Notes on Riemann Integral

An annex to H104 etc.

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1 Cells

1.1 Intervals

1.1.1 Connected subsets of $\mathbb{R}$

Definition 1.1 *A connected subset I of the topological space $\mathbb{R}$ is called an **interval**.*

Exercise 1 *Show that any connected subset $I \in \mathbb{R}$ contains (a, b) where $a = \inf S$ and $b = \sup S$. (Hint: prove that, for any $s, t \in S$, if $s < t$, then $[s, t] \subseteq S$.)*

1.1.2

It follows from Exercise 1 that any interval is of the form

$$\langle a, b \rangle \qquad (-\infty \leq a \leq b \leq \infty) \tag{1}$$

where '$\langle$' stands for either '$[$' or '$($' and '$\rangle$' stands for either '$]$' or '$)$'.

Definition 1.2 *We shall say that an interval I is*

*(a) **nondegenerate** if $a < b$,*

*(b) **open** if $I = (a, b)$,*

*(c) **closed** if $I = [a, b]$ or $(-\infty, b]$, or $[a, \infty)$, or $i = \mathbb{R}$,*

*(d) **bounded** if $-\infty < a$ and $b < \infty$.*

1.1.3

For any interval $I = \langle a, b \rangle$ we shall denote its closure by $\bar{I}$ and its interior by $\mathring{I}$.

1.1.4 Boundary

The set $\partial I := I \setminus \mathring{I}$ will be called the **boundary** of I. It consists of at most two points, and $\partial I = \emptyset$ precisely when I is open.

1.1.5 Length

The **length** of $I = \langle a, b \rangle$ will be denoted $|I| := b - a$.

1.1.6

For an interval $I = \langle a, b \rangle$ and $\delta > 0$, we shall denote by I_δ the δ-neighborhood of I:

$$I_\delta := (a - \delta, b + \delta) \tag{2}$$

1.2 Cells

1.2.1

Definition 1.3 *An n-**cell** is the Cartesian product of n intervals*

$$\mathbb{I} := I_1 \times \cdots \times I_n \tag{3}$$

It is naturally a subset of metric space $\mathbb{R}^n$. A 1-cell is the same as an interval.

Definition 1.4 *We shall say that an n-cell is*

*(a) **nondegenerate** if each I_j is nondegenerate,*

*(b) **open** if each I_j is open,*

*(c) **closed** if each I_j is closed,*

*(d) **bounded** if each I_j is bounded.*

1.2.2

A nonempty degenerate n-cell is isometric, as a metric space, to an m-cell for $m = n - l$ where l is the number of factors in (3) which are degenerate.

1.2.3

For any cell $\mathbb{I}$ we shall denote its closure by $\bar{\mathbb{I}}$ and its interior by $\mathring{\mathbb{I}}$.

Exercise 2 *Show that the intersection $\mathbb{I} \cap \mathbb{I}'$ of two n-cells is again an n-cell—possibly degenerate or empty.*

1.2.4 Boundary

The set $\partial\mathbb{I} := \mathbb{I} \setminus \mathring{\mathbb{I}}$ will be called the **boundary** of $\mathbb{I}$.

Exercise 3 *Let $\mathbb{I} = [a_1, b_1] \times \cdots \times [a_n, b_n]$ with $a_j < b_j$, $j = 1, \ldots, n$. Show that $\partial\mathbb{I}$ is the union of $2n$ degenerate cells, each isometric to an $(n-1)$-cell. The latter are called the* ***faces*** *of $\mathbb{I}$.*

1.2.5 Volume

The n-dimensional **volume** of a *bounded* cell, (3), is defined as

$$\|\mathbb{I}\| := |I_1| \cdots |I_n|. \tag{4}$$

It is greater than zero precisely when $\mathbb{I}$ is nondegenerate.

1.2.6 δ-thickening

For a cell $\mathbb{I}$ and $\delta > 0$, we shall denote by I_δ the open cell

$$\mathbb{I}_\delta := (I_1)_\delta \times \cdots \times (I_n)_\delta. \tag{5}$$

It is the smallest cell containing the δ-neighborhood of $\mathbb{I}$.

The volume of $\mathbb{I}_\delta$ satisfies the following obvious estimate

$$\|\mathbb{I}_\delta\| \leq \|\mathbb{I}\| + 2\delta \left(\sum_{l=0}^{l=n-1} \binom{n}{l} (2\delta)^l \operatorname{diam} \mathbb{I})^{l-1} \right). \tag{6}$$

In particular, by selecting δ sufficiently small, one can make $\|\mathbb{I}_\delta\|$ as close to $\|\mathbb{I}\|$ as desired.

2 Riemann Integral

2.1 Outer contents and measure

2.1.1

For a family $\mathscr{I}$ of bounded n-cells we define $\|\mathscr{I}\|$ as

$$\|\mathscr{I}\| := \sum_{\mathbb{I}\in\mathscr{I}} \|\mathbb{I}\|. \tag{7}$$

The quantity defined in (7) makes sense for any, even uncountable, family provided we define the sum in (7) as

$$\sup\{\|\mathscr{I}'\| \mid \mathscr{I}' \subseteq \mathscr{I} \text{ is finite}\}$$

In particular, the values that $\|\mathscr{I}\|$ can take belong to $[0,\infty]$.

Exercise 4 *Let S be an arbitrary set. Suppose that, for a function $f\colon S \longrightarrow [0,\infty)$,*

$$\sup\Big\{\sum_{s\in S'} f(s) \mid S' \subseteq S \textit{ is finite}\Big\} < \infty.$$

Prove that the set

$$\operatorname{supp} f := \{s \in S \mid f(s) \neq 0\} \tag{8}$$

is countable[1] (*the set defined in* (8) *is called the* ***support*** *of f*).

2.1.2 Cell covers

Let A be a subset of $\mathbb{R}^n$.

Definition 2.1 *A **cell cover** of A is a family $\mathscr{I}$ of bounded nondegenerate cells such that $\bigcup \mathscr{I} \supseteq A$.*

2.1.3 Outer contents

Definition 2.2 *The infimum over all finite cell covers of A,*

$$\bar{m}(A) := \inf\{\|\mathscr{I}\| \mid \mathscr{I} \textit{ is a finite cell cover of } A\}. \tag{9}$$

*will be called the **outer contents** of subset $A \subseteq \mathbb{R}^n$. When A is not bounded, A cannot be covered by finitely many bounded cells. In this case, we set $\bar{m}(A) = \infty$.*

[1]By *countable* we mean in these notes any set that can be embedded into the set of natural numbers. In particular, finite sets are 'countable' according to this definition.

2.1.4 Outer measure

Definition 2.3 *The infimum over all countable cell covers of A,*

$$\bar{\mu}(A) := \inf\{\|\mathscr{I}\| \mid \mathscr{I} \text{ is a countable cell cover of } A\}. \tag{10}$$

*will be called the **outer measure** of subset $A \subseteq \mathbb{R}^n$.*

2.1.5

Note that, in view of Exercise 4, we could have defined $\bar{\mu}(A)$ as the infimum of $\|\mathscr{I}\|$ over *all* cell covers since $\|\mathscr{I}\| = \infty$ for any uncountable cover.

2.1.6

It follows directly from the definition that for any subsets of $\mathbb{R}^n$:

$$\bar{\mu}(A) \leq \bar{m}(A) \tag{11}$$

and

$$\bar{m}(A) \leq \bar{m}(B) \quad \text{as well as} \quad \bar{\mu}(A) \leq \bar{\mu}(B) \tag{12}$$

whenever $A \subseteq B$.

Exercise 5 *Let $A = \mathbb{Q} \cap [a, b]$. Show that*

$$\bar{\mu}(A) = 0 \qquad \text{while} \qquad \bar{m}(A) = b - a.$$

Exercise 6 *Prove that*

$$\bar{m}\left(\bigcup_{A \in \mathscr{A}} A\right) \leq \sum_{A \in \mathscr{A}} \bar{m}(A), \tag{13}$$

for any finite family $\mathscr{A}$ of subsets of $\mathbb{R}^n$, and

$$\bar{\mu}\left(\bigcup_{A \in \mathscr{A}} A\right) \leq \sum_{A \in \mathscr{A}} \bar{\mu}(A), \tag{14}$$

for any countable family $\mathscr{A}$ of subsets of $\mathbb{R}^n$.

2.1.7

It follows directly from inequalities (12) and (13) that

$$\bar{m}(A \cap A') = \bar{m}(A) = \bar{m}(A') = \bar{m}(A \cup A') \tag{15}$$

whenever

$$\bar{m}(A \setminus A') = \bar{m}(A' \setminus A) = 0.$$

Indeed, one has

$$\begin{aligned} \bar{m}(A \cap A') &\leq \bar{m}(A) \leq \bar{m}(A \cup A') \\ &\leq \bar{m}(A \cap A') + \bar{m}(A \setminus A') + \bar{m}(A' \setminus A) = \bar{m}(A \cap A'). \end{aligned}$$

2.1.8

Similarly,

$$\bar{\mu}(A \cap A') = \bar{\mu}(A) = \bar{\mu}(A') = \bar{\mu}(A \cup A') \tag{16}$$

whenever

$$\bar{\mu}(A \setminus A') = \bar{\mu}(A' \setminus A) = 0.$$

Exercise 7 *Prove that in the definition of $\bar{m}(A)$ one could consider exclusively open (respectively, closed) cell covers:*

$$\begin{aligned} \bar{m}(A) &= \inf\{\|\mathscr{I}\| \mid \mathscr{I} \text{ is a finite open cell cover of } A\} & (17) \\ &= \inf\{\|\mathscr{I}\| \mid \mathscr{I} \text{ is a finite closed cell cover of } A\} & (18) \end{aligned}$$

and, similarly, for $\bar{\mu}(A)$. (Hint: for a cell cover $\mathscr{I}$ consider the family of closures, $\{\bar{\mathbb{I}} \mid \mathbb{I} \in \mathscr{I}\}$, and the family of δ-thickenings, $\{\mathbb{I}_\delta \mid \mathbb{I} \in \mathscr{I}\}$, for sufficientlly small $\delta > 0$.)

2.1.9 Removal of overlaps

For any finite family of closed cells $\mathscr{I}$, one can decompose each $\mathbb{I} \in \mathscr{I}$ into a union of finitely many closed subcells so that the distinct subcells, $\mathbb{I}$ and $\mathbb{I}'$, do not overlap, i.e., if $\mathbb{I} \cap \mathbb{I}'$ is either empty or a degenerate cell.

Denote a family obtained this way by $\mathscr{J}$. Since every cell $\mathbb{I} \in \mathscr{I}$ is the union of cells from $\mathscr{J}$, one has

$$\bigcup \mathscr{J} = \bigcup \mathscr{I} \tag{19}$$

and

$$\|\mathscr{J}\| \leq \|\mathscr{I}\| \tag{20}$$

since the volume of every cell $J \in \mathscr{J}$ contributes to $\|\mathscr{J}\|$ only once while to $\|\mathscr{I}\|$ it contributes as many times as there are cells $\mathbb{I} \in \mathscr{I}$ which contain it.

In particular, in the definition of $\bar{m}(A)$ one could replace arbitrary finite covers by finite families of closed nonoverlapping cells.

Exercise 8 *Produce an example showing that one cannot do the same in the case of $\bar{\mu}(A)$: the latter is generally smaller than the infimum of $\|\mathscr{J}\|$ over all closed nonoverlapping covers.*

2.1.10 The case of closed bounded subsets

Compactness of bounded closed subsets of $\mathbb{R}^n$ implies that that the outer measure and the outer contents of such sets coincide.

Proposition 2.4 *For a bounded closed subset $A \subseteq \mathbb{R}^n$, one has*

$$\bar{\mu}(A) = \bar{m}(A). \tag{21}$$

Exercise 9 *Prove Proposition 2.4.*

2.2 Oscillation of a mapping

2.2.1 Oscillation on a subset

Let $f\colon X \longrightarrow M$ be a mapping from a topological space X to a metric space M.

Definition 2.5 *For $A \subseteq X$, we set*

$$\operatorname{osc}_A(f) := \operatorname{diam} f(A) = \sup_{p,q \in X} \rho(f(p), f(q)) \tag{22}$$

2.2.2 Oscillation at a point

2.2.3

If $A \subseteq B$, then

$$\operatorname{osc}_A(f) \leq \operatorname{osc}_B(f).$$

2.2.4

In particular, the net

$$N \longmapsto \operatorname{osc}_N(f) \qquad (N \in \mathscr{N}_p)$$

indexed by the neighborhood filter $\mathscr{N}_p$ of a point $p \in X$ is nonincreasing and thus the limit

$$\lim_{N \in \mathscr{N}_p} \operatorname{osc}_N(f)$$

exists and equals

$$\operatorname{osc}_p(f) := \inf\{\operatorname{osc}_N(f) \mid N \in \mathscr{N}_p\}. \tag{23}$$

Exercise 10 *Prove that a function $f\colon X \longrightarrow M$ from a topological space X into a metric space M is continuous at a point $p \in X$ if and only if $\operatorname{osc}_p(f) = 0$.*

2.2.5 The set of discontinuity of a function

It follows that the set

$$\operatorname{Disc} f := \{p \in X \mid f \text{ is discontinuous at } p\} \tag{24}$$

coincides with the set

$$\{p \in X \mid \operatorname{osc}_p(f) > 0\} = \bigcup_{\delta > 0} D_\delta(f) = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}(f) \tag{25}$$

where

$$D_\delta(f) := \{p \in X \mid \operatorname{osc}_p(f) \geq \delta\} \tag{26}$$

Proposition 2.6 *For any $\delta > 0$, the set*

$$\{p \in X \mid \operatorname{osc}_p(f) < \delta\} \tag{27}$$

is open. Equivalently, for any $d \geq 0$, set $D_\delta(f)$ is closed.

Proof. Let $p \in X$ be a point where $\operatorname{osc}_p(f) < \delta$. Choose $\epsilon > 0$ so that $\epsilon < \delta - \operatorname{osc}_p(f)$. By definition, cf. (23), there exists an open neighborhood of p such that

$$\operatorname{osc}_U(f) \leq \operatorname{osc}_p(f) + \epsilon.$$

But then

$$\operatorname{osc}_q(f) \leq \operatorname{osc}_U(f) \leq \operatorname{osc}_p(f) + \epsilon < \delta$$

for any $q \in U$. This shows that p is an interior point of set (27). $\square$

Corollary 2.7 *The set of discontinuity of any function* $f\colon X\longrightarrow M$ *is the union of countably many closed sets* $D_{\frac{1}{n}}(f)$.

2.2.6 F_σ-sets

Even though countable unions of closed subsets of a topological space need not be closed they constitute an important class of subsets on their own: they are called F_σ-sets.

The complement of an F_σ-set is a countable intersection of open subsets. The latter are called G_δ-sets. Here, subscripts δ and σ are just Greek letters—they do not refer to any quantity.

2.2.7

Corollary 2.7 says that the set of discontinuity of any function with values in a metric space is an F_σ-set. And dually, the set of points where f is continuous is a G_δ-set.

2.2.8 Semicontinuous functions

We encounter in Proposition 2.6 an interesting property that a real valued function may possess.

If a function $h\colon X\longrightarrow[\alpha,\beta]$ is continuous then

$$h^{-1}((c,\beta)) \quad \text{and} \quad h^{-1}((c,\beta]) \quad \text{are open for any } c\in(\alpha,\beta), \tag{28}$$

and similarly

$$h^{-1}((\alpha,c)) \quad \text{and} \quad h^{-1}([\alpha,c)) \quad \text{are open for any } c\in(\alpha,\beta). \tag{29}$$

This explains the terminology employed in the following double definition.

Definition 2.8 *We say that a function* $h\colon X\longrightarrow[\alpha,\beta]$ *is* ***lower semicontinuous*** *if it satisfies* (28)*, and* ***upper semicontinuous****), if it satisfies* (29).

Exercise 11 *Show that a function* $h\colon X\longrightarrow[\alpha,\beta]$ *is continuous iff and only if it is both lower and upper semicontinuous.*

2.2.9

Proposition 2.6 thus can be also stated as saying that the oscillation of *any* function $f\colon X \longrightarrow M$, viewed as a function

$$\operatorname{osc}(f)\colon X \longrightarrow [0,\infty], \qquad p \longmapsto \operatorname{osc}_p(f),$$

is upper semicontinuous.

Exercise 12 *Show that the characteristic function $\chi_A\colon X \longrightarrow \mathbb{R}]$ of a subset $A \subseteq X$,*

$$\chi_A(p) := \begin{cases} 1 & \text{if } p \in A \\ 0 & \text{otherwise} \end{cases}$$

is lower semicontinuous if and only if A is open. Similarly, χ_A is upper semicontinuous if and only if A is closed.

Exercise 13 *By analogy with continuous functions, formulate an appropriate definition of a function* ***lower semicontinuous at a point*** *$p \in X$ and then prove that a function is lower semicontinuous if and only if it is lower semicontinuous at any point $p \in X$.*

2.2.10

We close this brief discussion of semicontinuous functions by noting their fundamental property:

On compact subsets lower semicontinuous functions attain their lower bounds while upper semicontinuous functions attain their upper bounds. (30)

Theorem 2.9 *If $K \subseteq X$ is compact subset of a topological space, then any lower semicontinuous function $h\colon X \longrightarrow [-\infty,\infty]$ attains its minimum on K, and any upper semicontinuous function attains its maximum on K.*

Proof. Suppose that h is lower semicontinuous and let

$$a = \inf h(K).$$

For any $c > a$, the sets $U_c = h^{-1}((c,\infty])$ are open. If there is no $p \in K$ such that $h(p) = a$, then $\{U_c\}_{c>a}$ forms an open cover of K. In view of compactness of K, there must then exist $c_0 > a$ such that $K \subseteq U_{c_0}$ since $U_c \subseteq U_{c'}$ if $c \leq c'$. But then

$$\inf h(K) = a < c_o \leq \inf h(K).$$

Contradiction.

The case of an upper semicontinuous function follows by applying the already proven part of the theorem to $-h$. □

3 Riemann Integral

3.1 Darboux sums

3.1.1 Partitions

Definition 3.1 *A finite closed cell cover $\mathscr{P}$ of a subset $A \subseteq \mathbb{R}^n$ is said to be a* ***partition of*** *A if:*

(a) $\bigcup \mathscr{P} = A$,

(b) $\mathbb{I} \cap \mathbb{I}'$ *is either empty or a degenerate cell if* $\mathbb{I} \neq \mathbb{I}'$.

3.1.2 Rectangular subsets

In order that A admits any partition A must be closed and representable as the union of finitely many cells. We shall refer to such subsets of $\mathbb{R}^n$ as **rectangular**.

3.1.3

If $\mathscr{P}$ and $\mathscr{P}'$ are two partitions we write $\mathscr{P} \preccurlyeq \mathscr{P}'$ and say that $\mathscr{P}$ is **inscribed** in $\mathscr{P}'$ or, equivalently, that $\mathscr{P}$ is **finer** than $\mathscr{P}'$, and that $\mathscr{P}'$ is **coarser** than $\mathscr{P}$, if

$$\text{for any } \mathbb{I} \in \mathscr{P} \text{ there exists } \mathbb{I}' \in \mathscr{P}' \text{ such that } \mathbb{I} \subseteq \mathbb{I}'. \tag{31}$$

Note that due to how we defined partitions, for any $\mathbb{I} \in \mathscr{P}$, such a cell $\mathbb{I}' \in \mathscr{P}'$ is unique.

3.1.4

Being finer is a partial ordering relation on the set of all partitions, $\operatorname{Part} A$, making $\operatorname{Part} A$ into a directed set. More precisely, for any two partitions $\mathscr{P}$ and $\mathscr{P}'$, their common refinement, $\mathscr{P} \vee \mathscr{P}'$, formed by all nonempty and nondegenerate intersections,

$$\mathbb{I} \cap \mathbb{I}' \qquad (\mathbb{I} \in \mathscr{P},\ \mathbb{I}' \in \mathscr{P}'),$$

is the supremum of the two-element set $\{\mathscr{P}, \mathscr{P}'\}$ in $\operatorname{Part} A$.

3.1.5 The nets of Darboux sums

With any function $f\colon A \longrightarrow \mathbb{R}$ we associate two nets indexed by directed set $\operatorname{Part} A$, the net of **lower Darboux sums**,

$$\underline{S}(f, \mathscr{P}) := \sum_{\mathbb{I} \in \mathscr{P}} \inf_{x \in \mathbb{I}} f(\boldsymbol{x}) \|\mathbb{I}\|, \tag{32}$$

and the net of **upper Darboux sums**,

$$\overline{S}(f, \mathscr{P}) := \sum_{\mathbb{I} \in \mathscr{P}} \sup_{x \in \mathbb{I}} f(\boldsymbol{x}) \|\mathbb{I}\|. \tag{33}$$

Lower Darboux sums take values in $[-\infty, \infty)$ while upper Darboux sums take values in $(-\infty, \infty]$.

3.1.6 Monotonicity of Darboux sums

It follows directly from the respective definitions that

$$\underline{S}(f, \mathscr{P}') \leq \underline{S}(f, \mathscr{P}) \leq \overline{S}(f, \mathscr{P}) \leq \overline{S}(f, \mathscr{P}') \tag{34}$$

whenever $\mathscr{P} \preccurlyeq \mathscr{P}'$. Thus, the net of lower Darboux sums is nondecreasing, while the net of upper Darboux sums is nonincreasing, and each is bounded by *any* term of the other net.

In particular, both nets converge and their limits are respectively called: the **lower Darboux integral** of f,

$$\underline{\int}_A f(\boldsymbol{x})\, dx_1 \cdots dx_n := \lim_{\mathscr{P} \in \operatorname{Part} A} \underline{S}(f, \mathscr{P}) = \sup_{\mathscr{P} \in \operatorname{Part} A} \underline{S}(f, \mathscr{P}), \tag{35}$$

and the **upper Darboux integral** of f,

$$\overline{\int}_A f(\boldsymbol{x})\, dx_1 \cdots dx_n := \lim_{\mathscr{P} \in \operatorname{Part} A} \underline{S}(f, \mathscr{P}) = \inf_{\mathscr{P} \in \operatorname{Part} A} \underline{S}(f, \mathscr{P}). \tag{36}$$

3.2 Riemann sums

3.2.1 Tagged partitions

Definition 3.2 *If $\mathscr{P}$ is a partition of a subset $A \subseteq \mathbb{R}^n$, then a function $\boldsymbol{x}^*\colon \mathscr{P} \longrightarrow \mathbb{R}^n$ is called a* ***tagging*** *of $\mathscr{P}$ if, for each $\mathbb{I} \in \mathscr{P}$, its value, $\boldsymbol{x}_{\mathbb{I}}$, belongs to $\mathbb{I}$. We shall refer to pairs $(\mathscr{P}, \boldsymbol{x})$ as* ***tagged partitions****.*

3.2.2

We shall write $(\mathscr{P}, \boldsymbol{x}^*) \preccurlyeq (\mathscr{P}', \boldsymbol{x}^{*\prime})$ if either $\mathscr{P}$ is *strictly* finer than $\mathscr{P}'$, or $(\mathscr{P}, \boldsymbol{x}^*) = (\mathscr{P}', \boldsymbol{x}^{*\prime})$. In other words, we ignore the tagging when comparing tagged partitions, and different taggings of the same partition we consider non-comparable.

3.2.3

The set of tagged partitions, $\mathrm{Part}^* A$ is directed: given two tagged partitions $(\mathscr{P}, \boldsymbol{x}^*)$ and $(\mathscr{P}', \boldsymbol{x}^{*\prime})$, take any partition strictly finer than either $\mathscr{P}$ or $\mathscr{P}'$ and equip it with *any* tagging.

3.2.4 The net of Riemann sums

With any function $f\colon A \longrightarrow \mathbb{R}$ we associate the net of **Riemann sums** which is indexed by directed set $\mathrm{Part}^* A$,

$$S(f, \mathscr{P}, \boldsymbol{x}^*) \coloneqq \sum_{\mathbb{I} \in \mathscr{P}} f(\boldsymbol{x}^*) \|\mathbb{I}\|, \tag{37}$$

By definition, the Riemann sum is sandwiched between the corresponding Darboux sums:

$$\underline{S}(f, \mathscr{P}) \leq S(f, \mathscr{P}, \boldsymbol{x}^*) \leq \overline{S}(f, \mathscr{P}) \tag{38}$$

Exercise 14 *Show that*

$$\liminf_{(\mathscr{P}, \boldsymbol{x}^*) \in \mathrm{Part}^* A} S(f, \mathscr{P}, \boldsymbol{x}^*) = \underline{\int}_A f(\boldsymbol{x})\, dx_1 \cdots dx_n \tag{39}$$

and

$$\limsup_{(\mathscr{P}, \boldsymbol{x}^*) \in \mathrm{Part}^* A} S(f, \mathscr{P}, \boldsymbol{x}^*) = \overline{\int}_A f(\boldsymbol{x})\, dx_1 \cdots dx_n. \tag{40}$$

3.3 Riemann integrability

3.3.1

Definition 3.3 *We say that a function $f\colon A\longrightarrow\mathbb{R}$ is* ***Riemann integrable*** *if the net of Riemann sums converges. in view of* (39) *and* (40)*, this is equivalent to the condition that*

$$\underline{\int}_A f(\boldsymbol{x})\,dx_1\cdots dx_n = \overline{\int}_A f(\boldsymbol{x})\,dx_1\cdots dx_n. \tag{41}$$

The common value of the limits of Riemann and Darboux sums is then denoted

$$\int_A f(\boldsymbol{x})\,dx_1\cdots dx_n \tag{42}$$

and called the ***Riemann integral of function f over set A****.*

Theorem 3.4 (Henri Lebesgue, 1907) *A function $f\colon A\longrightarrow\mathbb{R}^n$ is Riemann integrable if and only if it is bounded and*

$$\bar{\mu}(\operatorname{Disc} f) = 0. \tag{43}$$

3.3.2

Above we have given the definition of Riemann integral only over closed subsets of $\mathbb{R}^n$ which can be decomposed into the union of finitely many cells.

This definition can be easily extended to arbitrary *bounded* subsets of $\mathbb{R}^n$: embed A into a closed cell C, extend $f\colon A\longrightarrow\mathbb{R}$ *by zero* to a function $\bar{f}\colon C\longrightarrow\mathbb{R}$,

$$\bar{f}(\boldsymbol{x}) := \begin{cases} f(\boldsymbol{x}) & \text{if } \boldsymbol{x}\in A \\ 0 & \text{otherwise} \end{cases}, \tag{44}$$

and set:

$$\int_A f(\boldsymbol{x})\,dx_1\cdots dx_n := \int_C f(\boldsymbol{x})\,dx_1\cdots dx_n. \tag{45}$$

It is clear that the expression on the righ-hand-side of (45) depends only on f and A, and not on the choice of cell C containing A.

3.3.3

The following is the immediate corollary of the criterion of integrability 3.4.

Corollary 3.5 *A function $f\colon A\longrightarrow\mathbb{R}$ is Riemann integrable if and only if it is bounded and*
$$\bar{\mu}(\operatorname{Disc}\bar{f}) = 0. \tag{46}$$

3.3.4

If $f\colon A\longrightarrow\mathbb{R}$ is continuous, then the set of discontinuity of $\bar{f}$ is contained in the union of two sets: $\operatorname{Disc} f$ and the 'boundary' $\operatorname{Bd} A := \overline{A}\cap\overline{\mathbb{R}^n\setminus A}$,
$$\operatorname{Disc}\bar{f}\subseteq \operatorname{Disc} f\cup \operatorname{Bd} A.$$

Thus we obtain the following sufficient condition for Riemann integrability of f over an arbitrary bounded subset $A\subseteq\mathbb{R}^n$.

Corollary 3.6 *If*
$$\bar{\mu}(\operatorname{Disc} f) = \bar{\mu}(\operatorname{Bd} A) = 0, \tag{47}$$
then $f\colon A\longrightarrow\mathbb{R}$ is Riemann integrable.

3.3.5

Theorem of Lebesgue is an immediate corollary of the following double inequality.

Theorem 3.7 *For any function $f\colon A\longrightarrow\mathbb{R}$ defined on a rectangular subset $A\subseteq\mathbb{R}^n$, and any $\delta>0$, one has the following double inequality*
$$\delta\bar{m}(D_\delta)\le \overline{\int_A} f(\boldsymbol{x})\,dx_1\cdots dx_n - \underline{\int_A} f(\boldsymbol{x})\,dx_1\cdots dx_n \le \operatorname{diam} f(A)\bar{m}(D_\delta)+\delta\bar{m}(A). \tag{48}$$

Proof. For any partition $\mathscr{P}$ of A, let
$$\mathscr{P}' := \{\mathbb{I}\in\mathscr{P} \mid \mathring{\mathbb{I}}\cap D_\delta \neq \emptyset\}.$$

For any cell $\mathbb{I}$ and any point $p \in \mathring{\mathbb{I}}$, one has

$$\operatorname{osc}_{\mathbb{I}}(f) \geq \operatorname{osc}_p(f).$$

It follows that

$$\overline{S}(f, \mathscr{P}) - \underline{S}(f, \mathscr{P}) \geq \sum_{\mathbb{I} \in \mathscr{P}'} \operatorname{osc}_{\mathbb{I}}(f) \|\mathbb{I}\| \geq \delta \|\mathscr{P}'\| \geq \delta \bar{m}(D_\delta)$$

where

$$D'_\delta = D_\delta \setminus \bigcup_{\mathbb{I} \in \mathscr{P}'} \partial \mathbb{I}$$

since $\mathscr{P}'$ is a cell cover of D'_δ. Note that

$$D_d \setminus D'_\delta \subseteq \bigcup_{\mathbb{I} \in \mathscr{P}'} \partial \mathbb{I}$$

and each $\partial \mathbb{I}$ consista of finitely many degenerate cells, hence

$$0 \leq \bar{m}(D_d \setminus D'_\delta) \leq \bar{m} \left(\bigcup_{\mathbb{I} \in \mathscr{P}'} \partial \mathbb{I} \right) = 0$$

and thus

$$\bar{m}(D'_\delta) = \bar{m}(D_\delta)$$

This yields

$$\overline{S}(f, \mathscr{P}) - \underline{S}(f, \mathscr{P}) \geq \delta \bar{m}(D_\delta). \tag{49}$$

Since the right-hand side does not depend on $\mathscr{P}$, we have

$$\begin{aligned} \overline{\int_A} f(\boldsymbol{x})\, dx_1 \cdots dx_n - \underline{\int_A} f(\boldsymbol{x})\, dx_1 \cdots dx_n \\ = \inf_{\mathscr{P} \in \operatorname{Part} A} \left(\overline{S}(f, \mathscr{P}) - \underline{S}(f, \mathscr{P}) \right) \geq \delta \bar{m}(D_\delta). \end{aligned}$$

The lower estimate in inequality (48) has been proven.

In order to prove the upper estimate we choose, for a given $\epsilon > 0$, a nonoverlapping finite closed cell family $\mathscr{P}'$ such that

$$A' := \bigcup \mathscr{P}' \subseteq A \qquad \text{and} \qquad D_\delta \subseteq \mathring{A}', \tag{50}$$

and

$$\bar{m}(A') - \bar{m}(D_\delta) = \|\mathscr{P}'\| - \bar{m}(D_\delta) \leq \epsilon$$

Such a family can be obtained by, first, selecting a thickening of any finite closed cover $\mathscr{I}$ of D_δ satisfying

$$\|\mathscr{I}\| - \bar{m}(D_\delta) < \epsilon \tag{51}$$

such that the thickening still satisfies estimate (51). Secondly, by removing overlaps as mentioned in 2.1.9.

If D_δ is empty, then set $A' = \emptyset$ and, accordingly, $\mathscr{P}' = \emptyset$.

Let A'' be the closure of $A \setminus A'$. It is a closed bounded subset of $\mathbb{R}^n$, hence compact.

At any point of A'' the oscillation of f is less than δ. Let us choose, for each $p \in A''$, a closed bounded cell $\mathbb{I}_p$ such that $p \in \mathring{\mathbb{I}}_p$ and

$$\operatorname{osc}_{\mathbb{I}_p}(f) < \delta.$$

The collection $\{\mathring{\mathbb{I}}_p\}_{p \in A''}$ forms an open cover of A''. In view of compactness of the latter, there exist finitely many closed cells covering A'' such that the oscillation of f on each is less than δ. Denote the resulting finite closed cover by $\mathscr{I}''$.

Subset A'' is rectangular, cf. 3.1.2, so $\mathscr{I} \cap A''$ is rectangular for every $\mathbb{I} \in \mathscr{I}''$. By subdividing each $\mathscr{I} \cap A''$ into subcells, we can produce a partition $\mathscr{P}''$ of A'' such that

$$\operatorname{osc}_{\mathbb{I}}(f) < \delta \qquad \text{for any } \mathbb{I} \in \mathscr{P}''. \tag{52}$$

Set $\mathscr{P} = \mathscr{P}' \cup \mathscr{P}''$ (if $A' = A$, then set $\mathscr{P} = \mathscr{P}'$). We split the sum defing $\overline{S}(f, \mathscr{P}) - \underline{S}(f, \mathscr{P})$ into two parts:

$$\overline{S}(f, \mathscr{P}) - \underline{S}(f, \mathscr{P}) = \sum_{\mathbb{I} \in \mathscr{P}'} \operatorname{osc}_{\mathbb{I}}(f) \|\mathbb{I}\| + \sum_{\mathbb{I} \in \mathscr{P}''} \operatorname{osc}_{\mathbb{I}}(f) \|\mathbb{I}\|.$$

Now,

$$\begin{aligned} \sum_{\mathbb{I} \in \mathscr{P}'} \operatorname{osc}_{\mathbb{I}}(f) \|\mathbb{I}\| &\leq \sum_{\mathbb{I} \in \mathscr{P}'} \operatorname{osc}_{A}(f) \|\mathbb{I}\| = \operatorname{diam} f(A) \|\mathscr{P}'\| && (53) \\ &\leq \operatorname{diam} f(A)(\bar{m}(D_\delta) + \epsilon) && (54) \end{aligned}$$

and

$$\begin{aligned}\sum_{\mathbb{I}\in\mathscr{P}''} \operatorname{osc}_{\mathbb{I}}(f)\|\mathbb{I}\| &\leq \sum_{\mathbb{I}\in\mathscr{P}''} \delta\|\mathbb{I}\| = \delta\|\mathscr{P}''\| &(55)\\ &= \delta\bar{m}(A'') \leq \delta\bar{m}(A). &(56)\end{aligned}$$

It follows that

$$\begin{aligned}\bar{\int_A} f(\boldsymbol{x})\,dx_1\cdots dx_n - \underline{\int_A} f(\boldsymbol{x})\,dx_1\cdots dx_n &\leq \overline{S}(f,\mathscr{P}) - \underline{S}(f,\mathscr{P})\\ &\leq \operatorname{diam} f(A)(\bar{m}(D_\delta)+\epsilon) + \delta\bar{m}(A).\end{aligned} \tag{57}$$

Since the inequality in (57) holds for any $\epsilon > 0$, we derive the upper estimate in Inequality (48).

This completes the proof of Theorem 3.7 and therefore also of the characterization of Riemann integrable functions due to Henri Lebesgue, cf. Theorem 3.4.

Corollary 3.8 *If, for a function* $f\colon A\longrightarrow[0,\infty)$*, one has*

$$\bar{\int_A} f(\boldsymbol{x})\,dx_1\cdots dx_n = 0, \tag{58}$$

then its support, cf. (8)*, has measure zero,*

$$\bar{\mu}(\operatorname{supp} f) = 0.$$

Proof. Since, obviously,

$$0 \leq \underline{\int_A} f(\boldsymbol{x})\,dx_1\cdots dx_n,$$

we infer from (58) that f is Riemann integrable on A and its integral equals zero. If $f(p) > 0$ at some point p where f is continuous, then there would exist a cell-neighborhood $\mathbb{I}$ of p and $\delta > 0$ such that $f(\boldsymbol{x}) \geq \delta$ for any $\boldsymbol{x}\in\mathbb{I}$. In particular,

$$\bar{\int_A} f(\boldsymbol{x})\,dx_1\cdots dx_n \geq \delta\|\mathbb{I}\| > 0.$$

It follows that the support of f is contained in the set of discontinuity, $\operatorname{Disc} f$, which has measure zero in view of Theorem 3.4. □

3.3.6 Fubini's Theorem

3.3.7

Let A' be a bounded subset of $\mathbb{R}^m$ and A'' be a bounded subset of $\mathbb{R}^n$. We can consider a function $f\colon A' \times A'' \longrightarrow \mathbb{R}$ also as a function of two variables: $\boldsymbol{x}' \in A'$ and $\boldsymbol{x}'' \in A''$. In particular, in addition to the integral

$$\int_{A'\times A''} f(\boldsymbol{x})\, dx_1 \cdots dx_{m+n}$$

we can also consider the *iterated* integrals

$$\int_{A'} \left(\int_{A''} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx'_1 \cdots dx'_m \right) dx''_1 \cdots dx''_n$$

and

$$\int_{A''} \left(\int_{A'} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx''_1 \cdots dx''_n \right) dx'_1 \cdots dx'_m.$$

Proposition 3.9 *For any function $f\colon A' \times A'' \longrightarrow \mathbb{R}$ one has the following multiple inequality*

$$\begin{aligned}
\underline{\int}_{A'\times A''} f(\boldsymbol{x})\, dx_1 \cdots dx_{m+n} &\le \underline{\int}_{A''} \left(\underline{\int}_{A'} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx'_1 \cdots dx'_m \right) dx''_1 \cdots dx''_n \\
&\le \left\{ \begin{array}{c} \underline{\int}_{A''} \left(\overline{\int}_{A'} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx'_1 \cdots dx'_m \right) dx''_1 \cdots dx''_n \\ \overline{\int}_{A''} \left(\underline{\int}_{A'} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx'_1 \cdots dx'_m \right) dx''_1 \cdots dx''_n \end{array} \right\} \\
&\le \overline{\int}_{A''} \left(\overline{\int}_{A'} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx'_1 \cdots dx'_m \right) dx''_1 \cdots dx''_n \\
&\le \overline{\int}_{A'\times A''} f(\boldsymbol{x})\, dx_1 \cdots dx_{m+n}
\end{aligned} \tag{59}$$

Proof. For any partition $\mathscr{P}$ of $A' \times A''$, there exist partitions $\mathscr{P}'$ of A' and $\mathscr{P}''$ of A'' such that

$$\mathscr{P}' \times \mathscr{P}'' := \{\mathbb{I}' \times \mathbb{I}'' \mid \mathbb{I}' \in \mathscr{P}' \text{ and } \mathbb{I}'' \in \mathscr{P}''\}$$

is finer than $\mathscr{P}$. Thus,

Exercise 15 *Show that*

$$\inf_{\boldsymbol{x}',\boldsymbol{x}'')\in A'\times A''} f(\boldsymbol{x}',\boldsymbol{x}'') = \inf_{\boldsymbol{x}''\in \mathbb{I}''} \left(\inf_{\boldsymbol{x}'\in \mathbb{I}'} f(\boldsymbol{x}',\boldsymbol{x}'') \right). \tag{60}$$

In view of (60), one has

$$\begin{aligned} \underline{S}(f, \mathscr{P}'\times\mathscr{P}'') &= \sum_{\mathbb{I}''\in\mathscr{P}''} \inf_{\boldsymbol{x}''\in\mathbb{I}''} \left(\underline{S}(f(\,.\,,\boldsymbol{x}''),\mathscr{P}'')\right) \|\mathbb{I}''\| \\ &\le \sum_{\mathbb{I}''\in\mathscr{P}''} \inf_{\boldsymbol{x}''\in\mathbb{I}''} \left(\underline{\int}_{A'} f(\boldsymbol{x}',\boldsymbol{x}'')\, dx_1'\cdots dx_m' \right) \|\mathbb{I}''\| \qquad (61) \\ &\le \underline{\int}_{A''} \left(\underline{\int}_{A'} f(\boldsymbol{x}',\boldsymbol{x}'')\, dx_1'\cdots dx_m' \right) dx_1''\cdots dx_n'' \end{aligned}$$

where $f(\,.\,,\boldsymbol{x}'')$ is, for every $\boldsymbol{x}''\in A''$, the function $A'\longrightarrow\mathbb{R}$ which sends $\boldsymbol{x}'$ to $f(\boldsymbol{x}',\boldsymbol{x}'')$.

In view of the remark opening the proof,

$$\sup_{(\mathscr{P}',\mathscr{P}'')\in \operatorname{Part} A'\times \operatorname{Part} A''} \underline{S}(f,\mathscr{P}'\times\mathscr{P}'') = \underline{\int}_{A'\times A''} f(\boldsymbol{x})\, dx_1\cdots dx_{m+n}. \tag{62}$$

By combining inequality (60) with equality (62), we obtain the first inequality in (59). The last inequality in (59) is obtained similarly, by replacing lower sums and integrals with upper sums and integrals, and by exchanging infima with suprema. The middle two inequalities in (59) are obvious. $\square$

As a corollary of inequality (59), we obtain so called Fubini's Theorem (for continuous functions proven by du Bois-Reymond already in 1872, 30 years before Fubini's published work).

Theorem 3.10 (Fubini's Theorem) *For any Riemann integrable function* $f\colon A'\times A''\longrightarrow\mathbb{R}^n$*, the functions*

$$A''\ni\boldsymbol{x}''\longmapsto \underline{\int}_{A'} f(\boldsymbol{x}',\boldsymbol{x}'')\, dx_1'\cdots dx_m'$$

and

$$A''\ni\boldsymbol{x}''\longmapsto \overline{\int}_{A'} f(\boldsymbol{x}',\boldsymbol{x}'')\, dx_1'\cdots dx_m'$$

are Riemann integrable. Similarly,

$$A' \ni \boldsymbol{x}' \longmapsto \underline{\int}_{A''} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx_1'' \cdots dx_n''$$

and

$$A' \ni \boldsymbol{x}' \longmapsto \overline{\int}_{A''} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx_1'' \cdots dx_n''$$

are Riemann integrable, and

$$\begin{aligned}
\int_{A' \times A''} f(\boldsymbol{x})\, dx_1 \cdots dx_{m+n} &= \int_{A''} \left(\underline{\int}_{A'} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx_1' \cdots dx_m' \right) dx_1'' \cdots dx_n'' \\
&= \int_{A''} \left(\overline{\int}_{A'} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx_1' \cdots dx_m' \right) dx_1'' \cdots dx_n'' \\
&= \int_{A'} \left(\underline{\int}_{A''} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx_1'' \cdots dx_n'' \right) dx_1' \cdots dx_m' \\
&= \int_{A'} \left(\overline{\int}_{A''} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx_1'' \cdots dx_n'' \right) dx_1' \cdots dx_m'
\end{aligned} \tag{63}$$

□

3.3.8

From (63) it follows that

$$\int_{A''} \left(\overline{\int}_{A'} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx_1' \cdots dx_m' - \underline{\int}_{A'} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx_1' \cdots dx_m' \right) dx_1'' \cdots dx_n'' = 0$$

and, similarly,

$$\int_{A'} \left(\overline{\int}_{A''} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx_1'' \cdots dx_n'' - \underline{\int}_{A''} f(\boldsymbol{x}', \boldsymbol{x}'')\, dx_1'' \cdots dx_n'' \right) dx_1' \cdots dx_m' = 0$$

Thus, in view of Corollary 3.8, we deduce from Fubini's Theorem the following corollary.

Corollary 3.11 *For any Riemann integrable function $f\colon A' \times A'' \longrightarrow \mathbb{R}^n$, the set of points $\boldsymbol{x}'' \in A''$ where the function*

$$A' \ni \boldsymbol{x}' \longmapsto f(\boldsymbol{x}', \boldsymbol{x}'')$$

is not integrable has measure zero.

Similarly, the set of points $\boldsymbol{x}' \in A'$ *where the function*

$$A'' \ni \boldsymbol{x}'' \longmapsto f(\boldsymbol{x}', \boldsymbol{x}'')$$

is not integrable has measure zero. □

3.3.9 Example 1

Let

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{x+y}} & \text{if } x, y > 0 \\ 0 & \text{if } x = 0 \text{ or } y = 0 \end{cases} \tag{64}$$

be a function on the unit square $A = [0,1] \times [0,1]$ in $\mathbb{R}^2$. This function is not bounded, hence it is not integrable. However, f is integrable on every horizontal interval $[0,1] \times \{y_0\}$, as well as on every vertical interval $\{x_0\} \times [0,1]$, and both iterated integrals exist and coincide

$$\int_0^1 \int_0^1 f(x,y)\, dx\, dy = \frac{8\sqrt{2}}{3} = \int_0^1 \int_0^1 f(x,y)\, dy\, dx.$$

3.3.10 Example 2

Let

$$g(x,y) = \begin{cases} \frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ -\frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{65}$$

be another function on the unit square. It is not bounded, hence not integrable. Both iterated integrals exist yet in this case their values differ since

$$\int_0^1 g(x,y)\, dx = \int_0^y \frac{dx}{y^2} - \int_y^1 \frac{dx}{x^2} = 1 \qquad (0 < y < 1)$$

while

$$\int_0^1 g(x,y)\, dy = -\int_0^x \frac{dy}{x^2} + \int_x^1 \frac{dy}{y^2} = -1. \qquad (0 < x < 1),$$

hence

$$\int_0^1 \int_0^1 g(x,y)\, dx\, dy = 1 = -\int_0^1 \int_0^1 g(x,y)\, dy\, dx.$$