Notes on Ordered Sets

An annex to H104, H113, etc.

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1 Vocabulary

1.1 Definitions

Definition 1.1 *A binary relation $\preceq$ on a set S is said to be a* ***partial order*** *if it is* ***reflexive****,*

$$x \preceq x, \tag{1}$$

weakly antisymmetric*,*

$$\text{if } x \preceq y \text{ and } y \preceq x, \text{ then } x = y, \tag{2}$$

and ***transitive****,*

$$\text{if } x \preceq y \text{ and } y \preceq z, \text{ then } x \preceq z \tag{3}$$

Above x, y, z, are arbitrary elements of S.

Definition 1.2 *Let $E \subseteq S$. An element $y \in S$ is said to be an* ***upper bound*** *for E if*

$$x \preceq y \qquad \text{for any } x \in E. \tag{4}$$

By definition, any element of S is declared to be an upper bound for $\emptyset$, the empty subset.

We shall denote by $U(E)$ the set of all upper bounds for E

$$U(E) := \{y \in S \mid x \preceq y \text{ for any } x \in E\}. \tag{5}$$

Note that $U(\emptyset) = S$.

Definition 1.3 *We say that a subset $E \subseteq S$ is* ***bounded (from) above****, if $U(E) \neq \emptyset$, i.e., when there exists at least one element $y \in S$ satisfying* (4).

Definition 1.4 *If $y, y' \in U(E) \cap E$, then*

$$y \preceq y' \quad \text{and} \quad y' \preceq y.$$

Thus, $y = y'$, and that unique upper bound of E which belongs to E will be denoted $\max E$ *and called the* ***largest element*** *of E.*

It follows that $U(E) \cap E$ is empty when E has no largest element, and consists of a single element, namely $\max E$, when it does.

If we replace $\preceq$ by $\succeq$ everywhere above, we shall obtain the definitions of a **lower bound** for set E, of the set of all lower bounds,

$$L(E) := \{y \in S \mid x \succeq y \text{ for any } x \in E\}, \tag{6}$$

and, respectively, of the **smallest element** of E. The latter will be denoted $\min E$.

1.1.1 The Principle of Duality

Note that the relation defined by

$$x \preceq^{\text{rev}} y \quad \text{if} \quad x \succeq y \tag{7}$$

is also an order relation on S. We will refer to it as the **reverse ordering**.

Any general statement about partially ordered sets has the corresponding *dual* statement that is obtained by replacing $(S, \preceq)$ with $(S, \preceq^{\text{rev}})$. Under this "duality" *upper* bounds become *lower* bounds, maxima becoma minima, suprema become infima, and vice-versa.

Therefore most general theorems about partially ordered sets possess the corresponding *dual* theorems. Below, I will be usually formulating those dual statements for the convenience of reference.

Exercise 1 *Show that if $E \subseteq S$ is bounded below and nonempty, then $L(E)$ is bounded above and nonempty.*

Dually, if E is bounded above and nonempty, then $U(E)$ is bounded below and nonempty.

1.1.2

If $E \subseteq E' \subseteq S$, then
$$\max E' \in U(E) \tag{8}$$
when $\max E'$ exists, and, dually,
$$\min E' \in L(E) \tag{9}$$
when $\min E'$ exists.

If both $\max E$ and $\max E'$ exist, then
$$\max E \preceq \max E'. \tag{10}$$

Dually, if both $\min E$ and $\min E'$ exist, then
$$\min E' \preceq \min E. \tag{11}$$

Exercise 2 (Sandwich Lemma for maxima) *Show that if* $E'' \subseteq E \subseteq E'$ *and both* $\max E'$ *and* $\max E''$ *exist and are equal, then* $\max E$ *exists and*
$$\max E'' = \max E = \max E'. \tag{12}$$

Dually, if both $\min E'$ and $\min E''$ exist and are equal, then $\min E$ exists and
$$\min E' = \min E = \min E''. \tag{13}$$

Definition 1.5 *When* $\min U(E)$ *exists it is called the* ***least upper bound*** *of* E*, or the* ***supremum*** *of* E*, and is denoted* $\sup E$*.*

Dually, when $\max L(E)$ *exists it is called the* ***largest lower bound*** *of* E*, or the* ***infimum*** *of* E*, and is denoted* $\inf E$*.*

For the supremum of E to exist, subset E must be bounded above. The supremum of E may exist for some bounded above subsets of S and may not exist for others.

1.1.3 Example

Let us consider $S = \mathbb{Q}$, the set of rational numbers, with the usual order. Both the following subset $E_1 \subseteq \mathbb{Q}$,

$$E_1 := \{x \in \mathbb{Q} \mid x^2 < 1\} \tag{14}$$

and the subset $E_1 \subseteq \mathbb{Q}$,

$$E_2 := \{x \in \mathbb{Q} \mid x^2 < 2\}, \tag{15}$$

are simultaneously bounded above and below. None of them has either the largest nor the smallest element but

$$\sup E_1 = 1 \qquad \text{and} \qquad \inf E_1 = -1$$

while neither $\sup E_2$ nor $\inf E_2$ exist in $S = \mathbb{Q}$.

Exercise 3 *Show that*

$$\sup \emptyset = \min S \qquad \textit{and} \qquad \inf \emptyset = \max S. \tag{16}$$

In particular, $\sup \emptyset$ exists if and only if S has the smallest element; similarly, $\inf \emptyset$ exists if and only if S has the largest element.

1.1.4 Intervals $\langle s]$ and $[s\rangle$.

Let $(S, \preceq)$ be a partially ordered set. For each $s \in S$, let

$$\langle s] := \{x \in S \mid x \preceq s\}. \tag{17}$$

and

$$[s\rangle := \{y \in S \mid s \preceq y\}. \tag{18}$$

Exercise 4 *Show that, for $E \subseteq S$, one has*

$$L(E) = \langle s] \qquad \textit{for some } s \in S \tag{19}$$

if and only if $\inf E$ *exists. In this case,*

$$L(E) = \langle \inf E]. \tag{20}$$

Dually,

$$U(E) = [s\rangle \qquad \text{for some } s \in S \tag{21}$$

if and only if $\sup E$ exists. In this case,

$$U(E) = [\sup E\rangle. \tag{22}$$

1.1.5 Example

Consider the set of natural numbers,

$$\mathbb{N} := \{0, 1, 2, \ldots\}, \tag{23}$$

equipped with the ordering given by

$$m \preceq n \qquad \text{if} \qquad m \mid n \tag{24}$$

("*m divides n*").

Exercise 5 *Does* $(\mathbb{N}, |\,)$ *have the maximum? the minimum? If yes, then what are they?*

Exercise 6 *For a given* $n \in \mathbb{N}$*, describe intervals* $(n]$ *and* $[n)$ *in* $(\mathbb{N}, |\,)$*.*

Exercise 7 *For given* $m, n \in \mathbb{N}$*, is set* $\{m, n\}$ *bounded below? Does it possess infimum? If yes, then describe* $\inf\{m, n\}$*.*

Exercise 8 *For given* $m, n \in \mathbb{N}$*, is set* $\{m, n\}$ *bounded above? Does it possess supremum? If yes, then describe* $\sup\{m, n\}$*.*

Exercise 9 *Does every subset of* $\mathbb{N}$ *possess infimum in* $(\mathbb{N}, |\,)$*? Does every subset of* $\mathbb{N}$ *possess supremum?*

1.1.6 Totally ordered sets

Definition 1.6 *We say that a partially ordered set* $(S, \preceq)$ *is* ***totally****, or* ***linearly****, ordered if any two elements* x *and* y *of* S *are* comparable

$$\text{either } x \preceq y \text{ or } y \preceq x. \tag{25}$$

Totally ordered subsets in any given partially ordered set are called **chains**.

Exercise 10 *Let* $(S, \preceq)$ *be a totally ordered set and* $E, E' \subseteq S$ *be two subsets. Show that*

$$\text{either } L(E) \subseteq L(E') \text{ or } L(E') \subseteq L(E). \tag{26}$$

1.2 Observations

1.2.1

For any subset $E \subseteq S$, one has

$$E \subseteq LU(E) := L(U(E)) \tag{27}$$

and

$$E \subseteq UL(E) := U(L(E)). \tag{28}$$

1.2.2

If $E \subseteq E' \subseteq S$, then

$$U(E) \supseteq U(E') \tag{29}$$

and

$$L(E) \supseteq L(E'). \tag{30}$$

Exercise 11 *Show that if $E \subseteq E'$ and both $\sup E$ and $\sup E'$ exist, then*

$$\sup E \preceq \sup E'. \tag{31}$$

Dually, if both $\inf E$ and $\inf E'$ exist, then

$$\inf E' \preceq \inf E. \tag{32}$$

Exercise 12 (Sandwich Lemma for infima) *Show that if $E'' \subseteq E \subseteq E'$ and both $\inf E'$ and $\inf E''$ exist and are equal, then $\inf E$ exists and*

$$\inf E'' = \inf E = \inf E'. \tag{33}$$

Dually, if both $\sup E'$ and $\sup E''$ exist and are equal, then $\sup E$ exists and

$$\sup E' = \sup E = \sup E''. \tag{34}$$

1.2.3

By applying (29) to the pair of subsets in (27), one obtains

$$U(E) \supseteq ULU(E) := U(L(U(E)))$$

while (28) applied to subset $U(E)$ yields

$$U(E) \subseteq ULU(E).$$

It follows that

$$U(E) = ULU(E). \tag{35}$$

Dually,

$$L(E) = LUL(E). \tag{36}$$

Note that equality (36) is nothing but equality (35) for the *reverse* ordering.

1.2.4

For any subsets $E \subseteq S$ and $E' \subseteq S$, one has

$$U(E \cup E') = U(E) \cap U(E') \tag{37}$$

and

$$L(E \cup E') = L(E) \cap L(E'). \tag{38}$$

1.2.5

For any $E \subseteq S$, $\max E$ *exists if and only if* $\sup E$ *exists and belongs to* E, *and they are equal*

$$\sup E = \max E. \tag{39}$$

Dually, $\min E$ *exists if and only if* $\inf E$ *exists and belongs to* E, *and they are equal*

$$\inf E = \min E. \tag{40}$$

Indeed, if $\max E$ exists, then it is the least element of $U(E)$, cf. Definition (1.4).

If $\sup E$ exists and is a member of E, then it belongs to $U(E) \cap E$ which as we established, cf. Definition (1.4), consists of the single element $\max E$ when $U(E) \cap E$ is nonempty.

The case of $\min E$ and $\inf E$ follows if we apply the already proven to $(S, \preceq^{\text{rev}})$.

1.2.6

For any $E \subseteq S$, $\inf U(E)$ *exists if and only if* $\sup E$ *exists, and they are equal*

$$\max LU(E) = \inf U(E) = \min U(E) = \sup E. \tag{41}$$

Indeed,

$$\inf U(E) := \max LU(E) \in U(E),$$

in view of $E \subseteq LU(E)$, cf., (27), combined with (8) where $F = LU(E)$. Thus,

$$\inf U(E) = \min U(E)$$

by (39).

Dually, $\sup L(E)$ *exists if and only if* $\inf E$ *exists, and they are equal*

$$\min UL(E) = \sup L(E) = \max L(E) = \inf E. \tag{42}$$

1.2.7 Example: the power set as a partially ordered set

Let $S = \mathscr{P}(A)$ be the *power set* of a set A:

$$\mathscr{P}(A) := \text{the set of all subsets of } A. \tag{43}$$

Containment $\subseteq$ is a partial order relation on $\mathscr{P}(A)$.

Subsets $\mathscr{E}$ of $\mathscr{P}(A)$ are the same as *families* of subsets of A. Since $S = \mathscr{P}(A)$ contains the largest element, namely A, and the smallest element, namely $\emptyset$, every subset of $\mathscr{P}(A)$ is bounded above and below.

The union of all members of a family $\mathscr{E}$,

$$\bigcup_{X \in \mathscr{E}} X := \{a \in A \mid a \in X \text{ for some } X \in \mathscr{E}\} \tag{44}$$

is the smallest subset of A which *contains every member* of family $\mathscr{E}$. Hence, $\sup \mathscr{E}$ exists and equals (44).

Dually, the intersection of all members of family $\mathscr{E}$,

$$\bigcap_{X \in \mathscr{E}} X := \{a \in A \mid a \in X \text{ for all } X \in \mathscr{E}\} \tag{45}$$

is the largest subset of A which is *contained in every member* of family $\mathscr{E}$. Hence, $\inf \mathscr{E}$ exists and equals (45).

The power set provides an example of a partially ordered set in which every subset (including the empty set) possesses both suppremum and infimum.

1.3 Partially ordered subsets

1.3.1

In $S \subseteq T$ is a subset of a partially ordered set $(T, \preceq)$, then it can be regarded as a partially ordered set in its own right. One has to be cautioned, however, that S with the induced order may have vastly different properties.

For a subset $E \subseteq S$, the sets of upper and lower bounds will generally depend on whether one considers E as a subset of S or T. In particular, E may be not bounded as a subset of S yet be bounded as a subset of T.

When necessary, we shall indicate this by subscript S or T. Thus,

$$L_T(E), \quad U_T(E), \quad \inf_T E, \quad \sup_T E,$$

will denote the set of lower bounds, the set of upper bounds, the infimum, and the supremum, when E is viewed as a subset of T.

Exercise 13 *Show that, for $E \subseteq S$, one has*

$$\sup_T E \preceq \sup_S E \tag{46}$$

whenever both suprema exist.

Dually, one has

$$\inf_S E \preceq \inf_T E \tag{47}$$

whenever both infima exist.

Exercise 14 *For* $E \subseteq S$*, suppose that* $\sup_T E$ *exists and belongs to* S*. Show that* $\sup_S E$ *exists and*

$$\sup_S E = \sup_T E. \tag{48}$$

Dually, if $\inf_T E$ exists and belongs to S, then $\inf_S E$ exists and

$$\inf_S E = \inf_T E. \tag{49}$$

Exercise 15 *Let* $E \subseteq S$*, and suppose that* $(S, \preceq)$ *is a partially ordered subset of* $(T, \preceq)$*. Show that, if* $\inf_T E$ *exists, then*

$$L(E) = \langle \inf_T E] \cap S = \{s \in S \mid s \preceq \inf_T E\}. \tag{50}$$

(*Here* $L(E) = L_S(E)$*.*)

Dually, one has

$$U(E) = [\sup_T E\rangle \cap S = \{s \in S \mid \sup_T E \preceq s\} \tag{51}$$

if $\sup_T E$ exists.

Exercise 16 *Find examples of pairs* $E \subseteq S$ *of subsets of* $\mathbb{Q}$ *such that:*

(a) E *is unbounded above in* S *yet bounded in* $\mathbb{Q}$*;*

(b) E *is bounded in* S*, and* $\sup_{\mathbb{Q}} E$ *exists but* $\sup_S E$ *does not;*

(c) E *is bounded in* S*, and* $\sup_S E$ *exists but* $\sup_{\mathbb{Q}} E$ *does not;*

(d) both $\sup_S E$ *and* $\sup_{\mathbb{Q}} E$ *exist but* $\sup_S E \neq \sup_{\mathbb{Q}} E$*.*

1.3.2 Density

Definition 1.7 *We say that a subset* $S \subseteq T$ *is* sup***-dense*** *if every element* $t \in T$ *equals*

$$t = \sup E$$

for some subset $E \subseteq S$*.*

By replacing sup *with* inf*, one obtains the definition of a* inf***-dense*** *subset.*

1.3.3 Example

Suppose that a partially ordered set $(S, \preceq)$ is the union of three subsets $S = X \cup Y \cup Z$ such that

$$x \preceq y \quad \text{and} \quad x \preceq z \qquad \text{for any } x \in X,\ y \in Y, \text{ and } z \in Z,$$

and no $y \in Y$ and $z \in Z$ are comparable. Let $T = S \cup \{v, \zeta\}$ where

$$x \prec v \prec y \qquad \text{for any } x \in X,\ y \in Y,$$

and

$$x \prec \zeta \prec y \qquad \text{for any } x \in X,\ z \in Z.$$

Note that v and ζ are not comparable. Finally, let T' be the subset $S \cup \{v\}$ of T.

a) $(S, \preceq)$ **as a subset of** $(T, \preceq)$**.** One has $L_T(Y) = \langle v]$ and $L_T(Z) = \langle \zeta]$. It follows that

$$v = \inf_T Y \qquad \text{and} \qquad \zeta = \inf_T Z,$$

and therefore S is inf-dense in T. In addition,

$$\langle v] \cap S = X = \langle \zeta] \cap S$$

but $v \neq \zeta$, they are not even comparable.

Exercise 17 *Show that neither v nor ζ equals $\sup_T E$ for any $E \subseteq S$. In particular, S is* inf*-dense in T but not* sup*-dense.*

b) $(S, \preceq)$ **as a subset of** $(T', \preceq)$**.** One has

$$v = \inf_{T'} Y$$

hence S is inf-dense in T'. In addition,

$$L(Y) = \langle v] \cap S = X = L(Z)$$

and $v = \inf_{T'} Y$ while $\inf_{T'} Z$ does not exist.

Exercise 18 *Let* $(S, \preceq)$ *be a partially ordered subset of* $(T, \preceq)$. *Show that if*

$$t = \inf_T E \tag{52}$$

for some $E \subseteq S$, *then*

$$t = \inf_T([t\rangle \cap S). \tag{53}$$

Dually, if

$$t = \sup_T E' \tag{54}$$

for some $E' \subseteq S$, then

$$t = \sup_T(\langle t] \cap S). \tag{55}$$

Exercise 19 (A criterion of equality for infima) *Let* $(S, \preceq)$ *be a partially ordered subset of* $(T, \preceq)$ *and* $E \subseteq S$. *Suppose that both* $\inf E$ *and* $t = \inf_T E$ *exist. Show that if there exists a subset* $E' \subseteq S$ *such that* (54) *holds, then*

$$\inf_T E = \inf E.$$

Exercise 20 *Formulate the dual criterion of equality for suprema.*

1.4 Morphisms

Definition 1.8 *Given two partially ordered sets* $(S, \preceq)$ *and* $(S', \preceq')$, *a mapping* $f\colon S \longrightarrow S'$ *which preserves order,*

$$\text{if } s \preceq t, \text{ then } f(s) \preceq' f(t) \qquad (s, t \in S), \tag{56}$$

is said to be a ***morphism*** $(S, \preceq) \longrightarrow (S', \preceq')$.

Definition 1.9 *A mapping* $\iota\colon S \longrightarrow S'$ *is said to be an* ***order embedding****,*

$$(S, \preceq) \hookrightarrow (S', \preceq'), \tag{57}$$

if it satisfies a stronger condition

$$s \preceq t \text{ if and only if } \iota(s) \preceq' \iota(t) \qquad (s, t \in S). \tag{58}$$

Exercise 21 *Show that an order embedding is injective.*

Definition 1.10 *A morphism* $f\colon (S,\preceq)\longrightarrow(S',\preceq')$ *is said to be an* ***isomorphism*** *if it has an inverse, i.e., if there is a morphism* $g\colon (S',\preceq')\longrightarrow(S,\preceq)$ *such that* $f\circ g = \mathrm{id}_{S'}$ *and* $g\circ f = \mathrm{id}_S$.

Exercise 22 *Show that an order embedding, (57), is an isomorphism onto its image,* $(\iota(S),\preceq')$.

Definition 1.11 *We say that a morphism* $f\colon (S,\preceq)\longrightarrow(S',\preceq')$ *is* sup***-continuous*** *if* f *preserves the suprema. More precisely, if it has the following property*

$$\text{for any } E\subseteq S\text{, if } \sup E \text{ exists, then } \sup f(E) \text{ exists, and} \quad \sup f(E) = f(\sup E). \tag{59}$$

Here as everywhere else, the subset $f(E)\subseteq S'$ is defined to be the **image** of $E\subseteq S$ under f

$$f(E) := \{y\in S' \mid y = f(x) \text{ for some } x\in E\}. \tag{60}$$

Exercise 23 *State the dual definition of an* inf***-continuous*** *morphism.*

Exercise 24 *For any partially ordered* $(S,\preceq)$*, let* S^* *denote the subset obtained by removing* $\max S$ *and* $\min S$ *if they exist. Show that, for any nonempty bounded subset* E *of* S^**,*

$$\sup_{S^*}E = \sup_S E \qquad \text{and} \qquad \inf_{S^*}E = \inf_S E. \tag{61}$$

Exercise 25 (A criterion of continuity) *Let* $(S,\preceq)$ *be a* sup*-dense subset of* $(T,\preceq)$*. Show that the canonical embedding*

$$(S,\preceq)\hookrightarrow(T,\preceq) \tag{62}$$

is inf*-continuous.*

Dually, if $(S,\preceq)$ is a inf-dense subset of $(T,\preceq)$, then the canonical embedding (62) is sup-continuous.

1.4.1 Example: the canonical embedding into the power set

Let $(S, \preceq)$ be a partially ordered set. For any $s, s' \in S$, one has

$$s \preceq s' \qquad \text{if and only if} \qquad \langle s] \subseteq \langle s']. \tag{63}$$

Thus, the correspondence

$$\langle] \colon S \longrightarrow \mathscr{P}(S), \qquad s \longmapsto \langle s], \tag{64}$$

is an order embedding of $(S, \preceq)$ onto the partially ordered subset of

$$(\mathscr{P}(S), \subseteq)$$

.

Exercise 26 *Show that, for any* $E \subseteq S$*,*

$$\sup\langle E] = \bigcup_{s \in E} \langle s] \tag{65}$$

and

$$\inf\langle E] = L(E) \tag{66}$$

in $\mathscr{P}(S)$*.*[1]

It follows that

$$\sup\langle E] = \langle \sup E]$$

in $\mathscr{P}(S)$ if and only if $\max E$ exists. In particular, the canonical embedding into the power set, (64), is nearly never *sup*-continuous: $\sup\langle E]$ always exists, it equals (65), but is a *proper* subset of $\langle \sup E]$, except when E is an interval of the form (17).

The canonical embedding is, however, inf-continuous.

Exercise 27 *Show that the canonical embedding,* (64)*, is* inf*-continuous.*

Exercise 28 *Explain why* $\langle S]$ *is, generally, neither* sup- *nor* inf*-dense in* $\mathscr{P}(S)$*.*

[1] Here $\langle E]$ denotes the **image** of a subset $E \subseteq S$ under the canonical embedding, (64), *not* the interval, $\langle E] = \{X \in \mathscr{P}(S) \mid X \subseteq E\}$, in partially ordered set $\mathscr{P}(S), \subseteq)$. The latter coincides with $\mathscr{P}(E)$, the set of all subsets of E.

1.5 Closure operations on a partially ordered set

Definition 1.12 *Let* $(S, \preceq)$ *be an ordered set. A self-mapping*

$$s \longmapsto \bar{s}, \qquad S \longrightarrow S, \tag{67}$$

is said to be a ***closure operation*** *if it enjoys the following three properties*

$$s \preceq \bar{s}, \tag{68}$$

$$\text{if } s \preceq t, \text{ then } \bar{s} \preceq \bar{t}, \tag{69}$$

and

$$\bar{\bar{s}} = \bar{s} \tag{70}$$

where s *and* t *denote arbitrary elements of* S.

In this case, we say that an element $s \in S$ *is* ***closed*** *if* $\bar{s} = s$.

1.5.1 Example: a topology on a set

To give a *topology* on a set A is the same as to equip $(\mathscr{P}(A), \subseteq)$ with the closure operation which is inf-*continuous*,

> for any family $\mathscr{E} \subseteq \mathscr{P}(A)$, one has
>
> $$\overline{\inf \mathscr{E}} = \inf \overline{\mathscr{E}} \tag{71}$$
>
> where $\overline{\mathscr{E}}$ is the family of the closures of subsets belonging to $\mathscr{E}$.

and *finitely* sup-*continuous*,

> for any *finite* family $\mathscr{E} \subseteq \mathscr{P}(A)$, one has
>
> $$\overline{\sup \mathscr{E}} = \sup \overline{\mathscr{E}}. \tag{72}$$

Exercise 29 *Show that for any closure operation on* $(\mathscr{P}(A), \subseteq)$ *which satisfies* (72)*, one has*

$$\overline{\emptyset} = \emptyset. \tag{73}$$

Hint. Consider the empty family of subsets.

1.5.2 Example: two closure operations on the power set of a partially ordered set

A partial ordering $(S, \preceq)$ induces two closure operations on $(\mathscr{P}(S), \subseteq)$, the *LU-closure*,

$$E \longmapsto \bar{E}^{LU} := LU(E), \tag{74}$$

and the *UL-closure*,

$$E \longmapsto \bar{E}^{UL} := UL(E). \tag{75}$$

Indeed, for the *LU-closure*, (27) is property (68), property (69) follows from the combination of (29) and (30), while (36) implies property (70).

Note that, according to (36), a subset F of S is LU-closed if and only if it is of the form $F = L(E)$ for some $E \subseteq S$.

Note that

$$LU(\emptyset) = \begin{cases} \emptyset & \text{if } S \text{ has no least element} \\ \{\min S\} & \text{otherwise.} \end{cases}$$

Dually,

$$UL(\emptyset) = \begin{cases} \emptyset & \text{if } S \text{ has no largest element} \\ \{\max S\} & \text{otherwise.} \end{cases}$$

It follows that $\emptyset$ is LU-closed (respectively, UL-closed) if and only if S has no least element (respectively, no largest element).

On the other hand, S is always both LU- and UL-closed.

2 Dedekind-MacNeille Completion of a Partially Ordered Set

2.1 Completeness

Definition 2.1 *We say that a partially ordered set* $(S, \preceq)$ *has the* ***largest-lower-bound property*** *if* $\inf E$ *exists for every subset* $E \subseteq S$ *which is nonempty and bounded below. We shall say in this case that* $(S, \preceq)$ *is* $\inf$***-complete***.

Dually, we say that S *has the* ***least-upper-bound property*** *if* $\sup E$ *exists for subset* $E \subseteq S$ *which is nonempty and bounded above. We shall say in this case that* $(S, \preceq)$ *is* $\sup$***-complete***.

Lemma 2.2 *A partially ordered set S has the largest-lower-bound property if and only if it has the least-upper-bound property.*

Proof. Suppose that S is inf-complete. If $E \subseteq S$ is bounded above and nonempty, then the set of upper bounds, $U(E)$ is nonempty. Since

$$L(U(E)) \supseteq E \neq \emptyset$$

subset $U(E)$ is also bounded below. Then $\inf U(E)$ exists in view of our assumption about $(S, \preceq)$. But then it coincides with $\sup E$ in accordance with (41). This shows that S is sup-complete.

The reverse implication,

$$\text{sup-completeness} \Rightarrow \text{inf-completeness}$$

follows by applying the already proven implication

$$\text{inf-completeness} \Rightarrow \text{sup-completeness}$$

to the reverse order on S. □

Since sup- and inf-completeness are equivalent we shall simply call such sets *complete*.

2.1.1 Lattices

Definition 2.3 *A partially ordered set $(S, \preceq)$ is called a **pre-lattice** if every nonempty finite subset $E \subseteq S$ has supremum and infimum.*

Exercise 30 *Show that $(S, \preceq)$ is a **pre-lattice** if and only if, for any $s, t \in S$, both $\sup\{s, t\}$ and $\inf\{s, t\}$ exist.*

Definition 2.4 *A partially ordered set $(S, \preceq)$ is called a **lattice** if every finite subset $E \subseteq S$, including $\emptyset \subseteq S$, has supremum and infimum.*

Complete partially ordered sets with the largest and the smallest elements are the same as *complete lattices*. Note that in such sets every subset is bounded below and above.

For example, the totally ordered set of rational numbers, $(\mathbb{Q}, \leq)$, is a pre-lattice but not a lattice, and it is not complete.

The power set of an arbitrary set, $(\mathscr{P}(A), \subseteq)$, is an example of a complete lattice. A less obvious example is the subject of the next section.

2.1.2 The set of LU-closed subsets of a partially ordered set

Let $(S, \preceq)$ be a partially ordered set. Let $\mathscr{X}(S) \subseteq \mathscr{P}(S)$ be the subset of the power set which consists of all LU-closed subsets of S. Recall that $F \subseteq S$ is LU-closed if and only if it has the form

$$F = L(E) \tag{76}$$

for some $E \subseteq S$. In fact, one can take $E = U(F)$.

Exercise 31 *Show that, for any family $\mathscr{E} \subseteq \mathscr{X}(S)$ of LU-closed subsets of S, the supremum of $\mathscr{E}$ in $\mathscr{X}(S)$ exists and equals*

$$\sup \mathscr{E} = LU\Bigl(\bigcup_{E \in \mathscr{E}} E\Bigr), \tag{77}$$

i.e., $\sup_{\mathscr{X}(S)} \mathscr{E}$ is the LU-closure of $\sup_{\mathscr{P}(S)} \mathscr{E}$,

$$\sup_{\mathscr{X}(S)} \mathscr{E} = \overline{\sup_{\mathscr{P}(S)} \mathscr{E}}^{LU}. \tag{78}$$

Exercise 32 *Show that*

$$\min \mathscr{X}(S) = \begin{cases} \emptyset & \text{if } S \text{ has no least element} \\ \{\min S\} & \text{otherwise} \end{cases}. \tag{79}$$

Thus, every family $\mathscr{E} \subseteq \mathscr{X}(S)$ is bounded below and, obviously, the infimum of the empty family exists and is equal to

$$\inf \emptyset = \max \mathscr{X}(S) = S.$$

By invoking Lemma 2.2 we deduce that the infimum of every family $\mathscr{E} \subseteq \mathscr{X}(S)$ exists in $\mathscr{X}(S)$, and we established the following fact.

Proposition 2.5 *For any partially ordered set $(S, \preceq)$, the set of LU-closed subsets, $(\mathscr{X}(S), \subseteq)$ is a complete lattice.* □

Exercise 33 *Show that in $\mathscr{X}(S)$ one has*

$$\sup \langle E] = LU(E) \tag{80}$$

and

$$\inf \langle E] = L(E). \tag{81}$$

2.1.3

Since intervals $\langle s]$ are LU-closed,

$$\langle s] = L(\{s\}),$$

the image of set S under the canonical embedding, (64), is contained in $\mathscr{L}(S)$.

Exercise 34 *Show that* $\langle S]$ *is both* sup- *and* inf-*dense in* $\mathscr{L}(S)$.

Exercise 35 *Show that the embedding*

$$\langle] \colon (S, \preceq) \hookrightarrow (\mathscr{L}(S), \subseteq) \tag{82}$$

is both sup- *and* inf-*continuous.*

In other words, show that in $\mathscr{L}(S)$ *one has*

$$\sup_{\mathscr{L}(S)} \langle E] = \langle \sup E] \tag{83}$$

whenever $\sup E$ *exists in* $(S, \preceq)$, *and*

$$\inf_{\mathscr{L}(S)} \langle E] = \langle \inf E] \tag{84}$$

whenever $\inf E$ *exists in* $(S, \preceq)$.

By combining the results of Exercises 34 and 35 with Proposition 2.5, we establish the following important fact.

Theorem 2.6 *Any partially ordered set* $(S, \preceq)$ *admits an embedding onto a dense subset of a complete lattice, and this embedding preserves suprema and infima (i.e., is* sup- *and* inf-*continuous).* □

The lattice $\mathscr{L}(S), \subseteq)$ is called the **Dedekind-Macneille envelope** of a partially ordered set $(S, \preceq)$, or the **Dedekind-Macneille completion** of $(S, \preceq)$. Strictly speaking, the latter should apply to the embedding, (82), of $(S, \preceq)$ into $\mathscr{L}(S), \subseteq)$.

It is worth noticing that for any subset $E \subseteq S$ which is not bounded above in $(S, \preceq)$, one has

$$\sup \langle E] = \max \mathscr{L}(S) = S \tag{85}$$

and, for a subset not bounded below

$$\inf \langle E] = \min \mathscr{L}(S) = \emptyset \tag{86}$$

(note that $\min \mathscr{L}(S) = \emptyset$ in this case).

2.1.4 A variant

Dedekind-MacNeille completion adds to any partially ordered set all the "missing" suprema and infima, including the suprema and infima for unbouded sets.

A simple variant of the above construction allows to "add" only the suprema of nonempty subsets that are *bounded above*, and the infima of nonempty subsets which are *bounded below* while leaving unbounded sets not bounded.

Let $(S, \preceq)$ be a partially ordered set. Let us consider the subset $\mathscr{L}^*(S) \subseteq \mathscr{L}(S)$ which consists of the LU-closures of nonempty subsets of S which are bounded above.

Exercise 36 *Show that* $\emptyset \notin \mathscr{L}^*(S)$.

Exercise 37 *Show that* $S \in \mathscr{L}^*(S)$ *if and only if* S *has the largest element.*

It follows that

$$\mathscr{L}(S) = \begin{cases} \mathscr{L}^*(S) & \text{if both } \min S \text{ and } \max S \text{ exist} \\ \mathscr{L}^*(S) \cup \{S\} & \text{if } \min S \text{ exists and } \max S \text{ does not} \\ \mathscr{L}^*(S) \cup \{\emptyset\} & \text{if } \max S \text{ exists and } \min S \text{ does not} \\ \mathscr{L}^*(S) \cup \{\emptyset, S\} & \text{if neither } \min S \text{ nor } \max S \text{ exist} \end{cases}. \tag{87}$$

Thus, $\mathscr{L}^*(S)$ has the same elements as $\mathscr{L}(S)$ except possibly for the two extremal ones: $\emptyset$ and S.

Exercise 38 *Show that* $\langle S] \subseteq \mathscr{L}^*(S)$.

By Exercise 24, suprema and infima in $\mathscr{L}^*(S)$ coincide for nonempty bounded subsets of $\mathscr{L}^*(S)$ with those in $\mathscr{L}(S)$, hence $(\mathscr{L}^*(S), \subseteq)$ is a complete partially ordered set.

In particular, the embedding

$$\langle]\colon (S, \preceq) \longrightarrow (\mathscr{L}^*(S), \subseteq) \tag{88}$$

is both sup- and inf-continuous.

Finally, since $\langle S]$ is sup- and inf-dense in $\mathscr{L}(S)$ it is dense also in $\mathscr{L}^*(S)$.

We established thus the following variant of Theorem 2.6.

Theorem 2.7 *Any partially ordered set $(S, \preceq)$ admits an embedding onto a dense subset of a complete partially ordered set, and this embedding preserves suprema and infima (i.e., is* sup- *and* inf*-continuous), and also unboundedness of subsets:*

$$\text{if } E \subseteq S \text{ is not bounded above (below) in } S\text{, then } E \text{ is not bounded above (resp., below) in the completion.} \tag{89}$$

□

2.1.5

It follows from Exercise 10 that $(\mathscr{Z}(S), \subseteq)$, and therefore also $(\mathscr{Z}^*(S), \subseteq)$, are totally ordered if $(S, \preceq)$ is totally ordered. Thus one can strengthen the statements of Theorems 2.6 and 2.7 by adding that the corresponding completions of totally ordered sets are themselves totally ordered.

2.1.6 $\mathbb{R}$ and $\overline{\mathbb{R}} = [-\infty, \infty]$

In the case of $(\mathbb{Q}, \leq)$ we obtain two totally ordered completions

$$(\mathscr{Z}^*(\mathbb{Q}), \subseteq) \text{ is a model for the set of real numbers,} \tag{90}$$

while

$(\mathscr{Z}(\mathbb{Q}), \subseteq)$ *is a model for the set of* extended *real numbers*

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]. \tag{91}$$

3 Universal Properties of the Dedekind-MacNeille Completion

In this section $(S, \preceq)$ denotes a subset of $(T, \preceq)$.

3.1 Extending the Canonical Embedding

3.1.1

The correspondence

$$t \longmapsto \langle t] \cap S = \{s \in S \mid s \preceq t\} \tag{92}$$

defines a morphism $(T, \preceq) \longrightarrow (\mathscr{P}(S), \subseteq)$ which extends the canonical embedding, $(S, \preceq) \hookrightarrow (\mathscr{P}(S), \subseteq)$, cf. (64), to $(T, \preceq)$.

Exercise 39 *Show that* (92) *is an order embedding if* $(S, \preceq)$ *is* sup*-dense in* $(T, \preceq)$.

In particular,

$$\textit{morphism (92) is injective if } (S, \preceq) \textit{ is } \sup\textit{-dense in } (T, \preceq). \tag{93}$$

If $(S, \preceq)$ is *not* sup-dense, then (92) is generally not injective. Thus,

$$\langle v] \cap S = \langle \zeta] \cap S$$

in Example 1.3.3.a but $v \neq \zeta$.

3.1.2

If $t = \inf_T E$ for some $E \subseteq S$, then

$$\langle t] \cap S = L(E) \in \mathscr{L}(S),$$

according to (50). It follows that

$$\begin{array}{l}\textit{the image of morphism (92) is contained in} \\ \mathscr{L}(S) \textit{ if } (S, \preceq) \textit{ is } \inf\textit{-dense in } (T, \preceq).\end{array} \tag{94}$$

The following example demonstrates that the image of morphism (92) may be contained in $\mathscr{L}(S)$ even though *no* element of $T \setminus S$ may be of the form $\inf_T E$ for some $E \subseteq S$.

3.1.3 Example

Let $T := \mathbb{Q}\{0, 1\}$ equipped with the usual order. Let

$$S := \{x \in \mathbb{Q} \mid x < 0 \text{ or } x > 1\} = \mathbb{Q}_{<0} \cup \mathbb{Q}_{>1}.$$

Then, all $t \in T \setminus S = \{x \in \mathbb{Q} \mid 0 < x < 1\}$ are being sent by morphism (92) to one and the same LU-closed subset of S:

$$\langle t] \cap S = \mathbb{Q}_{<0} = L(\mathbb{Q}_{>1}).$$

Exercise 40 *Show that every* $t \in T \setminus S$ *is neither of the form* $\inf_T E$ *nor of the form* $\sup_T E$ *for some* $E \subseteq S$.

3.1.4

When $(S, \preceq)$ is inf-dense in $(T, \preceq)$, then one can easily describe the image of T under morphism (92):

> *a subset F of $\mathscr{L}(S)$ is in the image* if and only if *it is of the form*
>
> $$\langle \inf_T E] \cap S = L_T(E) \cap S = L(E) \tag{95}$$
>
> *for some $E \subseteq S$ which possesses infimum in T.*

This is so since every element of T equals $\inf_T E$ for a suitable subset $E \subseteq S$ and then $\langle t] = L_T(E)$.

Note, however, that $L(E)$, *for a particular* subset E, may belong to the image of morphism (92) while $\inf_T E$ may not exist. We only know that there must be another subset $E' \subseteq S$ such that

$$L(E) = L(E') \qquad \text{and} \qquad \inf_T E' \text{ exists.}$$

In Example 1.3.3.b, one has

$$L(Z) = \langle v] \cap S = L(Y)$$

where Y possesses infimum in the larger set, which is denoted there T', while Z does not.

3.1.5

By combining (95) with the assertion of Exercise 39, we deduce the following important universal property of the Dedekind-MacNeille completion.

Theorem 3.1 *If $(S, \preceq)$ is a subset of $(T, \preceq)$ which is both* inf- *and* sup-*dense, then morphism* (92) *embeds $(T, \preceq)$ isomorphically onto the subset of $\mathscr{L}(S)$:*

$$\{F \in \mathscr{L}(S) \mid F = L(E) \text{ for some } E \subseteq S \text{ which possesses infimum in } T\}.$$

In particular, if every *subset of S possesses infimum in T, then* (92) *establishes a canonical isomorphism between $(T, \preceq)$ and the Dedekind-MacNeille completion, $(\mathscr{L}(S), \subseteq)$ which extends the embedding of $(S, \preceq)$ into $(\mathscr{L}(S), \subseteq)$.* □

3.2 Completing a subset in a bigger partially ordered set

3.2.1

In general, there is no largest subset $T' \subseteq T$ such that $(S, \preceq)$ would be sup-dense, inf-dense, or both sup- and inf-dense in $(T', \preceq)$.

3.2.2 Example

Let $S = \mathbb{Q} \setminus \{0\}$ be equipped with usual order and $T = S \cup \{a, b\}$ with

$$x < a < y \qquad \text{and} \qquad x < b < y$$

for any $x \in \mathbb{Q}_{<0}$ and $y \in \mathbb{Q}_{>0}$. In this case, S is both sup- and inf-dense in $T_1 = S \cup \{a\}$ and $T_2 = S \cup \{b\}$ but is neither sup-dense nor inf-dense in $T = T_1 \cup T_2$.

Exercise 41 *Show that neither a nor b is of the form $\inf_T E$ or $\sup_T E$ for any subset $E \subseteq S$.*

Thus, T_1 and T_2 are two distinct *maximal* subsets of T in which S is dense in any of the three spelled out senses.

3.2.3

For any partially ordered set $(T, \preceq)$ containing $(S, \preceq)$, there is, however, a subset that is a perfect analog of the Dedekind-MacNeille completion:

$$\widehat{S_T} := \{t \in T \mid t = \inf_T U(E) \text{ for some } E \subseteq S\}. \tag{96}$$

If we identify S with its isomorphic image $\langle S]$ in $\mathscr{P}(S)$, then

$$\widehat{S_{\mathscr{P}(S)}} = \mathscr{L}(S) \tag{97}$$

since every $F \in \mathscr{L}(S)$ is of the form

$$F = LU(E) = \inf_{\mathscr{P}(S)} U(E), \tag{98}$$

cf. equalities (36) and (66).

3.2.4

If $t = \inf_T U(E)$, then

$$E \subseteq LU(E) = L_T(U(E)) \cap S \subseteq L_T(U(E)) = \langle t]$$

This shows that $t \in U_T(E)$.

If another element of $\widehat{S_T}$, say $t' = \inf_T U(E')$, belongs to $U_T(E)$, then

$$x \preceq t' \preceq y$$

for any $x \in E$ and $y \in U(E')$. It follows that

$$U(E') \subseteq U(E)$$

and hence

$$t = \inf_T E \preceq \inf_T E' = t',$$

cf. (32). Thus, t is the smallest element of $U_T(E) \cap \widehat{S_T} = U_{\widehat{S_T}}(E)$, i.e., $\sup_{\widehat{S_T}} E$ exists and equals t:

$$\sup_{\widehat{S_T}} E = \inf_T U(E). \tag{99}$$

Note that, in view of (49), the infimum of $U(E)$ in $\widehat{S_T}$ exists and coincides with $\inf_T U(E)$,

$$\inf_{\widehat{S_T}} U(E) = \inf_T U(E). \tag{100}$$

In particular,

$$\sup_{\widehat{S_T}} E = \inf_{\widehat{S_T}} U(E). \tag{101}$$

We established the following fact.

Proposition 3.2 *Any subset S of a partially oredered set $(T, \preceq)$ is both* sup- *and* inf-*dense in $(\widehat{S_T}, \preceq)$.* □

3.2.5

By the Universal Property of the Dedekind-MacNeille completion cf. Theorem 3.1, the order embedding of $(S, \preceq)$ into $\mathscr{L}(S), \subseteq)$ then extends to an order embedding

$$(\widehat{S_T}, \preceq) \hookrightarrow (\mathscr{L}(S), \subseteq). \tag{102}$$

3.2.6

If $(T, \preceq)$ is a complete lattice, then embedding (102) is *surjective* since, for every $E \subseteq S$, one has

$$LU(E) = \langle \inf_T U(E)] \cap S,$$

cf. Exercise 15.

In particular, $(\widehat{S_T}, \preceq)$ is isomorphic to the Dedekind-MacNeille completion.

This way we arrive at our final result.

Theorem 3.3 *Any order embedding $(S, \preceq) \hookrightarrow (T, \preceq)$ into a complete lattice induces an order embedding of the Dedekind-MacNeille completion,*

$$(\mathscr{L}(S), \subseteq) \hookrightarrow (T, \preceq) \tag{103}$$

which identifies $(\mathscr{L}(S), \subseteq)$ with $(\widehat{S_T}, \preceq)$. In particular, $(\widehat{S_T}, \preceq)$ is itself a complete lattice. □