Groups of order 36

Groups with no normal subgroups of order 9

Let G be a group of order 36 with no normal subgroup of order 9. If P is any subgroup of order 9, the action of G on the set of left cosets G/P defines a homomorphism

$$\phi\colon G \longrightarrow \mathrm{Sym}(G/P) \simeq S_4. \tag{1}$$

The order of $\phi(G)$ divides both $|\mathrm{Sym}(G/P)| = 24$ and $|G| = 36$, and is divisible by $|G/P| = 4$ since G acts transitively on G/P. This leaves us with only two possibilities: $|\phi(G)| = 4$, or 12. In the former case, $\mathrm{Ker}\,\phi$ would be a normal subgroup of order 9, hence contradicting the hypothesis. Thus, $|\phi(G)| = 12$, and $\mathrm{Ker}\,\phi$ is a normal subgroup of order 3.

The alternating group A_4 does not admit a nontrivial homomorphism into a group of order 2. Since $|\mathrm{Sym}(G/P)/\phi(G)| = 2$, the composite homomorphism

$$\mathrm{Alt}(G/P) \hookrightarrow \mathrm{Sym}(G/P) \twoheadrightarrow \mathrm{Sym}(G/)/\phi(G) \tag{2}$$

is trivial showing that $\phi(G)$ coincides with $\mathrm{Alt}(G/P) \simeq A_4$.

We have thus established that G fits into an extension

$$C_3 \rightarrowtail G \overset{\pi}{\twoheadrightarrow} A_4 \ . \tag{3}$$

The adjoint action of G on $\mathrm{Ker}\,\phi$ induces a homomorphism from $A_4 \simeq G/C_3$ to the group $\mathrm{Aut}\,C_3$ whose order equals 2. This being trivial, implies that G acts trivially on C_3. In other words, C_3 is contained in the center $Z(G)$. Since $Z(G)/C_3 \subseteq Z(G/C_3) \simeq Z(A_4) = 1$, we infer that $Z(G) = C_3$.

Group A_4 has a unique subgroup W of order 4 and the latter is of the form C_2^2. Since $\mathrm{Ker}\,\pi$ is central of order 3, for any $w \in W$, the cube $\widetilde{w}^3$ of an *arbitrary* lift $\widetilde{w} \in \pi^{-1}(w)$ depends only on w, and defines a well defined map

$$s\colon V \longrightarrow G \tag{4}$$

which splits π. In particular, $s(w)$ is the *only* element $v \in G$ such that $\pi(v) = w$ and $v^2 = \left(\widetilde{w}^2\right)^3 = 1$. This immediately implies that map (4) is a homomorphism embeding W into G, and that $V := s(W)$ is the unique subgroup of order 4 in G.

In particular, V is normal and the abelian group $C_3V \simeq C_3 \times V$ is a normal subgroup of G of order 12.

CASE 1: G HAS AT LEAST THREE ELEMENTS OF ORDER 3. In this case, at least one of them, u, does not belong to center C_3. The subgroup $H := V\langle u\rangle \subset G$, which has exactly 12 elements, is generated by the set $X := V \cup \{u\}$, and its image $\pi(X)$ generates group A_4 that has the same number of elements as H. It follows, that π restricted to H is an isomorphism of H with A_4. Since $|G| = |C_3H|$, and C_3 is central, we conclude that

$$G = C_3H \simeq C_3 \times A_4. \tag{5}$$

CASE 2: G HAS EXACTLY TWO ELEMENTS OF ORDER 3. In this case, any lift to G of any element of order 3 in A_4 has order 9. In particular, by considering the cyclic subgroup of G generated by any element t of order 9, we obtain a representation of G as a *nontrivial*[1] semidirect product of a cyclic group of order 9 and the Klein group:

$$G = C_9V = C_9 \ltimes V \simeq C_9 \ltimes C_2^2. \tag{6}$$

Since there are only two nontrivial homomorphisms

$$\rho_i \colon C_9 \longrightarrow \operatorname{Aut} C_2^2 \simeq S_3, \qquad (i = 1, 2), \tag{7}$$

and $\rho_2(t) = \rho_1(t^{-1})$, there exists only one *nonabelian* semidirect product (6) up to isomorphism. This is thus a *unique nontrivial* central extension of A_4 by C_3.

In particular, we have proved that **there are only two groups of order** 36, up to isomorphism, **with no normal subgroup of order** 9.

Note that in CASE 1, there are exactly 3 elements of order 2 in G, $3 \cdot 8 + 2 = 26$ elements of order 3, $2 \cdot 3 = 6$ elements of order 6, and no elements of order 9.

In CASE 2, there are exactly 2 elements of order 3 in G and $4 \cdot 6 = 24$ elements of order 9. The number of elements of order 2 and 6 is the same as in the other case.

The set of subgroups of order 9 is freely operated by the Klein group V.

[1]Otherwise G would have a normal subgroup of order 9.

Groups with a normal subgroup of order 9

There are four cases:

Case 1: $G \simeq C_9 \rtimes C_4$. There is only one nontrivial homomorphism

$$\rho\colon C_4 \longrightarrow \operatorname{Aut} C_9, \qquad m \longmapsto \rho_m, \tag{8}$$

where ρ_m sends a generator $g \in C_9$ to $g^{(-1)^m}$. Thus, in this case G is either abelian and then cyclic, or nonabelian, and then isomorphic to the unique nontrivial central extension

$$C_2 \rightarrowtail G \xrightarrow{\pi} D_9 \tag{9}$$

of dihedral group D_9 by C_2.

Case 2: $G \simeq C_9 \rtimes C_2^2$. There is only one, up to an automorphism of C_2^2, nontrivial homomorphism

$$\rho\colon C_2^2 \longrightarrow \operatorname{Aut} C_9, \qquad (m,n) \longmapsto \rho_{(m,n)}, \tag{10}$$

where $\rho_{(1,0)}$ sends a generator $g \in C_9$ to g^{-1} while $\rho_{(0,1)} = \mathrm{id}_{C_9}$. Thus, in this case G is either abelian and then isomorphic to $C_{18} \times C_2$, or nonabelian, and then isomorphic to $D_{18} \simeq D_9 \times C_2$.

Case 3: $G \simeq C_3^3 \rtimes C_4$.

Case 4: $G \simeq C_3^3 \rtimes C_2^2$. In this and the previous case G is the semidirect product of a Sylow 2-subgroup Q with any two-dimensional representation of Q over the three-element field $\mathbb{F}_3$.