^\circ)$, the interiors $\{(D_p(p)^\circ)\}_{p \in X}$ form an open cover of X. In view of compactness of X, one has

$$X = D_{p_1}(p_1) \cup \cdots \cup D_{p_n}(p_n) \tag{50}$$

for certain points $p_1, \ldots, p_n \in X$.

Set $D := D_{p_1} \cap \cdots \cap D_{p_n}$. The latter is an entourage of X.

Let x and x' be arbitrary points of X. Suppose that $x \sim_D x'$. In view of (50), one has $x' \sim_{D_{p_i}} p_i$ for some p_i. It follows that $x \sim_{D \circ D_{p_i}} p_i$. Since $D_{p_i} \subseteq D'_{p_i}$ and $D \circ D_{p_i} \subseteq D_{p_i} \circ D_{p_i} \subseteq D'_{p_i}$ we obtain

$$x \sim_{D'_{p_i}} p_i \qquad \text{and} \qquad x' \sim_{D'_{p_i}} p_i. \tag{51}$$

By combining (51) with (49) we obtain

$$f(x) \sim_{E'} f(p_i) \qquad \text{and} \qquad f(x') \sim_{E'} f(p_i).$$

and, since E' is symmetric, $f(x) \sim_{E' \circ E'} f(x')$. Recalling that $E' \circ E' \subseteq E$, we deduce that $f(x) \sim_E f(x')$. □

5 Uniformization

5.1 The neighborhood filter of the diagonal

5.1.1

Let X be a topological space. The neighborhood filter $\mathscr{N}_\Delta$ of the diagonal obviously satisfies Axiom $(\mathbf{U_1})$.

Exercise 19 *Show that $\mathscr{N}_\Delta$ satisfies Axiom $(\mathbf{U_2})$.*

5.1.2

Exercise 20 *Show that a family $\mathscr{U} \subseteq \mathscr{P}(X)$ is an open cover of X if and only if $\Delta_{\mathscr{U}}$ is a neighborhood of the diagonal.*

5.1.3

For any point $p \in X$ and $E \in \mathscr{N}_\Delta$, the set of E-relatives of of p is a neighborhood of p. Indeed, since E is a neighborhood of Δ, there exists a pair of open neighborhoods U and V of p such that

$$U \times V \subseteq E.$$

It follows that

$$U = (U \times V)(p) \subseteq E(p).$$

Thus, filter-base

$$\{E(p) \mid E \in \mathscr{N}_\Delta\}$$

generates a filter not finer than $\mathscr{N}_p$.

Proposition 5.1 *If a point $p \in X$ possesses a fundamental system of closed neighborhoods, then any open neighborhood of p is of the form $E(p)$ for some $E \in \mathscr{N}_\Delta$.*

In particular,

$$\{E(p) \mid E \in \mathscr{N}_\Delta\} \tag{52}$$

is a fundamental system of neighborhoods of point p.

Proof. For an open neighborhood $U \in \mathscr{N}_p$ let $N \in \mathscr{N}_p$ be a closed neighborhood such that $N \subseteq U$. Then

$$\mathscr{U} := \{U, N^c\}$$

is an open cover of X, and thus $\Delta_{\mathscr{U}}$ is an open neighborhood of the diagonal, Δ. Since $p \in U$ and $p \notin N^c$, we have

$$\Delta_{\mathscr{U}}(p) = U,$$

cf. (20). □

5.2 Uniformizable spaces

5.2.1

A topological space $(X, \mathscr{T})$ is said to be **uniformizable**, if there exists a uniform structure $\mathcal{U}$ on X such that $\mathscr{T}$ is the associated uniform topology $\mathscr{T}^{\mathcal{U}}$, cf. (41).

Exercise 21 *Show that a uniformizable space is necessarily regular.*

5.2.2

In view of Proposition 5.1, A regular topological space is uniformizable if and only if the neighborhood filter of the diagonal, $\mathscr{N}_\Delta$, satisfies Axiom $(\mathbf{U}_3)$.

Exercise 22 *Show that the family*

$$\mathscr{B} = \{\Delta_{\mathscr{U}} \mid \mathscr{U} \text{ is an open cover of } X\} \tag{53}$$

is a base of filter $\mathscr{N}_\Delta$.

5.2.3

The family

$$\mathscr{B}^{\circ 2} := \{\Delta_{\mathscr{U}} \circ \Delta_{\mathscr{V}} \mid \mathscr{U} \text{ and } \mathscr{V} \text{ are open covers of } X\} \tag{54}$$

is similarly a filter-base and, in view of (23), it generates a filter not finer than $\mathscr{N}_\Delta$.

The neighborhood filter of the diagonal, $\mathscr{N}_\Delta$, satisfies Axiom $(\mathbf{U}_3)$, and thus is a uniformity on X, precisely when (54) generates $\mathscr{N}_\Delta$.

5.2.4

We shall now show that in a regular topological space pairs of separable points (p, q) can be separated from the diagonal, Δ, using neighborhoods from $\mathscr{B}^{\circ 2}$.

Lemma 5.2 *Let p and q be a pair of points in a regular topological space X such that $\mathscr{N}_p \neq \mathscr{N}_q$. Then, there exists an open cover $\mathscr{U}$ of X and a neighborhood W of (p, q), such that*

$$W \cap (\Delta_{\mathscr{U}} \circ \Delta_{\mathscr{U}}) = \emptyset. \tag{55}$$

Proof. In a regular space any pair of points p and q with $\mathscr{N}_p \neq \mathscr{N}_q$ can be separated by a pair of open neighborhoods $U \in \mathscr{N}_p$ and $V \in \mathscr{N}_q$:

$$U \cap V = \emptyset.$$

In a regular space, closed neighborhoods of a point form a fundamental system of neighborhoods of that point. Hence, there exist closed neighborhoods $M \in \mathcal{N}_p$ and $N \in \mathcal{N}_q$ such that

$$M \subseteq U \qquad \text{and} \qquad N \subseteq V.$$

In particular,

$$\mathcal{U} := \{U, V, (M \cup N)^c\}$$

is an open cover of X. The final two steps of the proof we leave as exercises.

Exercise 23 *Show that*

$$\mathcal{U} \diamond \mathcal{U} = \{M^c, N^c\}.$$

In particular,

$$\Delta_{\mathcal{U}} \circ \Delta_{\mathcal{U}} = \Delta_{\mathcal{U} \diamond \mathcal{U}} = (M^c \times M^c) \cup (N^c \cup N^c).$$

Exercise 24 *Describe*

$$X \times X \setminus \big((M^c \times M^c) \cup (N^c \cup N^c)\big)$$

and derive from it that

$$(M \times N) \cap \Delta_{\mathcal{U} \diamond \mathcal{U}} = \emptyset.$$

Thus, set $W = M \times N$ is a desired neighborhood of (p, q) in $X \times X$.

□

Theorem 5.3 (Compact Uniformization Theorem) *The neighborhood filter, $\mathcal{N}_\Delta$, of the diagonal $\mathcal{D} \subset X \times X$, is a uniform structure on X if X is a compact regular space.*

Proof. If $\mathcal{N}_\Delta$ does not satisfy Axiom $(\mathbf{U_3})$, then there exists an open neighborhood $E \in \mathcal{N}_\Delta$ such that

$$(D \circ D) \setminus E \neq \emptyset \qquad (D \in \mathcal{N}_\Delta). \tag{56}$$

Thus, the family

$$\mathcal{C} := \{(D \circ D) \setminus E \mid D \in \mathcal{N}_\Delta\} \tag{57}$$

consists of non-empty subsets, and is directed by reverse inclusion. In other words, (57) is a filter-base on $X \times X \setminus E$.

The latter being a closed subset of the product of two compact spaces is compact by Tichonov's Theorem. It follows that $\mathscr{C}$ adheres to a certain point $(p, q) \in X \times X \setminus E$.

Note that p is not E-close to q, i.e., the neighborhood $E(q)$ of q does not contain p and, similarly, the neighborhood $E^{-1}(p)$ of p does not contain q.

By Lemmma 5.2, there exists an open neighborhood D of Δ such that $D \circ D$ is disjoint with a certain neighborhood of (p, q). It follows that (p, q) cannot be a cluster point of (57).

The contradiction proves that $\mathscr{N}_\Delta$ satisfies Axiom $(\mathbf{U_3})$. □

6 Completeness

6.1 Cauchy filters

6.1.1

Definition 6.1 *A filter $\mathscr{F}$ on a uniform space $(X, \mathfrak{U})$ is called a* ***Cauchy filter*** *if, for any $E \in \mathfrak{U}$, there exists an E-small subset $A \in \mathscr{F}$.*

Exercise 25 *Show that, for any filter $\mathscr{F}$ on X, the collection of sets*

$$\mathscr{B} := \{E(A) \mid E \in \mathfrak{U} \text{ and } A \in \mathscr{F}\} \tag{58}$$

is a base of a filter.

Definition 6.2 *The filter generated by $\mathscr{B}$, cf.* (58)*, will be denoted $F^{\mathfrak{U}}$.*

Exercise 26 *Show that $\mathscr{F}^{\mathfrak{U}}$ is Cauchy if F is Cauchy.*

Exercise 27 *Suppose that $\mathscr{G} \subseteq \mathscr{F}$ and $\mathscr{G}$ is a Cauchy filter. Show that $\mathscr{F}^{\mathfrak{U}} \subseteq \mathscr{G}$.*

It follows from Exercises 26 and 27 that $\mathscr{F}^{\mathfrak{U}}$ is the smallest Cauchy subfilter contained in a given Cauchy filter $\mathscr{F}$. We shall therefore say that $\mathscr{F}$ is a **minimal** Cauchy filter if $\mathscr{F} = \mathscr{F}^{\mathfrak{U}}$.