Calculus 214

Lecture notes

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Contents

0 Vocabulary

0.1 Γ-diagrams and categories

Recall that an **oriented graph**

$$\Gamma : \quad \Gamma_0 \underset{t}{\overset{s}{\leftleftarrows}} \Gamma_1$$

consists of a pair of sets: Γ_0, whose elements are called *vertices*, and Γ_1, whose elements are called *edges*, and of a pair of maps s and t; for an edge $e \in \Gamma_1$, $s(e)$ is called the *source* of e, while $t(e)$ is called the *target* of e. Morphisms between graphs are defined naturally as pairs of maps

$$(\Gamma_0 \xrightarrow{f_0} \Gamma_0', \ \Gamma_1 \xrightarrow{f_1} \Gamma_1')$$

which are compatible with the corresponding source and target maps.

By reversing the direction of all edges we obtain the **opposite** graph Γ^{op}. One has $\Gamma_i^{op} = \Gamma_i$, $i = 0, 1$, but $s^{op} = t$ and $t^{op} = s$.

Definition 0.1 *A **category** structure on a graph Γ, consists of an associative law of composition for arrows*

$$\mu : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \longrightarrow \Gamma_1$$

*where the **fibered product** of two copies of Γ_1 over Γ_0 is defined as follows:*

$$\Gamma_1 \times_{\Gamma_0} \Gamma_1 := \{(e_1, e_2) \mid s(e_1) = t(e_2)\}.$$

*The elements of Γ_0 are then referred to as **objects** and the elements of Γ_1 as **morphisms**, or **arrows**, of the category.*

The original definition of a category requires, in addition, the existence of "the identity" morphisms, i.e., that multiplication μ has a neutral element $1 : \Gamma_0 \longrightarrow \Gamma_1$, $\nu \longmapsto 1_\nu$.

Formally speaking, we defined above the so called *small* categories. General categories are defined similarly, but Γ_i, $i = 0, 1$, are allowed to be *classes* instead of being sets. Needless to say, caution is required when operations on classes are involved.

The *underlying graph* $|\mathscr{C}|$ of a given category $\mathscr{C}$ is obtained by forgetting multiplication μ.

Morphisms between categories are called **functors**. They are defined naturally as morphisms between the underlying graphs $|\mathscr{C}| \longrightarrow |\mathscr{C}'|$ which are compatible with the corresponding multiplication laws (and take the identity morphisms to the corresponding identity morphisms).

When we think of functors as morphisms between categories, then small categories form a category themselves. It is denoted *Cat* and called the **category of all (small) categories**.

Semigroups or, more properly, monoids, are nothing but categories with a single object (in that case the whole structure reduces to the multiplication on the set of arrows).

By formally reversing the direction of all arrows in a given category $\mathscr{C}$, one obtains the **opposite category** $\mathscr{C}^{op}$. Note that $|\mathscr{C}^{op}| = |\mathscr{C}|^{op}$.

Functors $F : \mathscr{C}^{op} \longrightarrow \mathscr{D}$ are usually referred to as **contravariant** functors from $\mathscr{C}$ to $\mathscr{D}$ in order to distinguish them from actual functors $\mathscr{C} \longrightarrow \mathscr{D}$ which are then referred to as **covariant** functors.

Definition 0.2 *Let Γ be a graph and $\mathscr{C}$ be a category. A morphism $\Delta : \Gamma \longrightarrow |\mathscr{C}|$ is called a Γ-**diagram** in category $\mathscr{C}$.*

Exercise. For a given graph Γ, Γ-diagrams in a category $\mathscr{C}$ naturally form a category, denoted $\mathit{Diag}_\Gamma(\mathscr{C})$. Give its definition.

A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ transforms Γ-diagrams in category $\mathscr{C}$ into Γ-diagrams in category $\mathscr{D}$,

$$\Delta \longmapsto F(\Delta) \qquad (\Delta \in \operatorname{Ob} \mathit{Diag}_\Gamma(\mathscr{C})),$$

whereas a *contravariant* functor $G : \mathscr{C} \longrightarrow \mathscr{D}$ transforms Γ-diagrams $\mathscr{C}$ into Γ^{op}-diagrams in $\mathscr{D}$.

Definition 0.3 *We shall say that a graph Γ is **sequential** if each vertex has at most one out-going and at most one in-going edge. The connected components of sequential graphs are the graphs*

$$\Sigma_n : \quad \bullet \leftarrow \cdots \leftarrow \bullet \quad (n \textit{ vertices}). \tag{1}$$

*Γ-diagrams in a category $\mathscr{C}$, where Γ is sequential, will be referred to as **sequences** in $\mathscr{C}$.*

0.2 Modules

Let R be a ring (always assumed to be associative, unless otherwise stated, but not necessarily unital), and $\mu : R \times R \longrightarrow R$ be the corresponding multiplication.

Definition 0.4 *The **opposite ring** R^{op} is defined as follows: as an additive abelian group it coincides with $(R, +)$, but the multiplication is new:*

$$\begin{tikzcd} R \times R \arrow[rr, "\mu^{op}"] \arrow[dr, "\tau"'] & & R \\ & R \times R \arrow[ur, "\mu"'] & \end{tikzcd}$$

where τ is the involution:

$$\tau(r_1, r_2) = \tau(r_2, r_2) \qquad (r_1, r_2 \in R)\,.$$

It is advisable to use the following convention: denote $r \in R$, when it is viewed as an element of opposite ring R^{op}, as r^{op}. Then multiplication in R^{op} is given by the formula:

$$r^{op} \cdot s^{op} = (sr)^{op}\,.$$

Note that $(R^{op})^{op} = R$.

Definition 0.5 *A ring R is **commutative** if $R^{op} = R$.*

Definition 0.6 *A* ***left*** *(respectively,* ***right****)* R*-****module*** *structure on an abelian group* M *is a ring homomorphism*

$$\lambda : R \longrightarrow \mathrm{End}_{Ab}(M), \qquad r \longmapsto \lambda_r,$$

(respectively, a ring homomorphism

$$\rho : R^{op} \longrightarrow \mathrm{End}_{Ab}(M), \qquad r^{op} \longmapsto \rho_{r^{op}}\ ).$$

Traditional notation:

$$rm := \lambda_r(m)$$

in the left R-module case, and

$$mr := \rho_{r^{op}}(m)$$

in the right R-module case.

Note that a right R-module structure on M is the same as a left R^{op}-module structure.

If $R \ni 1$ and $\lambda : 1 \longmapsto \mathrm{id}_A$, then the left module is said to be **unitary** (similarly for right modules). All modules over a unital ring are tacitly assumed to be unitary unless explicitly stated otherwise.

In the rest of this chapter R is assumed to have 1 unless otherwise stated. The category of (unitary) left R-modules is denoted R-mod while the category of (unitary) right R-modules is denoted mod-R.

Definition 0.7 *A right* R*-module* M *is said to be:*

(a) ***projective*** *if, for any diagram of right* R*-modules*

$$\begin{tikzcd} L_0 & & L_1 \arrow[ll, two heads, "\pi"'] \\ & M \arrow[ul, "f"] & \end{tikzcd}$$

there exists a morphism $\tilde{f} : M \longrightarrow L_1$ *such that the triangle*

$$\begin{tikzcd} L_0 & & L_1 \arrow[ll, two heads, "\pi"'] \\ & M \arrow[ul, "f"] \arrow[ur, dotted, "\tilde{f}"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

*commutes ($\tilde{f}$ is called a **lifting** of f). In other words, if the functor*

$$\mathrm{Hom}_{\text{mod-R}}(M, \ ) : \text{mod-R} \longrightarrow \mathrm{Ab}$$

***preserves epimorphisms**, i.e., if*

$$\mathrm{Hom}_{\text{mod-R}}(M, L_0) \xleftarrow{\pi_*} \mathrm{Hom}_{\text{mod-R}}(M, L_1)$$

is an epimorphism whenever π is one; here $\pi_*(\phi) := \pi \circ \phi$.

(b) ***injective** if, for any diagram of right R-modules*

$$\begin{tikzcd} L_0 \arrow[r, tail, "\iota"] \arrow[dr, "g"'] & L_1 \\ & M \end{tikzcd}$$

there exists a morphism $\tilde{g} : L_1 \longrightarrow M$ such that the triangle

$$\begin{tikzcd} L_0 \arrow[r, tail, "\iota"] \arrow[dr, "g"'] & L_1 \arrow[d, dotted, "\tilde{g}"] \\ & M \end{tikzcd}$$

*commutes ($\tilde{g}$ is an **extension** of g). In other words, if the functor*

$$\mathrm{Hom}_{\text{mod-R}}(\ , M) : (\text{mod-R})^{op} \longrightarrow \mathrm{Ab}$$

***preserves epimorphisms**, i.e., if*

$$\mathrm{Hom}_{\text{mod-R}}(L_0, M) \xleftarrow{\iota^*} \mathrm{Hom}_{\text{mod-R}}(L_1, M)$$

is an epimorphism whenever ι^{op} is one (this happens precisely when ι is a monomorphism in mod-R); here $\iota^*(\phi) := \phi \circ \iota$.

(c) *a **generator** (of category mod-R) if, for any right R-module L and any $\ell \in L$, there exists a morphism $f : M \longrightarrow L$ such that $\ell \in f(M)$. In other words, if any right R-module L is isomorphic to a quotient module of $\bigoplus_{x \in X} M$ for some set X (take, e.g., $X = \mathrm{Hom}_{\text{mod-R}}(M, L)$).*

(d) ***a cogenerator*** *(of category* mod-R*) if, for any right* R*-module* L *and any* $\ell \in L$*, there exists a morphism* $g : L \longrightarrow M$ *such that* $g(\ell) \neq 0$*. In other words, if any right* R*-module* L *is isomorphic to a submodule of* $\prod_{x \in X} M$ *for some set* X *(take, e.g.,* $X = \mathrm{Hom}_{\text{mod-R}}(L, M)$*).*

(e) ***free*** *if there exists a subset* $X \subseteq M$ *such that any element* $m \in M$ *can be expressed as* $m = \sum_{x \in X} x r_x$ *for a* ***unique*** *collection* $r_x \in R$ *of finite support. In this situation,* X *is called a* ***basis*** *of* M *and the correspondence* $m \longmapsto (r_x)_{x \in X}$ *defines an isomorphism of right* R*-modules* $M \simeq \bigoplus_{x \in X} R$*.*

The proof of the following lemma is a simple exercise

Lemma 0.8 *A right* R*-module* M *is:*

(a) *projective* ***if and only if*** M *is isomorphic to a direct summand of free module* $\bigoplus_{x \in X} R$ *for some set* X*;*

(b) *a generator* ***if and only if*** *rank 1 free module* R *is isomorphic to a direct summand of* $\bigoplus_{x \in X} M$ *for some set* X*.* □

0.2.1 Case $R = \mathbb{Z}$

Definition 0.9 *An abelian group is said to be* ***divisible*** *if, for any* $a \in A$ *and positive integer* n*, there exists* $a' \in A$ *such that* $a = na'$*.*

Proposition 0.10 *An abelian group is injective* ***if and only if*** *it is divisible.* □

The multiplicative group of complex roots of identity $\mu_\infty := \bigcup_{n \geqslant 1} \mu_n$, where

$$\mu_n := \{\zeta \in \mathbb{C} \mid \zeta^n = 1\},$$

is divisible; the correspondence $q \longmapsto e^{2\pi q}$, identifies the additive group $\mathbb{Q}/\mathbb{Z}$ with μ_∞.

Proposition 0.11 μ_∞ *is an injective cogenerator of the category of abelian groups.* □

0.2.2 Case of a general ring

For any abelian group A and any left (respectively, right) R-module L, abelian group $\mathrm{Hom}_{\mathrm{Ab}}(L, A)$ is equipped with a canonical structure of a right R-module:

$$(\phi r)(\ell) := \phi(r\ell) \qquad (r \in R; \ell \in L)$$

(respectively, left R-module:

$$(r\phi)(\ell) := \phi(\ell r) \qquad (r \in R; \ell \in L)\,.$$

Definition 0.12 *For any R-module L, the module $L^* := \mathrm{Hom}_{\mathrm{Ab}}(L, \mu_\infty)$ is called the* ***character module*** *of L.*

Note that R^* is both a left and a right R-module.

Proposition 0.13 *For any ring R, character module R^* is an injective cogenerator in categories R-mod and mod-R.* $\square$

The following is the counterpart of Lemma 0.8

Corollary 0.14 *A right R-module M is:*

(a) injective ***if and only if*** *M is isomorphic to a direct summand of the character module of some free R-module $\prod_{x\in X} R^* = (\bigoplus_{x\in X} R)^*$;*

(b) a cogenerator ***if and only if*** *character module R^* is isomorphic to a direct summand of $\prod_{x\in X} M$ for some set X.* $\square$

0.3 Exactness

Definition 0.15 *A connected sequence of R-modules*

$$M_0 \xleftarrow{f_1} M_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} M_n$$

is ***exact*** *if $\mathrm{Im}\, f_i = \mathrm{Ker}\, f_{i-1}$ for all $i = 1, \ldots, n$. A general sequence S of R-modules is* ***exact*** *if all of its connected components are such.*

The following definitions is particularly important.

Definition 0.16 *A functor* F *from* R-mod, *or* mod-R, *to category of abelian groups* Ab*:*

(a) ***preserves exactness*** *(such functors are called* ***exact****) if, for any exact sequence* S *of* R*-modules, sequence* $F(S)$ *is exact;*

(b) ***reflects exactness*** *if, for any sequence* S *of* R*-modules, the latter is exact if* $F(S)$ *is such.*

In the above definition it suffices to consider only Σ_3-sequences (cf. (1)):

$$M_0 \overset{f_1}{\leftarrow} M_1 \overset{f_2}{\leftarrow} M_2 \tag{2}$$

Exercise. Verify that the *covariant* Hom-functor:

$$\mathrm{Hom}_{\text{mod-R}}(M,\ ) : \text{mod-R} \longrightarrow \mathrm{Ab}, \qquad L \longmapsto \mathrm{Hom}_{\text{mod-R}}(M, L), \tag{3}$$

and the *contravariant* Hom-functor:

$$\mathrm{Hom}_{\text{mod-R}}(\ , M) : \text{mod-R} \longrightarrow \mathrm{Ab}, \qquad L \longmapsto \mathrm{Hom}_{\text{mod-R}}(L, M), \tag{4}$$

preserve the exactness of sequences

$$0 \longrightarrow L^0 \xrightarrow{f^0} L^1 \xrightarrow{f^1} L^2.$$

Such functors are called **left exact** and form a very important class of functors.

It follows directly from the respective definitions of a projective and of an injective module, see Definitions 0.7(a) and 0.7(b), that functor (3) is *not exact* if M is not projective, and that functor (4) is *not exact* if M is not injective. It turns out that the exactness of the corresponding Hom-functors characterizes projectivity and, respectively, injectivity.

Proposition 0.17 *A right* R*-module* M *is:*

(a) projective $\Leftrightarrow$ *functor* $\mathrm{Hom}_{\text{mod-R}}(M,\ )$ *preserves exactness;*

(b) injective $\Leftrightarrow$ *functor* $\mathrm{Hom}_{\text{mod-R}}(\ , M)$ *preserves exactness;*

(c) a generator $\Leftrightarrow$ *functor* $\mathrm{Hom}_{\text{mod-R}}(M,\ )$ *reflects exactness;*

(d) a cogenerator $\Leftrightarrow$ *functor* $\mathrm{Hom}_{\text{mod-R}}(\ , M)$ *reflects exactness.* □

1 Derivations

1.1 Non-graded case

1.1.1 Split k-algebra extensions

An extension of k-algebras

$$\mathscr{E}: \qquad A \overset{\pi}{\twoheadleftarrow} B \overset{\iota}{\hookleftarrow} J$$

is said to be *split* if there exists a morphism of k-algebras

$$\sigma : A \longrightarrow B$$

such that $\pi \circ \iota = \mathrm{id}_A$; σ is then called a *splitting* of $\mathscr{E}$. The set of splittings of extension $\mathscr{E}$ will be denoted $\mathit{Split}(\mathscr{E})$.

When $J^2 = 0$, the extension ideal J is a bimodule over $B/J \simeq A$:

$$ax := \sigma(a)x \quad \text{and} \quad xa := x\sigma(a) \qquad (a \in A; x \in J)$$

for any splitting σ (the result does not depend on the choice of σ).

For any $\sigma, \sigma' \in \mathit{Split}(\mathscr{E})$, their difference $\delta = \sigma' - \sigma$ has the following properties:

$$\delta(a) \in J \qquad (a \in A)\,,$$

since $\pi \circ \delta = \pi \circ \sigma' - \pi \circ \sigma = \mathrm{id}_A - \mathrm{id}_A = 0$, and

$$\begin{aligned}
\delta(a_1 a_2) &= \sigma'(a_1 a_2) - \sigma(a_1 a_2) = \sigma'(a_1)\sigma'(a_2) - \sigma(a_1)\sigma(a_2) \\
&= (\sigma'(a_1) - \sigma(a_1))\sigma'(a_2) + \sigma(a_1)(\sigma'(a_2) - \sigma(a_2)) \\
&= \delta(a_1)\sigma'(a_2) + \sigma(a_1)\delta(a_2) = \delta(a_1)a_2 + a_1\delta(a_2)
\end{aligned}$$

Definition 1.1 *A k-linear map $\delta : A \longrightarrow M$ of a k-algebra A into an A-bimodule M is called **derivation** if*

$$\delta(a_1 a_2) = \delta(a_1)a_2 + a_1\delta(a_2) \qquad (a_1, a_2 \in A)\,. \tag{5}$$

The set of k-linear derivations, denoted $\mathit{Der}_{A/k}(M)$, is a k-module.

Example For any element $m \in M$, the correspondence

$$a \longmapsto [a, m] := am - ma \qquad (a \in A; m \in M) \tag{6}$$

is a derivation. Such derivations are called **inner**. They form a k-submodule of $Der_{A/k}(M)$. The quotient

$$H^1(A;M) := Der_{A/k}(M)/Der^{inn}_{A/k}(M) \tag{7}$$

is the first **Hochschild cohomology group** of k-algebra A with coefficients in bimodule M.

Returning to our discussion of split extensions with $J^2 = 0$, we note that, for any $\sigma \in Split(\mathscr{E})$ and $\delta \in Der_{A/k}(M)$, we have

$$\begin{aligned}(\sigma+\delta)(a_1a_2) &= \sigma(a_1)\sigma(a_2) + \delta(a_1)a_2 + a_1\delta(a_2)\\ &= \sigma(a_1)\sigma(a_2) + \delta(a_1)\sigma(a_2) + \sigma(a_1)\delta(a_2)\\ &= (\sigma(a_1)+\delta(a_1))(\sigma(a_2)+\delta(a_2)),\end{aligned}$$

i.e., $\sigma+\delta$ is another splitting of $\mathscr{E}$.

Definition 1.2 *Let G be a group and X be a set on which G operates (i.e., X is a G-set). We say that X is a* ***G-torsor*** *if, for any $x, x' \in X$, there exists a* ***unique*** *$g \in G$ such that $x' = gx$.*

Thus, we have established

Lemma 1.3 *For any split extension $\mathscr{E}$ with $J^2 = 0$, the set of splittings $Split(\mathscr{E})$ is a $Der_{A/k}(M)$-torsor.* □

For any A-bimodule M, there exists a canonically split extension of A by M, called the **semidirect** product of A by M, and denoted $A \ltimes M$. As a k-module, it coincides with $A \bigoplus M$ while the multiplication is given by

$$(a,m)(a',m') = (aa', ma' + am').$$

Corollary 1.4 *A map $\delta : A \longrightarrow M$ is a derivation if and only if the map*

$$h_\delta : A \longrightarrow A \ltimes M, \quad a \longmapsto (a, \delta(a)) \qquad (a \in A; M \in M),$$

is a k-algebra morphism. Every splitting of the extension

$$A \twoheadleftarrow A \ltimes M \hookleftarrow M$$

is of this form. Thus, the correspondence $\delta \longmapsto h_\delta$ establishes a natural isomorphism of k-modules

$$Der_{A/k}(M) \simeq Split(A \ltimes M).$$

□

1.1.2 The Lie algebra structure on $\mathit{Der}_{A/k}(A)$

For any δ_1, $\delta_2 \in \mathit{Der}_{A/k}(A)$, their commutator $[\delta_1, \delta_2]$ is a derivation too:

$$\begin{aligned}[\delta_1, \delta_2](a_1 a_2) &= \delta_1(\delta_2(a_1)a_2 + a_1\delta_2(a_2)) - \delta_2(\delta_1(a_1)a_2 + a_1\delta_1(a_2)) \\ &= ((\delta_1\delta_2)(a_1))a_2 + \delta_2(a_1)\delta_1(a_2) + \delta_1(a_1)\delta_2(a_2) + a_1((\delta_1\delta_2)(a_2)) \\ &\quad - ((\delta_2\delta_1)(a_1))a_2 - \delta_2(a_1)\delta_1(a_2) - \delta_1(a_1)\delta_2(a_2) - a_1((\delta_2\delta_1)(a_2)) \\ &= ([\delta_1, \delta_2](a_1))a_2 + a_1([\delta_1, \delta_2](a_2)).\end{aligned}$$

Note that we did not use above associativity of multiplication. Thus, for *any binary* k-algebra A, k-module of derivations $\mathit{Der}_{A/k}(A)$ is a Lie subalgebra of Lie algebra $\mathfrak{gl}_k(A)$ of k-linear endomorphisms of k-module A.

1.1.3 Functorial properties of derivations

A homomorphism of k-algebras $f : A \longrightarrow B$ makes any B-bimodule M into an A-bimodule, and also induces the obvious map of k-modules

$$f^* : \mathit{Der}_{B/k}(M) \longrightarrow \mathit{Der}_{A/k}(M), \qquad \delta \longmapsto f^*(\delta) := \delta \circ f\,,$$

whose kernel, $\mathit{Der}_{B/A/k}(M)$, consists of those derivations $\delta : B \longrightarrow M$ which vanish on the image of A in B. We record this fact as

Lemma 1.5 *For any homomorphism of k-algebras $f : A \longrightarrow B$ and any B-bimodule M, one has the following exact sequence*

$$0 \longrightarrow \mathit{Der}_{B/A/k}(M) \longrightarrow \mathit{Der}_{B/k}(M) \xrightarrow{f^*} \mathit{Der}_{A/k}(M)\,. \tag{8}$$

All derivations $\delta : B \longrightarrow M$ which vanish on A are automatically A-bimodule maps. If $B \ni 1$ then also the opposite is true:

$$\delta(a) = \delta(a \cdot 1) = a\delta(1) = 0\,,$$

since

$$0 = \delta(1^2) - \delta(1) = \delta(1)1 + 1\delta(1) - \delta(1) = \delta(1)\,.$$

If A is an extension of B with an ideal J:

$$B \overset{f}{\twoheadleftarrow} A \leftarrowtail J\,, \tag{9}$$

then $\mathit{Der}_{B/A/k}(M) = 0$ and

$$f^* : \mathit{Der}_{B/k}(M) \xrightarrow{\sim} \{\delta \in \mathit{Der}_{A/k}(M) \mid \delta_{|J} = 0\}\,.$$

1.1.4 Example: Derivations from the tensor algebra

Let E be a bimodule over a unital k-algebra B and M be a bimodule over the tensor algebra $A = T^*_B E$. The restriction of any derivation $\delta : T^*_B E \longrightarrow M$ to $T^0_B E = B$ is a derivation $\delta_0 : B \longrightarrow M$ and thus

$$h_0 : B \longrightarrow A \ltimes M, \quad B \longmapsto (b, \delta_0(b)) \qquad (b \in B),$$

is a homomorphism of algebras. This endows $A \ltimes M$ with a new structure of a B-bimodule:

$$b(a, m) := (b, \delta_0(b))(a, m) = (ba, bm + \delta_0(b)a) \tag{10}$$

and

$$(a, m)b := (a, m)(b, \delta_0(b)) = (ab, mb + a\delta_0(b)) \tag{11}$$

$(b \in B; a \in T^*_B M; m \in M)$.

The restriction of δ to $T^1 BE = E$, denoted δ_1, satisfies the pair of identities

$$\delta_1(be) = \delta_0(b)e + b\delta_1(e) \tag{12}$$

and

$$\delta_1(eb) = \delta_1(e)b + e\delta_0(b) \tag{13}$$

$(b \in B; e \in E)$ which express the fact that the map

$$h_1 : E \longrightarrow A \ltimes M, \quad e \longmapsto (e, \delta_1(e)) \qquad (e \in E), \tag{14}$$

is a morphism of B-bimodules *if* $A \ltimes M$ is given the bimodule structure described in (12) and (13).

The pair of maps

$$(B \xrightarrow{\delta_0} M, E \xrightarrow{\delta_1} M) \tag{15}$$

determines derivation $\delta : T^*_B E \longrightarrow M$ uniquely, since $T^*_B E$ is generated by $B \cup E$ as a k-algebra.

Vice-versa, given a pair like (15), consisting of a derivation $\delta_0 : B \longrightarrow M$ and of a k-module map $\delta_1 : E \longrightarrow M$ which satisfies identities (12) and (13), we obtain B-bimodule map (14) which, by the universal property of the tensor algebra, extends to a unique B-algebra homomorphism

$$h : T^*_B E \longrightarrow T^*_B E \ltimes M$$

which splits the extension

$$T_B^* E \twoheadleftarrow T_B^* E \ltimes M \leftarrowtail M\,.$$

In other words, $h(a)$ has the form $(a, \delta(a))$ where $\delta : T_B^* E \longrightarrow M$ is a (unique) derivation that equals δ_0 on B and δ_1 on E.

Proposition 1.6 *For any $T_B^* E$-bimodule M, there is a natural k-module isomorphism between $Der_{(T_B^* E)/k}(M)$ and*

$$\{(\delta_0, \delta_1) \in Der_{B/k}(M) \times \mathrm{Hom}_{k-\mathrm{bimod}}(E, M) \mid \delta_1 \text{ satisfies } (12) \text{ and } (13)\}\,. \tag{16}$$

□

In the special case $B = k$, one has $Der_{k/k}(M) = 0$, so $\delta_0 = 0$ and identities (12) and (13) together mean that $\delta_1 : E \longrightarrow M$ is a map of k-bimodules. Hence the following

Corollary 1.7 *For any $T_k^* E$-bimodule M, the correspondence $\delta \longmapsto \delta_1$ defines a natural k-module isomorphism*

$$Der_{(T_k^* E)/k}(M) \simeq \mathrm{Hom}_{k\text{-bimod}}(E, M) \tag{17}$$

1.1.5 Variant: Derivations from the symmetric algebra

Let E be a module over a unital *commutative* k-algebra B and M be a bimodule over the symmetric algebra $A = S_B^* E$.

Derivations $S_B^* E \longrightarrow M$ are the same as derivations $T_B^* E \longrightarrow M$ satisfying the identity

$$[\delta(e_1), e_2] + [e_1, \delta(e_2)] = 0 \qquad (e_1, e_2 \in E) \tag{18}$$

which expresses the fact that $\delta([e_1, e_2]) = \delta(e_1 \otimes e_2) - \delta(e_2 \otimes e_1) = 0$. Thus, the following statement is a corollary of Proposition 1.6.

Proposition 1.8 *For any $S_B^* E$-bimodule M, there is a natural k-module isomorphism between $Der_{(S_B^* E)/k}(M)$ and*

$$\{(\delta_0, \delta_1) \in Der_{B/k}(M) \times \mathrm{Hom}_{k-\mathrm{bimod}}(E, M) \mid \delta_1 \text{ satisfies } (12), (13) \text{ and } (18)\}\,. \tag{19}$$

□

In the special case $B = k$, we have the following analog of Corollary 1.7

Corollary 1.9 *For any $S^*_k E$-bimodule M, the correspondence $\delta \longmapsto \delta_1$ defines a natural k-module isomorphism*

$$Der_{(S^*_k E)/k}(M) \simeq \{\delta_1 \in \mathrm{Hom}_{k\text{-bimod}}(E, M) \mid \delta_1 \textit{ satisfies } (18)\}. \tag{20}$$

1.1.6 Variant: Derivations from the exterior algebra

Let E be a module over a unital *commutative* k-algebra B and M be a bimodule over the exterior algebra $A = \Lambda^*_B E$.

Derivations $\Lambda^*_B E \longrightarrow M$ are the same as derivations $T^*_B E \longrightarrow M$ satisfying the identity

$$\delta(e)e + e\delta(e) = 0 \qquad (e \in E) \tag{21}$$

which expresses the fact that $\delta(e \otimes e) = 0$. Thus, the following statement is a corollary of Proposition 1.6.

Proposition 1.10 *For any $\Lambda^*_B E$-bimodule M, there is a natural k-module isomorphism between $Der_{(\Lambda^*_B E)/k}(M)$ and*

$$\{(\delta_0, \delta_1) \in Der_{B/k}(M) \times \mathrm{Hom}_{k-\mathrm{bimod}}(E, M) \mid \delta_1 \textit{ satisfies } (12), (13) \textit{ and } (21)\}. \tag{22}$$

□

1.1.7 The universal derivation

For any k-algebra A, the kernel of the multiplication map

$$I_\Delta(A) := \mathrm{Ker}(A \otimes_k A \xrightarrow{\mu} A), \quad \mu(a_1 \otimes a_2) := a_1 a_2. \tag{23}$$

is an A-sub-bimodule of A. We will call it the **diagonal ideal** of A. The terminology stems from the fact that $I_\Delta = I_\Delta(A)$ is a left ideal in algebra $A \otimes A^{op}$ (all tensor products over k unless indicated otherwise).

Assumption: *in this subsection A is assumed to be unital.*

Lemma 1.11 *The correspondence*

$$d_\Delta : a \longmapsto d_\Delta a := 1 \otimes a - a \otimes 1 \tag{24}$$

is a k-linear derivation $A \longrightarrow I_\Delta$.

Indeed,

$$\begin{aligned} d_\Delta(a_1a_2) &= 1\otimes(a_1a_2) - (a_1a_2)\otimes 1 \\ &= (1\otimes(a_1a_2) - a_1\otimes a_2) + (a_1\otimes a_2 - (a_1a_2)\otimes 1) \\ &= (1\otimes a_1 - a_1\otimes 1)a_2 + a_1(1\otimes a_2 - a_2\otimes 1) \\ &= d_\Delta(a_1)a_2 + a_1 d_\Delta(a_2)\,. \end{aligned}$$

Observation 1.12 *If* $\alpha = \sum_{i=1}^{\ell} a_i' \otimes a_i'' \in I_\Delta$, *then*

$$\alpha = \sum_{i=1}^{\ell} a_i' d_\Delta(a_i'')\,. \tag{25}$$

Indeed,

$$\sum_{i=1}^{\ell} a_i' \otimes a_i'' = \sum_{i=1}^{\ell} a_i' d_\Delta(a_i'') + (\sum_{i=1}^{\ell} a_i' a_i'')\otimes 1\,.$$

Proposition 1.13 *For any derivation* $\delta : A \longrightarrow M$ *from* A *into an* A*-bimodule* M*, there exists a unique* A*-bimodule map* $\bar\delta : I_\Delta \longrightarrow M$ *such that the following triangle commutes:*

$$\begin{tikzcd} A \arrow[rr, "\delta"] \arrow[dr, "d_\Delta"'] & \circlearrowleft & M \\ & I_\Delta \arrow[ur, dotted, "\bar\delta"'] & \end{tikzcd}$$

Proof. The pairing $A\times A \longrightarrow M$, $(a_0, a_1) \longmapsto a_0\delta(a_1)$, is biadditive and k-balanced, hence induces a map

$$A\otimes A \longrightarrow M, \quad a_0\otimes a_1 \longmapsto a_0\delta(a_1), \tag{26}$$

which is clearly left A-linear. The restriction of (26) to I_Δ is also right A-linear. Indeed,

$$\sum_{i=1}^{\ell} a_i'\delta(a_i''b) = \sum_{i=1}^{\ell} a_i'\delta(a_i'')b + (\sum_{i=1}^{\ell} a_i'a_i'')\delta b \tag{27}$$

and the right hand sum in (27) vanishes if $\sum_{i=1}^{\ell} a_i'\otimes a_i'' \in I_\Delta$.

The uniqueness of $\bar\delta$ follows from the fact that I_Δ is generated by $d_\Delta A$, the image of A in I_Δ, as a left A-module, while $\bar\delta$ is requested to be A-linear. $\square$

Corollary 1.14 *The correspondence* $f \longmapsto f \circ d_\Delta$ *defines a canonical isomorphism of* k*-modules:*

$$\mathrm{Hom}_{A\text{-bimod}}(I_\Delta, M) \simeq \mathit{Der}_{A/k}(M)\,. \tag{28}$$

□

1.1.8 Point derivations

Definition 1.15 *A* ***point*** *of a* k*-algebra* A *(more precisely, a* k*–**point**) is a* k*-algebra homomorphism* $p : A \longrightarrow k$*. The set of* k*-points will be denoted* $\mathrm{Spec}_k(A)$ *and called the* k***-spectrum*** *of* A*.*

When talking about points, it is customary to write $a(p)$ instead of $p(a)$, for $a \in A$, as if the elements of A where functions on the spectrum of A and the homomorphism $A \longrightarrow k$ were the "evaluation at point p".

For a given point p, the set

$$\mathfrak{m}_p := \{a \in A \mid a(p) = 0\} \tag{29}$$

is an ideal in A ('ideal' will always mean 'two-sided ideal', unless stated otherwise).

A choice of a point p equips k with a structure of an A-bimodule:

$$ac := a(p)c, \quad ca := ca(p) \qquad (a \in A; c \in k)\,.$$

The corresponding derivations $v : A \longrightarrow k$ form the **tangent space**, T_pA, to A at point p. We shall call them **point derivations** (at p), or **tangent vectors** to A at p.

From now on we assume A *to be unital.*

The correspondence

$$a \longmapsto (a(p), \Delta_p a) \qquad (a \in A)\,,$$

where $\Delta_p a := a - a(p)$, defines a splitting of k-module A into a direct sum $k \oplus \mathfrak{m}_p$.

Proposition 1.16 *(a) For any point* p *of* A*, the correspondence*

$$d_p : a \longmapsto \textit{ class of } \Delta_p a \textit{ modulo } \mathfrak{m}_p^2$$

defines a derivation $A \longrightarrow \mathfrak{m}/\mathfrak{m}_p^2$.

(b) Derivation d_p *is universal for all point derivations at point* p. *More precisely, for any tangent vector* $v \in T_pA$, *there exists a unique* A*-bimodule map* $\bar{v}$ *such that the triangle commutes:*

$$\begin{tikzcd} A \arrow[rr, "v"] \arrow[dr, "d_p"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & k \\ & \mathfrak{m}_p/\mathfrak{m}_p^2 \arrow[ur, dotted, "\bar{v}"'] & \end{tikzcd}$$

Proof. Part (a) is an immediate consequence of the identity

$$\Delta_p(a_1a_2) = (\Delta_p a_1)a_2 + a_1(\Delta_p a_2) - (\Delta_p a_1)(\Delta_p a_2) \qquad (a_1, a_2 \in A)\,.$$

For a tangent vector $v \in T_pA$ and $a \in A$,

$$v(a) = v(a(p)) + v(\Delta_p a) = v(\Delta_p a)$$

and the right hand term depends only on the class of $\Delta_p a$ modulo $\mathfrak{m}_p^2$, since v vanishes on $\mathfrak{m}_p^2$:

$$v(ab) = v(a)b(p) + a(p)v(b) = v(a)0 + 0v(b) = 0\,.$$

Since any element of $\mathfrak{m}_p/\mathfrak{m}_p^2$ is of the form $d_p a$ for some $a \in A$, this implies that the correspondence $\bar{v} : d_p a \longmapsto v(a)$ produces a well defined k-linear map $\bar{v} : \mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow k$. Noting that the action of A on both $\mathfrak{m}_p/\mathfrak{m}_p^2$ and k factorizes through $A/\mathfrak{m}_p \simeq k$, we obtain the canonical isomorphism:

$$\mathrm{Hom}_{k\text{-mod}}(\mathfrak{m}_p/\mathfrak{m}_p^2, k) = \mathrm{Hom}_{A\text{-bimod}}(\mathfrak{m}_p/\mathfrak{m}_p^2, k) \simeq T_pA\,. \tag{30}$$

□

Definition 1.17 *We call* $d_p a \in \mathfrak{m}_p/\mathfrak{m}_p^2$ *the* ***differential*** *of* $a \in A$ *at point* p. *We shall denote* $\mathfrak{m}_p/\mathfrak{m}_p^2$ *either* $\Omega_{A/k,p}$ *or* $\Omega_{A/k}(p)$.

The space of differentials at a given point p plays the role of the **cotangent** space in Analysis on Manifolds. There, it is defined as the *dual*, $(T_p)^*$, of the tangent space. As we have seen, it is rather the tangent space which appears as the dual of the space of differentials of $a \in A$ ("functions") at point p.

The canonical A-bimodule map $I_\Delta \longrightarrow \mathfrak{m}_p/\mathfrak{m}_p^2$, which sends $a_0 d_\Delta a_1$ to $a(p) d_p a_1$, is the restriction to I_Δ of the tensor product of the canonical quotient maps

$$(A \twoheadrightarrow A/\mathfrak{m}_p) \otimes_k (A \twoheadrightarrow A/\mathfrak{m}_p^2)$$

viewed as an A-bimodule map $A \otimes A \longrightarrow k \otimes (A/\mathfrak{m}_p^2) \simeq A/\mathfrak{m}_p^2$.

1.2 Graded case

Suppose that algebra $\mathcal{A}$ is $\mathbb{Z}$-graded, i.e.,

$$\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i \quad \text{and} \quad \mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j} \qquad (i, j \in \mathbb{Z}),$$

and that $\mathcal{M}$ is a $\mathbb{Z}$-graded $\mathcal{A}$-bimodule, i.e.,

$$\mathcal{M} = \bigoplus_{i \in \mathbb{Z}} \mathcal{M}_i \quad \text{and} \quad \mathcal{A}_i \mathcal{M}_j \subseteq \mathcal{M}_{i+j} \supseteq \mathcal{M}_i \mathcal{A}_j .$$

A nonzero element $a \in \mathcal{A}$ is said to be *homogeneous of degree* i if $a \in \mathcal{A}_i$ (by $\tilde{a}$ we shall denote the parity of the degree of a; thus $\tilde{a} = 0$ or 1). Similarly for elements of $\mathcal{M}$.

Definition 1.18 *For any graded $\mathcal{A}$-bimodule $\mathcal{M}$ and $j \in \mathbb{Z}$, the j-**shifted** bimodule $[j]\mathcal{M}$ is defined as follows:*

$$([j]\mathcal{M})_i = \mathcal{M}_{i-j} \qquad (i \in \mathbb{Z})$$

with the left and right actions of $\mathcal{A}$ given by

$$([j]m)a = [j](ma), \quad a([j]m) = (-1)^{ij}[j](am) \qquad (a_i \in \mathcal{A}_i; m \in \mathcal{M}).$$

Definition 1.19 *A map between $\mathbb{Z}$-graded modules $f : \mathcal{L} \longrightarrow \mathcal{M}$ is said to have **degree** $d \in \mathbb{Z}$ if $f(\mathcal{L}_i) \subseteq \mathcal{M}_{i+d}$, $i \in \mathbb{Z}$. Maps of degree 0 are also called **graded** maps.*

Note that the shift map $[j] : \mathcal{M} \longrightarrow [j]\mathcal{M}$, $m \longmapsto [j]m$, has degree j, and that the composition $f \circ g$ of maps of degree d and, respectively, e has degree $d + e$.

Definition 1.20 *A map* $\delta : \mathcal{A} \longrightarrow \mathcal{M}$ *of degree* d *is said to be a* k*-**linear derivation of degree*** d *if,* $[-d]\delta$*, i.e., the composition with shift* $[-d] \circ \delta$*, is a* k*-linear derivation* $[-d]\delta : \mathcal{A} \longrightarrow [-d]\mathcal{M}$.

Equivalently, $\delta\mathcal{A}_i \subseteq \mathcal{M}_{i+d}$ *for all* $i \in \mathbb{Z}$ *and*

$$\delta(a_1a_2) = (\delta(a_1))a_2 + (-1)^{id}a_1(\delta(a_2)) \qquad (a_i \in \mathcal{A}_i, a_2 \in \mathcal{A}).$$

The following is a version of Corollary 1.4 for graded bimodules.

Corollary 1.21 *A map* $\delta : \mathcal{A} \longrightarrow \mathcal{M}$ *is a derivation of degree* d *if and only if*

$$a \longmapsto (a, [-d]\delta(a))$$

is a graded k*-algebra homomorphism* $\mathcal{A} \longrightarrow \mathcal{A} \ltimes [-d]\mathcal{M}$. □

1.2.1 Example: Derivations from the exterior algebra

Let E be a module over a unital commutative k-algebra A and $\mathcal{M}$ be a graded module over the exterior algebra $\mathcal{A} = \Lambda^*_A(E)$.

Note that $\Lambda^*_A(E) = S^*_A([1]E)$, provided $\frac{1}{2} \in k$; here E is treated as a graded module whose i-components, for $i \neq 0$, are zero.

Let $\delta : \Lambda^*_A(E) \longrightarrow \mathcal{M}$ be a derivation of degree d. Its 0-th component $\delta_0 : A = \Lambda^0_A(E) \longrightarrow \mathcal{M}_d$ satisfies the condition that the correspondence

$$a \longmapsto (a, \delta_0(a)) \tag{31}$$

is a homomorphism of k-algebras

$$A \longrightarrow A \ltimes \mathcal{M}_d$$

(note that $A \ltimes \mathcal{M}_d$ is the 0-th component of graded algebra $\Lambda^*_A(E) \ltimes [-d]\mathcal{M}$). Equivalently, δ_0 is a derivation.

The component $\delta_1 : E = \Lambda^1_A(E) \longrightarrow \mathcal{M}_{d+1}$ satisfies the identity

$$\delta_1(ae) = \delta_0(a)e + a\delta_1(e) \qquad (a \in A; e \in E) \tag{32}$$

and, since $e^2 = 0$ for any $e \in E$, also the identity

$$\delta_1(e)e + (-1)^{-d}e\delta_1(e) = 0 \qquad (e \in E). \tag{33}$$

Conversely, for any derivation $\delta_0 : A \longrightarrow \mathcal{M}_d$, correspondence (31) defines a morphism of k-algebras $A \longrightarrow \Lambda^*_A(E) \ltimes [-d]\mathcal{M}$ which makes

$\Lambda^*_A(E) \ltimes [-d]\mathcal{M}$ an A-algebra, and thus an A-module. Identity (32) then esxpresses the fact that the correspondence

$$e \longmapsto (e, [-d]\delta_a(e)) \tag{34}$$

which maps E into

$$E \oplus \mathcal{M}_{d+1} = (\Lambda^*_A(E) \ltimes [-d]\mathcal{M})_1\,,$$

defines an A-linear map $h_1 : E \longrightarrow \Lambda^*_A(E) \ltimes [-d]\mathcal{M}$. The unique extension of h_1 to a homomorphism of graded A-algebras

$$h : T^*_A(E) \longrightarrow \Lambda^*_A(E) \ltimes [-d]\mathcal{M}$$

annihilates elements $e \otimes e \in T^2_A(E)$, This follows from the formula precisely when δ_1 satisfies identity (33).

$$\begin{aligned} h(e \otimes e) &= h_1(e)^2 = (e, [-d]\delta_1(e))^2 \\ &= (e \wedge e, [-d](\delta_1(e)e + (-1)^{-d}e\delta_1(e))) \\ &= (0, [-d](\delta_1(e)e + (-1)^{-d}e\delta_1(e)))\,, \end{aligned}$$

If so, then homomorphism h passes to the quotient algebra $\Lambda^*_A(E)$ and the obtained graded k-algebra homomorphism

$$h : \Lambda^*_A(E) \longrightarrow \Lambda^*_A(E) \ltimes [-d]\mathcal{M}$$

is then of the form

$$h(\alpha) = (\alpha, [-d]\delta'(\alpha))$$

for some derivation $\delta' : \Lambda^*_A(E) \longrightarrow \mathcal{M}$ of degree d. Since $\delta'_i = \delta_i$, for $i = 0, 1$, we have

$$\begin{aligned} \delta'(ae_1 \wedge \cdots \wedge e_n) &= \delta_0(a) \wedge e_1 \wedge \cdots \wedge e_n \\ &\quad + \sum_{i=1}^{n} (-1)^{(i-1)d} ae_1 \wedge \cdots \wedge \delta_1(e_i) \wedge \cdots \wedge e_n \\ &= \delta(ae_1 \wedge \cdots \wedge e_n)\,, \end{aligned} \tag{35}$$

i.e., $\delta' = \delta$. Identities (35) mean, in particular, that any derivation δ of degree d is *uniquely* determined by its components $\delta_0 : A \longrightarrow \mathcal{M}_d$ and $\delta_1 : E \longrightarrow \mathcal{M}_{d+1}$.

We have thus established

Proposition 1.22 *For any graded $\Lambda^*_A(E)$-bimodule $\mathcal{M}$, there is a natural bijective correspondence between derivations $\delta : \Lambda^*_A(E) \longrightarrow \mathcal{M}$ of degree d and pairs of k-linear maps*

$$(A \xrightarrow{\delta_0} \mathcal{M}_d,\ E \xrightarrow{\delta_1} \mathcal{M}_{d+1}) \tag{36}$$

such that δ_0 is a derivation and δ_1 satisfies identities (32) and (33). □

Exercise. Prove the following variant of Proposition 1.22:

For any graded $T^*_A(E)$-bimodule $\mathcal{M}$, there is a natural bijective correspondence between derivations $\delta : T^*_A(E) \longrightarrow \mathcal{M}$ of degree d and pairs of k-linear maps (36) such that δ_0 is a derivation and δ_1 satisfies identity (32).

2 Differential forms

Unless otherwise stated, A represents in this section and the next section a unital commutative k-algebra and M an A-module treated as a ***symmetric*** *A-bimodule:*

$$am = ma \qquad (a \in A; m \in M).$$

We shall give in this chapter three realizations of the derivation that is **universal** in the class of unital commutative algebras and modules over them.

2.1 Kähler's realization

Let $A\langle da \mid a \in A\rangle$ be the A-module freely generated by the set whose elements are *formal symbols* da, for all $a \in A$.

Definition 2.1 *The A-module of* ***Kähler differentials*** *(or,* ***Kähler differential*** *1-**forms**) of a unital commutative k-algebra A is the quotient module*

$$\Omega_{A/k} := A\langle da \mid a \in A\rangle / N \tag{37}$$

by the A-submodule N generated by elements of three types

(a) $d(a_1 + a_2) - da_1 - da_2$,

(b) $d(ca) - cda$,

(c) $d(a_1 a_2) - a_2 da_1 - a_1 da_2$,

where a_1, $a_2 \in A$, *and* $c \in k$.

It follows tautologically from the definition that

$$d : A \longrightarrow \Omega_{A/k}, \quad d : a \longmapsto da\,, \tag{38}$$

is a derivation universal in the class of unital commutative algebras and modules over them.

2.2 Hochschild's realization

2.2.1 A-linearization of k-linear maps

Any k-linear map f from a k-module E into an A-module M induces a unique A-linear map $\tilde{f} : A \otimes_k E \longrightarrow M$ such that the triangle commutes:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\epsilon \oplus \mathrm{id}_E"'] & {} \arrow[d, phantom, "\circlearrowleft"] & M \\ & A \otimes_k E \arrow[ur, dotted, "\tilde{f}"'] & \end{tikzcd}$$

where $\epsilon : k \longrightarrow A$ is the structural homomorphism $1_k \longmapsto 1_A$. We can express this also by saying that there is a canonical isomorphism of k-modules

$$\mathrm{Hom}_{k\text{-mod}}(E, M) \simeq \mathrm{Hom}_{A-\mathrm{mod}}(A \otimes_k E, M))\,. \tag{39}$$

Definition 2.2 *An* A*-module is* ***relatively free*** *if there exists a* k*-module* E *and a* k*-module map* $f : E \longrightarrow M$ *such that* $\tilde{f}$ *is an isomorphism.*

Consider the linearization, $\tilde{\delta} : A \otimes A \longrightarrow M$, of a k-module map $\delta : A \longrightarrow M$. Note that:

$$(\delta a_1)a_2 = a_2(\delta a_1) = \tilde{\delta}(a_2 \otimes a_1)\,,$$

and

$$a_1(\delta a_2) = \tilde{\delta}(a_1 \otimes a_2)\,,$$

as well as

$$\delta(a_1 a_2) = \tilde{\delta}(1_A \otimes a_1 a_2)\,.$$

It follows that

$$(\delta a_1)a_2 + a_1(\delta a_2) - \delta(a_1a_2) = \tilde{\delta}(a_1 \otimes a_2 - 1_A \otimes a_1a_2 + a_2 \otimes a_1)\,. \quad (40)$$

The three-term expression in parentheses on the right hand side of (40) is related to a very important object introduced below.

2.2.2 Hochschild homology

For any A-bimodule M over an arbitrary k-algebra A, let

$$C_n(A;M) := M \otimes A^{\otimes n} \quad (41)$$

and maps $b_n : C_n(A;M) \longrightarrow C_{n-1}(A;M)$, $n \geqslant 1$, be defined as follows:

$$\begin{aligned} b_n(m \otimes a_1 \otimes \cdots \otimes a_n) = {} & ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ & + \sum_{i=1}^{n-1} m \otimes a_1 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n a_nm \otimes a_1 \otimes \cdots \otimes a_{n-1}\,. \end{aligned} \quad (42)$$

The following lemma is established by a straightforward calculation.

Lemma 2.3 *Sequence of* k*-modules and* k*-linear maps* $(C_*(A;M), b_*)$ *is a chain complex, i.e.,* $b_{n-1}b_n = 0$ *for all* $n \geqslant 1$. □

Definition 2.4 $(C_*(A;M), b_*)$ *is called the* ***Hochschild complex of a*** k***-algebra*** A ***with coefficients in an*** A ***bimodule*** M*, and its homology groups* $H_*(A;M)$ *are called the* ***Hochschild homology groups*** *of* A *with coefficients in* M*. Maps* b_n *are called the* ***Hochschild boundary maps,***

When $M = A$ *(and* A *is unital), then* $HH_*(A) := H_*(A;A)$ *is called the* ***Hochschild homology*** *of algebra* A.

One has

$$HH_0(A;M) = M/[A,M] \quad (43)$$

where $[A,M]$ denotes the *commutator space*

$$[A,M] := \{\textstyle\sum_{i=1}^{\ell}(a_im_i - m_ia_i) \mid a_i \in A, m_i \in M\} \quad (44)$$

Comment When A is commutative and M is a symmetric A-bimodule, then the Hochschild boundary maps are A-linear and, consequently, groups $H_*(A;M)$ are A-modules.

Now, returning to the case of commutative A and symmetric module M, identity (40) can be rewritten as

$$(\delta a_1)a_2 + a_1(\delta a_2) - \delta(a_1 a_2) = (\tilde{\delta} \circ b_2)(1_A \otimes a_1 \otimes a_2) \tag{45}$$

where $b_2 : A^{\otimes 3} \longrightarrow A^{\otimes 2}$ is the corresponding Hochschild boundary map in $C_*(A;A)$.

Since A is commutative, we have

$$a_0 b_2(1_A \otimes a_1 \otimes a_2) = b_2(a_0 \otimes a_1 \otimes a_2) \qquad (a_i \in A)\,,$$

and thus we conclude that δ is a derivation if and only if $\tilde{\delta}$ vanishes on $b_2 A^{\otimes 3}$. In the latter case, $\tilde{\delta}$ induces an A-linear map $\bar{\delta} : HH_1(A) \longrightarrow M$.

For any $a \in A$, let

$$d_H a := \text{ the class of } 1_A \otimes a \text{ modulo } b_2 A^{\otimes 3}\,. \tag{46}$$

We note that $1 \otimes a = 1 \otimes a - a \otimes 1 + b_2(a \otimes 1 \otimes 1)$, i.e., $d_H a$ is the image in $HH_1(A)$ of *diagonal* differential $d_\Delta a$, cf. (24).

We also note that $a_1 d_H a_2 - d_H(a_1 a_2) + a_2 d_H a_1$ coincides with the class *modulo* $b_2 A^{\otimes 3}$ of

$$a_1 \otimes a_2 - 1_A \otimes a_1 a_2 + a_1 \otimes a_2 = b_2(1_A \otimes a_1 \otimes a_2)\,,$$

i.e., vanishes. Thus we have

Proposition 2.5 *The map* $d_H : A \longrightarrow HH_1(A)$, $a \longrightarrow d_H a$, *is a derivation and, for any derivation* $\delta : A \longrightarrow M$, *there exists a unique* A*-linear map* $\bar{\delta} : HH_1(A) \longrightarrow M$ *such that the triangle*

$$\begin{tikzcd}
A \arrow[rr, "\delta"] \arrow[dr, "d_H"'] & & M \\
& HH_1(A) \arrow[ur, dotted, "\bar{\delta}"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

commutes. In other words, $d_H : A \longrightarrow HH_1(A)$ *is a universal derivation in the class of unital commutative algebras and modules over them.* $\square$

The uniqueness, up to a *unique* isomorphism, of the universal derivation (in the class of unital commutative algebras and modules), plus the fact that both Kähler's and Hochschild's derivations are universal, result in the following

Corollary 2.6 *The correspondence* $a_0 da_1 \longmapsto a_0 d_H a_1$ *defines a canonical isomorphism of* A*-modules*

$$\Omega_{A/k} \simeq HH_1(A)\,. \tag{47}$$

2.3 Serre's realization

Let $\tilde{d}_\Delta : A \longrightarrow I_\Delta/I_\Delta^2$ be the diagonal derivation, $d_\Delta : a \longrightarrow I_\Delta$, followed by the canonical quotient map $I_\Delta \twoheadrightarrow I_\Delta/I_\Delta^2$. Since the latter is an A-bimodule map, the result is a derivation $A \longrightarrow I_\Delta/I_\Delta^2$.

Lemma 2.7 I_Δ/I_Δ^2 *is a symmetric* A*-bimodule; more precisely:*

$$I_\Delta^2 = [A, I_\Delta] \tag{48}$$

(cf. (44)*). In particular,*

$$I_\Delta/I_\Delta^2 = I_\Delta^2/[A, I_\Delta] = HH_0(A; I_\Delta)\,. \tag{49}$$

Proof. One has the following identity in $A \otimes A$:

$$\begin{aligned}
[a_1, d_\Delta a_2] &= a_1 \otimes a_2 - a_1 a_2 \otimes 1 - 1 \otimes a_2 a_1 + a_2 \otimes a_1 \\
&= -(1 \otimes a_1 a_2 - a_1 \otimes a_2 - a_2 \otimes a_1 + a_1 a_2 \otimes 1) \\
&= -(d_\Delta a_1)(d_\Delta a_2)\,,
\end{aligned} \tag{50}$$

which, in view of commutativity of A, implies that

$$[a, \alpha] = (d_\Delta a)\alpha \in I_\Delta^2 \quad (a \in A; \alpha \in I_\Delta)$$

(cf. representation (25) of elements of I_Δ).

Vice-versa,

$$a_0(d_\Delta a_1)(d_\Delta a_2) = [a_0(d_\Delta a_1), a_2] \in [I_\Delta, A] = [A, I_\Delta]\,,$$

and I_Δ^2 is additively spanned by products

$$a_0(d_\Delta a_1)(d_\Delta a_2)\,. \tag{51}$$

This proves equality (48), and (49) follows with help of equality (43). $\square$

Proposition 2.8 *Derivation* $\tilde{d}_\Delta : A \longrightarrow I_\Delta/I_\Delta^2$ *is universal in the class of unital commutative algebras and modules over them. In particular, the correspondence* $a_0\tilde{d}_\Delta a_1 \longmapsto a_0 d_H a_1$ *establishes a canonical isopmorphism of* A*-modules:*

$$I_\Delta/I_\Delta^2 = H_0(A; I_\Delta) \simeq HH_1(A)\,. \tag{52}$$

Proof. In view of the universality of $d_H : A \longrightarrow HH_1(A)$, the correspondence $a_0 d_H a_1 \longmapsto a_0\tilde{d}_\Delta a_1$ yields a well defined a-module map $HH_1(A) \longrightarrow I_\Delta/I_\Delta^2$. The universality of $d_\Delta : A \longrightarrow I_\Delta$ in the class of all A-bimodules yields an A-bimodule map $I_\Delta \longrightarrow HH_1(A)$ which sends $a_0 d_\Delta a_1$ to $a_0 d_H a_1$. It remains to show that this last map passes to I_Δ/I_Δ^2 or, equivalently, that $d_\Delta(I_\Delta^2) \subseteq b_2 A^{\otimes 3}$. This follows from the identity

$$\begin{aligned} a_0(d_\Delta a_1)(d_\Delta a_2) &= a_0 \otimes a_1 a_2 - a_2 a_0 \otimes a_1 + a_0 a_1 a_2 \otimes 1 - a_0 a_1 \otimes a_2 \\ &= b_2(a_0 a_1 a_2 \otimes 1 \otimes 1 - a_0 \otimes a_1 \otimes a_2)\,. \end{aligned} \tag{53}$$

□

In what follows we shall be freely identifying all three models of the module of differential 1*-forms, and we shall be using notation* $\Omega_{A/k}$ *and* da *irrespective of the model chosen.*

2.4 Functorialities

2.4.1 First Fundamental Exact Sequence

Let $f : A \longrightarrow B$ be a homomorphism of unital commutative k-algebras. The composite $d_{B/k} \circ f : A \longrightarrow \Omega_{B/k}$ is a derivation (where $\Omega_{B/k}$ is given an A-module structure via $f : A \longrightarrow B$). In view of the universal property of derivation $d_{A/k} : A \longrightarrow \Omega_{A/k}$, there exists a unique A-module morphism

$$f_\bullet : \Omega_{A/k} \longrightarrow \Omega_{B/k}$$

such that the following square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "d_{A/k}"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "d_{B/k}"] \\ \Omega_{A/k} \arrow[r, dotted, "f_\bullet"] & \Omega_{B/k} \end{tikzcd}$$

commutes. Let $f_* : B \otimes_A \Omega_{A/k} \longrightarrow \Omega_{B/k}$ be its B-linearization:

$$f_*(b \otimes a_0 da_1) = a_0 b\, d(f(a)) \qquad (a_0, a_1 \in A; b \in B)\,,$$

see Section 2.2.1. The image of f_* coincides with the B-submodule generated by dA, while a glance at Kähler's definition of the module of differential forms shows that the quotient $\Omega_{B/k}/BdA$ canonically identifies with $\Omega_{B/A}$. Thus, we obtain the **First Fundamental Exact Squence:**

$$B \otimes_A \Omega_{A/k} \xrightarrow{f_*} \Omega_{B/k} \longrightarrow \Omega_{B/A} \longrightarrow 0 \tag{54}$$

whose dual form we already encountered in more general situation, cf. (8).

2.4.2 Second Fundamental Exact Sequence

Suppose f is an epimorphism, i.e., A is an extension of B by some ideal $J \subset A$, see (9). In this case, $\Omega_{B/A} = 0$ and therefore f_* is surjective. By tensoring $\Omega_{A/k}$ with exact sequence (9), we obtain the following exact sequence:

$$J \otimes_A \Omega_{A/k} \longrightarrow A \otimes_A \Omega_{A/k} \longrightarrow B \otimes_A \Omega_{A/k} \longrightarrow 0 \tag{55}$$

which shows that B-module $B \otimes_A \Omega_{A/k}$ is canonically isomorphic to the quotient module $\Omega_{A/k}/JdA$. Since $d(J^2) \subseteq JdJ \subset JdA$, the restriction of $d_{A/k}$ to J induces a map

$$\bar{d} : J/J^2 \longrightarrow \Omega_{A/k}/JdA \simeq B \otimes_A \Omega_{A/k}\,,$$

and

$$\operatorname{Coker} \bar{d} \simeq \Omega_{A/k}/(JdA + dJ)\,.$$

Note that $\Omega_{A/k}/(JdA+dJ)$ is a B-module and that the composite derivation

$$A \xrightarrow{d_{A/k}} \Omega_{A/k} \twoheadrightarrow \Omega_{A/k}/(JdA + dJ)$$

vanishes on J, and thus induces a derivation

$$\delta : B \longrightarrow \Omega_{A/k}/(JdA + dJ)\,.$$

The latter induces a B-module map

$$\bar{\delta} : \Omega_{B/k} \longrightarrow \Omega_{A/k}/(JdA + dJ)\,. \tag{56}$$

For any $a \in A$, let $\overline{da}$ denote the class of da modulo $JdA + dJ$. Then f_* sends $\overline{da}$ to $d(f(a)) \in \Omega_{B/k}$, and $\bar{\delta}$ sends $d(f(a))$ to $\overline{da}$. In other words, B-linear maps f_* and $\bar{\delta}$ supply mutually inverse correspondences between set $\overline{dA}$, which generates B-module $\Omega_{A/k}/(JdA + dJ$, and set dB, which generates B-module $\Omega_{B/k}$.

We conclude that f_* induces an isomorphism

$$\operatorname{Coker} \bar{d} \simeq \Omega_{B/k}$$

which establishes the **Second Fundamental Exact Sequence**

$$J/J^2 \xrightarrow{\bar{d}} B \otimes_A \Omega_{A/k} \xrightarrow{f_*} \Omega_{B/k} \longrightarrow 0\,. \tag{57}$$

We shall sometimes be using it in the form

Proposition 2.9 *For any extension* (9) *of unital commutative* k*-algebras, the epimorphism* $\Omega_{A/k} \twoheadrightarrow \Omega_{B/k}$ *induces a canonical isomorphism of* B*-modules*

$$\Omega_{A/k}/(J\, dA + dJ) \simeq \Omega_{B/k}\,. \tag{58}$$

2.5 Examples

2.5.1 The symmetric algebra

Let $A = S_k^* E$ be the symmetric algebra of a k-module E and M be any $S_k^* E$-module. Corollary 1.9, combined with canonical isomorphism (39), yields in this case the canonical isomorphisms

$$\operatorname{Hom}_{S_k^* E\text{-mod}}(\Omega_{S_k^* E/k}, M) \simeq \operatorname{Hom}_{k\text{-mod}}(E, M) \simeq \operatorname{Hom}_{S_k^* E\text{-mod}}(S_k^* E \otimes E, M)\,,$$

which show that

$$\Omega_{S_k^* E/k} \simeq S_k^* E \otimes E \tag{59}$$

(it suffices to take $M = \Omega_{S_k^* E/k}$). Under (59), differentials de, $e \in E$, correspond to elements $1 \otimes e \in S_k^* E \otimes E$. We notice that $\Omega_{S_k^* E/k}$ is a *relatively free* $S_k^* E$-module, cf. Definition 2.2.

If E is a free k-module with basis $((e_i)_{i \in I})$, then $S_k^* E \otimes E$ is a free $S_k^* E$-module with the same basis. In this case, $\Omega_{S_k^* E/k}$ is a free $S_k^* E$-module with basis $((de_i)_{i \in I})$:

$$\Omega_{S_k^* E/k} = S_k^* \langle de_i \mid i \in I \rangle\,. \tag{60}$$

In the special case $E = k^n$, symmetric algebra $S^*_k E$ becomes the k-algebra of polynomials in n variables:

$$S^*_k E = k[X_1, \ldots, X_n],$$

and the formula for the differential

$$d : k[X_1, \ldots, X_n] \longrightarrow \Omega_{k[X_1,\ldots,X_n]/k}$$

takes the familiar form

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} dX_i \tag{61}$$

for $f \in k[X_1, \ldots, X_n]$.

When k-module E is projective, $S^*_k E$-module $S^*_k E \otimes E$ is projective, and thus also $\Omega_{S^*_k E/k}$.

2.5.2 Affine algebra $A = k[X_1, \ldots, X_n]/(F_1, \ldots, F_m)$.

Let $k_n := k[X_1, \ldots, X_n]$ and $\Omega_n := \Omega_{k_n/k}$. As a k_n-module, the latter is freely generated by differentials of variables dX_1, …, dX_n. According to (58), $\Omega_{A/k}$ is canonically isomorphic to the quotient of Ω_n by the k_n-submodule

$$(F_1, \ldots, F_m)\Omega_n + k_n dF_1 + \ldots + k_n dF_m. \tag{62}$$

Example: Plane curve $XY = c$ For a given $c \in k$, let $A_c := k[X, Y]/(XY - c)$. If c is invertible then the correspondence $T \longmapsto c^{-1}X$ induces a k-algebra isomorphism:

$$k[T, T^{-1}] \simeq k[X, Y]/(XY - c)$$

where $k[T, T^{-1}]$ is the algebra of Laurent polynomials with coefficients in k.

2.6 The algebra of differential forms

Definition 2.10 *The exterior algebra $\Omega^*_{A/k} := \Lambda^*_A \Omega_{A/k}$ is called the* ***Kähler-de Rham algebra*** *of a unital commutative k-algebra A. Elements of $\Omega^q_{A/k} = \Lambda^q_A \Omega_{A/k}$ are called* ***differential q-forms.***

Note that

$$\Omega^0_{A/k} = A \quad \text{and} \quad \Omega^1_{A/k} = \Omega_{A/k}.$$

Proposition 2.11 *There exists a unique extension of derivation* $d : A \longrightarrow \Omega^1_{A/k}$ *to a derivation of degree* 1 *of* $\mathbb{N}$*-graded* k*-algebra*

$$d : \Omega^*_{A/k} \longrightarrow \Omega^*_{A/k} \tag{63}$$

such that

$$d \circ d : A \longrightarrow \Omega^2_{A/k} \text{ is zero.} \tag{64}$$

Proof. According to Proposition 1.22, derivations $d_* : \Omega^*_{A/k} \longrightarrow \Omega^*_{A/k}$ of degree 1 are in bijective correspondence with pairs:

(d_0) a k-linear derivation $d_0 : A \longrightarrow \Omega^1_{A/k}$;

(d_1) a k-linear map $d_1 : \Omega^1_{A/k} \longrightarrow \Omega^2_{A/k}$ which satisfies the identity

$$d_1(a\varphi) = d_0 a \wedge \varphi + a d_1 \varphi$$

Identity (33) is automatically satisfied:

$$d_1\varphi \wedge \varphi + (-1)^{-1}\varphi \wedge d_1\varphi = d_1\varphi \wedge \varphi - d_1\varphi \wedge \varphi = 0\,,$$

since $d_1\varphi$ has degree 2, and thus commutes with all elements of $\Omega^*_{A/k}$.

We shall construct d_1 as follows. The k-bilinear pairing

$$A \times A \longrightarrow \Omega^2_{A/k}, \quad (a_0, a_1) \longmapsto da_0 \wedge da_1 \qquad (a_0, a_1 \in A)\,,$$

induces a k-linear map

$$A \otimes A \longrightarrow \Omega^2_{A/k}\,. \tag{65}$$

Since $a_0(d_\Delta a_1)(d_\Delta a_2)$ is sent by (65) to

$$\begin{aligned} da_0 \wedge d(a_1a_2) - d(a_0a_2) \wedge da_1 - d(a_0a_1) \wedge da_2 + d(a_0a_1a_2) \wedge d1 \\ = (a_2 da_0 \wedge da_1 + a_1 da_0 \wedge da_2) \\ - (a_2 da_0 \wedge da_1 + a_0 da_2 \wedge da_1) \\ - (a_1 da_0 \wedge da_2 + a_0 da_1 \wedge da_2) = 0\,, \end{aligned}$$

and elements (51) additively span I^2_Δ, map (65) vanishes on I^2_Δ. Denote the induced map

$$\Omega^1_{A/k} \simeq I_\Delta / I^2_\Delta \hookrightarrow A^{\otimes 2}/I^2_\Delta \longrightarrow \Omega^2_{A/k} \tag{66}$$

by d_1 and notice that $d_1(a_0(a_1 da_2))$ is the image of $a_0a_1 \otimes a_2 - a_0a_1a_2 \otimes 1$ under map (65):

$$\begin{aligned} d(a_0a_1) \wedge da_2 - d(a_0a_1a_2) \wedge d1 &= da_0 \wedge (a_1 da_2) + a_0 da_1 \wedge da_2 \\ &= da_0 \wedge (a_1 da_2) + a_0 \wedge (a_1 da_2)\,. \end{aligned}$$

Thus, our $d_1 : \Omega^1_{A/k} \longrightarrow \Omega^2_{A/k}$ satisfies condition (32), and

$$(d_1 \circ d_0)(a) = d_1(1 \otimes a - a \otimes 1) = d1 \wedge da - da \wedge d1 = 0\,,$$

which proves the existence of the desired derivation $d : \Omega^*_{A/k} \longrightarrow \Omega^*_{A/k}$.

The uniqueness follows from Leibniz' identity combined with condition (64):

$$\begin{aligned} d(a_0 da_1 \wedge \cdots \wedge da_q) &= da_0 \wedge da_1 \wedge \cdots \wedge da_q \\ &\quad + a_0 d^2 a_1 \wedge \cdots \wedge da_q \\ &\quad - a_0 da_1 \wedge d^2 a_2 \wedge \cdots \wedge da_q + \cdots \\ &= da_0 \wedge da_1 \wedge \cdots \wedge da_q\,. \end{aligned} \tag{67}$$

□

Comment One has $d \circ d = 0$ on forms of any degree as follows immediately from formula (67). In particular, $(\Omega^*_{A/k}, d)$ is a cochain complex of k-modules.

Definition 2.12 *We will refer to* $(\Omega^*_{A/k}, d)$ *as the* ***Kähler-de Rham complex*** *(or, just the* ***de Rham complex****) of a unital commutative* k*-algebra* A*. Its cohomology,* $H^*_{dR}(A/k)$*, will be called the* ***de Rham cohomology*** *of* A*.*

De Rham cohomology $H^*_{dR}(A/k)$ is a very important invariant of algebra A.