Notes on Topology

An annex to H104, H113, etc.

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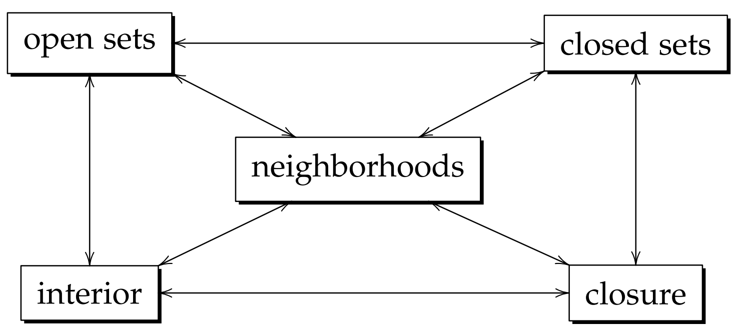
1 Five basic concepts



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1.1 Introduction

1.1.1

A set X can be made into a *topological space* in five different ways, each corresponding to a certain basic concept playing the role of a primitive notion in terms of which the other four are expressed.

Thus, one can talk of the topology on a set X in the language of *open sets*, or the language of *closed sets*, or the language of the *interior* operation, or the language of the *closure* operation or, finally, the language of *neighborhoods* of an arbitrary point $p \in X$.

The five 'languages' are completely equivalent which means, in particular, that one can faithfully translate from any one of them into any other one.

1.1.2 The closure operation

A mapping $\mathscr{P}(X) \longrightarrow \mathscr{P}(X)$, $E \longmapsto \bar{E}$, is said to be a **(topological) closure operation** if it satisfies the following conditions for any $E, F \subseteq X$:

($\mathbf{C_1}$) $E \subseteq \bar{E}$

($\mathbf{C_2}$) $\bar{\bar{E}} = \bar{E}$

($\mathbf{C_3}$) $\overline{E \cup F} = \bar{E} \cup \bar{F}$

($\mathbf{C_4}$) $\overline{\varnothing} = \varnothing$

Exercise 1 *Show that* $\bar{E} \subseteq \bar{F}$ *whenever* $E \subseteq F$.

1.1.3 The interior operation

A mapping $\mathscr{P}(X) \longrightarrow \mathscr{P}(X)$, $E \longmapsto \mathring{E}$, is said to be an **interior operation** if it satisfies the following conditions for any $E, F \subseteq X$:

($\mathbf{I_1}$) $\mathring{E} \subseteq E$

($\mathbf{I_2}$) $\mathring{\mathring{E}} = \mathring{E}$

($\mathbf{I_3}$) $(E \cap F)^{\circ} = \mathring{E} \cap \mathring{F}$

($\mathbf{I_4}$) $\mathring{X} = X$

Exercise 2 *Show that* $\mathring{E} \subseteq \mathring{F}$ *whenever* $E \subseteq F$.

1.1.4

If $E \longmapsto \bar{E}$ is a closure operation on X, then

$$E \longmapsto \mathring{E} := \left(\overline{(E^c)}\right)^c \tag{2}$$

is an interior operation. And vice-versa, if $E \longmapsto \mathring{E}$ is an interior operation on X, then

$$E \longmapsto \bar{E} := ((E^c)^\circ)^c \tag{3}$$

is a closure operation.

Exercise 3 *Prove the above statement.*

1.1.5

The two operations are *conjugate* to each other. This is represented by the commutativity of the following diagram

$$\begin{array}{ccc} \mathscr{P}(X) & \xrightarrow{(\ )^\circ} & \mathscr{P}(X) \\ {\scriptstyle (\ )^c} \updownarrow & & \updownarrow {\scriptstyle (\ )^c} \\ \mathscr{P}(X) & \xrightarrow{\overline{(\ )}} & \mathscr{P}(X) \end{array} \tag{4}$$

The conjugating operation, $E \longmapsto E^c$, is an example of *anti-automorphism* of a partially ordered set: it is invertible (in fact, it is equal to its own inverse), but *reverses* the order instead or preserving it:

$$E^c \supseteq F^c \qquad \text{if} \qquad E \subseteq F. \tag{5}$$

Moreover, it interchanges the operations of union and intersection:

$$(E \cup F)^c = E^c \cap F^c \qquad \text{and} \qquad (E \cap F)^c = E^c \cup F^c. \tag{6}$$

1.1.6 Open sets

A subset $\mathscr{T} \subseteq \mathscr{P}(X)$ is said to be **a topology** on a set X if it satisfies the following conditions

($\mathbf{T_1}$) for any $\mathscr{U} \subseteq \mathscr{T}$, one has $\bigcup \mathscr{U} \in \mathscr{T}$

($\mathbf{T_2}$) for any finite $\mathscr{U} \subseteq \mathscr{T}$, one has $\bigcap \mathscr{U} \in \mathscr{T}$

($\mathbf{T_3}$) $X \in \mathscr{T}$

($\mathbf{T_4}$) $\emptyset \in \mathscr{T}$

Members of $\mathscr{T}$ are referred to as **open** sets or, more precisely, open *subsets* of X.

1.1.7 Closed sets

Complements of members of any topology on X form a subset $\mathscr{Z} \subseteq \mathscr{P}(X)$ which possesses the following properties

($\mathbf{Z_1}$) for any $\mathscr{W} \subseteq \mathscr{Z}$, one has $\bigcap \mathscr{W} \in \mathscr{Z}$

($\mathbf{Z_2}$) for any finite $\mathscr{W} \subseteq \mathscr{Z}$, one has $\bigcup \mathscr{W} \in \mathscr{Z}$

($\mathbf{Z_3}$) $\emptyset \in \mathscr{Z}$

($\mathbf{Z_4}$) $X \in \mathscr{Z}$

Members of $\mathscr{Z}$ are then referred to as **closed** sets or, more precisely, closed *subsets* of X.

1.1.8 Neighborhoods

A family $\{\mathscr{N}_p\}_{p\in X}$, indexed by elements of X, of subsets $\mathscr{N}_p \subseteq \mathscr{P}(X)$ is said to be a *neighborhood system* on a set X if it satisfies the following conditions

($\mathbf{N_1}$) if $M, N \in \mathscr{N}_p$, then $M \cap N \in \mathscr{N}_p$

($\mathbf{N_2}$) if $N \in \mathscr{N}_p$ and $N \subseteq N'$, then $N' \in \mathscr{N}_p$

($\mathbf{N_3}$) if $N \in \mathscr{N}_p$, then $\{q \in N \mid N \in \mathscr{N}_q\} \in \mathscr{N}_p$

($\mathbf{N_4}$) $p \in \bigcap \mathscr{N}_p$

($\mathbf{N_5}$) $X \in \mathscr{N}_p$

Members of $\mathscr{N}_p$ are referred to as **neighborhoods** of point p.

1.2 An interlude: Filters and filter-bases

1.2.1 Directed sets

A nonempty partially ordered set $(\Lambda, \preceq)$ is said to be **directed** if

$$\textit{for any } \lambda, \mu \in \Lambda \textit{ there exists } \nu \in \Lambda \textit{ such that } \lambda \preceq \nu \textit{ and } \lambda \preceq \nu \tag{7}$$

or, equivalently, if any nonempty finite subset of I is bounded above.

1.2.2 Filter-bases

A nonempty subset $\mathscr{F} \subseteq \mathscr{P}(X)$ is said to be a **filter-base** on a set X if it satisfies the following conditions

$(\mathbf{F_1})$ for any $E, F \in \mathscr{F}$, there exists $G \in \mathscr{F}$ such that $G \subseteq E \cap F \in \mathscr{F}$

$(\mathbf{F_2})$ $\emptyset \notin \mathscr{F}$.

In other words, a filter-base on a set X is a directed subset $\mathscr{F} \subset \mathscr{P}(X)$ which does not contain $\emptyset$, the smallest element of $(\mathscr{P}(X) \subseteq)$.

Exercise 4 *Show that the intersection of finitely many members of a filter-base is nonempty:*

$$E_1 \cap \cdots \cap E_n \neq \emptyset \qquad (E_1, \ldots, E_n \in \mathscr{B}).$$

1.2.3 Filters

A filter-base that satisfies one more condition:

$(\mathbf{F_3})$ for any subsets $E \subseteq F \subseteq X$, if $E \in \mathscr{F}$, then also $F \in \mathscr{F}$,

is called a **filter** on a set X.

1.2.4 The filter generated by a filter-base

For a given filter-base $\mathscr{B} \subseteq \mathscr{P}(X)$ on a set X, define

$$\mathscr{B}_* := \{F \subseteq X \mid F \supseteq E \text{ for some } E \in \mathscr{B}\} \tag{8}$$

Exercise 5 *Show that $\mathscr{B}_*$ satisfies condtitions $(\mathbf{F_1})$-$(\mathbf{F_3})$ above. Show that any filter $\mathscr{F}$ containing $\mathscr{B}$ contains $\mathscr{B}_*$ as well.*

Thus, $\mathscr{B}_*$ is the smallest filter containing $\mathscr{B}$. We shall refer to it as **the filter generated by** $\mathscr{B}$.

1.2.5 Example: principal filters

For any *nonempty* subset $E \subseteq X$, the family $\mathscr{P}_E$ of all subsets of X which contain $\mathscr{E}$, is a filter. In fact, it is the filter generated by the filter-base consisting of one single set E:

$$\mathscr{P}_E := \{E\}_*. \tag{9}$$

We shall call such filters *principal*. In case, E is a singleton set $\{p\}$ for some $p \in X$, we may write $\mathscr{P}_p$ instead of $\mathscr{P}_{\{p\}}$.

Exercise 6 *Show that any filter on a finite set X is principal.*

Exercise 7 *Show that if $F \in \mathscr{F}$, then $\mathscr{P}_F \subseteq \mathscr{F}$.*

Exercise 8 *Show that any filter $\mathscr{F}$ on a set X is the union of principal filters. More precisely,*

$$\mathscr{F} = \bigcup_{F \in \mathscr{F}} \mathscr{P}_F. \tag{10}$$

1.2.6 Example: the Fréchet filter of a directed set

Let $(S, \preceq)$ be a partially ordered set.

Exercise 9 *Show that the image of S under the canonical emebedding of $(S, \preceq^{rev})$ into $\mathscr{P}(S)$,*

$$\{[s\rangle \mid s \in S\}, \tag{11}$$

is a filter-base on S if and only if $(S, \preceq)$ is directed.

In that case, the filter generated by (11) is called the *Fréchet filter* on directed set $(S, \preceq)$. We shall denote it $\mathrm{Fr}(S, \preceq)$ or, simply, $\mathrm{Fr}(S)$, if the order on S is clear.

Exercise 10 *Show that* $\mathrm{Fr}(\mathbb{N}, \leq)$ *consists of all subsets $E \subseteq \mathbb{N}$ such that the complement, E^c, is finite.*

1.2.7 Example: the direct image of a filter

Exercise 11 *Let $\mathscr{B}$ be a filter-base on a set X and $f\colon X\longrightarrow Y$ be a mapping. Show that the images of all members of $\mathscr{B}$ under f,*

$$f(\mathscr{B}) := \{f(B) \mid B \in \mathscr{B}\}, \tag{12}$$

form a filter-base on Y.

In other words, filter-bases on X are sent by mappings $f\colon X\longrightarrow Y$ to filter-bases on Y. If $\mathscr{F}$ is a filter on X, its image, $f(\mathscr{F})$, is only a filter-base however: a subset G of Y which contains $f(E)$ does not have to be of the form $f(F)$ for some $F \in \mathscr{F}$. It does not even have to be of the form $f(F)$ for any $F \in \mathscr{P}(X)$.

For this reason, we consider instead the generated filter

$$f_*\mathscr{F} := (f(\mathscr{F}))_* \tag{13}$$

and call it the **direct image** of filter $\mathscr{F}$ under mapping f.

1.2.8 Example: the elementary filter associated with a net

A **net** in a set X is just a family of elements of X, indexed by a certain directed set Λ. In other words, a net $\xi = \{\xi_\lambda\}_{\lambda\in\Lambda}$ in X is the same as a mapping

$$\xi\colon \Lambda\longrightarrow X, \qquad i \longmapsto \xi_\lambda.$$

1.2.9

A net indexed by the set of natural numbers, $\mathbb{N}$, or by any of its subsets, is called a *sequence* in X.

1.2.10 The elementary filter associated with a net

The direct image of the Fréchet filter,

$$\xi_*(\mathrm{Fr}(\Lambda)) \tag{14}$$

is called the *elementary filter associated with* net ξ.

1.2.11 Example: the inverse image of a filter

In contrast with the image of a filter-base, the *preimage* of a filter-base is generally not a filter-base, it is not even contained in any filter, and the reason is clear: the preimage of a nonempty subset may be empty.

Exercise 12 *Let* $\mathscr{C}$ *be a filter-base on a set* Y *and* $f\colon X\longrightarrow Y$ *be a mapping. Show that the preimages of all members of* $\mathscr{C}$ *under* f*,*

$$f^{-1}(\mathscr{C}) := \{f^{-1}(C) \mid C \in \mathscr{C}\}, \tag{15}$$

form a filter-base on X *if and only if*

$$C \cap f(X) \neq \emptyset \qquad \text{for every member } C \in \mathscr{C}. \tag{16}$$

1.2.12

Let $\mathscr{G}$ be a filter on Y. If it satisfies condition (16), then the filter generated by $f^{-1}(\mathscr{G})$,

$$f^{*}(\mathscr{G}) := \big(f^{-1}(\mathscr{G})\big)_{*} \tag{17}$$

is called the **inverse image** of filter $\mathscr{G}$.

1.3 The neighborhood filters

1.3.1

Let us return to the definition of a system of neighborhoods, cf. 1.1.8.

Exercise 13 *Show conditions* $(\mathbf{N_1})$, $(\mathbf{N_2})$, $(\mathbf{N_4})$, *and* $(\mathbf{N_5})$ *imply that every* $\mathscr{N}_p$ *is a filter.*

Vice versa, show that any filter $\mathscr{F}$ *on* X *such that*

$$p \in \bigcap \mathscr{F}$$

satisfies conditions $(\mathbf{N_1})$, $(\mathbf{N_2})$, $(\mathbf{N_4})$, *and* $(\mathbf{N_5})$.

1.3.2 A fundamental system of neighborhoods

Any filter-base of $\mathscr{N}_p$ is called a **fundamental system of neighborhoods** of X at point p.

The name reflects the fact that the neighborhood filters are often defined as the filters generated by certain explicitly given filter-bases.

2 Different approaches to equipping a set with a topological structure

2.1 Sets with a closure operation

Let X be a set equipped with a closure operation, cf. 1.1.2. The associated interior operation can be defined exactly as in 1.1.4.

2.1.1

Define a subset $Z \subseteq X$ to be *closed* if $\bar{Z} = Z$, and a subset $U \subseteq X$ to be *open* if its complement, U^c, is closed.

Exercise 14 *Show that so defined family of closed subsets of X satisfies axioms* $(\mathbf{Z_1})$-$(\mathbf{Z_4})$.

2.1.2

Neighborhoods are defined in terms of the associated interior operations, cf. 2.2.2 below.

2.2 Sets with an interior operation

Let X be a set equipped with an interior operation, cf. 1.1.3. The associated closure operation has been defined in 1.1.4.

2.2.1

Define a subset $U \subseteq X$ to be *open* if $\mathring{U} = U$, and a subset $Z \subseteq X$ to be *closed* if its complement, Z^c, is open.

2.2.2

Let us say that a subset $N \subseteq X$ is a *neighborhood* of a point $p \in X$ if $p \in \mathring{N}$.

Exercise 15 *Show that so defined family* $\{\mathcal{N}_p\}_{p \in X}$ *satisfies axioms* $(\mathbf{N_1})$-$(\mathbf{N_5})$.

2.3 Sets with a topology

Let X be a set equipped with a topology, cf. 1.1.6.

2.3.1

Define a subset $Z \subseteq X$ to be *closed* if its complement, Z^c, is open.

2.3.2

For any $E \subseteq X$, define its interior as the union of all the open subsets of E,

$$\mathring{E} := \bigcup_{\substack{U \subseteq E \\ U \text{ is open}}} U. \tag{18}$$

2.3.3

Let us say that a subset $N \subseteq X$ is a *neighborhood* of a point $p \in X$ if $p \in U$ for some open subset U of N.

2.3.4 A base of the topology

A subset $\mathscr{S} \subseteq \mathscr{T}$ of the topology is said to be a **base** of the topology if every open subset is the union of members of $\mathscr{S}$.

Exercise 16 *Show that $\mathscr{S}$ is a base of the topology if and only if the family $\{\mathscr{B}_p\}_{p \in X}$,*

$$\mathscr{B}_p := \{V \in \mathscr{S} \mid V \in \mathscr{N}_p\} \tag{19}$$

is a fundamental system of neighborhoods, i.e. is a family of filter-bases generating the neighborhood filters $\mathscr{N}_p$ defined in 2.3.3.

2.3.5 Comparing topologies

Topologies on a a given set X form a set, denoted $\mathrm{Top}(X)$, which is partially ordered by inclusion. It is a subset of $(\mathscr{P}(\mathscr{P}(X)), \subseteq)$.

When $\mathscr{T} \subseteq \mathscr{T}'$ it is common to say that topology $\mathscr{T}'$ is *finer*, or *stronger*, than $\mathscr{T}$. In the same situation one also says that $\mathscr{T}$ is *coarser*, or *weaker* than $\mathscr{T}'$.

2.3.6 The discrete topology

The set of all topologies on X has the greatest element, namely

$$\mathscr{T}^{\mathrm{discr}} := \mathscr{P}(X). \tag{20}$$

This is the *finest* topology on X. It is called the **discrete topology**. In discrete topology every subset of X is open. In paricular, every subset is also closed.

2.3.7 The trivial topology

The set of all topologies on X has also the smallest element,

$$\mathscr{T}^{\mathrm{triv}} := \{\emptyset, X\}. \tag{21}$$

This is the *coarsest* topology on X. It is called the **trivial topology**. In trivial topology the closure of any nonempty subset equals X, and the interior of any proper subset is empty.

2.3.8

Let $\mathcal{T} \subseteq \mathrm{Top}(X)$ be any family of topologies on a set X. It follows immediately from the definition that subsets of X which belong to each member $\mathscr{T} \in \mathcal{T}$ form a topology on X. In other words, the intersection of any family of topologies on X,

$$\bigcap \mathcal{T} = \bigcap_{\mathscr{T} \in \mathcal{T}} \mathscr{T} \tag{22}$$

is again a topology on X. In particular, the infimum of any subset of $\mathcal{T}$ exists and equals (22).

It follows that $\mathrm{Top}(X)$ is a complete lattice.

2.3.9

We can understand better the structure of the set of all topologies on a given set X by considering two operations on the set of all families of subsets of X:

$$(\ )^{\wedge}\colon \mathscr{P}(\mathscr{P}(X)) \longrightarrow \mathscr{P}(\mathscr{P}(X)), \qquad \mathscr{E} \longmapsto \mathscr{E}^{i}, \tag{23}$$

where $\mathscr{E}^{\wedge}$ is the family formed by intersections of arbitrary *finite* subfamilies $\mathscr{E}_0 \subseteq \mathscr{E}$ of a given family $\mathscr{E}$,

$$\mathscr{E}^{\wedge} := \left\{\bigcap \mathscr{E}_0 \mid \mathscr{E}_0 \subseteq \mathscr{E} \text{ and } \mathscr{E}_0 \text{ is finite}\right\}, \tag{24}$$

and

$$(\ )^{\cup} \colon \mathscr{P}(\mathscr{P}(X)) \longrightarrow \mathscr{P}(\mathscr{P}(X)), \qquad \mathscr{E} \longmapsto \mathscr{E}^{\cup}, \tag{25}$$

where $\mathscr{E}^{\cup}$ is the family formed by intersections of arbitrary subfamilies $\mathscr{E}' \subseteq \mathscr{E}$ of a given family $\mathscr{E}$,

$$\mathscr{E}^{\cup} := \left\{\bigcup \mathscr{E}' \mid \mathscr{E}' \subseteq \mathscr{E}\right\}. \tag{26}$$

2.3.10

By considering the intersection and the union of the empty subfamily $\emptyset \subseteq \mathscr{E}$, we deduce that, for any $\mathscr{E} \subseteq \mathscr{P}(X)$,

$$X = \inf_{\mathscr{P}(X)} \emptyset = \bigcap \emptyset \in \mathscr{E}^{\wedge}$$

and, similarly,

$$\emptyset = \sup_{\mathscr{P}(X)} \emptyset = \bigcup \emptyset \in \mathscr{E}^{\cup}.$$

2.3.11

Topologies on a set X are precisely the subsets of $\mathscr{P}(X)$ which are both ${}^{\wedge}$- and ${}^{\cup}$-*closed*,

$$\mathrm{Top}(X) = \{{}^{\wedge}\text{-closed subsets of } \mathscr{P}(X)\} \cap \{{}^{\cup}\text{-closed subsets of } \mathscr{P}(X)\}. \tag{27}$$

2.3.12

Both of the above operations on $\mathscr{P}(\mathscr{P}(X))$ are morphisms of partially ordered sets, and possess the following two properties of a (formal) closure operation, namely,

$$\mathscr{E} \subseteq \mathscr{E}^{-} \qquad \text{and} \qquad (\mathscr{E}^{-})^{-} = \mathscr{E}^{-}$$

where $\mathscr{E}^{-}$ denotes either $\mathscr{E}^{\wedge}$ or $\mathscr{E}^{\cup}$.

Exercise 17 *Show that for any family* $\mathscr{E} \subseteq \mathscr{P}(X)$*, one has*

$$((\mathscr{E}^{\wedge})^{\cup})^{\wedge} = (\mathscr{E}^{\wedge})^{\cup}.$$

Solution. *For any subfamilies* $\mathscr{D}$ *and* $\mathscr{D}'$ *of* $\mathscr{E}^{\wedge}$*, one has*

$$\left(\bigcup \mathscr{D}\right) \cap \left(\bigcup \mathscr{D}'\right) = \bigcup_{D \in \mathscr{D},\, D' \in \mathscr{D}'} D \cap D' \tag{28}$$

and, since the intersection of any two memebers of $\mathscr{E}^{\wedge}$*, belongs to* $\mathscr{E}^{\wedge}$*, the right-hand-side of* (28) *belongs to* $(\mathscr{E}^{\wedge})^{\cup}$*.*

Exercise 18 *Using the previous exercise, show that, for any family* $\mathscr{E} \subseteq \mathscr{P}(X)$*, family* $(\mathscr{E}^{\wedge})^{\cup}$ *is a topology on set* X*.*

2.3.13 The topology generated by a family of subsets

For any family $\mathscr{E} \subseteq \mathscr{P}(X)$, the family $(\mathscr{E}^{\wedge})^{\cup}$ is the coarsest topology containing $\mathscr{E}$ and $\mathscr{E}^{\wedge}$, the $^{\wedge}$-closure of $\mathscr{E}$, is its base.

2.3.14 A pre-base of a topology

Given a topology $\mathscr{T}$ on a set X, any family $\mathscr{E} \subseteq \mathscr{P}(X)$ such that

$$(\mathscr{E}^{\wedge})^{\cup} = \mathscr{T}$$

is referred to as a *pre-base* of $\mathscr{T}$.

Exercise 19 *For any family* $\mathcal{T}$ *of topologies on a set* X*, describe* $\sup_{\mathrm{Top}(X)} \mathcal{T}$*.*

2.4 Sets with a family of closed sets

Let X be a set equipped with a family of closed sets, cf. 1.1.7.

2.4.1

Define a subset $U \subseteq X$ to be *open* if its complement, U^c, is closed.

Exercise 20 *Show that so defined family of subsets of* X *is a topology on* X*.*

2.4.2

Neighborhoods are then defined as in 2.3.3 above.

2.4.3

For any $E \subseteq X$, define its closure as the intersection of all the closed oversets of E,

$$\bar{E} := \bigcap_{\substack{E \subseteq Z \\ Z \text{ is closed}}} Z. \tag{29}$$

2.5 Sets with a system of neighborhoods

Let X be a set equipped with a system of neighborhoods, cf. 1.1.8.

2.5.1

Define a subset $U \subseteq X$ to be *open* if U is a neighborhood of every $p \in U$.

Exercise 21 *Show that so defined family of subsets of X is a topology on X.*

2.5.2

For any $E \subseteq X$, define its *interior*, $\mathring{E}$, as the set of points $p \in X$ such that N is a neighborhood of p.

Exercise 22 *Show that so defined operation, $E \longmapsto \mathring{E}$, satisfies conditions* $(\mathbf{I_1})$-$(\mathbf{I_4})$.

2.5.3

Define a subset $Z \subseteq X$ to be *closed* if for any $p \notin Z$, the complement Z^c is a neighborhood of p.

2.5.4

For any $E \subseteq X$, define its *closure* as the set of points $p \in X$ such that *every* neighborhood N of p has a nonempty intersection with E.

Exercise 23 *Show that so defined operation, $E \longmapsto \bar{E}$, satisfies conditions* $(\mathbf{C_1})$-$(\mathbf{C_4})$.

2.5.5

Define a subset $Z \subseteq X$ to be *closed* if its complement, Z^c, is open.

2.6 Topological spaces

2.6.1

A set equipped with any of the five structures described above: a closure operation, an interior operation, a topology, a family of closed subsets, or a system of neighborhoods—is called a *topological space*. Whichever notion is taken to be primitive, the other four are associated with it exactly as described in Sections 2.1-2.5 above.

2.6.2

The most common definition of a topological space in use today is 2.3. This is probably the reason why the family of all open sets of a topological space X is referred to as *the topology* of X.

2.7 Induced topology

2.7.1

Let $(Y, \mathscr{T})$ be a topological space and X be an arbitrary subset of Y.

Exercise 24 *Show that the family*

$$\mathscr{T}_{|X} := \{V \cap X \mid V \in \mathscr{T}\} \tag{30}$$

satisfies conditions $(\mathbf{T_1})$-$(\mathbf{T_4})$*, and thus is a topology on set* X*.*

Family $\mathscr{T}_{|X}$ is called the *induced* topology or, the topology *induced by the topology of* Y.

2.7.2

When vieved as subsets of Y, members of $\mathscr{T}_{|X}$ are said to be *open relative to* X or, *relatively open*—if the subset, X, is clear from the context.

Exercise 25 *Show that a subset* $Z' \subseteq X$ *is closed in the induced topology if and only if it is of the form* $Z' = Z \cap X$ *for some closed subset* Z *of* Y*.*

2.7.3 Axioms of Countability

A topological space X such that every point $p \in X$ admits a countable fundamental system of neigborhoods is said to satisfy the so called *First Axiom of Countability*. Informally, such spaces are referred to as being *first-countable*.

If the topology, $\mathscr{T}$, of X admits a countable base, then X is said to satisfy the so called *Second Axiom of Countability*. Informally, such spaces are referred to as being *second-countable*.

Exercise 26 *Show that a second-countable space is first-countable.*

2.7.4 Isolated points of a subset of a topological space

A point $p \in E$ is said to be *isolated* if there exists a neighborhood $N \in \mathscr{N}_p$ such that p is the only member of E which belongs to it:

$$N \cap E = \{p\}.$$

Exercise 27 *Show that $p \in E$ is not isolated if and only if*

$$p \in \overline{E \setminus \{p\}}$$

or, equivalently, if there exists a net in $E \setminus \{p\}$ which converges to p.

Exercise 28 *Let $p \in X$. Show that $p \in \overline{E \setminus \{p\}}$ if and only if the family of subsets of $E^* := E \setminus \{p\}$,*

$$N^* := N \cap E^* \qquad (N \in \mathscr{N}_p), \tag{31}$$

is a filter in E^*.

3 An interplay between filters and the topology

3.1 The partialy ordered sets of filters and filter-bases on an arbitrary set

3.1.1 The Finite Intersection Property

Let us start from the following question:

Under what conditions a family $\mathscr{E}$ of subsets of a given set X is contained in a filter on X? (32)

The answer is readily found:

A family $\mathscr{E} \subseteq \mathscr{P}(X)$ *is contained in a filter, if and only if the intersection of any* finite *subfamily* $\mathscr{E}_0 \subseteq \mathscr{E}$ *is nonempty:*

$$\bigcap \mathscr{E}_0 \neq \emptyset \qquad \textit{for any finite subfamily } \mathscr{E}_0 \subseteq \mathscr{E}. \tag{33}$$

Condition (33) is obviously necessary. It is also sufficient: the family of intersections of finite collections of members of $\mathscr{E}$,

$$\mathscr{B}_{\mathscr{E}} := \left\{ \bigcap \mathscr{E}_0 \mid \mathscr{E}_0 \text{ a finite nonempty subfamily of } \mathscr{E} \right\}, \tag{34}$$

is clearly a filter-base. We shall call it the filter-base *associated with* family $\mathscr{E}$. We shall also denote by $\mathscr{E}_*$ the filter generated by $\mathscr{B}_{\mathscr{E}}$.

3.1.2

The set of all filter-bases on a given set X is partially ordered by inclusion and contains as a subset the set of all filters. We shall denote the latter by $\mathrm{Filt}(X)$ and the former by $\mathrm{FB}(X)$.

3.1.3

It is common to say that $\mathscr{B}'$ is *finer or equal* than $\mathscr{B}$, and that $\mathscr{B}$ is *coarser or equal* than $\mathscr{B}'$, if

$$\mathscr{B} \subseteq \mathscr{B}'.$$

In this case one can also say that $\mathscr{B}'$ is a *refinement* of $\mathscr{B}$.

In practice, the phrase *'or equal'* is often dropped and one hears instead *finer*, or *coarser*, even though the case $\mathscr{B} = \mathscr{B}'$ is not excluded.

Exercise 29 *Show that the intersection of any family of filters* $\mathcal{F}$,

$$\bigcap \mathcal{F} = \bigcap_{\mathscr{F} \in \mathcal{F}} \mathscr{F},$$

is a filter.

3.1.4

It follows that the partially ordered set of filters on a set X, $\text{Filt}(X)$, is complete. The principal filter $\mathscr{P}_X$, consisting of just one set X,

$$\mathscr{P}_X = \{X\},$$

is the smallest element of $\text{Filt}(X)$.

3.1.5

The situation is different if we consider the set of all filter-bases on X.

Let $B, B' \subseteq X$ be two nenempty and not equal subsets of X. Then the singleton sets $\mathscr{B} = \{B\}$ and $\mathscr{B}' = \{B'\}$ are filter-bases whose intersection is empty

$$\mathscr{B} \cap \mathscr{B}' = \emptyset$$

which means that the set

$$\{\mathscr{B}, \mathscr{B}'\}$$

is not bouned below in $\text{FB}(X)$. It is bounded above precisely when $B \cap B' \neq \emptyset$.

Exercise 30 *Let $B, B' \subseteq X$ and suppose that $B \not\subseteq B'$ and $B' \not\subseteq B$.*

Show that a filter base $\mathscr{C} \in \text{FB}(X)$ is a minimal element of the set of upper bounds of $\{\mathscr{B}, \mathscr{B}'\}$ if and only if

$$\mathscr{C} = \{B, B', C\}$$

where C is a nonempty subset of $B \cap B'$.

Exercise 31 *Deduce from this that $\sup_{\text{FB}(X)}\{\mathscr{B}, \mathscr{B}'\}$ exists if and only if $B \cap B'$ is a singleton set.*

In particular, the partially ordered set of filter-bases, $\text{FB}(X)$, is not complete if X has at least 4 elements.

Exercise 32 *Let X be a 4-element set $\{a, b, c, d\}$. Find two filter-bases $\mathscr{B}$ and $\mathscr{B}'$ on X such that the set $\{\mathscr{B}, \mathscr{B}'\}$ is bounded below yet $\inf_{\text{FB}(X)}\{\mathscr{B}, \mathscr{B}'\}$ does not exist.*

3.1.6 Suprema in **Filt**(X)

A subset $\mathcal{F} \subseteq \mathrm{Filt}(X)$ is bounded above if and only if the union of all members of $\mathcal{F}$,

$$\mathscr{V} = \bigcup \mathcal{F} = \bigcup_{\mathscr{F} \in \mathcal{F}} \mathscr{F} \tag{35}$$

is a fimily of subsets of X which satisfies the Finite Intersection Property, cf. (33) above. In that case, the supremum of the family of filters, $\mathcal{F}$, is the filter generated by (35),

$$\sup_{\mathrm{Filt(X)}} \mathcal{F} = \mathscr{V}_*.$$

3.1.7 The join of two filters

In the special case of $\mathcal{F}$ consisting of just two filters,

$$\mathcal{F} = \{\mathscr{F}, \mathscr{G}\},$$

we shall call the supremum of $\{\mathscr{F}, \mathscr{G}\}$ — the *join* of $\mathscr{F}$ and $\mathscr{G}$, and will denote it

$$\mathscr{F} \vee \mathscr{G} := \sup_{\mathrm{Filt(X)}} \mathcal{F}. \tag{36}$$

The join of $\mathscr{F}$ and $\mathscr{G}$ exists if and only if

$$F \cap G \neq \emptyset \qquad \text{for any } F \in \mathscr{F} \text{ and } G \in \mathscr{G}. \tag{37}$$

Exercise 33 *Let E and F be two nenempty subsets of a set X. Show that $\mathscr{P}_E \vee \mathscr{P}_F$ exists if and only if $E \cap F \neq \emptyset$. What is it equal to?*

3.1.8

We are ready now to make the following important observation.

Proposition 3.1 *For any filter F on a set X and any subset $A \subseteq X$, there exists a refinement $\mathscr{F}'$ of $\mathscr{F}$ which contains A or its complement A^c as a member.*

Proof. Suppose that no refinement of $\mathscr{F}$ contains A. This happens precisely when

$$A \cap E = \emptyset \tag{38}$$

for some member $E \in \mathscr{F}$. One can restate (38) as

$$E \subseteq A^c$$

which in turn implies that $A^c \in \mathscr{F}$. □

3.1.9

The above propositions asserts that for any filter $\mathscr{F}$ on a set X and any subset $A \subseteq X$, always at least one of the following two refinements of $\mathscr{F}$ exists: $\mathscr{F} \vee \mathscr{P}_A$ or $\mathscr{F} \vee \mathscr{P}_{A^c}$.

3.1.10 Example: the preimage of a filter-base revisited

If $f\colon X \longrightarrow Y$ is a mapping and $\mathscr{C}$ a filter-base on Y, then $f^{-1}(\mathscr{C})$ is a filter-base on X precisely when the the set $\{\mathscr{C}, \mathscr{P}_{f(X)}\}$ is *bounded above* in $\mathrm{FB}(Y)$i. This is equivalent to the existence of the join $\mathscr{C}_* \vee \mathscr{P}_{f(X)}$. cf. 1.2.11.

3.1.11 Two representations of an arbitrary filter

We have seen, cf. (10), that any filter is the supremum in $\mathscr{P}(\mathscr{P}(X))$ of the family of *principal* filters

$$\left\{\mathscr{P}_F\right\}_{F\in\mathscr{F}}.$$

In particular, the set of principal filters

$$\left\{\mathscr{P}_E\right\}_{E\in\mathscr{P}(X)\setminus\{\emptyset\}}$$

is sup-dense in the set of all filters, $\mathrm{Filt}(X)$.

We shall see next that the set of *elementary* filters is inf-dense in $\mathrm{Filt}(X)$.

3.1.12

The Axiom of Choice asserts guarantees that, for any filter-base $\mathscr{B}$ on a set X, the set of all choice functions (also known as *selectors*) for family $\mathscr{B}$,

$$\mathrm{Sel}(\mathscr{B}) := \{\xi\colon \mathscr{B} \longrightarrow X \mid \xi_B \in B \text{ for each } B \in \mathscr{B}\}, \tag{39}$$

is nonempty. Recall that $(\mathscr{B}, \supseteq)$ is a directed set, thus every choice function for $\mathscr{B}$ is a $\mathscr{B}$-indexed net in X.

Proposition 3.2 *For any filter-base on a set X, the filter generated by $\mathscr{B}$ is the intersection of the elementary filters associated with the choice function for $\mathscr{B}$:*

$$\mathscr{B}_* = \bigcap_{\xi\in\mathrm{Sel}(\mathscr{B})} \xi_*(\mathrm{Fr}(\mathscr{B})). \tag{40}$$

Proof. By definition, $\xi_B \in B$ for any $B \in \mathscr{B}$ and any choice function $\xi \in \mathrm{Sel}(\mathscr{B})$. Hence the set

$$\xi([B\rangle) = \{\xi_{B'} \mid B' \subseteq B\} \tag{41}$$

is contained in B. Sets (41) form a base of the elementary filter, $\xi_*(\mathrm{Fr}(\mathscr{B}))$, associated with net ξ, and therefore

$$\mathscr{B}_* \subseteq \bigcap_{\xi \in \mathrm{Sel}(\mathscr{B})} \xi_*(\mathrm{Fr}(\mathscr{B})), \tag{42}$$

If $E \notin \mathscr{B}_*$, then, for every $B \in \mathscr{B}$, one has $B \nsubseteq E$. In particular, the family of sets

$$\mathscr{B}' := \{B \setminus E \mid B \in \mathscr{B}\}$$

consists of nonempty subsets of X. Let χ be any choice function for family $\mathscr{B}'$. Since

$$B \setminus E \subseteq B \qquad (B \in \mathscr{B}),$$

χ induces a choice function for $\mathscr{B}$:

$$\phi : B \longmapsto \chi_{B \setminus E}.$$

Since

$$(B \setminus E) \cap E = \emptyset,$$

set E does not belong to the elementary filter, $\phi_*(\mathrm{Fr}(\mathscr{B}))$, associated with with ϕ. In particular,

$$E \notin \bigcap_{\xi \in \mathrm{Sel}(\mathscr{B})} \xi_*(\mathrm{Fr}(\mathscr{B})). \tag{43}$$

This proves the reversed containment

$$\mathscr{B}_* \supseteq \bigcap_{\xi \in \mathrm{Sel}(\mathscr{B})} \xi_*(\mathrm{Fr}(\mathscr{B})). \tag{44}$$

□

3.1.13 Ultrafilters

Maximal elements in $\text{Filt}(X)$ are traditionally referred to as **ultrafilters**.

Proposition 3.1 applied to an ultrafilter produces the following corollary.

Corollary 3.3 *Let $\mathscr{M}$ be an ultrafilter on a set X and A any subset of X. Then either A, or its complement, A^c, belongs to $\mathscr{M}$.*

Indeed, either $\mathscr{M} \vee \mathscr{P}_A$ or $\mathscr{M} \vee \mathscr{P}_{A^c}$ exists by Proposition 3.1. Each is a refinement of $\mathscr{M}$ when it exists, but $\mathscr{M}$ is maximal, hence

$$\mathscr{M} = \mathscr{M} \vee \mathscr{P}_A$$

if $\mathscr{M} \vee \mathscr{P}_A$ exists, and

$$\mathscr{M} = \mathscr{M} \vee \mathscr{P}_{A^c}$$

if $\mathscr{M} \vee \mathscr{P}_{A^c}$ exists. In the former case, $\mathscr{P}_A \subseteq \mathscr{M}$ which means that A is a member of $\mathscr{M}$, in the latter, $\mathscr{P}_{A^c} \subseteq \mathscr{M}$ which means that A^c is a member of $\mathscr{M}$.

Exercise 34 *Show the the hypothesis and conclusion in Corollary 3.3 can be reversed:*

> *Let $\mathscr{E}$ be any family of subsets of a set X which satisfies the Finite Intersection Property,* (33)*. If, in addition,*
>
> $$\text{for any subset } A \subseteq X, \text{ either } A \in \mathscr{E} \text{ or } A^c \in \mathscr{E}, \qquad (45)$$
>
> *then $\mathscr{E}$ is an ultrafilter.*

Exercise 35 *Let $\mathscr{M}$ be an ultrafilter on a set X and $f\colon X \longrightarrow Y$ be any mapping. Show that the image, $f(\mathscr{M})$, is an ultrafilter on set Y.* (*Hint: Show that, for any subset $B \subseteq Y$, either B or B^c belongs to $f(\mathscr{M})$.*)

Exercise 36 *Show that, for any $p \in X$, the principal filter $\mathscr{P}_p$ is an ultrafilter.*

Exercise 37 *Show that any ultrafilter on a finite set X is of the form $\mathscr{P}_p$ for a unique $p \in X$. Thus, there exists a bijective correspondence between a finite set X and the set of ultrafilters on X.*

Principal ultrafilters are sometimes referred to as *trivial* ultrafilters.

3.1.14

By using Axiom of Choice one can show that any element $\mathscr{F}$ of $\mathrm{Filt}(X)$ admits a refinement which is an ultrafilter, i.e., there exeists an ultrafilter $\mathscr{M}$ on X such that $\mathscr{F} \subseteq \mathscr{M}$.

3.2 Convergence and limits

3.2.1 A limit of a filter-base

Let $\mathscr{B}$ be a filter-base on a topological space X. We say that $\mathscr{B}$ **converges** to a point $p \in X$ if $\mathscr{B}$ contains a fundamental system of neighborhoods of p. Equivalently: *if the generated filter $\mathscr{B}_*$ contains the filter of neighborhoods, $\mathscr{N}_p$, of point p:*

$$\mathscr{N}_p \subseteq \mathscr{B}_*. \tag{46}$$

In this case, we say that point p is a **limit** of filter-base $\mathscr{B}$, and denote this by

$$\mathscr{B} \longrightarrow p. \tag{47}$$

The set of such points will be denoted $\mathrm{Lim}(\mathscr{B})$ since a filter-base may converge to more than one point.

3.2.2

In fact, a filter-base can even converge to *every* point of the topological space. This is so if X is equipped with trivial topology: in this case, the neighborhood filter of every point $p \in X$ is the smallest filter $\{X\}$, and therefore any filter-base on X converges to every point of X.

3.2.3

The filter of neighborhoods, $\mathscr{N}_p$, is obviously the smallest filter converging to p, and the set of filters in X which converge to a point p is the interval

$$[\mathscr{N}_p\rangle$$

in the set of all filters on X which is partially ordered by inclusion.

Exercise 38 *Let p and q be arbitrary points of a topological space X. Prove that the following three conditions are equivalent* *(a)* $\mathscr{N}_p \subseteq \mathscr{N}_q$;

(b) $p \in \overline{\{q\}}$;

(c) $q \in \bigcap \mathscr{N}_p$.

3.2.4 A limit of a mapping along a filter-base

Suppose that $f\colon X \longrightarrow Y$ be a mapping of a *set* X into a *topological space* Y. Let $\mathscr{B}$ be a filter-base on X. We say that a point $q \in Y$ is a **limit** of f *along* filter-base $\mathscr{B}$ if $f(\mathscr{B})$ converges to q, i.e., when

$$\mathscr{N}_q \subseteq f_*(\mathscr{B}). \tag{48}$$

3.2.5

Suppose that X is a topological space. We say that a point $p \in X$ is a **limit** of a net ξ if p is a limit of mapping $\xi\colon \Lambda \longrightarrow X$ along the Fréchet filter $\mathrm{Fr}(\Lambda)$. In this case, we also say that net $\xi = \{\xi_\lambda\}_{\lambda\in\Lambda}$ **converges** to point $p \in X$.

Exercise 39 *Using the above definition, show that a net ξ converges to p if and only if*

$$\textit{for any neighborhood } N \textit{ of } p\textit{, there exists } \mu \in \Lambda \textit{ such that } \xi_\lambda \in N \textit{ for all } \lambda \succeq \mu. \tag{49}$$

In the above formulation one can replace the filter of *all* neighborhoods of p by any fundamental system of neighborhoods of p.

Exercise 40 *Let $\mathscr{B}$ be a fundamental system of neighborhoods of a point $p \in X$. Show that a net ξ converges to p if and only if*

$$\textit{for any } B \in \mathscr{B}\textit{, there exists } \mu \in \Lambda \textit{ such that } \xi_\lambda \in B \textit{ for all } \lambda \succeq \mu. \tag{50}$$

3.2.6

Since a net may converge to more than one point, we cannot use the familar notation

$$\lim_{\lambda\in\Lambda} \xi_\lambda = p.$$

Sometimes notation $\xi \longrightarrow p$, or even $\xi_\lambda \longrightarrow p$, is used. One has to be careful not to confuse it with the notation employed to denote mappigs.

3.3 Two characterizations of the topological structure in terms of net convergence

3.3.1

The following proposition shows that the filter of neighborhoods of a point p in a topological space is the infimum of the elementary filters of the nets that converge to p. In particular, the neighborhood structure of a topological space can be fully recovered if we are told which nets converge to which points.

Proposition 3.4 *Let $\mathscr{B}$ be a fundamental system of neighborhoods of a point $p \in X$. The neighborhood filter, $\mathscr{N}_p$, is the intersection,*

$$\mathscr{N}_p = \bigcap_{\substack{\xi \in \mathrm{Map}(\mathscr{B}, X) \\ \xi \longrightarrow p}} \xi_*(\mathrm{Fr}(\mathscr{B})) \tag{51}$$

of the elementary filters associated with nets converging to p and indexed by the directed set $(\mathscr{B}, \supseteq)$.

Proof. Since $\mathscr{N}_p \subseteq \xi_*(\mathrm{Fr}(\mathscr{B}))$, for each $\mathscr{B}$-indexed net that converges to p, we have

$$\mathscr{N}_p \subseteq \bigcap_{\substack{\xi \in \mathrm{Map}(\mathscr{B}, X) \\ \xi \longrightarrow p}} \xi_*(\mathrm{Fr}(\mathscr{B})).$$

Any choice function of family $\mathscr{B}$ is a net convergent to p and indexed by $(\mathscr{B}, \supseteq)$. Hence

$$\mathscr{N}_p \subseteq \bigcap_{\substack{\xi \in \mathrm{Map}(\mathscr{B}, X) \\ \xi \longrightarrow p}} \xi_*(\mathrm{Fr}(\mathscr{B})) \subseteq \bigcap_{\xi \in \mathrm{Sel}(\mathscr{B})} \xi_*(\mathrm{Fr}(\mathscr{B})) = \mathscr{N}_p \tag{52}$$

where the last equality in (52) is supplied by Proposition 3.2. □

3.3.2

If space X is *first-countable* at point p, then there exists a nested sequence of neighborhoods:

$$C_0 \supsetneq C_1 \supsetneq C_2 \supsetneq \cdots$$

which forms a fundamental system of neighborhoods of p. Indeed, if

$$\mathscr{B} = \{B_n \mid n \in \mathbb{N}\}$$

is any countable base of filter $\mathscr{N}_p$, then consider the nested sequence

$$B_0 \supseteq B_0 \cap B_1 \supseteq \cdots \supseteq (B_0 \cap \cdots \cap B_n) \supseteq \cdots$$

and remove from it all the 'duplicates'.

Thus obtained partially ordered set $(\{C_n \mid n \in \mathbb{N}\}, \supseteq)$ is then isomorphic to the set of natural naumbers with its standard ordering. In particular, nets indexed by $(\{C_n \mid n \in \mathbb{N}\}, \supseteq)$ are the same as sequences, and we obtain the following corollary of Proposition 3.4.

Corollary 3.5 *If a topological space X is first-countable at a point $p \in X$, then the neighborhood filter, $\mathscr{N}_p$, is the intersection of the elementary filters associated with all sequences $\xi = \{\xi_n\}_{n \in \mathbb{N}}$ which converge to p,*

$$\mathscr{N}_p = \bigcap_{\xi_n \longrightarrow p} \xi_*(\mathrm{Fr}(\mathscr{B})). \tag{53}$$

3.3.3

Our next Proposition demonstrates how, using the knowledge of which nets converge to which points, to recover the closure operation.

Proposition 3.6 *Let E be a subset of a topological space X. A point p belongs to $\bar{E}$ if and only if there exists a net ξ in E which converges to p in X.*

Proof. Let B be any fundamental system of neighborhoods of p and suppose that $p \in \bar{E}$. Then, for every $B \in \mathscr{B}$, set $B \cap E$ is nonempty, and any choice function χ for the family

$$\{B \cap E \mid B \in \mathscr{B}\}$$

induces a net in E,

$$\xi \colon B \longmapsto \chi_{B \cap E} \qquad (B \in \mathscr{B}),$$

which converges to p since $\xi_B \in B$ for every $B \in \mathscr{B}$.[1]

Vice-versa, if $\xi = \{\xi_\lambda\}_{\lambda \in \Lambda}$ is any net in E which converges to p in X, then any neighborhood N of p contains at least one ξ_i, cf. Exercise 39. In particular, $N \cap E$ is not empty, i.e., $p \in \bar{E}$. □

[1]Note the similarities with the proof of Proposition 3.2.

Corollary 3.7 *Let E be a subset of a topological space X. Its closure consists of all points $p \in X$ such that there exists a net in E which converges to p in X.*

Corollary 3.8 *Let E be a subset of a topological space X which is first-countable at a point $p \in X$. Then $p \in \bar{E}$ if and only if there exists a sequence $\xi = \{\xi_n\}_{n \in \mathbb{N}}$ in E which converges to p in X.*

Corollary 3.9 *Let E be a subset of a second-countable topological space X. Its closure consists of all points $p \in X$ such that there exists a sequence in E which converges to p in X.*

3.4 Adherence and cluster points

3.4.1 A cluster point of a filter-base

A filter-base $\mathscr{B}$ on a topological space X may not converge to a point $p \in X$, but some *refinement* $\mathscr{C}$ of $\mathscr{B}$ may do so.

In this case we say that $\mathscr{B}$ *adheres* to p, and call p a *cluster point* of $\mathscr{B}$.

3.4.2

If $\mathscr{C}$ is a refinement of $\mathscr{B}$ which converges to p, then the generated filter, $\mathscr{C}_*$, is a common refinement of $\mathscr{B}_*$ and the neighborhood filter, $\mathscr{N}_p$. In particular, the join $\mathscr{B}_* \vee \mathscr{N}_p$ exists. Vice-versa, if $\mathscr{B}_* \vee \mathscr{N}_p$ exists, then it is a refinement of $\mathscr{B}$ which converges to point p.

We arrive at the following reformulation of the definition of adherence:

A filter-base $\mathscr{B}$ on a topological space X adheres to a point $p \in X$ if and only if the join of filters $\mathscr{B}_*$ and $\mathscr{N}_p$ exists. (54)

Equivalently, we have

A filter-base $\mathscr{B}$ on a topological space X adheres to a point $p \in X$ if and only if

$$B \cap N \neq \emptyset \qquad \text{for any } B \in \mathscr{B} \text{ and } N \in \mathscr{N}_p. \tag{55}$$

3.4.3

Note that the condition in (55) means that $p \in \bar{B}$ for any $B \in \mathscr{B}$. Thus we obtain one more characterization of adherence:

> *A filter-base $\mathscr{B}$ on a topological space X adheres to a point $p \in X$ if and only if*
>
> $$p \in \bigcap_{b \in \mathscr{B}} \bar{B}. \tag{56}$$

In particular, the set $\mathrm{Adh}(\mathscr{B})$ of cluster points of a filter-base $\mathscr{B}$ equals

$$\mathrm{Adh}(\mathscr{B}) = \bigcap_{b \in \mathscr{B}} \bar{B}. \tag{57}$$

Exercise 41 *Show that if $\mathscr{B} \subseteq \mathscr{B}'$, then*

$$\mathrm{Adh}(\mathscr{B}) \supseteq \mathrm{Adh}(\mathscr{F}).$$

Exercise 42 *Show that the set of cluster points of a filter-base $\mathscr{B}$ coincides with the set of cluster points of the generated filter*

$$\mathrm{Adh}(\mathscr{B}) = \mathrm{Adh}(\mathscr{B}_*).$$

Exercise 43 *Show that the closure of a subset $E \subseteq X$ coincides with the set of cluster points of the associated principal filter*

$$\bar{E} = \mathrm{Adh}(\mathscr{P}_E).$$

Exercise 44 *Prove that $p \in \mathrm{Adh}(\mathscr{N}_q)$ if and only if $q \in \mathrm{Adh}(\mathscr{N}_p)$.*

3.4.4 Cluster points of a net

A point p in a topological space X is said to be a *cluster point* of a net $\xi = \{\xi_\lambda\}_{\lambda \in \Lambda}$ in X, if it is a cluster point of the associated elementary filter, $\xi_*(\mathrm{Fr}(\Lambda))$.

The set of cluster points of net ξ will be denoted $\mathrm{Adh}(\xi)$.

3.4.5 Adherence for ultrafilters

If p is a cluster point of an ultrafilter $\mathscr{M}$, then every neighborhood $N \in \mathscr{N}_p$ has nonempty intersection with every member of $\mathscr{M}$. It follows that $\mathscr{M} \vee \mathscr{P}_N$ exists which is then a refinement of $\mathscr{M}$. In view of maximality of $\mathscr{M}$, one has $\mathscr{M} = \mathscr{M} \vee \mathscr{P}_N$ which implies in turn that $N \in \mathscr{M}$.

We have established the following important fact.

Proposition 3.10 *A point $p \in X$ of a topological space X is a cluster point of an ultrafilter if and only $\mathscr{M}$ converges to p.*

3.4.6

In other words, for ultrafilters their cluster points are precisely their limit points, and *adherence* is equivalent to *convergence*.

3.5 Continuity of mappings

3.5.1 Mappings continuous at a point

Let $f\colon X \longrightarrow Y$ be a mapping between topological spaces. We say that f ic **continuous** at point $p \in X$ if the image of $\mathscr{N}_p$ under f generates a filter *finer* than $\mathscr{N}_{f(p)}$:

$$\mathscr{N}_{f(p)} \subseteq f_*(\mathscr{N}_p). \tag{58}$$

3.5.2

Note that (58) is equivalent to saying that

for any neighborhood N of $f(p)$, there exists
a neighborhood M of p such that $f(M) \subseteq N$. (59)

3.5.3

For any subsets $A \subseteq X$ and $B \subseteq Y$, one has

$$f(A) \subseteq B \qquad \text{if and only if} \qquad A \subseteq f^{-1}(B). \tag{60}$$

Thus, we can rephrase the definition of continuity at point $p \in X$ as follows:

the preimage $f^{-1}(N)$ of any neighborhood
N of $f(p)$ is a neighborhood of p. (61)

3.5.4

As usual, one can replace the full neighborhood filters by any fundamental systems of neighborhoods.

Exercise 45 *Let $\mathscr{E}$ be an arbitrary fundamental system of neighborhoods of point $f(p) \in Y$. Show that $f\colon X \longrightarrow Y$ is continuous at p if and only if*

$$\text{the preimage } f^{-1}(N) \text{ of any } N \in \mathscr{E} \text{ is a neighborhood of } p. \tag{62}$$

Exercise 46 *Let $\mathscr{D}$ and $\mathscr{E}$ be arbitrary fundamental systems of neighborhoods of points $p \in X$ and $f(p) \in Y$, respectively. Show that $f\colon X \longrightarrow Y$ is continuous at p if and only if*

$$\text{for any } N \in \mathscr{E}, \text{ there exists } M \in \mathscr{D} \text{ such that } f(M) \subseteq N. \tag{63}$$

3.5.5

Exercise 47 *Let $f\colon X \longrightarrow Y$ be a mapping between topological spaces. Show that if f is continuous at $p \in X$ and ξ is a net convergent to p, then $f(\xi)$ is a net convergent to $f(p)$.*

Exercise 48 *Let $f\colon X \longrightarrow Y$ be a mapping between topological spaces. Show that if f is continuous at $p \in X$ and $p \in \bar{E}$ for some subset $E \subseteq X$, then $f(p) \in \overline{f(E)}$.*

3.5.6

The assertion of Exercise 47 can be reversed:

$$\text{if } f \text{ sends any net which converges to } p \text{ in } X \text{ to a net which converges to } f(p) \text{ in } Y, \text{ then } f \text{ is continuous at point } p. \tag{64}$$

Indeed, if f is *not* continuous, then there exists a neighborhood E of $f(p)$ such that

$$f(N) \not\subseteq E$$

for *every* neighborhood N of p. Equivalently,

$$N \not\subseteq f^{-1}(E)$$

for all $N \in \mathscr{N}_p$. It follows that the family of subsets of X,

$$\{N \setminus f^{-1}(E)\}_{N \in \mathscr{N}_p}, \tag{65}$$

which is indexed by elements of the neighborhhod filter, $\mathscr{N}_p$, consists of nonempty sets. Let ξ be any choice function for family (65). It is a net in X which converges to p but since

$$f(\xi_N) \in f(N) \setminus E$$

no $f(\xi_N)$ belongs to E. In particular, $f(\xi)$ does not converge to $f(p)$.

3.5.7

Exercise 49 *Consider a pair of composable mappings $f\colon X \longrightarrow Y$ and $g\colon W \longrightarrow X$ between topological spaces such that g is continuous at $p \in W$ and f is continuous at $q = g(p)$. Show that $f \circ g$ is continuous at p.*

3.5.8 Continuous mappings

When $f\colon X \longrightarrow Y$ is continuos at every point $p \in X$, then we say that f is *continuous everywhere* or, simply, that f is a *continuous* mapping.

Proposition 3.11 *For any mapping $f\colon X \longrightarrow Y$ between topological spaces X and Y, the following conditions are equivalent:*

(a) f is continuous;

(b) the preimage, $f^{-1}(V)$, of any open subset $V \subseteq Y$ is open in X;

(c) the preimage, $f^{-1}(W)$, of any closed subset $W \subseteq Y$ is closed in X;

(d) for any $E \subseteq X$, one has $f(\bar{E}) \subseteq \overline{f(E)}$.

Condition (b) can be also expressed as:

$$f^{-1}(\mathscr{T}_Y) \subseteq \mathscr{T}_X \tag{66}$$

while Condition (c) can be expressed as:

$$f^{-1}(\mathscr{Z}_Y) \subseteq \mathscr{Z}_X \tag{67}$$

where $\mathscr{T}_X$ and $\mathscr{T}_Y$ denote the topologies, and $\mathscr{Z}_X$ and $\mathscr{Z}_Y$ denote the families of all closed subsets — of X and Y, respectively.

Proof. Suppose that the preimage of any open subset of Y is open in X. Let p be a point of X. It follows that the preimage of any open neighborhood V of $f(p)$ is a neighborhood of p. Open neighborhoods form a base of the neighborhood filter. Thus (b) implies (a).

For any subset $V \subseteq Y$, one has

$$(f^{-1}(V))^c = (f^{-1}(V))^c \quad \text{in} \quad Y. \tag{68}$$

This establishes equivalence of (b) and (c).

Suppose f satisfy Condition (d). Then, for any closed subset W of Y, one has

$$f\left(\overline{f^{-1}(W)}\right) \subseteq \overline{f(f^{-1}(W))}, \tag{69}$$

while

$$f(f^{-1}(W))) \subseteq W$$

implies that

$$\overline{f(f^{-1}(W))} \subseteq \overline{W} = W. \tag{70}$$

By combining (69) with (70), we obtain

$$f\left(\overline{f^{-1}(W)}\right) \subseteq W$$

which is equivalent to

$$\overline{f^{-1}(W)} \subseteq f^{-1}(W).$$

Since $f^{-1}(W)$ is obviously contained in its closure, this shows that $\overline{f^{-1}(W)} = f^{-1}(W)$, i.e., $f^{-1}(W)$ is closed. We proved that Condition (d) implies Condition (c).

Earlier we established that (a) implies (d), cf. Exercise 48. □

Exercise 50 *Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces, and $\mathscr{E}$ be a prebase of $\mathscr{T}_Y$. Show that a mapping $f\colon X \longrightarrow Y$ is continuous if and only if*

$$f^{-1}(V) \textit{ is open for any } V \in \mathscr{E}. \tag{71}$$

3.5.9

If we consider two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on a set X, then the identity mapping

$$\mathrm{id}_X\colon (X, \mathscr{T}_1) \longrightarrow (X, \mathscr{T}_2), \qquad \mathrm{id}_X\colon x \longmapsto x,$$

is continuous if and only if $\mathscr{T}_2 \subseteq \mathscr{T}_1$, i.e., when the topology on the source, $\mathscr{T}_1$, is *finer* than $\mathscr{T}_2$, the topology on the target.

3.5.10 Homeomorphisms

A mapping between topological spaces $f\colon X \longrightarrow Y$ is said to be a **homeomorphism**, if there exists a continuous mapping $g\colon Y \longrightarrow X$ such that

$$f \circ g = \mathrm{id}_Y \qquad \text{and} \qquad g \circ g = \mathrm{id}_X\,.$$

This is equivalent to f being bijective and both f and f^{-1} being continuous.

3.5.11

Continuity of f usually does not guarantee that f^{-1} is continuous. Take for example, f to be the identity mapping, id_X, where the topology on the *source* X is strictly finer than the topology on the target X.

3.5.12 Open mappings

A mapping between topological spaces is said to be **open** if

$$f\big(\mathscr{T}_X\big) \subseteq \mathscr{T}_Y, \tag{72}$$

i.e., when the image of any open subset in X is an open subset in Y.

A bijective continuous mapping $f\colon X \longrightarrow Y$ is a homeomorhism if and only if it is an open mapping.

3.5.13

It follows directly from the definition that if a mapping $f\colon X \longrightarrow Y$ is continuous for given topologies $\mathscr{T}_X$ and $\mathscr{T}_Y$ on X and Y, respectively, then it is also continuous for any topology $\mathscr{T}$ on set X which is *finer* than $\mathscr{T}_X$, and for any topology $\mathscr{T}'$ on set y which is *coarser* than $\mathscr{T}_Y$.

3.6 Natural topologies

3.6.1

Let $f\colon X\longrightarrow Y$ be a continuous mapping between two topological spaces. The same mapping remains continuous if we replace the topology on the source, $\mathscr{T}_X$, by any *finer* topology,

$$\mathscr{T}'_X \supseteq \mathscr{T}_X,$$

or, if we replace the topology on the target, $\mathscr{T}_X$, by any *coarser* topology

$$\mathscr{T}'_Y \subseteq \mathscr{T}_Y.$$

Exercise 51 *Let X be a set, and $(Y, \mathscr{T}_Y)$ be a topological space. Show that, for any mapping $f\colon X\longrightarrow Y$, the family*

$$f^{-1}(\mathscr{T}_Y) = \{f^{-1}(V) \mid V \in \mathscr{T}_Y\} \tag{73}$$

is a topology on X. (*Hint: Use the result of Exercise 50.*)

3.6.2 Initial topologies

Topology (73) is the coarsest topology on X so that f is continuous.

3.6.3

Its generalization involves a family of topological spaces $\{(Y_i, \mathscr{T}_i)\}_{i\in I}$ and an arbitrary family of mappings $f_i\colon X\longrightarrow Y_i$:

$$\mathscr{T} := \sup_{\mathrm{Top}(X)}\{f^{-1}(\mathscr{T}_i) \mid i \in I\} = \left(\left(\bigcup_{i\in I} f^{-1}(\mathscr{T}_i)\right)^{\wedge}\right)^{\cup} \tag{74}$$

Since $f^{-1}(\mathscr{T}_i) \subseteq \mathscr{T}$, each f_i is continuous in topology (74). The latter is usually referred to as the *coarsest topology on X such that all $f_i\colon X\longrightarrow Y_i$ are continuous*.

Exercise 52 *Let W be a topological space and X be equipped with topology* (74)*. Show that a map $g\colon W\longrightarrow X$ is continuous if and only if g composed with each $f_i\colon X\longrightarrow Y_i$,*

$$f_i \circ g\colon W\longrightarrow Y_i \qquad (i \in I),$$

is continuous.

3.6.4 Example: the topology induced on a subset revisited

If $X \subseteq Y$ is a subset of a topological space $(Y, \mathscr{T})$ and $\iota\colon X \hookrightarrow Y$ denotes the canonical inclusion mapping, then $\iota^{-1}(\mathscr{T})$ is the *induced* topology, $\mathscr{T}_{|X}$, cf. 2.7.

3.6.5 Example: the product of topological spaces

The product of a family of topological spaces $\{(X_i, \mathscr{T}_i)\}_{i\in I}$ is defined as the product of *sets*,

$$X = \prod_{i\in I} X_i, \tag{75}$$

equipped with the coarsest topology for which the canonical projections

$$\pi_j\colon \prod_{i\in I} X_i \twoheadrightarrow X_j \qquad (j \in I),$$

are continuous.

3.6.6

Suppose a base $\mathscr{S}_i \subseteq \mathscr{T}_i$ is given for each member-topology $\mathscr{T}_i$. Consider the family of products

$$\prod_{i\in I} U_i \tag{76}$$

where

$$U_i \in \mathscr{S}_i,$$

for *finitely* many $i \in I$, and

$$U_i = X_i$$

for all the remaining $i \in I$. Family (76) is a base of the product topology.

3.6.7 The neighborhood filters of the product topology

By the defition of the product topology, the neighborhood filter, $\mathscr{N}_p$, of a point $p = (p_i)_{i\in I}$ in the product space,[2] (75), is the supremum of the inverse images of all filters $\mathscr{N}_{p_i}$:

$$\mathscr{N}_p = \sup_{\mathrm{Filt}(X)}\{\pi_i^*(\mathscr{N}_{p_i}) \mid i \in I\} \tag{77}$$

[2] Elements of the product $\prod_{i\in I} X_i$ will be denoted $(p_i)_{i\in I}$ rather than $\{p_i\}_{i\in I}$. This will help us to distinguish them notationally from nets.

3.6.8 Convergence of filters in the product topology

In particular, a filter $\mathscr{F}$ on the product space contains $\mathscr{N}_p$ if and only if it contains each $\pi_i^*(\mathscr{N}_{p_i})$ or, equivalently, if and only if

$$(\pi_i)_*(\mathscr{F}) \supseteq \mathscr{N}_{p_i} \qquad (i \in I).$$

In other words, we obtain the following convenient characterization of convergence of filters in the product topology

a filter $\mathscr{F}$ on $\prod_{i \in I} X_i$ converges to a point $p = (p_i)_{i \in I}$ if and only if each 'projection' $(\pi_i)_*(\mathscr{F})$ converges to the corresponding component of p:

$$(\pi_i)_*(\mathscr{F}) \longrightarrow p_i \qquad (i \in I). \tag{78}$$

3.6.9 Convergence of nets in the product topology

As a corollary of (78), we obtain the following simple characterization of convergence of nets in the product topology

a net $\xi = \{\xi_\lambda\}_{\lambda \in \Lambda}$ in $\prod_{i \in I} X_i$, where $\xi_\lambda = (\xi_{\lambda i})_{i \in I}$, converges to a point $p = (p_i)_{i \in I}$ if and only if it converges componentwise, i.e., for every $i \in I$, the i-th component of ξ converges to p_i (79)

$$\xi_i = \{\xi_{\lambda i}\}_{\lambda \in \Lambda} \longrightarrow p_i.$$

3.6.10 Final topologies

Let X be a set, let $\{W_j\}_{j \in J}$ be a family of topological spaces, and $\{g_j \colon W_j \longrightarrow X\}_{j \in J}$ be a family of mappings with target X.

Exercise 53 *Show that the family of subsets of X*

$$\mathscr{T} := \{U \subseteq X \mid g_j^{-1}(U) \text{ is open in } W_j \text{ for every } j \in J\} \tag{80}$$

is a topology on X.

3.6.11

Topology (80) is the finest topology on X for which all mappings $g_j\colon W_j \longrightarrow X$ are continuous.

Exercise 54 *Let Y be a topological space and X be equipped with topology* (80). *Show that a map $f\colon X \longrightarrow Y$ is continuous if and only if f composed with each $g_j\colon W_j \longrightarrow Y$,*

$$f \circ g_j\colon W_j \longrightarrow Y \qquad (j \in J),$$

is continuous.

3.6.12 Example: the quotient topology

Let $(X, \mathscr{T})$ be a topological space and $\sim$ be an equivalence relation on *set* X. If $\pi\colon X \twoheadrightarrow X/_\sim$ denotes the canonical quotient map,

$$\pi(x) := \text{the equivalence class of } x,$$

then $\pi_\bullet(\mathscr{T})$ is called the *quotient topology*. It is the finest topology on $X/_\sim$ for which the canonical quotient map, π, is continuous.

3.6.13 Example: the union of topological spaces

The union

$$\bigcup_{j \in J} X_j \tag{81}$$

of a family of $\{X_j\}_{j \in J}$ of topological spaces, can be equipped with the finest topology for which all inclusion mappings

$$\epsilon_i\colon X_i \hookrightarrow \bigcup_{j \in J} X_j \tag{82}$$

are continuous.

Despite its name, the resulting topology may be very weak if few subsets of intersections

$$X_{j_1} \cap \cdots \cap X_{j_n}$$

are simultaneosly *relatively open* with respect to $X_{j_1}, \ldots X_{j_n}$.

3.6.14 Example: the disjoint union of topological spaces

For a family of $\{X_j\}_{j\in J}$ of topological spaces, their disjoint union is the disjoint union of the corresponding sets,

$$\bigsqcup_{j\in J} X_j, \tag{83}$$

equipped with the finest topology for which the canonical inclusions

$$\iota_i \colon X_i \hookrightarrow \bigsqcup_{j\in J} X_j$$

are continuous.

3.6.15

Note that (81) is the quotient of the disjoint union, (83), by the obvious equivalence relation:

> $x \in X_i$ *is* equivalent *to* $x' \in X_j$ *if both* x *and* x' *represent the same element of* $X_i \cap X_j$.

Thus, one can equip (81) also with the quotient topology obtained from the topology of the disjoint union. The result is the same topology, since each is the finest topology on (81), for which all inclusion mappings, (82), are continuous.

3.7 Metric spaces

3.7.1 A semi-metric

A *semi-metric* on a set X is a function $\rho\colon X\times X\longrightarrow[0,\infty)$ which satisfies the following three conditions

$(\mathbf{M_1})$ $\rho(p,r) \leq \rho(p,q)+\rho(q,r)$

$(\mathbf{M_2})$ $\rho(p,q) = \rho(q,p)$

$(\mathbf{M_3})$ $\rho(p,p) = 0$

for any elements $p,q,r \in X$.[3]

[3] A function satisfying Conditions $(\mathbf{M_1})$–$(\mathbf{M_3})$ is often called a *pseudo-metric* in literature.

3.7.2 A metric

If ρ additionally *separates points*, i.e., if

$(\mathbf{M_4})$ $\rho(p,q) = 0$ implies $p = q$,

then it is called a **metric**.

3.7.3

Axiom $(\mathbf{M_1})$ is by far the most important one. It is referred to as the *Triangle Inequality*.

3.7.4

Given any function $f\colon X\times X\longrightarrow[0,\infty)$ which vanishes on the diagonal

$$\Delta = \{(p,q) \in X\times X \mid p = q\}, \tag{84}$$

one can produce a semi-metric on X by first symmetrizing it:

$$f \longmapsto f^s, \qquad f^s(p,q) := \frac{1}{2}\big(f(p,q) + f(q,p)\big), \tag{85}$$

and then enforcing the Triangle Inequality:

$$f \longmapsto f^t, \qquad f^t(p,q) = \inf\left\{\sum_{i=1}^{l} f(x_{i-1},x_i) \mid x_0 = p, x_l = q\right\}, \tag{86}$$

where the infimum is taken over all finite sequences $\{x_i\}_{i\in\{0,\dots,l\}}$ of elements of X of any length which start at p and terminate at q.

3.7.5

Note that

$$f^s = f \qquad \text{if and only if } f \text{ is symmetric.}$$

3.7.6

Note also that, by definition,

$$f^t(p,q) \leq f(p,q) \qquad (p,q \in X). \tag{87}$$

In particular,

$$f^s = f \qquad \text{if and only if } f \text{ satisfies Triangle Inequality.}$$

3.7.7

It follows that

$$f \longmapsto f^{st} := \left(f^s\right)^t = \left(f^t\right)^s$$

is a *retraction* of the set of all functions $f\colon X\times X \longrightarrow [0,\infty)$ vanishing on Δ, onto the subset of semi-metrics on X.

3.7.8

A set X equipped with a metric is called a **metric space**.

3.7.9 The family of open balls

For any $p \in X$ and a positive number $\epsilon > 0$ define the open ball

$$B_p(\epsilon) := \{q \in X \mid \rho(p,q) < \epsilon\}. \tag{88}$$

Point p is called its *center* and number ϵ its *radius*.

3.7.10 The topology associated with a semi-metric

Exercise 55 *Show that the family of balls with center at p,*

$$\mathscr{B}_p := \{B_p(\epsilon)\},$$

is a filter-base.

Proposition 3.12 *The family of filters $\{\mathscr{N}_p\}_{p\in X}$ generated by the family of filter-bases $\{\mathscr{B}_p\}_{p\in X}$ is a neighborhood system.*

Proof. We only need to show that Axiom $(\mathbf{N}_3)$ is satisfied. If $N \in \mathscr{N}_p$, then N contains a ball $B_p(\epsilon)$ for some $\epsilon > 0$.

For a point $q \in B_p(\epsilon)$, put

$$\delta := \epsilon - \rho(p,q).$$

Note that $\delta > 0$. Then for any point $r \in B_q(\delta)$,

$$\rho(p,r) \leq \rho(p,q) + \rho(q,r) < \rho(p,q) + \delta = \epsilon,$$

which shows that

$$B_q(\delta) \subseteq B_p(\epsilon).$$

This proves that $B_p(\epsilon) \in \mathscr{N}_q$ for each $q \in B_p(\epsilon)$. In particular, the set

$$\{q \in N \mid N \in \mathscr{N}_q\} \tag{89}$$

contains $B_p(\epsilon)$ which in turn implies that (89) belongs to $\mathscr{N}_p$. □

3.7.11

Note that in the proof of Proposition 3.12 we used exclusively the Triangle Inequality, $(\mathbf{M_1})$.

According to Proposition 3.12, families of balls $\{B_p(\epsilon)\}_{\epsilon>0}$ form fundamental systems of neighborhoods of a topology. We call this topology the *topology associated with the semi-metric*, or the *metric topology*.

3.7.12

We can also introduce the metric topology by defining the corresponding interior operation:

$$\mathring{E} := \{p \in E \mid B_p(\epsilon) \subseteq E \text{ for some } \epsilon > 0\}. \tag{90}$$

Exercise 56 *Using only the axioms of a semi-metric show that* (90) *satisfies the axioms of an interior operation.*

3.7.13

Alternatively, we can define the family $\mathscr{T}$ of open sets by declaring a subset of X to be *open* if it can be represented as the union of a family of balls.

Exercise 57 *Using only the axioms of a semi-metric show that so defined family T satisfies the axioms of a topology.*

By definition, then, the family of all balls $\{B_p(\epsilon)\}_{p\in X,\,\epsilon>0}$ forms a *base* of the metric topology.

3.7.14 Example: Euclidean spaces $\mathbb{R}^n$

Euclidean n-dimensional space $\mathbb{R}^n$ equipped with the Euclidean distance function,

$$\rho_{\mathrm{Eucl}}(\boldsymbol{x}, \boldsymbol{y}) := \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \tag{91}$$

is a metric space.

Exercise 58 *Show that the topology of $\mathbb{R}^n$ coincides with the product topology of*

$$\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}.$$

3.7.15 Limits of functions in Calculus

Limits of functions $f\colon E \longrightarrow \mathbb{R}$ in Calculus,

$$\lim_{\boldsymbol{x} \longrightarrow \boldsymbol{a}} f(\boldsymbol{x}), \tag{92}$$

where E, the domain of f, is a subset of $\mathbb{R}^n$, are precisely limits of f (restricted to E^*) *along* the filter

$$\mathcal{N}_p^* := \{N^*\}$$

defined in (31). For (92) to be well defined, point $\boldsymbol{a}$ must belong to the *closure* of $E^* = E \setminus \{\boldsymbol{a}\}$.

4 Separability properties

4.1 T_0-spaces

4.1.1

A topological space X which satisfies the following property:

for any points $p, q \in X$, if $p \neq q$, then there exists an open set $U \subset X$ which contains only one of those points, (93)

is referred to as a space of type T_0, or as a T_0*-space*.

Exercise 59 *Let p and q be two points in a topological space X. Show that*

$$\text{there exists an open set } U \subset X \text{ which contains only one of those points,} \tag{94}$$

if and only if

$$\mathscr{N}_p \neq \mathscr{N}_q. \tag{95}$$

4.2 T_1-spaces

4.2.1

A topological space X which satisfies the following property:

> *for any points $p, q \in X$, if $p \neq q$, then there exists an open set $U \subset X$ which contains p but not q, and an open set $U \subset X$ which contains q but not p,* (96)

is referred to as a space of type T_1, or as a T_1-*space*.

Exercise 60 *Show that X is a T_1-space if and only if any singleton subset $\{p\}$ is closed.*

Exercise 61 *Provide an example of a finite space $(X, \mathscr{T})$ which is T_0 but not T_1, and another example of a finite space $(X', \mathscr{T}')$ which is not T_0.*

4.2.2

In an arbitrary space X, if $\mathscr{N}_p \subseteq \mathscr{N}_q$, then any neighborhood N of p contains q, and thus $p \in \overline{\{q\}}$.

Vice-versa, if $p \in \overline{\{q\}}$, then any neighborhood N of p intersects $\{q\}$ which implies that $N \ni q$. Since

$$\{x \in X \mid N \in \mathscr{N}_x\} \tag{97}$$

is a neighborhood of p, we infer that set (97) contains q which means that N is a neighborhood of q.

4.2.3

We demonstrated the following fact:

$$p \in \overline{\{q\}} \text{ if and only of } \mathcal{N}_p \subseteq \mathcal{N}_q. \tag{98}$$

In other words,

$$p \in \overline{\{q\}} \text{ if and only of filter } \mathcal{N}_q \text{ converges to point } p. \tag{99}$$

4.2.4

Equipped with (98), we can provide an equivalent definition of T_1-spaces:

$$\begin{array}{l}\textit{a topological space } X \textit{ is } T_1 \textit{ if and only if, for every} \\ p \in X, \textit{ point } p \textit{ is the only limit of filter } \mathcal{N}_p.\end{array} \tag{100}$$

4.3 T_2-spaces

4.3.1

A topological space X which satisfies the following property:

$$\begin{array}{l}\textit{for any points } p, q \in X, \textit{ if } p \neq q, \textit{ then there exist open} \\ \textit{sets } U, V \subset X \textit{ such that} \\ \qquad U \ni p, \quad V \ni q, \qquad \textit{and} \qquad U \cap V = \emptyset,\end{array} \tag{101}$$

is referred to as a space of type T_2, or as a T_2*-space*.

4.3.2

Spaces of type T_2 are also frequently referred to as **Hausdorff** spaces, in honor of one of the pioneers of Topology, or, simply, as *separable* spaces.[4]

Exercise 62 *Show that a finite* T_1*-space* X *is automatically a* T_2*-space.*

[4]The term *separable* is used in several other situations in Mathematics, so T_2- or *Hausdorff*, is to be preferred.

Exercise 63 *Let X be an arbitrary set equipped with the coarsest T_1-topology:*

$$\mathscr{T} = \{U \subseteq X \mid U^c \text{ is finite}\} \cup \{\emptyset\}. \qquad (102)$$

Show that $(X, \mathscr{T})$ is T_1 and, if X is infinite, is not T_2.

Exercise 64 *Show that X is a T_2-space if and only if $\mathscr{N}_p \vee \mathscr{N}_q$ exists for no $p \neq q$. In other words, if no subset of the set of all neighborhood filters, $\{\mathscr{N}_p \mid p \in\}$, which has at least wto elements, is bounded above in $\mathscr{P}(\mathscr{P}(X))$.*

Exercise 65 *Show that*

a topological space X is T_2 if and only if, for each $p \in X$, point p is the only cluster point of filter $\mathscr{N}_p$. (103)

4.3.3

Compare the above characterisation of T_2-spaces with the characterisation of T_1-spaces provided in (100).

4.3.4

We finish with the following interesting characterisation of the T_2-property.

Exercise 66 *Show that a topological space is T_2 if and only if the diagonal, Δ, cf.* (84)*, is closed in the product topology of the Cartesian square $X \times X$.*

4.4 T_3-spaces

4.4.1

A topological T_1-space X which satisfies the following property:

for any point $p \in X$ and a closed subset $Q \subset X$ which does not contain p, there exist open sets $U, V \subset X$ such that

$$U \ni p, \quad V \supseteq Q, \qquad \text{and} \qquad U \cap V = \emptyset, \qquad (104)$$

is referred to as a space of type T_3, or as a *T_3-space*.

4.5 Regular spaces

Condition (104) does not guarantee that X is even a T_0-space since singleton subsets $\{p\}$ need not be closed. For example, (104) is obviously satisfied when $\emptyset$ and X are the only closed subsets of X.

Spaces satisfying (104) are sometimes referred to as *regular*, and we will follow this practice here. More often, however, *regular* is used as a synomym for T_3.

4.6 Inseparable pairs of points in a regular space

If $p \notin \overline{\{q\}}$, which we know is equivalent to $\mathcal{N}_p \not\subseteq \mathcal{N}_q$, then the Regularity Condition, (104), guarantees existence of a pair of open subsets $U, V \subset X$ such that

$$U \ni p, \quad V \supseteq \{q\}, \qquad \text{and} \qquad U \cap V = \emptyset.$$

In particular, pair (p, q) satisfies the separability condition in (101).

This shows that if a pair of points in a regular space does *not* satisfy the T_2-separability condition, then both $p \in \overline{\{q\}}$ and $q \in \overline{\{p\}}$ which means that the closures of the singleton sets $\{p\}$ and $\{q\}$ coincide,

$$\overline{\{p\}} = \overline{\{q\}},$$

or, equivalently, that

$$\mathcal{N}_p = \mathcal{N}_q.$$

4.6.1

We have thus proved that in a regular space:

the partially ordered set $(\{\mathcal{N}_p \mid p \in X\}, \subseteq)$ is discrete, i.e., no two $\mathcal{N}_p \neq \mathcal{N}_q$ are comparable. (105)

Compare this to the characterisation of T_1-spaces given in (100).

4.6.2

The language of filters provides again a key to understanding the meaning of the condition of *regularity*, (104).

Proposition 4.1 *A point p of a topological space X possesses a fundamental system of closed neighborhoods if and only*

$$\begin{gathered}\text{for any closed subset } Q \subset X \text{ which does not contain } p, \\ \text{there exist open sets } U, V \subset X \text{ such that} \\ U \ni p, \quad V \supseteq Q, \qquad \text{and} \qquad U \cap V = \emptyset.\end{gathered} \tag{106}$$

Proof. Sufficiency. Let N be any open neighborhood of p. Its complement $Q := U^c$ is closed and does not contain p.

Let U and V be disjoint open subsets containing p and Q, respectively. Then

$$U \subseteq V^c \subseteq Q^c = (N^c)^c = N$$

which shows that V^c is both a neighborhood of p and is contained in the given open neighborhood N.

Necessity. Closed neighborhoods of any point p obviously always form a filter-base. Let us assume that this filter-base generates the neighborhood filter $\mathcal{N}_p$.

A closed subset $Q \subseteq X$ does not contain p precisely when its complement, Q^c, is an open neighborhood of p. Take any closed neighborhood $Z \subseteq Q^c$, and set

$$U := \mathring{Z} \qquad \text{and} \qquad V := Z^c.$$

Sets U and V are open and obviously disjoint:

$$U \cap V = \mathring{Z} \cap Z^c \subseteq Z \cap Z^c = \emptyset,$$

and

$$V = Z^c \supseteq (Q^c)^c = Q.$$

□

Corollary 4.2 *A topological space is regular if and only if every point $p \in X$ possesses a fundamental system of closed neighborhoods.*

□

4.7 T_4-spaces

4.7.1

A topological T_1-space X which satisfies the following property:

for any disjoint closed subsets $P, Q \subseteq X$, there exist open sets $U, V \subset X$ such that

$$U \supseteq P, \quad V \supseteq Q, \qquad \textit{and} \qquad U \cap V = \emptyset, \tag{107}$$

is referred to as a space of type T_4, or as a T_4*-space*.

Spaces satisfying (107) are sometimes referred to as *normal*. More often, however, *normal* is used as a synomym for T_4.

4.8 Normal spaces

Condition (107) does not guarantee that X is even a T_0-space since singleton subsets $\{p\}$ need not be closed. For example, (104) is obviously satisfied when $\emptyset$ and X are the only closed subsets of X.

Spaces satisfying (107) are sometimes referred to as *normal*, and we will follow this practice here. More often, however, *normal* is used as a synomym for T_4.

4.9 Neighborhood filter of a subset of a topological space

Let us call a subset N of a topological space X a *neighborhood* of a subset $P \subseteq X$ if

$$A \subseteq \mathring{N}.$$

Exercise 67 *Show that the set of neighborhoods of any subset $P \subseteq X$,*

$$\mathscr{N}_P := \{N \in \mathscr{P}(X) \mid \mathring{N} \supseteq P\}, \tag{108}$$

is a filter.

Exercise 68 *Show that a topological space satisfies* Regularity Condition (104) *if and only if every closed subset $Z \subseteq X$ is the intersection of its closed neighborhoods*

$$Z = \bigcap_{\substack{N \in \mathscr{N}_Z \\ N \text{ is closed}}} N.$$

Exercise 69 *A subset P of a topological space X possesses a fundamental system consisting of closed neighborhoods if and only*

for any closed subset $Q \subset X$ which is disjoint with P, there exist open sets $U, V \subset X$ such that

$$U \ni P, \quad V \supseteq Q, \qquad \text{and} \qquad U \cap V = \emptyset. \tag{109}$$

4.9.1

In particular, we obtain a characterisation of normal spaces analogous to the characterisation of regular spaces, cf. Corollary 4.2:

a topological space X is normal if and only if the neighborhood filter, $\mathscr{N}_P$, of any closed subset $P \subseteq X$ is generated by closed sets. (110)

5 Compactness

5.1 Compact subsets

5.1.1 Covers

We say that a family $\mathscr{C} \subseteq \mathscr{P}(X)$ of subsets of X *covers* a subset $K \subseteq X$ if

$$K \subseteq \bigcup \mathscr{C} = \bigcup_{C \in \mathscr{C}} C.$$

5.1.2 Subcovers

A subfamily $\mathscr{C}'$ of $\mathscr{C}$ which covers K, is referred to as a *subcover* of $\mathscr{C}$.

5.1.3

Note that

$$K \nsubseteq \bigcup \mathscr{C} = \bigcup_{C \in \mathscr{C}} C$$

precisely when

$$\emptyset \neq \left(\bigcup_{C \in \mathscr{C}} C \right)^c \cap K = \bigcap_{C \in \mathscr{C}} (C^c \cap K). \tag{111}$$

It follows that $\mathscr{C} \subseteq \mathscr{P}(X)$ is a cover of K without finite subcovers if and only if the family of complements,

$$\mathscr{C}^c = \{C^c \mid C \in \mathscr{C}\},$$

satisfies the Finite Intersection Property, (33), of Section 3.1.1.

The latter is equivalent to saying that the family

$$\{C^c \cap K \mid C \in \mathscr{C}\}$$

generates a certain filter $\mathscr{F}$ on K, and $\mathscr{C}$ covers K precisely when

$$\bigcap \mathscr{F} = \bigcap_{C \in \mathscr{C}} (C^c \cap K) = \varnothing.$$

5.1.4 Open covers

Suppose X is a topological space. A family $\mathscr{U} \subseteq \mathscr{T}$ of open subsets of X which covers a subset $K \subseteq X$ is called an **open cover** of K.

5.1.5 Compact subsets of a topological space

We arrive at a very important definition.

Definition 5.1 (Borel-Lebesgue) *A subset K of a topological space X is said to be **compact** if any open cover $\mathscr{U}$ of K admits a finite subcover.*

5.1.6

Finite subsets of a topological space are obviously compact. In particular, every subset of a finite topological space is compact.

5.1.7 Compact Spaces

When we apply the above definition to $K = X$, then we obtain the definition of a compact space.

5.1.8 Compact subsets *versus* compact spaces

Any subset K of a topological space becomes a topological space on its own when equipped with the induced topology, $\mathscr{T}_{|K}$, cf. 2.7.

If $\mathscr{U}$ is any open cover of *subset* K, then the family of intersections with K

$$\mathscr{U}' := \{U \cap K \mid U \in \mathscr{U}\}$$

still covers K since

$$K \subseteq \bigcup_{U \in \mathscr{U}} U$$

implies

$$K = K \cap K \subseteq \left(\bigcup_{U \in \mathscr{U}} U \right) \cap K = \bigcup_{U \in \mathscr{U}} (U \cap K),$$

and every $U \cap K$ is *relatively open*, i.e., is open in the topology, $\mathscr{T}_{|K}$, induced by $\mathscr{T}$ on K.

5.1.9

Vice-versa, every open cover of *space* K arises this way. Indeed, if $\mathscr{U}'$ is a cover of K by relatively open subsets $U' \subseteq K$, then each U' is of the form $U \cap K$ for some open subset U of X. By chosing one such U for each member $\mathscr{U}'$ of $\mathscr{U}'$ we obtain a family $\mathscr{U} := \{U\}$ of open sets which covers K if $\mathscr{U}'$ does.[5]

5.1.10

By combining together the above remarks, we infer that any cover of K by open subsets of X admits a finite subcover if and only if any cover of K by relatively open subsets of K admits a finite subcover.

In other words, we established the following fact:

A subset K of a topological space $(X, \mathscr{T})$ is compact if and only if topological space $(K, \mathscr{T}_{|K})$ is compact. (112)

Compactness of a subset $K \subseteq X$ is thus a property that depends only on the induced topology $\mathscr{T}_{|K}$.

[5] Note that the existence of such a family $\mathscr{U} := \{U\}$ follows from the Axiom of Choice.

5.2 Elementary properties of compact subsets

5.2.1

$$\textit{The union of two compact subsets is compact.} \tag{113}$$

Indeed, for arbitrary subsets K_1 and K_2, any open cover $\mathscr{U}$ of their union, $K \cup L$, is automatically an open cover of K_1 and of K_2. If K_1 and K_2 are compact, then there are finite subsets $\mathscr{U}_1, \mathscr{U}_2 \subseteq \mathscr{U}$, such that $\mathscr{U}_1$ covers K_1 and U_2 covers K_2. It follows that $\mathscr{U}_1 \cup \mathscr{U}_2 \subseteq \mathscr{U}$ is finite and covers $K_1 \cup K_2$.

5.2.2

$$\textit{The intersection of a compact subset with a closed subset is compact.} \tag{114}$$

Let $K \subseteq X$ be compact and $Z \subseteq X$ be closed. If $\mathscr{U}$ is an open cover of $K \cap Z$, then by adding just one more open subset, namely the complement of Z, we obtain an open cover of K:

$$\mathscr{U} \cup \{Z^c\}. \tag{115}$$

Indeed, K is the disjoint union of $K \cap Z$ and $K \cap Z^c = K \setminus Z$. The latter is contained in Z^c, the former is contained in $\bigcup \mathscr{U}$.

Let $\mathscr{U}'$ be a finite subfamily of (115) which covers K. Let $\mathscr{U}'' = \mathscr{U}' \setminus \{Z^c\}$. Then

$$K \subseteq \bigcup \mathscr{U}' = \left(\bigcup \mathscr{U}''\right) \cup Z^c,$$

implies that

$$K \cap Z \subseteq \left(\left(\bigcup \mathscr{U}''\right) \cap Z\right) \cup (Z^c \cap Z) = \left(\bigcup \mathscr{U}''\right) \cap Z \subseteq \bigcup \mathscr{U}''.$$

Thus, $\mathscr{U}''$ is a finite subfamily of $\mathscr{U}$ which covers $K \cap Z$, and assertion (114) is proven.

5.2.3

As a corollary we obtain, that in a compact space X all closed subsets are compact. On the other hand a compact subset may not be closed: consider, for example, a finite set X equipped with a non-discrete topology. Then every subset of X is compact but not every one is closed.

5.2.4

Let $f\colon X \longrightarrow Y$ be a continuous mapping and $K \subseteq X$ be a compact subset. For any family $\mathscr{V} = \{V_i\}_{i\in I}$ of subsets of Y, the family

$$f^{-1}(\mathscr{V}) = \{f^{-1}(V) \mid V \in \mathscr{V}\}$$

covers K if and only if $\mathscr{V}$ covers $f(K)$.

In particular, if $\mathscr{U} = \{f^{-1}(V_1), \dots, f^{-1}(V_n)\}$ is a finite subcover of $f^{-1}(\mathscr{V})$ then $\{V_1, \dots, V_n\}$ is a finite subfamily of $\mathscr{V}$ which covers $f(K) \subseteq Y$.

If $\mathscr{V}$ consists of open subsets of Y, then $f^{-1}(\mathscr{V})$ consists of open subsets of X.

5.2.5

We established another important fact:

The image, $f(K)$, of a compact subset $K \subseteq X$ under a continuous mapping $f\colon X \longrightarrow Y$ is a compact subset of Y. (116)

5.3 Characterisation of compactness via filters

5.3.1

Let X be a non-compact topological space. There exists an open cover $\mathscr{U}$ of X without a finite subcover. Then the family of complements,

$$\mathscr{U}^c = \{U^c \mid U \in \mathscr{U}\},$$

satisfies the Finite Intersection Property, cf. 5.1.3.

In particular, $\mathscr{U}^c$ generates a certain filter $\mathscr{F}$ on X, and

$$\bigcap_{F\in\mathscr{F}} \bar{F} \subseteq \bigcap_{U\in\mathscr{U}} \overline{U^c} = \bigcap_{U\in\mathscr{U}} U^c = \left(\bigcup_{U\in\mathscr{U}} U\right)^c = \emptyset$$

in view of the fact that each U^c is closed and $\mathscr{U}$ covers X. In other words, filter F has no cluster points.

5.3.2

Vice-versa, if B is any filter-base without cluster points on X, then the family of complements of the closures of members of $\mathscr{B}$,

$$\mathscr{U} := \{\overline{B^c} \mid B \in \mathscr{B}\},$$

is an open cover of X without a finite subcover.

5.3.3

We have established an important characterisation of compact spaces.

Proposition 5.2 *A topological space X is compact if and only if any filter-base $\mathscr{B}$ on X has a cluster point.*

Corollary 5.3 *A topological space X is compact if and only if any ultrafilter $\mathscr{M}$ on X converges to some point of X.*

Proof. If X is compact, then any filter, in particular any ultrafilter, has a cluster point. But an ultrafilter converges to each of its cluster points.

Vice-versa, if $\mathscr{F}$ is a filter on X, it is contained in some ultrafilter $\mathscr{M}$, and

$$\operatorname{Adh}(\mathscr{F}) \supseteq \operatorname{Adh}(\mathscr{M}) \neq \emptyset.$$

□

5.3.4

If $\mathscr{B}$ is any filter-base, and $\xi\colon \mathscr{B}\colon \longrightarrow X$ is any choice function for $\mathscr{B}$, then the filter associated with ξ is finer than than the filter generated by $\mathscr{B}$:

$$\mathscr{B}_* \subseteq \xi_*(\operatorname{Fr}(\mathscr{B})).$$

Hence,

$$\operatorname{Adh}(\mathscr{B}) = \operatorname{Adh}(\mathscr{B}_*) \supseteq \operatorname{Adh}(\xi_*(\operatorname{Fr}(\mathscr{B})) = \operatorname{Adh}(\xi),$$

which demonstrates that if ξ has a cluster point, so does filter-base $\mathscr{B}$.

The above remark combined with Proposition 5.2 provides a characterisation of compact spaces in terms of nets.

Proposition 5.4 *A topological space X is compact if and only if any net ξ in X has a cluster point.*

5.3.5 Compactness of the product: Tichonov's Theorem

Corollary 5.3 leads to a remarkably simple proof of the following celebrated theorem of Tichonov.

Theorem 5.5 (Tichonov) *The product $\prod_{i\in I} X_i$ of a family of topological spaces $\{X_i\}_{i\in I}$ is compact if and only if each component-space X_i is compact.*

Proof. Each component-space X_j is the image of the product space, $\prod_{i\in I} X_i$, under the corresponding canonical projection mapping

$$\pi_j \colon \prod_{i\in I} X_i \twoheadrightarrow X_j. \tag{117}$$

Projection mappings (117) are continuous, so, if the product space is compact, each component-space X_j is compact.

Vice-versa, an ultrafilter $\mathscr{M}$ on the product space converges if and only if its direct image under projection (117), $\mathscr{M}_i = (\pi_i)_*\mathscr{M}$, converges in the corresponding component-space X_i for every $j \in I$, cf. (79).

Each $\mathscr{M}_i$, being the direct image of an ultrafilter, is an ultrafilter itself. So, if component-space X_j is compact, then $\mathscr{M}_i$ converges in X_i, in view of Corollary 5.3. This in turn implies that $\mathscr{M}$ converges in the product space. By Corollary 5.3 again, this implies that the product space is compact. □

5.4 Separation properties of compact spaces

5.4.1

A compact space need not have any reasonable separation properties: it suffices to invoke that arbitrary finite spaces as well as the space equipped with the coarsest topology $\mathscr{T} = \{\emptyset, X\}$ are compact.

Just by looking at finite sets we encounter compact T_0-spaces which are not T_1, and any infinite set equipped with the coarsest T_1-topology, cf. Exercise 63, is compact but not T_2.

Exercise 70 *Show that the topological space of Exercise 63, is compact. Show that it is not T_2 if set X is infinite.*

5.4.2

We shall prove, however, that a compact T_2-space is automatically T_4.

Theorem 5.6 *Let X be a Hausdorff space and $K, L \subseteq X$ be a pair of disjoint compact subsets. Then, there exist open subsets*

$$U \supseteq K, \quad V \supseteq L, \qquad \text{and} \qquad U \cap V = \varnothing.$$

5.4.3

In a Hausdorff space every point is closed, and of course compact. By applying the above theorem to the case $L = \{p\}$, for $p \notin K$, we conclude that if $p \notin K$, then $p \notin \bar{K}$.

Corollary 5.7 *Every compact subset of a Hausdorff space is closed.*

□

5.4.4

In fact, a direct proof of the above corollary will form the first step in the proof of Theorem 5.6.

Step 1. Suppose that $K \subseteq X$ is compact and $p \notin K$. For each point $q \in K$, there exist open sets such that

$$V_q \ni p, \quad U_q \ni q, \qquad \text{and} \qquad V_q \cap U_q = \varnothing.$$

The family

$$\mathscr{U} := \{U_q \mid q \in K\}$$

is an open cover of K. Let $\mathscr{U}_0 = \{U_{q_1}, \ldots, U_{q_n}\}$ be a finite subcover. Denote by U its union

$$U := U_{q_1} \cup \cdots \cup U_{q_n},$$

and by V the intersection

$$V := V_{q_1} \cap \cdots \cap V_{q_n}.$$

Both are open, U contains K, point p is an element V, and their intersection is empty:

$$\begin{aligned} V \cap U &= (V_{q_1} \cap \cdots \cap V_{q_n}) \cap (U_{q_1} \cup \cdots \cup U_{q_n}) \\ &= \bigcup_{i=1}^{n} (V_{q_1} \cap \cdots \cap V_{q_n}) \cap U_{p_i} \subseteq \bigcup_{i=1}^{n} V_{q_i} \cap U_{p_i} = \emptyset. \end{aligned}$$

This demonstrates the desired separation property in the case when one of the two sets consists of a single point.

Step 2. Suppose that K and L form a pair of disjoint compact subsets of X.

By Step 1, for any point $p \in L$, there exists a pair of disjoint open neighborhoods of K and p, respectively,

$$U_p \supseteq K, \quad V_p \ni p, \qquad \text{and} \qquad U_p \cap V_p = \emptyset.$$

Like in Step 1, the family

$$\mathscr{V} := \{V_p \mid p \in L\}$$

is an open cover of K, and by following what we did in Step 1, we construct desired open neighborhoods U of K and V of L.

Exercise 71 *Provide the missing details needed to complete the proof of Theorem 5.6.*

5.4.5 Automatic continuity of the inverse

As an application of Theorem 5.6 we shall now demonstrate that the inverse of the bijective continuous mapping $f\colon X \longrightarrow Y$ is automatically continuous if the source space, X, is compact, and the target space, Y, is Hausdorff.

Theorem 5.8 *Let $f\colon X \longrightarrow Y$ be a bijective continuous mapping between a compact space X and a Hausdorff space Y. Then the inverse mapping*

$$f^{-1}\colon Y \longrightarrow X, \tag{118}$$

is continuous.

Proof. Let Z be a closed subset of X. It is compact, in view of (114). Its preimage under f^{-1} coincides with the image $f(Z)$ and the latter is compact in view of (116).

Since Y is Hausdorff, the set

$$f(Z) = (f^{-1})^{-1}(Z)$$

is closed. Now, invoking Proposition 3.11 completes the proof of continuity of (118). □

5.4.6

Theorem 5.8 says that a bijective continuous mapping between a compact and a Hausdorff spaces is automatically a homeomorphism.

5.4.7

By applying Theorem 5.8 to the identity mapping id_X when the 'source' X and the 'target' X may have different topologies, we deduce the following two corollaries.

Corollary 5.9 *Let* $(X, \mathscr{T})$ *be a Hausdorff space. If* X *is compact in some topology* $\mathscr{T}'$ *which is finer than* $\mathscr{T}$*, then* $\mathscr{T}' = \mathscr{T}$*.*

Corollary 5.10 *Let* $(X, \mathscr{T})$ *be a compact space. If* X *is Hausdorff in some topology* $\mathscr{T}''$ *which is coarser than* $\mathscr{T}$*, then* $\mathscr{T}'' = \mathscr{T}$*.*