Notes on Sets, Mappings, and Cardinality

An annex to H104, H113, etc.

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1 Vocabulary

1.1 Families of sets

1.1.1

In use in Mathematics there are two types of families of sets, which are always assumed to be subsets of some common set U: *indexed* and *non-indexed* ones. The latter are just subsets $\mathscr{E} \subseteq \mathscr{P}(U)$. The former are mappings

$$I \longrightarrow \mathscr{P}(U), \qquad i \longmapsto E_i, \tag{1}$$

from a certain set I, called the *indexing set*. Members of I are referred to as *indices*. Indexed families are often denoted $\{E_i\}_{i \in I}$, or even $\{E_i\}$, if the indexing set is clear from the context.

1.1.2

Non-indexed families, $\mathscr{E}$, can be thought of as indexed ones if one uses set $\mathscr{E}$ to index members of $\mathscr{E}$. In that case, mapping (1) is just the inclusion map

$$\mathscr{E} \hookrightarrow \mathscr{P}(U).$$

1.1.3 Direct product

For any indexed family $\{E_i\}_{i\in I}$ of subsets of a set U, the set

$$\prod_{i\in I} E_i := \{\tau\colon I \longrightarrow U \mid \tau(i) \in E_i\} \tag{2}$$

is called the **(direct) product** of the family. subsets $\{E_i\}_{i\in I}$.

1.1.4

The Cartesian product of sets $E_1, \dots, E_n$,

$$E_1 \times \cdots \times E_n,$$

is naturally identified with the direct product:

$$\prod_{i\in\{1,\dots,n\}} E_i.$$

1.1.5 The disjoint union

For sets X_1 and X_1 their **disjoint union**, $X_1 \sqcup X_1$, is constructed as

$$E_1 \sqcup E_2 := \{(u,i) \in (E_1 \cup E_2) \times \{1,2\} \mid u \in X_i\}. \tag{3}$$

This generalizes to any family $\{E_i\}_{i\in I}$ of subsets of a set U:

$$\bigsqcup_{i\in I} E_i := \{(u,i) \in U \times I \mid u \in E_i\}. \tag{4}$$

1.2 Terminology applicable to mappings

1.2.1

Let X and Y be sets.

Definition 1.1 *We say that a mapping $f\colon X \longrightarrow Y$ is **injective** if*

$$\textit{for any } x, x' \in X, \textit{ if } f(x) = f(x'), \textit{ then } x = x'.$$

1.2.2 Example: the canonical inclusion mapping

Any subset $E \subseteq X$ provides the injective mapping

$$\iota_E \colon E \longrightarrow X, \qquad e \longmapsto \iota(e) = e, \tag{5}$$

which is called the *canonical inclusion mapping*.

Definition 1.2 *We say that a mapping $f \colon X \longrightarrow Y$ is **surjective** if*

$$\textit{for any } y \in Y, \textit{ there exists } x \in X \textit{ such that } f(x) = y.$$

1.2.3 Example: the canonical quotient mapping

Any equivalence relation $\sim$ on a set X provides the surjective mapping of X onto the set of equivalence classes $X_{/\sim}$,

$$\pi_\sim \colon X \longrightarrow X_{/\sim}, \qquad x \longmapsto \pi_\sim(x) = [x], \tag{6}$$

which sends an element x of X to the equivalence class

$$[x] := \{\xi \in X \mid \xi \sim x\}.$$

Mapping (6) is called the *canonical quotient mapping*.

1.2.4 Example: the evaluation map

For any sets X and Y, mappings $f \colon X \longrightarrow Y$ form a set, denoted $\mathrm{Map}(X,Y)$, which is a subset of the set of all binary relations $\mathscr{P}(X \times Y)$.

Let $p \in X$ be a fixed element of X. Evaluation at p defines the mapping

$$\mathrm{ev}_p \colon \mathrm{Map}(X,Y) \longrightarrow Y, \qquad f \longmapsto \mathrm{ev}_p(f) := f(x). \tag{7}$$

We will refer to (7) as the *evaluation map (at point p)*.

Exercise 1 *Show that the evaluation map ev_p is surjective.*

Proposition 1.3 (Cantor) *For any set X, no mapping $f \colon X \longrightarrow \mathscr{P}(X)$ is surjective.*

More precisely, the set

$$E_f := \{x \in X \mid x \notin f(x)\}$$

is not in the image of f. Indeed, if $E_f = f(e)$ for some $e \in X$, then $e \notin E_f$ which means that $e \in f(e) = E_f$. Contradiction.

1.2.5

The above argument is a reflection of the *Liar's Paradox*: 'Is the man who is saying "I am lying" speaking truth or not?'

1.2.6

Definition 1.4 *We say that a mapping* $f\colon X \longrightarrow Y$ *is* ***bijective*** *if it is both injective and surjective.*

1.2.7

Surjective mappings are often called **surjections** while *bijective mappings* are called **bijections**.

Exercise 2 *Show that if the composition* $f \circ g$ *of mappings* $f\colon X \longrightarrow Y$ *and* $g\colon W \longrightarrow X$ *is injective, then* g *is injective.*

Exercise 3 *Show that if the composition* $f \circ g$ *of mappings* $f\colon X \longrightarrow Y$ *and* $g\colon W \longrightarrow X$ *is surjective, then* f *is surjective.*

Exercise 4 *Show that if the composition* $f \circ g$ *of mappings* $f\colon X \longrightarrow Y$ *and* $g\colon W \longrightarrow X$ *is bijective, then* f *is surjective and* g *is injective.*

Exercise 5 *Show that the canonical mapping*

$$\bigsqcup_{i \in I} E_i \longrightarrow \bigcup_{i \in I} E_i \qquad (u, i) \longmapsto u, \tag{8}$$

is surjective.

Exercise 6 *State a sufficient and necessary condition for* (8) *to be injective, and then prove it.*

Exercise 7 *State a sufficient and necessary condition for the canonical mapping,*

$$\bigsqcup_{i \in I} E_i \longrightarrow I \qquad (u, i) \longmapsto i, \tag{9}$$

to be surjective, and then prove it.

1.2.8 Canonical representation of a mapping

Any mapping $f\colon X \longrightarrow Y$ can be represented in a canonical manner as a composite of a surjection folowed by a bijection followed by an injective mapping. This is represented by the following commuting diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "\pi"'] & Y \\ X_{/\sim} \arrow[r, dashed, "\bar{f}"] & f(X) \arrow[u, hook, "\iota"'] \end{tikzcd} . \tag{10}$$

Here $\sim$ is the equivalence relation:

$$x \sim x' \quad \text{if} \quad f(x) = f(x') \qquad (x, x' \in X),$$

π denotes the canonical quotient map, and ι denotes the canonical inclusion of $f(X)$, the image of f, into Y.

The mapping $\bar{f}\colon X_{/\sim} \longrightarrow f(X)$ is defined by

$$\bar{f}([x]) := f(x). \tag{11}$$

This definition makes sense since the value, $f(x)$, depends only on the equivalence class $[x]$, not on x.

Exercise 8 *Show that $\bar{f}$ is a bijection.*

1.2.9

We say that a mapping $g\colon Y \longrightarrow X$ is a **right inverse** of $f\colon X \longrightarrow Y$ if

$$f \circ g = \mathrm{id}_Y .$$

Similarly, we say that g is a **left inverse** of f if

$$g \circ f = \mathrm{id}_X .$$

Exercise 9 *Show that if f possesses a right inverse, say g, and a left inverse, say h, then they are equal: $g = h$. In that case we call it* the *inverse of f, and denote it f^{-1}.*

Exercise 10 *Show that f possesses a left inverse if and only if f is injective.*

Exercise 11 *Show that if f possesses a right inverse, then f is surjective.*

1.2.10 The Axiom of Choice

The reverse implication,

$$\text{if } f \text{ is surjective, then } f \text{ possesses a right inverse,} \tag{12}$$

cannot be proven on the basis of ordinary axioms of Set Theory and is one of the equivalent formulations of the additional axiom called the **Axiom of Choice**:

> Any family $\mathscr{E}$ of nonempty subsets of a set U admits a mapping
> $$\sigma\colon \mathscr{E} \longrightarrow U \tag{13}$$
> such that, for any $E \in \mathscr{E}$, one has $\sigma(E) \in E$.

Mapping σ in (13) is sometimes called a *choice function* for family $\mathscr{E}$.

Exercise 12 *Show that the Axiom of Choice,* (13)*, implies the apparently stronger statement to the effect that the direct product,* (2)*, of any family of nonempty sets* $\{E_i\}_{i\in I}$ *is nonempty.*

Hint. Consider the family of disjoint subsets $\mathscr{E} = \{E_i \times \{i\}\}$ of $U \times I$ and use the existence of the mappings

$$\rho\colon I \longrightarrow \mathscr{E}, \qquad i \longmapsto \rho(i) := E_i \times \{i\},$$

and (8).

Exercise 13 *Show that the Axiom of Choice,* (13)*, implies that, for any set* X *and any equivalence relation* $\sim$ *on* X*, there exists a subset* $T \subseteq X$ *which has exactly one element in common with* every *equivalence class.*

1.2.11 The fiber of a mapping

For any mapping $f\colon X \longrightarrow Y$ and an element $y \in Y$, the subset

$$f^{-1}(y) := f^{-1}(\{y\}) = \{x \in X \mid f(x) = y\} \tag{14}$$

is called the **fiber of** f **at** (or, **over**) $y \in Y$.

1.2.12

We can reformulate Definitions 1.1–1.2 as follows:

$$f \text{ is } \textit{injective} \text{ if and only if every fiber } f^{-1}(y) \\ \text{has no more than one element,} \tag{15}$$

and

$$f \text{ is } \textit{surjective} \text{ if and only if every fiber } f^{-1}(y) \\ \text{has at least one element.} \tag{16}$$

Exercise 14 *Show that the Axiom of Choice,* (13)*, implies that any surjection admits a right inverse, cf.* (12)*.*

Hint. Consider the family of fibers $\mathscr{E} = \{f^{-1}(y) \mid y \in Y\}$.

2 Cardinality

2.1 'Same cardinality'

2.1.1

Definition 2.1 *We say that sets X and Y* ***have the same cardinality*** *if there exists a bijection $f\colon X \longrightarrow Y$. We express this symbolically by writing*

$$\mid X \mid = \mid Y \mid .$$

Note that in Definition 2.2 we *do not* define the *cardinality*, $\mid X \mid$, of a set X.

2.2 'Not greater cardinality'

2.2.1

Definition 2.2 *Similarly, we could say that a set X* ***has not greater cardinality*** *than a set Y if there exists an injective mapping $f\colon X \longrightarrow Y$. We could express this symbolically by writing*

$$\mid X \mid \leq \mid Y \mid .$$

2.2.2

If X is a subset of Y, then $\mid X \mid \leq \mid Y \mid$: the canonical inclusion ι_X is injective.

2.2.3

Since any bijection is injective, $\mid X \mid = \mid Y \mid$ implies $\mid X \mid \leq \mid Y \mid$.

Moreover, since the composite of two injective mappings is injective we infer that

$$\text{if } \mid X \mid \leq Y \mid \text{ and } \mid Y \mid \leq Z, \text{ then } \mid X \mid \leq Z \mid.$$

Theorem 2.3 (Bernstein–Schröder Theorem) *If $\mid X \mid \leq \mid Y \mid$ and $\mid Y \mid \leq X$, then $\mid X = \mid Y \mid$.*

2.2.4

It is enough to prove the theorem in the case when X is a subset of Y.

Indeed, if $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ are injective, then $f \circ g\colon Y \longrightarrow Y$ is an injective mapping whose image is obviously contained in $f(X)$. Hence, if there exists a bijection $\chi\colon f(X) \longrightarrow Y$, then $\chi \circ f$ is the desired bijection betwen X and Y.

2.2.5 Invariant subsets

Let $\phi\colon Y \longrightarrow Y$ be any self-mapping of a set Y. A subset $W \subseteq Y$ is said to be *invariant under* ϕ, or ϕ-invariant, if $\phi(W) \subseteq W$.

Exercise 15 *Suppose that both W and its complement, $Y \setminus W$, are ϕ-invariant. Show that if $\phi(y) = \phi(y')$, for some $y, y' \in Y$, then either y and y' belong both to W or they belong both to $Y \setminus W$.*

In particular, if ϕ is injective both on W and on $Y \setminus W$, then it is injective.

Exercise 16 *Show that, for any subset $Z \subseteq Y$, the set*

$$\bar{Z} := \bigcup_{n=0}^{\infty} \phi^n(Z) = Z \cup \phi(Z) \cup \phi^2(Z) \cup \cdots, \tag{17}$$

where $\phi^0 = \mathrm{id}_Y$ *and, for* $n > 0$,

$$\phi^n = \phi \circ \cdots \circ \phi \qquad (n \text{ times}),$$

is ϕ*-invariant. Show that if a* ϕ*-invariant subset* $W \subseteq Y$ *contains* Z*, then it also contains* $\bar{Z}$*.*

2.2.6 Invariant closure of a subset

In other words, $\bar{Z}$ is the *smallest* ϕ-invariant subset of Y which contains Z. We shall refer to it as the *ϕ-invariant closure* of Z.

2.2.7 The proof of Bernstein–Schröder Theorem

Let $\phi\colon Y \longrightarrow Y$ be an injective self-mapping whose image is contained in a subset X. Let us set $Z := Y \setminus X$ and define the maping

$$h\colon Y \longrightarrow X, \qquad h(y) := \begin{cases} \phi(y) & \text{if } y \in \bar{Z} \\ y & \text{otherwise.} \end{cases} \tag{18}$$

The following exercises complete the proof.

Exercise 17 *Show that both* $\bar{Z}$ *and* $Y \setminus \bar{Z}$ *are* h*-invariant.*

Exercise 18 *Show that* h *is injective.*

Exercise 19 *Show that* $\bar{Z} \cap X = \phi(\bar{Z})$*.*

Exercise 20 *Show that* h *is surjective.*

2.2.8

Exercise 21 *For any set* X*, show that no mapping* $f\colon \mathscr{P}(X) \longrightarrow X$ *is injective.*