Differential Calculus

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December 19, 2015

1 Vocabulary

1.1 Terminology

1.1.1 A ground ring of coefficients

Let k be a fixed unital commutative ring. We shall be refering to it throughout as the *ground ring* and practically all objects will be expected to be k-modules.

1.1.2 A-Modules

In order to simplify terminology used below, the meaning of the term *A-module* will depend on the context. A k-module M equipped with a k-bilinear pairing

$$\alpha\colon A\times M \longrightarrow M, \tag{1}$$

where A is another k-module, will be referred to as an *A-module*. The value $\alpha(a,m)$ is usually denoted am or ma, depending on circumstances. An A-module structure on a k-module M is thus the same as a k-linear map

$$A \longrightarrow \mathrm{End}_{k\text{-mod}}(M). \tag{2}$$

1.1.3 k-algebras

An A-module structure on A itself, i.e., a k-bilinear pairing

$$\mu\colon A\times A \longrightarrow A, \tag{3}$$

will be called a k-algebra.

1.1.4 The opposite k-algebra functor

Given a k-algebra A, the same k-module equipped with the multiplication $\mu^{\mathrm{op}} := \mu \circ \tau$, where

$$\tau\colon A\times A \longrightarrow A\times A, \qquad (a,b) \longmapsto (b,a), \tag{4}$$

is called the *opposite algebra* and denoted A^{op}. In order to distinguish between a viewed as an element of algebra A and a viewed as an element of algebra A^{op} is advisable to denote the latter by a^{op}. In this notation the formula for multiplication in A^{op} becomes

$$a^{\mathrm{op}} b^{\mathrm{op}} = (ba)^{\mathrm{op}}.$$

Note that

$$(\ )^{\mathrm{op}} \colon a \longmapsto a^{\mathrm{op}}$$

itself becomes an *anti-isomorphism* of A with A^{op}.

1.1.5

The correspondence

$$A \longmapsto A^{\mathrm{op}}, \qquad f \longmapsto (\ )^{\mathrm{op}} \circ f(\ )^{\mathrm{op}},$$

becomes a functorial involution of the category of k-algebras. The latter means that when applied twice, it produces the identity functor.

1.1.6

An algebra is commutative if and only if $A = A^{\mathrm{op}}$.

1.1.7 Associative k-algebras

If multiplication (3) is expected to be associative, we shall refer to A as an *associative algebra*.

1.1.8 Unital k-algebras

If A is expected to have an identity element, we shall refer to A as a *unital algebra*. The identity element will be denoted 1_A. Homomorphisms between unital algebras are expected to preserve the identity.

1.1.9 Unitalization

For any k-algebra A, the k-module $\tilde{A} = k \oplus A$ equipped with multiplication

$$(c, a)(c', a') := (cc', ca' + ac) \qquad (c, c' \in k,\ a, a' \in A),$$

is called the *unitalization* of A. The correspondence $A \longmapsto \tilde{A}$ defines a functor from the category of k-algebras to the category of unital k-algebras.

1.1.10 The universal property of unitalization

Exercise 1 *Show that any k-algebra homomorphism $f\colon A \longrightarrow B$ into a unital k-algebra extends in a unique manner to a unital homomorphism $f'\colon \tilde{A} \longrightarrow B$.*

1.1.11 Example: the convolution algebra of a binary structure

Given a binary structure $(G, \cdot)$, let kG be the *free* k-module with basis G. Its elements are usually represented as formal linear combinations of elements of G

$$\sum_{g \in G} c_g g$$

with finitely many nonzero coefficients $c_g \in k$. Multiplication on G extends by k-linearity to multiplication of such formal linear combinations. The resulting product

$$\left(\sum_{g \in G} a_g g\right) * \left(\sum_{g \in G} b_g g\right) := \left(\sum_{g \in G} c_g g\right), \tag{5}$$

where

$$c_g := \sum a_{g'} b_{g''} \quad (\text{summation over } (g', g'') \in G \times G \text{ such that } g'g'' = g),$$

is called the *convolution product*.

1.1.12 The universal property of the convolution algebra

The assignment

$$G \longrightarrow kG$$

extends to a functor from the category of binary structures to the category of k-algebras.

Exercise 2 *Show that any homomorphism*

$$f\colon G \longrightarrow A^{\times} \tag{6}$$

from a binary structure G into the multiplicative binary structure of a k-algebra A factorizes

$$\begin{tikzcd} & T \\ G \arrow[ur, "f"] \arrow[dr, tail, "i"'] & \\ & kG \arrow[uu, dashed, "\tilde{f}"'] \end{tikzcd}$$

for a unique homomorphism of k-algebras

$$\tilde{f}\colon kG \longrightarrow A. \tag{7}$$

Here $i\colon G \hookrightarrow kG$ denotes the inclusion of G into kG.

1.1.13

The convolution algebra is unital if and only if $(G, \cdot)$ has identity. It is associative if and only if $(G, \cdot)$ is a semigroup. If $(G, \cdot)$ is a group, it is referred to as the *group algebra* of G.

1.1.14

If (6) is a unital homomorphism, then (7) is a unital homomorphism.

1.2 Augmented k-algebras

1.2.1 An augmentation

A homomorphism

$$\epsilon\colon A \longrightarrow k \tag{8}$$

from a unital k-algebra A to k is said to be an *augmentation* if the composition with the *unit* map

$$\eta\colon k \longrightarrow A, \qquad 1_k \longmapsto 1_A, \tag{9}$$

is the identity on k,

$$\epsilon \circ \eta = \mathrm{id}_k\,. \tag{10}$$

1.2.2 Example: point augmentations on function algebras

Let $\mathcal{O} \subseteq k^X$ be a unital subalgebra of the algebra k^X of all k-valued functions on a set X. We shall refer to elements of X as *points*. For any $P \in X$, evaluation at point P,

$$\mathrm{ev}_P\colon \mathcal{O} \longrightarrow k, \qquad f \longmapsto f(P), \tag{11}$$

is an augmentation of $\mathcal{O}$. The correspondence

$$P \longmapsto \mathrm{ev}_P$$

defines a map

$$X \longrightarrow \mathrm{Hom}(\mathcal{O}, k) \tag{12}$$

where $\mathrm{Hom}(\mathcal{O}, k)$ denotes the set of morphisms in the category of unital associative k-algebras. This map is injective if and only if $\mathcal{O}$ *separates points* of X, i.e., if for any pair of distinct points $P \neq Q$, there exists a function $f \in \mathcal{O}$ such that $f(P) \neq f(Q)$.

1.2.3

A unital algebra equipped with an augmentation (8) is said to be *augmented*. The kernel of the augmentation is called *the augmentation ideal*.

1.3 Lie k-algebras

1.3.1

A k-algebra satisfying the Jacobi identity

$$(ab)c + (bc)a + (ca)b = 0 \qquad (a, b, c \in A) \tag{13}$$

and the identity

$$a^2 = 0 \qquad (a \in A) \tag{14}$$

is called a *Lie k-algebra*.

1.3.2 The Lie algebra associated with an associative multiplication

Given an algebra A, the pairing

$$(a, b) \longmapsto [a, b] := ab - ba \qquad (a, b \in A) \tag{15}$$

is referred to as the *commutator operation*. Operation (15) satisfies identity (14). It also satisfies (13), if the multiplication in A is associative. In particular, $(A, [\ ,\ ])$ is a Lie k-algebra. We shall denote it A_{Lie}.

1.3.3 Bracket notation

It is a standard practice to denote the product of elements in *any* Lie algebra not ab but $[a, b]$.

1.4 Modules over k-algebras

1.4.1

A module over a *nonunital nonassociative* k-algebra A has the same meaning as when A is expected to be equipped only with a k-module structure: it is a k-module equipped with a k-bilinear pairing (1).

1.4.2 Modules over unital k-algebras

In the context of unital algebras, M is supposed to be *unitary*, i.e., it is expected that

$$1_A m = m \qquad (m \in M). \tag{16}$$

Exercise 3 *Show that any module structure over a nonunital k-algebra A extends in a unique manner to a unitary $\tilde{A}$-module structure.*

In particular, the category of A-modules is isomorphic to the category of unitary $\tilde{A}$-modules.

1.4.3 Modules over associative k-algebras

In the context of associative algebras, M is supposed to be *associative*. The condition itself has two forms, one — for *left* A-modules, another one — for *right* A-modules.

1.4.4 Left A-modules

In a left A-module $\alpha(a, m)$ is denoted am and associativity of the A-action on M reads as the following identity

$$(ab)m = a(bm) \qquad (a, b \in A,\ m \in M). \tag{17}$$

Associativity of a left A-module is equivalent to k-linear map (2) being a homomorphism of k-algebras.

1.4.5 Right A-modules

In a left A-module $\alpha(a, m)$ is denoted ma and associativity of the A-action on M reads as the identity

$$m(ab) = (ma)b \qquad (m \in M;\ a, b \in A). \tag{18}$$

Associativity of a right A-module is equivalent to k-linear map (2) being an anti-homomorphism of k-algebras.

1.4.6

The category of associative right A-modules is isomorphic to the category of associative left A^{op}-modules.

1.4.7

In the context of unital associative algebras, identities (16) and (17) are both expected to hold.

1.4.8 Associative modules over not associative algebras

Distinction between left and right A-modules exists only when one of the two associativity conditions (17)–(18) is expected to hold. In this case k-linear map (2) is a homomorphism of k-algebras, algebra A itself does not have to be associative.

1.4.9 Associativization of a k-algebra

Any homomorphism of A into an associative k-algebra uniquely factorizes through the quotient $A^{\mathrm{ass}} = A_{/\sim}$ by the congruence relation generated by

$$(ab)c \sim a(bc) \qquad (a, b, c \in A).$$

The associative algebra A^{ass} is a *reflection* of a nonassociative algebra A in the subcategory of associative k-algebras.

In particular, the category of associative left A-modules is isomorphic to the category of left A^{ass}-modules, and similarly for right modules.

1.4.10 Modules over Lie k-algebras

Lie algebras with nonzero bracket operation are almost never associative themselves yet the Jacobi identity plays a role similar to associativity. This, however requires adopting the notion of a module: over a Lie algebra it is expected to satisfy the identity

$$a(bm) - b(am) = [a, b]m \qquad (a, b \in A;\ m \in M) \tag{19}$$

which is an analog of identity (17). For *right* modules the corresponding condition reads

$$(ma)b - (mb)a = m[a, b] \qquad (a, b \in A;\ m \in M). \tag{20}$$

1.4.11

Identity (19) is equivalent to k-linear map (2) being a homomorphism of Lie k-algebras, while (20) is equivalent to it being an anti-homomorphism.

1.4.12

Note that f is an anti-homorphism of Lie algebras if and only if $-f$ is a homomorphism. Thus, the distinction between left and right module structures over a Lie algebra is exclusively in the sign. This explains why, on one hand, there is little need to use the concept of a right module over a Lie algebra, but it also explains certain otherwise mysterious sign changes in various formulae involving dualization of Lie modules.

1.4.13 Notation

In order to simplify notation, in all cases we shall be denoting by A-mod and mod-A the corresponding categories of left and, respectively, right A-modules, with no distinction in meaning when the associativity conditions (or their analogs for Lie algebras) are not expected.

1.5 Bimodules

1.5.1

Given a pair of k-modules A and B, an (A, B)*-module* is a k-module M equipped with a structure of an A-module and of a B-module. Expecting morphisms to be simultaneaously A and B-linear, (A,B)-modules obviously form a category that will be denoted (A,B)-mod.

1.5.2

A pair of module structures on a k-module M attains particular importance if A acts on M by B-linear maps or, equivalently, if B acts on M by A-linear maps. Either of these condition can be expressed by the same identity that is particularly suggestive if the right multiplication notation is adopted for the action of B on M

$$(am)b = a(mb) \qquad (a \in A, m \in M, b \in B). \tag{21}$$

We shall refer to such (A,B)-modules as *bimodules* and will denote the corresponding full subcategory of (A,B)-mod by (A,B)-bimod or A-mod-B.

1.5.3

When $A = B$, we shall use A-bimod instead of (A,A)-mod or A-mod-A, and shall refer to (A,A)-bimodules as A-bimodules.

1.5.4

In the context of associative algebras M is expected to be an (associative) left A-module and an associative right B-module.

1.5.5

Note that any left A-module is automatically an (A,k)-bimodule and any right B-module is automatically a (k,B)-bimodule with k-bimodules being precisely k-modules. This underlies the approach according to which all modules, algebras and bimodules carry preexistent k-module structures and those additional structures respect the "ground" k-module structure.

2 Tensor product

2.1 Tensor product of k-modules

2.1.1

Given a k-bilinear map

$$\beta\colon M\times N \longrightarrow X \tag{22}$$

and a k-linear map $f\colon X\longrightarrow X'$, the composite

$$f\circ\beta\colon M\times N \longrightarrow X'$$

is k-bilinear. We say in such a situation that this bilinear map is *produced* from β by a k-linear map f.

2.1.2

A tensor product of k-modules M and N consists of a K-bilinear map

$$\tau\colon M\times N \longrightarrow T \tag{23}$$

such that *every* k-bilinear map from $M\times N$, af. (22), is produced from τ by a *unique* k-linear map $T\longrightarrow X$.

2.1.3

If

$$\tau'\colon M\times N \longrightarrow T'$$

is another bilinear map possessing this universal property, then T and T' are isomorphic by a *unique* isomorphism such that the diagram

$$\begin{tikzcd} & T \arrow[dd, dashed, "\simeq"] \\ M\times N \arrow[ur, "\tau"] \arrow[dr, "\tau'"'] & \\ & T' \end{tikzcd}$$

commutes.

2.1.4

The actual model for the tensor product of two k-modules is inessential: all properties of tensor product are encoded in its universal property of "converting" bilinear into linear maps.

2.1.5 Example: tensor product of free modules

Given sets X and Y, let kX and kY be the corresponding *free* k-modules with bases X and Y.

Exercise 4 *Show that the bilinear map*

$$\nu\colon kX\times kY \longrightarrow k(X\times Y), \qquad \left(\sum_{x\in X} a_x x, \sum_{y\in Y} b_y y\right) \longmapsto \sum_{(x,y)\in X\times Y} a_x b_y (x,y), \tag{24}$$

is a tensor product of kX and kY.

2.1.6 Notation

Generic notation used for the tensor product is

$$\otimes_k\colon M\times N \longrightarrow M\otimes_k N, \qquad (m,n)\longmapsto m\otimes_k n. \tag{25}$$

2.1.7 Caveat

Elements of $M\otimes_k N$ are referred to as *tensors*. Tensors of the form

$$t = m\otimes_k n$$

are said to be of *rank one*. General tensors, i.e., elements of $M\otimes_k N$, are sums of tensors of rank one,

$$t = \sum m_i \otimes_k n_i \tag{26}$$

yet such a representation is *highly nonunique*. Even a representation of rank one tensors is generally highly nonunique. For this reason, defining k-linear maps

$$f\colon M\otimes_k N \longrightarrow X \tag{27}$$

by defining them on tensors of rank one and then extending the "definition" to all tensors by additivity is incorrect even though it is a common practice.

The correct way to define a k-linear map (27) is by defining a k-bilinear map

$$\varphi\colon M\times N\longrightarrow X$$

that would induce it.

2.1.8

Given an (A,B)-bimodule M, the bilinear pairing

$$A\times B^{\mathrm{op}} \longrightarrow \mathrm{End}_{k\text{-mod}} M, \qquad \big(a,b^{\mathrm{op}}\big) \longmapsto {}_a\lambda_b,$$

where

$${}_a\lambda_b\colon m \longmapsto amb,$$

defines a k-linear map

$$A\otimes_k B^{\mathrm{op}} \longrightarrow \mathrm{End}_{k\text{-mod}} M$$

making M an $A\otimes_k B^{\mathrm{op}}$-module with

$$\big(a\otimes_k b^{\mathrm{op}}\big)m = amb.$$

2.1.9

If M is a unitary bimodule over unital k-algebras, then both the left A-module structure and the right B-module structure are encoded in the $A\otimes_k B^{\mathrm{op}}$-module structure:

$$am = \big(a\otimes_k 1_B^{\mathrm{op}}\big)m \qquad \text{and} \qquad mb = \big(1_A\otimes_k b^{\mathrm{op}}\big).$$

In particular, the categories A-mod-B and $A\otimes_k B^{\mathrm{op}}$-mod are in this case isomorphic.

2.2 Comodules and coalgebras

2.2.1

Tensor product allows us to express A-module structures on k-modules as k-linear maps

$$\lambda\colon A\otimes_k M \longrightarrow M, \tag{28}$$

and k-algebra structures as k-linear maps

$$\mu\colon A \otimes_k A \longrightarrow A. \tag{29}$$

If we reverse the direction of arrows in (28) and (29), we obtain the definition of an *A-comodule* and a *k-coalgebra*. The corresponding maps

$$M \longrightarrow A \otimes_k M \tag{30}$$

and

$$A \longrightarrow A \otimes_k A \tag{31}$$

are referred to as the *coaction of A on M* and the *comultiplication* on A. It is customary to denote the comultiplication by Δ.

2.2.2 The category of A-comodules

A map $f\colon M \longrightarrow M'$ of A-comodules is said to be a homomorphism if the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"'] & A \otimes_k M \arrow[d, "\mathrm{id}_A \otimes_k f"] \\ M' \arrow[r] & A \otimes_k M \end{tikzcd}$$

with the horizontal arrows being the corresponding coactions, commutes.

2.2.3 The category of k-coalgebras

A map $f\colon A \longrightarrow A'$ of A-coalgebras is said to be a homomorphism if the diagram

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "f"'] & A \otimes_k A \arrow[d, "f \otimes_k f"] \\ A' \arrow[r, "\Delta'"] & A \otimes_k A \end{tikzcd}$$

with the horizontal arrows being the corresponding comultiplications, commutes.

2.2.4 The "coconvolution" coalgebra

Given a set X equipped with a binary *cooperation* (referred to as *comultiplication*), i.e., wth a map

$$\kappa\colon X \longrightarrow X{\times}X, \tag{32}$$

the induced k-linear map of free k-modules

$$\tilde{\kappa}\colon kX \longrightarrow k(X{\times}X) \tag{33}$$

equips kX with a structure of k-algebra since, according to Exercise 4, $k(X{\times}X)$ is a model for $kX \otimes_k kX$.

2.2.5

If $\kappa(x) = (x', x'')$, then

$$x \longmapsto x' \otimes_k x''. \tag{34}$$

Since the multiplication on kX induced by a multiplication on X is called *convolution*, the comultiplication on kX could possibly be referred to as *coconvolution* (or even *volution*).

2.2.6 The diagonal k-coalgebra

Every set X is equipped with a canonical comultiplication, namely the diagonal map

$$\Delta\colon X \longrightarrow X{\times}X, \qquad x \longmapsto (x, x). \tag{35}$$

It induces the canonical k-coalgebra structure on the free module spanned by X,

$$\tilde{\Delta}\colon kX \longrightarrow kX \otimes_k kX, \qquad \sum_{x\in X} c_x x \longmapsto \sum_{x\in X} c_x \big(x \otimes_k x\big) \tag{36}$$

We shall denote $\big(kX, \tilde{\Delta}\big)$ by $\mathrm{Diag}_k X$ and refer to it as the *diagonal k-coalgebra* of a set X.

2.3 q-tuple tensor products

2.3.1

The definition of a tensor product of k-modules $M_1, \dots, M_q$ is obtained by replacing in the definition of a tensor product bilinear with q-linear maps. Generic notation is $M_1 \otimes_k \cdots \otimes_k M_q$ and the same caveat applies: linear maps

$$M_1 \otimes_k \cdots \otimes_k M_q \longrightarrow X$$

must be defined via q-linear maps

$$\beta\colon M_1 \times \cdots \times M_q \longrightarrow X. \tag{37}$$

2.3.2 q-tuple versus iterated tensor products

Note that the map

$$M_1 \times \cdots \times M_{p+q} \longrightarrow \big(M_1 \otimes \cdots \otimes M_p\big) \otimes \big(M_{p+1} \otimes \cdots \otimes M_{p+q}\big)$$

that sends $(m_1, \ldots, m_{p+q})$ to

$$\big(m_1 \otimes \cdots \otimes m_p\big) \otimes \big(m_{p+1} \otimes \cdots \otimes m_{p+q}\big)$$

is $(p+q)$-linear. Thus, it induces a k-module map

$$M_1 \otimes \cdots \otimes M_{p+q} \longrightarrow \big(M_1 \otimes \cdots \otimes M_p\big) \otimes \big(M_{p+1} \otimes \cdots \otimes M_{p+q}\big)$$

(from now on the unmarked symbol $\otimes$ will always stand for $\otimes_k$).

2.3.3 Existence of the inverse map

Bilinear maps

$$\big(M_1 \otimes \cdots \otimes M_p\big) \times \big(M_{p+1} \otimes \cdots \otimes M_{p+q}\big) \longrightarrow X$$

correspond to linear maps

$$M_1 \otimes \cdots \otimes M_p \longrightarrow \mathrm{Hom}_{k\text{-mod}}\big(M_{p+1} \otimes \cdots \otimes M_{p+q},\, X\big),$$

the latter correspond to p-linear maps

$$M_1 \times \cdots \times M_p \longrightarrow \mathrm{Hom}_{k\text{-mod}}\big(M_{p+1} \otimes \cdots \otimes M_{p+q},\, X\big),$$

and those correspond to p-linear maps into the k-module of q-linear maps

$$M_1 \times \cdots \times M_p \longrightarrow \mathrm{Hom}_{k\text{-mod}}\big(M_{p+1}, \cdots {}_{p+q};\, X\big).$$

Exercise 5 *Show that*

$$(m_1, \ldots, m_p) \longmapsto t_{m_1,\ldots,m_p}$$

where

$$t_{m_1,\ldots,m_p}\colon (m_{p+1}, \ldots, m_{p+q}) \ \longmapsto\ m_1 \otimes \cdots \otimes m_p \otimes m_{p+1} \otimes \cdots \otimes m_{p+q}$$

is such a map.

2.3.4 Commutativity of tensor product

Given a permutation σ of the set $\{1,\ldots,q\}$, the correspondence

$$(m_1,\cdots,m_q) \longmapsto m_{\sigma(1)}\otimes\cdots\otimes m_{\sigma(q)},$$

defines a q-linear map

$$M_1\times\cdots\times M_q \longrightarrow M_{\sigma(1)}\otimes\cdots\otimes M_{\sigma(q)}.$$

The latter induces a k-module homomorphism

$$\sigma_*\colon M_1\otimes\cdots\otimes M_q \longrightarrow M_{\sigma(1)}\otimes\cdots\otimes M_{\sigma(q)}.$$

The inverse permutation σ^{-1} induces the inverse map.

2.3.5 The action of the permutation groups on tensor powers

In particular, the permutation group Σ_q acts on the q-th tensor power of a k-module

$$M^{\otimes q} = M\otimes\cdots\otimes M \qquad (q \text{ times}). \tag{38}$$

2.4 Existence of tensor product

2.4.1

Any map (37) uniquely extends by k-linearity to a k-linear map

$$\tilde{\beta}\colon k(M_1\times\cdots\times M_q) \longrightarrow X$$

from the free k-module with basis $M_1\times\cdots\times M_q$ to X.

2.4.2

If β is q-linear, then $\tilde{\beta}$ vanishes on elements

$$c(m_1,\ldots,m_q) - (m_1,\ldots,cm_i,\ldots,m_q) \qquad (c\in k) \tag{39}$$

and

$$(m_1,\ldots,m_i'+m_i'',\ldots,m_q) - (m_1,\ldots,m_i',\ldots,m_q) - (m_1,\ldots,m_i',\ldots,m_q) \tag{40}$$

where m_i, m_i', m_i'' are arbitrary elements of M_i and $1\leq i\leq q$.

2.4.3

Thus, $\tilde{\beta}$ vanishes on the k-submodule

$$R \subseteq k(M_1 \times \cdots \times M_q)$$

generated by these elements and therefore it induces a k-linear map $\bar{\beta}$ from the quotient module to X. We obtain the following diagram with commutative triangles

$$\begin{tikzcd}
M_1 \times \cdots \times M_q \arrow[rr, "\beta"] \arrow[dr, tail, "\iota"] \arrow[ddr, "\tau := \pi \circ \iota"'] & & X \\
& k(M_1 \times \cdots \times M_q) \arrow[ur, dashed, "\tilde{\beta}"] \arrow[d, two heads, "\pi"] & \\
& k(M_1 \times \cdots \times M_q)/R \arrow[uur, dashed, "\bar{\beta}"'] &
\end{tikzcd}$$

2.5

The map $\tau := \pi \circ \iota$ obtained by composing the inclusion map

$$\iota \colon M_1 \times \cdots \times M_q \hookrightarrow k(M_1 \times \cdots \times M_q)$$

with the quotient map

$$\pi \colon k(M_1 \times \cdots \times M_q) \twoheadrightarrow k(M_1 \times \cdots \times M_q)/R$$

is obviously q-linear.

Exercise 6 *Show that if β' and β'' are two k-linear maps*

$$k(M_1 \times \cdots \times M_q)/R \longrightarrow X$$

such that $\beta' \circ \tau = \beta = \beta' \circ \tau$, then $\beta' = \beta''$.

This demonstrates that

$$\tau \colon M_1 \times \cdots \times M_q \longrightarrow k(M_1 \times \cdots \times M_q)/R$$

is a tensor product of $M_1, \ldots, M_q$.

2.6 Properties of algebras and coalgebras via diagrams

2.6.1 Unitality and counitality

Unitality of a k-algebra structure (29) can be expressed as an existence of a k-module map $\eta\colon k \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
 & A & \\
k \otimes A \arrow[ur, "\simeq"] \arrow[dr, "\eta\otimes\mathrm{id}_A"'] & & A \otimes k \arrow[ul, "\simeq"'] \arrow[dl, "\mathrm{id}_A\otimes\eta"] \\
 & A \otimes A \arrow[uu, "\mu"'] &
\end{tikzcd}$$

commutes. Dually, counitality of a k-coalgebra structure (29) can be expressed as an existence of a k-module map $\epsilon\colon A \longrightarrow k$ such that the diagram

$$\begin{tikzcd}
 & A \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] \arrow[dd, "\Delta"] & \\
k \otimes A & & A \otimes k \\
 & A \otimes A \arrow[ul, "\epsilon\otimes\mathrm{id}_A"] \arrow[ur, "\mathrm{id}_A\otimes\epsilon"'] &
\end{tikzcd}$$

commutes.

2.6.2 Associativity and coassociativity

Associativity of a k-algebra structure (29) can be expressed as commutativity of the diagram

$$\begin{tikzcd}
A^{\otimes 3} \arrow[r, "\mathrm{id}_A\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}_A"'] & A^{\otimes 2} \arrow[d, "\mu"] \\
A^{\otimes 2} \arrow[r, "\mu"] & A
\end{tikzcd}$$

and, dually, commutativity of the diagram

$$\begin{tikzcd}
A^{\otimes 3} & A^{\otimes 2} \arrow[l, "\mathrm{id}_A \otimes \Delta"'] \\
A^{\otimes 2} \arrow[u, "\Delta \otimes \mathrm{id}_A"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

expresses coassociativity of a k-coalgebra structure (31).

2.6.3 Commutativity and cocommutativity

Commutativity of a k-algebra structure (29) can be expressed as commutativity of the diagram

$$\begin{tikzcd}
A^{\otimes 2} \arrow[dr, "\mu"] \arrow[dd, leftrightarrow, "\mathrm{flip}"'] & \\
& A \\
A^{\otimes 2} \arrow[ur, "\mu"'] &
\end{tikzcd}$$

and, dually, commutativity of the diagram

$$\begin{tikzcd}
A^{\otimes 2} \arrow[dd, leftrightarrow, "\mathrm{flip}"'] & \\
& A \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\
A^{\otimes 2} &
\end{tikzcd}$$

expresses cocommutativity of a k-coalgebra structure (31).

2.6.4 A coaugmentation

Note that the isomorphism

$$k \longrightarrow k \otimes_k k, \qquad 1_k \longmapsto 1_k \otimes_k 1_k,$$

is a coassociative cocommutative comultiplication with the identity map id_k serving the double purpose of supplying the unit and the counit.

2.6.5

A homomorphism $\eta\colon k\longrightarrow C$ of counital k-coalgebras is said to be a *coaugmentation* if $\epsilon\circ\eta = \mathrm{id}_k$. This is the same condition that was used to define augmentations of unital algebras.

2.6.6 The module of primitive elements

A coalgebra equipped with a coaugmentation $\eta\colon k\longrightarrow C$ is said to be *coagmentated*. We shall denote $\eta(1_k)$ by 1_C even though no multiplication on C is specified. Elements $c\in C$ such that

$$\Delta(c) = 1_C\otimes c + c\otimes 1_C \tag{41}$$

are said to be *primitive*. They form a k-submodule of C denoted $\mathrm{Prim}\, C$.

2.6.7

If one replaces in the above diagrams a *k-module* A by a *set* X, the k-module k by the single element set

$$X^0 := \{\emptyset \hookrightarrow X\}, \tag{42}$$

and $\otimes_k$ by $\times$, then one obtains the corresponding notions of unitality and counitality, associativity and coassociativity, commutativity and cocommutativity—for binary operations and, respectively, cooperations on sets.

2.6.8 Cosemigroups and comonoids

Sets equipped with a coassociative comultiplication are referred to as *cosemigroups*. Counital cosemigroups are referred to as *comonoids*. They form the corresponding categories denoted Cosgr and, respectively, Comon.

2.6.9 Integral $\int_X$ as the counit map in the diagonal coalgebra

Exercise 7 *Show that the map*

$$\int_X\colon kX \longrightarrow k, \qquad \sum_{x\in X} c_x x \longmapsto \sum_{x\in X} c_x, \tag{43}$$

is induced by the counit map

$$X \longrightarrow X^{\mathrm{o}} \tag{44}$$

of the diagonal comonoid (X, Δ).

Accordingly, $\int_X$ is a counit map in the coalgebra $\mathrm{Diag}_k X$.

2.7 Tensor product of k-algebras

2.7.1

The tensor product of k-algebras A and B is naturally equipped with a k-algebra structure. Since the map

$$A \times B \times A \times B \longrightarrow A \otimes B, \qquad (a, b, a', b') \longmapsto (aa') \otimes (bb')$$

is k-linear in each argument, it induces a k-linear map

$$(A \otimes B) \otimes (A \otimes B) \simeq A \otimes B \otimes A \otimes B \;\longrightarrow\; A \otimes B.$$

Note that $A \otimes B$ is associative or commutative if *both* algebras have these properties. Note also

$$1_{A \otimes B} = 1_A \otimes 1_B.$$

2.7.2 The universal property of the tensor product of k-algebras

Given a pair of unital k-algebras, maps

$$A \longrightarrow A \otimes B, \quad a \longrightarrow a \otimes 1_B \qquad \text{and} \qquad B \longrightarrow A \otimes B, \quad b \longrightarrow 1_A \otimes b,$$

are unital k-algebra homomorphisms. They commute:

$$(a \otimes 1_B)(1_A \otimes b) = a \otimes b = (1_A \otimes b)(a \otimes 1_B).$$

We say that maps

$$f\colon A \longrightarrow C \qquad \text{and} \qquad g\colon B \longrightarrow C \tag{45}$$

into a k-algebra C *commute* if

$$f(a)g(b) = g(b)f(a) \qquad (a \in A, b \in B). \tag{46}$$

Exercise 8 *Show that any commuting pair* (45) *of homomorphisms of unital k-algebras uniquely factorizes through* $A \otimes B$.

In other words, commuting pairs of unital homomorphisms (45) correspond to unital homomorphisms

$$A \otimes B \longrightarrow C.$$

2.8 The $k(\,)$-functor

2.8.1

The assignment

$$(X,\kappa) \longmapsto (kX,\tilde{\kappa}) \tag{47}$$

gives rise to a functor

$$k(\,)\colon \mathrm{Comult} \longrightarrow k\text{-coalg} \tag{48}$$

from the category Comult of sets equipped with a comultiplication (32) to the category of k-coalgebras.

2.8.2

It also gives rise to a number of related functors:

$$\mathrm{Comult}_{\mathrm{un}} \longrightarrow k\text{-coalg}_{\mathrm{un}}, \quad \mathrm{Comult} \longrightarrow k\text{-coalg}_{\mathrm{com}},$$

as well as

$$\mathrm{Cosgr} \longrightarrow k\text{-coalg}_{\mathrm{ass}} \qquad \text{and} \qquad \mathrm{Comon} \longrightarrow k\text{-coalg}_{\mathrm{un,ass}},$$

where subscripts $_{\mathrm{un}}$, $_{\mathrm{com}}$ and $_{\mathrm{ass}}$ signify *counitality*, *cocommutativity* and *coassociativity* of the corresponding structures.

2.8.3 The diagonal k-coalgebra functor

The assignment

$$X \longmapsto \mathrm{Diag}_k X \tag{49}$$

gives rise to a functor

$$\mathrm{Diag}_k\colon \mathrm{Set} \longrightarrow k\text{-coalg}_{\mathrm{un,ass,com}} \tag{50}$$

from the category of sets to the category of counital, associative and commutative k-coalgebras.

2.8.4 Tensor product in the category of sets

Cartesian product in the category of sets behaves like tensor product of k-modules and plays the same role: it converts *bimorphisms* into *morphisms*. One is therefore advised to think of it as of being *the* tensor product on Set.

2.8.5 Monoidal categories

Both $(\mathrm{Set}, \times)$ and $(k\text{-mod}, \otimes)$ are examples of *symmetric monoidal categories*. Monoidal categories are categories equipped with bifunctors

$$\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

satisfying conditions reflecting the unitality and associativity properties of tensor product on the category of k-modules. The epithet *symmetric* refers to the additional property reflecting commutativity property of $\otimes$.

2.8.6 The free k-module functor as a monoidal functor

Exercise 24 expresses the fact that the functor,

$$\mathrm{Set} \longrightarrow k\text{-mod}, \qquad X \longmapsto \text{the free } k\text{-module spanned by } X,$$

is a *monoidal functor*: it preserves *multiplication* on these categories provided by Cartesian product in the case of Set, and by tensor product—in the case of k-mod.

2.8.7

The free k-module functor is the composite

$$\mathrm{Set} \xrightarrow{\mathrm{Diag}} \mathrm{Comon}_{\mathrm{com}} \xrightarrow{k(\ )} k\text{-coalg}_{\mathrm{un,ass,com}} \xrightarrow{F} k\text{-mod}$$

where Diag sends a set X to the diagonal comonoid (X, Δ) and F is a forgetful functor: it sends a k-coalgebra to the underlying k-module.

2.8.8

In fact, each of the above functors is monoidal: both $(\mathrm{Comon}_{\mathrm{com}}, \times)$ and $(k\text{-coalg}_{\mathrm{un,ass,com}}, \otimes)$ are symmetric monoidal categories and the functors preserve multiplication.

2.8.9 Tensor product of k-coalgebras

Exercise 9 *Show that a tensor product $C \otimes C'$ of a k-coalgebra (C, Δ) and of a k-coalgebra (C', Δ') is naturally equipped with a structure of a k-coalgebra.*

Tensor product of counital coalgebras is counital, of coassociative coalgebras is coassociative and, finally, of cocommutative coalgebras is cocommutative.

2.9 Bialgebras

2.9.1

Suppose a k-module B is equipped with a k-algebra structure μ and a k-coalgebra structure Δ. We say that (B, μ, Δ) is a *k-bialgebra* if the diagram

$$\begin{tikzcd} A^{\otimes 2} \arrow[r, "\mu"] \arrow[d, "\Delta^{\otimes 2}"'] & A \arrow[d, "\Delta"] \\ A^{\otimes 4} \arrow[r, "\mu^{\otimes 2}"] & A^{\otimes 2} \end{tikzcd}$$

commutes. Commutativity of this diagram simultaneously expresses the fact that the multiplication is a homomorphism of k-coalgebras and that the comultiplication is a homomorphism of algebras.

2.9.2 The Lie algebra of primitive elements

If 1_B is an identity element for multiplication, then we can adopt the definition of primitive elements previously given in the case of coaugmented coalgebras. Let us call an element $b \in B$ *primitive* if

$$\Delta(b) = 1_B \otimes b + b \otimes 1_B.$$

Exercise 10 *Show that the k-module* $\operatorname{Prim} B$ *of primitive elements in a bialgebra B is closed with respect to the commutator operation* (15).

If multiplication in B is associative, then $\operatorname{Prim} B$ becomes a Lie subalgebra of $\big(B, [\ ,\ ]\big)$.

2.9.3 Unital and counital k-bialgebras

A bialgebra equipped with a unit map $\eta\colon k \longrightarrow B$ that is a homomorphism of coalgebra structures will be called *unital*.

Dually, a bialgebra equipped with a counit $\epsilon\colon B \longrightarrow k$ that is a homomorphism of algebra structures will be called *counital*.

2.9.4 The convolution k-bialgebra

For a binary structure G, the diagonal map

$$\Delta\colon G \longrightarrow G \times G, \qquad g \longmapsto (g, g),$$

is a homomorphism of binary structures, hence

$$\tilde{\Delta}\colon kG \longrightarrow k(G \times G)$$

is a homomorphism of counital k-algebras. In particular, kG is a counital coassocitaive and cocommutative bialgebra. If multiplication in G is unital, associative, or commutative, the convolution operation $*$ ois likewise unital, associative or, respectively, commutative.

2.10 Tensor product of A-modules

2.10.1 Balanced bilinear maps

If M and N are modules over a k-module A, then a k-bilinear map (22) is said to be *balanced*, if it satisfies the identity

$$\beta(ma, n) = \beta(m, an) \qquad (m \in M, a \in A, n \in N). \tag{51}$$

We use the right multiplication notation for M even though neither M nor N are assumed to be associative A-modules.

If A is expected to be an associative k-algebra, however, then M is expected to be an associtive *right*, and N is expected to be an associtive *left* A-module. For this reason we shall use the terminology reflecting this notation and will speak of M as a right A-module and of N as a left A-module.

2.10.2

A tensor product of A-modules is an A-balanced k-bilinear map (23) from which any A-balanced k-bilinear map (22) is produced by a unique k-linear map $T \longrightarrow X$.

2.10.3 Notation

Generic notation used for the tensor product of A-modules is

$$\otimes_k \colon M \times N \longrightarrow M \otimes_A N, \qquad (m, n) \longmapsto m \otimes_A n. \tag{52}$$

This is consistent with the fact that every k-bilinear map is automatically k-balanced.

Exercise 11 *Show that any k-bilinear map is k-balanced.*

2.10.4 Functoriality

The tensor product is not unique yet any assignment of a tensor product (23) to every pair (M, N) of right and left A-modules *uniquely* extends to a functor

$$T \colon \text{mod-}A \times A\text{-mod} \longrightarrow k\text{-mod}.$$

Such functors are also viewed as bifunctors, i.e., functors of 2 arguments. Generic notation for T is $\otimes_A$.

Exercise 12 *If $f \colon M \longrightarrow M'$ is an A-module map, show that*

$$M \times N \longrightarrow M' \otimes_A N, \qquad (m, n) \longmapsto f(m) \otimes_A n, \qquad (m \in M, n \in N),$$

is a balanced k-bilinear map.

2.10.5

If M is an (A', A)-bimodule, then $M \otimes_A N$ is equipped with an A'-module structure in view of functoriality of $\otimes_A$.

In particular, $\otimes_A$ gives rise to a bifunctor

$$\otimes_A \colon A'\text{-mod-}A \times A\text{-mod} \longrightarrow A'\text{-mod}.$$

2.10.6

One similarly has bifunctors

$$\otimes_A : \text{mod-}A \times A\text{-mod-}A'' \longrightarrow \text{mod-}A''.$$

and

$$\otimes_A : \ A'\text{-mod-}A \times A\text{-mod-}A'' \longrightarrow A'\text{-mod-}A''.$$

2.10.7

In the special case $A' = A = A''$, tensor product bifunctors define a "multiplication" on the category of A-bimodules

$$\otimes_A : \ A\text{-bimod} \times A\text{-bimod} \longrightarrow A\text{-bimod}.$$

It makes the category of A-bimodules into a *monoidal* category, i.e., a "monoid in the category of categories".

2.10.8 A-algebras

A bimodule B over a k-module A, equipped with a balanced k-bilinear pairing

$$B \times B \longrightarrow B, \tag{53}$$

will be referred to as an A-algebra. Equivalently, it is an A-bimodule with an A-bimodule map

$$B \otimes_A B \longrightarrow B.$$

2.10.9 q-tuple tensor products of (A_{i-1}, A_i)-modules

Given an A_1-module M_1, an (A_1, A_2)-module M_2, ..., an (A_{q-2}, A_{q-1})-module M_{q-1}, and an A_{q-1}-module M_q, one defines *balanced* q-linear maps (37) in a similar way as we did for $q = 2$. Then, a tensor product is an $(A_1, \ldots, A_{q-1})$-balanced q-linear map

$$M_1 \times \cdots \times M_q \longrightarrow T$$

from which any $(A_1, \ldots, A_{q-1})$-balanced k-bilinear map (37) is produced by a unique k-linear map $T \longrightarrow X$.

Exercise 13 *Prove the existence of a tensor product.*

2.10.10

Generic notation:

$$M_1 \times \cdots \times M_q \longrightarrow M_1 \otimes_{A_1} \cdots \otimes_{A_{q-1}} M_q.$$

2.10.11

If M_1 is an (A_0, A_1)-bimodule and M_q is an (A_q, A_{q+1})-bimodule, then q-tuple tensor product of (A_{i-1}, A_i)-modules gives rise to a q-functor from

$$A_0\text{-mod-}A_1 \times (A_1, A_2)\text{-mod} \times \cdots \times (A_{q-1}, A_q)\text{-mod} \times A_q\text{-mod-}A_{q+1}$$

to the category of (A_0, A_{q+1})-bimodules

$$A_0\text{-mod-}A_{q+1}.$$

Exercise 14 *Explain why the left A_0 action on $M_1 \otimes_{A_1} \cdots \otimes_{A_{q-1}} M_q$ commutes with the right A_{q+1}-action.*

2.11 The tensor algebra

2.11.1 The tensor algebra of a k-module

Given a k-module, the direct sum of all positive tensor powers,

$$\tilde{T}_k M := \bigoplus_{q>0} M^{\otimes q}, \tag{54}$$

(cf. (38), is naturally equipped with an associative multiplication such that

$$(m_1 \otimes \cdots \otimes m_p)(m'_1 \otimes \cdots \otimes m'_q) = m_1 \otimes \cdots \otimes m_p \otimes m'_1 \otimes \cdots \otimes m'_q.$$

We shall refer to (54) as the *nonunital tensor algebra* of a k-module M. Its unitalization,

$$T_k M = \bigoplus_{q \geq 0} M^{\otimes q}, \tag{55}$$

will be referred to as the *tensor algebra* of M.

2.11.2 Universal properties of $\bar{T}_k M$ and $T_k M$

Exercise 15 *Show that any k-linear map $f\colon M \longrightarrow B$ from a k-module M to an associative k-algebra B has a unique extension to a homomorphism of k-algebras*

$$\hat{f}\colon \bar{T}_k M \longrightarrow B. \tag{56}$$

Show that a unital homomorphism has a unique extension to a unital homomorphism $T_k M \longrightarrow B$.

2.11.3

If M is a module over a k-module A, then the action of A on M being a k-linear map (2), extends in a unique manner to a unitary associative $T_k A$-module structure. Thus the categories A-mod and $T_k A$-mod are naturally isomorphic.

Exercise 16 *Show that, for any unital associative k-algebra A, there exists a surjective homomorphism*

$$T_k A \longrightarrow A$$

such that

$$a_1 \otimes \cdots \otimes a_q \longmapsto a_1 \cdots a_q.$$

2.11.4 The symmetric algebra of a k-module

Let $S_k M$ be the quotient of $T_k M$ by the congruence generated by the relation

$$t \otimes t' \simeq t' \otimes t \qquad (t, t' \in T_k M). \tag{57}$$

As a consequence of Exercise 15, any k-linear map $f\colon M \longrightarrow A$ into a unital associative and *commutative* algebra uniquely factorizes as

$$\begin{tikzcd} & A \\ M \arrow[ur, "f"] \arrow[dr, "\check{\iota}"'] & \\ & S_k M \arrow[u, dashed, "\check{f}"'] \end{tikzcd} \tag{58}$$

where $\check{\iota}$ is obtained by composing the inclusion $M \hookrightarrow T_k M$ with the quotient epimorphism $T_k M \longrightarrow S_k M$, and $\check{f}$ is a unital k-algebra homomorphism.

2.11.5

The equivalence class of

$$m_1 \otimes \cdots \otimes m_p$$

is denoted $m_1 \cdots m_p$ and the class of

$$m \otimes \cdots \otimes m \qquad (q \text{ times})$$

is denoted m^q.

2.11.6 The algebra of polynomial functions

Elements of the symmetric algebra $S_k(M^\vee)$ of the *dual k-module*

$$M^\vee := \mathrm{Hom}_{k\text{-mod}}(M, k) \tag{59}$$

are referred to as *polynomial functions* on a k-module M.

2.11.7 The exterior algebra of a k-module

Let $\Lambda_k M$ be the quotient of $T_k M$ by the congruence generated by the relation

$$t \otimes t \simeq 0 \qquad (t \in T_k M). \tag{60}$$

As a consequence of Exercise 15, any k-linear map $f\colon M \longrightarrow A$ into a unital associative algebra such that

$$f(M)^2 = 0,$$

uniquely factorizes as

$$\begin{tikzcd} & A \\ M \arrow[ur, "f"] \arrow[dr, "\hat{\imath}"'] & \\ & \Lambda_k M \arrow[u, dashed, "\hat{f}"'] \end{tikzcd} \tag{61}$$

where $\hat{\imath}$ is obtained by composing the inclusion $M \hookrightarrow T_k M$ with the quotient epimorphism $T_k M \longrightarrow \Lambda_k M$, and $\hat{f}$ is a unital k-algebra homomorphism.

2.11.8

The equivalence class of

$$m_1 \otimes \cdots \otimes m_p$$

is denoted $m_1 \wedge \cdots \wedge m_p$.

2.11.9

By considering the inclusion $f\colon M \hookrightarrow \tilde{M}$ of M into unitalization of M considered as an algebra with zero multiplication, we deduce that both

$$M \xrightarrow{\ \check{\imath}\ } S_k M \qquad \text{and} \qquad M \xrightarrow{\ \check{\imath}\ } \Lambda_k M$$

are injective.

2.11.10 Augmentation

Sending all tensors of positive degree to 0

$$\epsilon\colon T_k M \longrightarrow k, \qquad m_1 \otimes \cdots \otimes m_q \longmapsto 0 \qquad (q > 0), \tag{62}$$

defines an augmentation of $T_k M$. he augmentation The augmentation ideal,

$$\bar{T}_k M := \bigoplus_{q>0} M^{\otimes q}, \tag{63}$$

is referred to as the *nonunital tensor algebra of a k-module M*.

2.11.11 Comultiplication of tensors

Exercise 17 *Show that there exists a k-linear map*

$$\Delta\colon T_k M \longrightarrow T_k M \otimes T_k M, \tag{64}$$

such that

$$\Delta(m_1 \otimes \cdots \otimes m_q) \longmapsto \sum_{i=0}^{q} \big(m_1 \otimes \cdots \otimes m_i\big) \otimes \big(m_{i+1} \otimes \cdots \otimes m_q\big).$$

Note that the corresponding summands for $i = 0$ and $i = q$ are

$$1_k \otimes \big(m_1 \otimes \cdots \otimes m_q\big) \qquad \text{and} \qquad \big(m_1 \otimes \cdots \otimes m_q\big) \otimes 1_k,$$

respectively. Show that (64) *is homomorphism of algebras.*

2.11.12 The tensor algebra of a bimodule

Given an A-bimodule M, the A-bimodule

$$\bar{T}_A M := \sum_{q>0} M^{\otimes_A q} \tag{65}$$

is a quotient of the nonunital tensor algebra $\beta T_k M$ and thus an associative k-algebra with balanced multiplication.

2.11.13

If A is equipped with an associative multiplication and both the left and the right actions of A on M are associative, then

$$T_A M := A \oplus \bar{T}_A M \tag{66}$$

is an associative A-algebra. If A is unital and M is unitary both as a left and a right module over A, then (66) is unital.

2.12 The universal enveloping algebra of a Lie algebra

2.12.1

Given a Lie k-algebra $\mathfrak{g}$, the quotient of $T_k\mathfrak{g}$ by the 2-sided ideal generated by elements

$$[g,h]_{T_k\mathfrak{g}} - [g,h]_{\mathfrak{g}} \tag{67}$$

is called the *universal enveloping algebra* of a Lie k-algebra $\mathfrak{g}$ and is denoted $\mathcal{U}_k\mathfrak{g}$. Note that the first commutator equals

$$g \otimes h - h \otimes g$$

and thus belongs to $\mathfrak{g}^{\otimes 2}$ while the second belongs to $\mathfrak{g}$.

2.12.2

The universal enveloping algebra is a unital and counital associative and coassociative cocommutative k-algebra. It inherits these structures from $T_k\mathfrak{g}$.

2.12.3

Exercise 18 *Show that, given a $\mathfrak{g}$-module M, the action of $\mathfrak{g}$ on M extends uniquely to a left $\mathcal{U}_k\mathfrak{g}$-structure and, vice-versa, restriction of a left $\mathcal{U}_k\mathfrak{g}$-structure to $\mathfrak{g}$ defines a $\mathfrak{g}$-module structure.*

In particular, the category of $\mathfrak{g}$-modules is isomorphic to the category $\mathcal{U}_k\mathfrak{g}$-mod.

2.12.4 The universal property of $\mathcal{U}_k\mathfrak{g}$

Exercise 19 *Show that any homomorphism of Lie k-algebras*

$$f\colon \mathfrak{g} \longrightarrow A_{\mathrm{Lie}}$$

into the Lie algebra $(A, [\,,\,])$ of a unital associative k-algebra A, extends in a unique manner to a homomorphism of unital k-algebras

$$\hat{f}\colon \mathcal{U}_k\mathfrak{g} \longrightarrow A.$$

2.12.5

Any unital counital associative coassociative k-bialgebra B, is obviously a $\operatorname{Prim} B$-module. The inclusion of the Lie k-algebra of primitive elements into B induces a homomorphism of unital associative algebras

$$\mathcal{U}_k \operatorname{Prim} B \;\longrightarrow\; B.$$

It is surjective precisely when B is *primitively generated*, i.e., is generated as an algebra by the k-submodule of primitive elements. Under quite general conditions this canonical homomorphism is an isomorphism. For example, when $B = \mathcal{U}_k\mathfrak{g}$ which is equivalent to saying that

$$\operatorname{Prim} \mathcal{U}_k\mathfrak{g} = \mathfrak{g}. \tag{68}$$

2.12.6 The symmetric algebra of a k-module

The universal enveloping algebra of the Lie algebra M with zero bracket operation coincides with the *symmetric algebra* of M.

2.13 Base change

2.13.1

Whether A and B are expected to be k-modules or k-algebras—unital, associative, or both—a morphism $\phi\colon A \longrightarrow B$ induces a functor

$$\phi^*\colon\ B\text{-mod} \longrightarrow A\text{-mod} \tag{69}$$

between the corresponding categories of modules. Given a B-module N, $\phi^* N$ is the k-module N equipped with the A-action

$$an \ \coloneqq\ \phi(a)n \qquad (a \in A, n \in N). \tag{70}$$

2.13.2

Functor (69) has a left adjoint

$$\phi_*\colon\ A\text{-mod} \longrightarrow B\text{-mod} \tag{71}$$

which means that there is a correspondence

$$\mathrm{Hom}_{A\text{-mod}}(M, \phi^* N) \longleftrightarrow \mathrm{Hom}_{B\text{-mod}}(\phi_* M, N)$$

natural in $M \in \mathrm{Ob}\, A\text{-mod}$ and $N \in \mathrm{Ob}\, B\text{-mod}$.

2.13.3 The left adjoint for modules over unital associative algebras

For any k-module map $f\colon M \longrightarrow N$ such that

$$f(am) = \phi(a)f(m) \qquad (a \in A, m \in M),$$

there exists a B-linear map ${}_B\tilde{f}\colon B \otimes_A M \longrightarrow N$, such that the diagram

$$\begin{tikzcd} & N \\ M \arrow[ur, "f"] \arrow[dr, "\eta_M"'] & \\ & B \otimes_A M \arrow[uu, dashed, "{}_B\tilde{f}"'] \end{tikzcd} \tag{72}$$

commutes where

$$\eta_M\colon m \longmapsto 1 \otimes_A m \qquad (m \in M). \tag{73}$$

This map is induced by the bilinear map

$$(b, m) \longmapsto bf(m).$$

Exercise 20 *Prove that a B-module map $\tilde{f}\colon B \otimes_A M \longrightarrow N$ making diagram* (72) *commute is unique.*

In particular, the pair of correspondences

$$M \longmapsto B \otimes_A M \qquad \text{and} \qquad f \longmapsto {}_B\tilde{f}, \tag{74}$$

where M denotes an A-module and f denotes an A-module map, defines a functor left adjoint to ϕ^*.

2.13.4 The left adjoint for modules over nonunital associative algebras

Modules over a nonunital algebra are the same as unitary modules over its unitalization, cf. Exercise 3. Therefore the left adjoint functor ϕ_* in nonunital associative case is provided by

$$M \longmapsto \tilde{B} \otimes_{\tilde{A}} M \qquad \text{and} \qquad f \longmapsto {}_{\tilde{B}}\tilde{f} \tag{75}$$

2.13.5 The left adjoint for modules over k-modules

Modules over a k-module are the same as unitary modules over its tensor algebra, cf. Section 2.11.3. Therefore in this case the left adjoint functor ϕ_* is provided by

$$M \longmapsto T_k B \otimes_{T_k A} M \qquad \text{and} \qquad f \longmapsto {}_{T_k B}\tilde{f}. \tag{76}$$

2.14 Linearization

2.14.1

When ϕ is the unit homomorphism $\eta\colon k \longrightarrow A$ and $f\colon M \longrightarrow N$ is a k-linear map from a k-module M to an A-module N, we shall refer to

$$\tilde{f}\colon A \otimes_k M \longrightarrow N \tag{77}$$

as the *linearization* of f.

2.14.2

Let us observe that the linearization of the unit homomorphism (9) is the identity map $\mathrm{id}_A\colon A{\longrightarrow}A$ while the linearization of id_A is the multiplication map on A expressed as a k-linear map

$$A \otimes A \longrightarrow A, \qquad \sum a_i \otimes b_i \longmapsto \sum a_i b_i. \tag{78}$$

2.15 The nonassociative tensor algebra

2.15.1

The forgetful functor

$$k\text{-alg}_{\text{un,ass}} \longrightarrow k\text{-mod} \tag{79}$$

that sends an algebra to the underlying k-module (it *forgets* the multiplication), has a left adjoint, namely

$$M \longmapsto T_k M,$$

cf. Exercise 15. Forgetful functor (79) is a restriction of the forgetful functor from the category k-alg of nonunital nonassociative k-algebras

$$k\text{-alg} \longrightarrow k\text{-mod}$$

that also has a left adjoint functor.

2.15.2

The nonunital tensor k-algebra $\bar{T}_k M$ is the direct sum of q-tuple tensor powers of M. One can employ $q-1$ iterations of the *binary* tensor product $\otimes_k$ to obtain a model for the q-tuple tensor product. The *nonassociative tensor algebra* replaces a single q-tuple tensor power of M by the direct sum of *all* realizations of such a power by means of the repeated use of $\otimes_k$.

There are as many such realizations as there are ways to place $q-1$ pairs of parentheses around q symbols in order to produce a recipe for multiplication of q elements in a binary structure. This number equals the number of words of length q in the *free nonassociative* binary structure $F\{\bullet\}$ generated by the set containing a single element $\bullet$. If $w \in F\{\bullet\}$, we shall denote by $M^{\otimes w}$ the corresponding realization of the tensor power of M. Thus,

$$M^{\otimes \bullet} = M, \qquad M^{\otimes \bullet\bullet} = M \otimes M,$$

and, continuing,

$$M^{\otimes (\bullet\bullet)\bullet} = (M \otimes M) \otimes M, \qquad M^{\otimes \bullet(\bullet\bullet)} = M \otimes (M \otimes M),$$

and

$$M^{\otimes ((\bullet\bullet)\bullet)\bullet} = ((M \otimes M) \otimes M) \otimes M, \quad M^{\otimes (\bullet\bullet)(\bullet\bullet)} = (M \otimes M) \otimes (M \otimes M), \ \ldots$$

2.15.3

Let

$$\bar{\mathfrak{T}}_k M := \bigoplus_{w \in F\{\bullet\}} M^{\otimes w}. \tag{80}$$

Multiplication on (80) is given by bilinear pairings

$$M^{\otimes w} \times M^{\otimes' w} \longrightarrow M^{\otimes ww'}, \qquad (t, t') \longmapsto t \otimes t',$$

where ww' denotes the product of w and w' in $F\{\bullet\}$.

2.15.4 Another realization of $\bar{\mathfrak{T}}_k M$

In Łukasiewicz's parentheses-free notation elements of $F\{\bullet\}$ become sequences of symbols $|$ and $\bullet$,

$$\bullet, \quad |\bullet\bullet, \quad ||\bullet\bullet\bullet, \quad |\bullet|\bullet\bullet, \quad |||\bullet\bullet\bullet\bullet, \quad ||\bullet\bullet|\bullet\bullet, \quad \ldots$$

with multiplication being just the concatenation of sequences with $|$ placed in front, e.g.,

$$\big(|\bullet\bullet, ||\bullet\bullet\bullet\big) \longmapsto ||\bullet\bullet||\bullet\bullet\bullet.$$

For every element of $F\{\bullet\}$ of length q, represented as a sequence of $q-1$ symbols $|$ and q symbols $\bullet$, let M_w denote the $2q-1$-tuple product

$$M_w := M'_1 \otimes \cdots \otimes M'_{2q-1} \tag{81}$$

where M'_i is either

$$M^{\otimes 0} = k \qquad \text{or} \qquad M,$$

depending on whether the i-th symbol in the sequence is $|$ or $\bullet$. For example,

$$M_{||\bullet\bullet||\bullet\bullet\bullet} = k \otimes k \otimes M \otimes M \otimes k \otimes k \otimes M \otimes M \otimes M.$$

2.15.5

The pairings

$$(t, t') \longmapsto 1 \otimes t \otimes t'$$

define a nonassociative multiplication on the k-module $\bar{T}_k\big(M^{\otimes 0} \oplus M\big)$. The subalgebra generated by $M = M_\bullet$ is

$$\bar{M}_* := \bigoplus_{w \in F\{\bullet\}} M_w. \tag{82}$$

Translating *nested-pairs-of-parentheses* notation into Łukasiewicz's parentheses free notation defines an isomorphism of k-modules

$$M^{\otimes w} \simeq M_w,$$

so that the resulting isomorphism

$$\bar{\mathfrak{T}}_k M \simeq \bar{M}_*$$

is an isomorphism of k-algebras.

2.15.6

Given a k-algebra $\big(A,\ A \otimes A \xrightarrow{\mu} A\big)$, the maps

$$A^{\otimes w} \longrightarrow A$$

that are induced by performing $q-1$ number of times operation μ according to the recipe encoded by $w \in F\{\bullet\}$ give rise to a canonical surjective homomorphism of k-algebras

$$\bar{\mathfrak{T}}_k A \twoheadrightarrow A \tag{83}$$

Its unital associative version we encountered in Exercise 16.

2.15.7 The universal property of $\bar{\mathfrak{T}}_k$

Any k-linear map f from a k-module M into a k-algebra A induces a homomorphism of k-algebras

$$\bar{\mathfrak{T}}_k f \colon \bar{\mathfrak{T}}_k M \longrightarrow \bar{\mathfrak{T}}_k A.$$

Let us denote by $\tilde{f}$ its composition with the canonical epimorphism of k-algebras (83). Then

$$f = \tilde{f} \circ \iota$$

where $\iota\colon M \hookrightarrow \bar{\mathfrak{T}}_k M$ is the canonical inclusion and $\tilde{f}$ is the unique k-algebra homomorphism with this property in view of the fact that $\iota(M)$ generates k-algebra $\bar{\mathfrak{T}}_k M$.

2.15.8 The unital nonassociative tensor algebra

If we denote by $\mathfrak{T}_k M$ and M_* the unitalizations of $\bar{\mathfrak{T}}_k M$ and $\bar{M}_*$, then we obtain the corresponding unital nonassociative k-algebras satisfying the universal property for k-linear maps from a k-module M into a unital k-algebra.

2.15.9

The tensor algebra $T_k M$ coincides, of course, with the *associativization* of $\mathfrak{T}_k M$. The surjective homomorphism

$$\mathfrak{T}_k M \longrightarrow\!\!\!\!\!\rightarrow T_k M$$

is induced by canonical isomorphisms

$$M^{\otimes w} \simeq M^{\otimes q}.$$

2.16 The universal Lie algebra of a module

2.16.1

For any k-linear map $f\colon M \longrightarrow \mathfrak{g}$, the induced k-algebra homomorphism

$$\bar{f}\colon \bar{\mathfrak{T}}_k M \longrightarrow \mathfrak{g}$$

is constant on the equivalence classes of the congruence $\sim_{\text{Lie}}$ on $\bar{\mathfrak{T}}_k M$ generated by the relation

$$(tu)v + (uv)t + (vt)u \sim 0 \qquad \text{and} \qquad tt \sim 0,$$

cf. (13)–(14). In particular, f uniquely factorizes through the quotient epimorphism

$$\bar{\mathfrak{T}}_k M \xrightarrow{\pi} L_k M := \bar{\mathfrak{T}}_k M_{/\sim_{\mathrm{Lie}}} .$$

If we denote the induced map by $\hat{f}\colon L_k M \longrightarrow \mathfrak{g}$, then it is a unique homomorphism of k-Lie algebras such that the diagram

$$\begin{tikzcd} & \mathfrak{g} \\ M \arrow[ur, "f"] \arrow[dr, "\hat{\iota}"'] & \\ & L_k M \arrow[uu, dashed, "\hat{f}"'] \end{tikzcd} \tag{84}$$

commutes where $\hat{\iota} := \pi \circ \iota$. By taking $\mathfrak{g}$ to be M with zero multiplication and f to be the identity map, we deduce that

$$\hat{\iota}\colon M \longrightarrow L_k M$$

is split injective.

2.16.2

The forgetful functor

$$k\text{-alg}_{\mathrm{un,ass}} \longrightarrow k\text{-mod} \tag{85}$$

is obtained by composing the functors

$$k\text{-alg}_{\mathrm{un,ass}} \xrightarrow{(\ )_{\mathrm{Lie}}} k\text{-alg}_{\mathrm{Lie}} \longrightarrow k\text{-mod} .$$

Each of these functors has a left adjoint,

$$k\text{-alg}_{\mathrm{un,ass}} \xleftarrow{\mathcal{U}_k} k\text{-alg}_{\mathrm{Lie}} \xleftarrow{L_k} k\text{-mod} ,$$

hence their composition yields a left adjoint to (85). Since the tensor algebra functor is left adjoint to (85), we obtain the canonical isomorphism of functors

$$T_k \simeq \mathcal{U}_k \circ L_k. \tag{86}$$

Combining this with (68), we deduce that the Lie algebra of *primitive* tensors is a universal Lie algebra of a k-module M,

$$\operatorname{Prim} T_k M \simeq L_k M.$$

3 Graded structures

3.1 Graded morphisms

3.1.1

Let G be a binary structure. Given two families of objects in a category $\mathcal{C}$,

$$\mathbf{c} = (c_g)_{g\in G} \qquad \text{and} \qquad \mathbf{c}' = (c'_g)_{g\in G},$$

a family $\mathbf{f} = (f_g)_{g\in G}$ of morphisms

$$\left(\ c_g \xrightarrow{\ f_g\ } c'_{dg}\ \right)_{g\in G}$$

will be referred to as a *graded morphism of degree d*.

3.1.2

Composition of graded morphisms of degree d and d',

$$\big(f_{d'g} \circ f_g\big)_{g\in G'}$$

is the morphism of degree dd' if multiplication in G is associative. In particular, G-graded objects of $\mathcal{C}$ form a *G-graded* category, i.e., a category whose Hom-sets are *graded sets*,

$$\mathrm{Hom}_{\mathcal{C}_{G\text{-gr}}}(\mathbf{c},\mathbf{c}') := \big(\mathrm{Hom}^d(\mathbf{c},\mathbf{c}')\big)_{d\in G}$$

where

$$\mathrm{Hom}^d(\mathbf{c},\mathbf{c}') := \prod_{g\in G} \mathrm{Hom}_{\mathcal{C}}(c_g, c'_{dg})$$

denotes the set of morphisms of degree d. If G has an identity, the resulting category of G-graded objects is a unital category with morphims of degree e being the families

$$\left(\ c_g \xrightarrow{\ f_g\ } c'_g\ \right)_{g\in G}.$$

The latter are well defined whether G has or lacks an identity element. We shall refer to them as morphisms of *degree zero*.

3.2 Graded modules

3.2.1

Any morphism of G-graded k-modules $\mathbf{f}\colon \mathbf{M}\longrightarrow\mathbf{M}'$ induces a k-linear map

$$\bigoplus_{g\in G} M_g \longrightarrow \bigoplus_{g\in G} M'_g \ .$$

It is customary to think of *graded modules* as precisely the modules with a decomposition

$$M = \bigoplus_{g\in G} M_g \tag{87}$$

with a *restricted* notion of a morphism as an element of

$$\bigoplus_{d\in G} \mathrm{Hom}^d(\mathbf{M},\mathbf{M}') \ = \bigoplus_{d\in G}\prod_{g\in G} \mathrm{Hom}_{k\text{-mod}}(M_g, M'_{dg})$$

rather than

$$\prod_{d\in G} \mathrm{Hom}^d(\mathbf{M},\mathbf{M}') \ = \prod_{d\in G}\prod_{g\in G} \mathrm{Hom}_{k\text{-mod}}(M_g, M'_{dg}).$$

One should be always aware of this distinction regarding what is considered a morphism when employing and refering to graded module categories.

3.2.2

On the other, hand it is of little difference whether one considers graded modules to be G-indexed families of modules, or single modules with a direct sum decomposition (87).

3.2.3

Let $\mathbf{A} = (A_g)_{g\in G}$ be a G-graded k-module. A G-graded k-module $\mathbf{M} = (M_g)_{g\in G}$ equipped with a family of k-bilinear pairings

$$A_g \times M_h \xrightarrow{\lambda_{g,h}} M_{gh} \qquad (g,h\in G)$$

or, equivalently, k-linear maps

$$A_g \otimes M_h \xrightarrow{\tilde{\lambda}_{g,h}} M_{gh} \qquad (g, h \in G),$$

will be referred to as a G-graded $\mathbf{A}$-module.

3.2.4

Graded $\mathbf{A}$-modules form a unital category $k\text{-mod}_{G\text{-gr}}$ with morphisms being graded degree *zero* morphisms $\boldsymbol{\phi} = (\phi_g)_{g \in G}$ such that the diagram

$$\begin{tikzcd} A_g \times M_h \arrow[r, "\lambda_{g,h}"] \arrow[d, "\mathrm{id}_{A_g} \times \phi_h"'] & M_{gh} \arrow[d, "\phi_{gh}"] \\ A_g \times M'_h \arrow[r, "\lambda'_{g,h}"] & M'_{gh} \end{tikzcd}$$

commutes.

3.2.5 Graded algebras

When $\mathbf{M} = \mathbf{A}$, we say that $\mathbf{A}$ is a G-graded k-algebra.

3.2.6

Graded k-algebras form a unital category $k\text{-alg}_{G\text{-gr}}$ with morphisms being graded degree *zero* morphisms $\boldsymbol{\phi} = (\phi_g)_{g \in G}$ such that the diagram

$$\begin{tikzcd} A_g \times A_h \arrow[r, "\mu_{g,h}"] \arrow[d, "\phi_g \times \phi_h"'] & A_{gh} \arrow[d, "\phi_{gh}"] \\ M'_g \times M'_h \arrow[r, "\mu'_{g,h}"] & M'_{gh} \end{tikzcd}$$

commutes.

3.2.7 $\mathcal{T}_k M$ as a graded algebra

Several of the algebra functors introduced above yield graded k-algebras. The nonassociative tensor algebra construction yields, e.g., a functor

$$k\text{-mod} \xrightarrow{\mathcal{T}_k} k\text{-alg}_{F\{\bullet\}\text{-gr}} .$$

Note that $\mathcal{T}_k M$ carries also a coarser $\mathbf{N}$-grading, where $\mathbf{N}$ denotes the additive monoid of natural numbers,

$$\mathcal{T}_k M = \bigoplus_{l \in \mathbf{N}} \mathcal{T}_k^q M. \tag{88}$$

Here

$$\mathcal{T}_k^q M := \bigoplus_{\substack{w \in F\{\bullet\} \\ |w| = q}} M^{\otimes w}$$

is the direct sum of iterated tensor powers of M over all words of length q.

3.2.8

The tensor algebra $T_k M$, the symmetric algebra $S_k M$ and the exterior algebra $\Lambda_k M$ are all naturally $\mathbf{N}$-graded,

$$T_k M = \bigoplus_{q \in \mathbf{N}} T_k^q M, \qquad S_k M = \bigoplus_{q \in \mathbf{N}} S_k^q M \quad \text{and} \quad \Lambda_k M = \bigoplus_{q \in \mathbf{N}} \Lambda_k^q M.$$

Here

$$T_k^q M = M^{\otimes q}, \qquad S_k^q M = \left(M^{\otimes q}\right)_{/\sim_{\mathrm{sym}}} \qquad \text{and} \qquad \Lambda_k^q M = \left(M^{\otimes q}\right)_{/\sim_{\mathrm{alt}}}$$

where the equivalence relation $\sim_{\mathrm{sym}}$ is generated by

$$m_1 \otimes \cdots \otimes m_i \otimes \cdots \otimes m_j \otimes \cdots \otimes m_q \ \sim_{\mathrm{sym}} \ m_1 \otimes \cdots \otimes m_j \otimes \cdots \otimes m_i \otimes \cdots \otimes m_q$$

and the equivalence relation $\sim_{\mathrm{alt}}$ is generated by

$$m_1 \otimes \cdots \otimes m_q \ \sim_{\mathrm{alt}} \ 0$$

whenever $m_i = m_j$ for some $1 \leq i \neq j \leq q$.

3.2.9

The tensor algebra T_k, the symmetric algebra S_k and the exterior algebra Λ_k all yield functors

$$k\text{-mod} \longrightarrow k\text{-alg}_{\mathrm{un,ass},\mathbf{N}\text{-gr}} .$$

The target of S_k is the full subcategory of commutative algebras while the target of Λ_k is the full subcategory of k-algebras A generated by the set of their square-zero elements

$$\{a \in A \mid a^2 = 0\}.$$

3.2.10

Unlike the symmetric and the exterior algebras, the universal enveloping algebra $\mathcal{U}_k\mathfrak{g}$ is only $\mathbf{N}$*-filtered*, i.e., it is represented as the union of an increasing sequence of k-submodules

$$A_0 \subseteq A_1 \subseteq \cdots$$

such that

$$A_iA_j \subseteq A_{i+j} \qquad (i, j \in \mathbf{N}).$$

In all three cases, the corresponding algebras

$$S_kM, \qquad \Lambda_kM \qquad \text{and} \qquad \mathcal{U}_k\mathfrak{g}$$

are quotients of the graded algebra T_kM by a congruence relation. Only in the first two cases, however, the congruence is compatible with the structure of the grading.

3.2.11

Most common are structures graded by a commutative monoid. In the latter case one usually employs additive notation for the operation. Since $\mathbf{N}$-graded k-modules ae the same as $\mathbf{Z}$-graded modules $(M_n)_{n\in\mathbf{Z}}$ with

$$M_n = 0 \qquad (n < 0),$$

we shall be always treating $\mathbf{N}$-graded modules as special cases of $\mathbf{Z}$-graded modules.

3.3 Chain and cochain complices

3.3.1

Chain complices of k-modules are $\mathbf{Z}$-graded k-modules equipped with a degree -1 k-linear map $\partial = (\partial_q)_{q \in \mathbf{Z}}$ satisfying

$$\partial \circ \partial = 0.$$

3.3.2

Cochain complices of k-modules are $\mathbf{Z}$-graded k-modules equipped with a degree +1 k-linear map $\delta = (\delta_q)_{q \in \mathbf{Z}}$ satisfying

$$\delta \circ \delta = 0.$$

3.3.3 Standard representation of complices

Chain complices are represented as sequences of k-module maps with arrows pointing to the left

$$\mathbf{C}_\bullet : \qquad \cdots \xleftarrow{\partial_{q-1}} C_{q-1} \xleftarrow{\partial_q} C_q \xleftarrow{\partial_{q+1}} \cdots$$

and the degrees appearing as *subscripts*.

3.3.4 Homology groups of a chain complex

The quotient module

$$H_q(\mathbf{C}_\bullet) := \frac{\operatorname{Ker} \partial_q}{\operatorname{Im} \partial_{q+1}} \tag{89}$$

is called the *q-th homology group* of $\mathbf{C}_\bullet$.

3.3.5

Cochain complices are represented as sequences of k-module maps with arrows pointing to the right

$$\mathbf{C}^\bullet : \qquad \cdots \xrightarrow{\delta^{q-2}} C^{q-1} \xrightarrow{\delta^{q-1}} C^q \xrightarrow{\delta^q} \cdots$$

and the degrees appearing as *superscripts*.

3.3.6 Cohomology groups of a cochain complex

The quotient module

$$H^q(\mathbf{C}_\bullet) := \frac{\operatorname{Ker} \delta^q}{\operatorname{Im} \partial^{q-1}} \tag{90}$$

is called the *q-th cohomology group* of $\mathbf{C}^\bullet$.

3.4 Tensor product of graded modules

3.4.1 Exterior tensor product

Given a G-graded k-module $\mathbf{M}$ and a G'-graded k-module $\mathbf{M}'$, the family of tensor products

$$\left(M_g \otimes M'_{g'}\right)_{(g,g')\in G\times G'} \tag{91}$$

is a $G\times G'$-graded k-module. We shall denote it

$$\mathbf{M} \boxtimes \mathbf{M}'$$

and call it the *exterior tensor product* of $\mathbf{M}$ and $\mathbf{M}'$.

3.4.2

Multiple exterior tensor products are defined similarly: the exterior tensor product of a G_1-graded module M_1, a G_2-graded module M_2, ..., a G_q-graded module M_q, is the $G_1\times\cdots\times G_q$-graded module

$$\mathbf{M}_1 \boxtimes \cdots \boxtimes \mathbf{M}_q := \left(M_{g_1} \otimes \cdots \otimes M_{g_q}\right)_{(g_1,\ldots,g_q)\in G_1\times\cdots G_q}$$

3.4.3 The totalization functors

Given a $G\times G$-graded k-module

$$\mathbf{M} = \left(M_{g_1,g_2}\right)_{(g_1,g_2)\in G\times G'}$$

one can form a G-graded module in 4 slightly different ways and all four are important:

$$\left(\bigoplus_{\substack{(g_1,g_2)\in G\times G\\ g_1g_2=g}} M_{g_1,g_2}\right)_{g\in G}, \qquad \left(\prod_{\substack{(g_1,g_2)\in G\times G\\ g_1g_2=g}} M_{g_1,g_2}\right)_{g\in G}$$

or

$$\left(\bigoplus_{\substack{(g_1,g_2)\in G\times G \\ g_1g_2=g}} M_{g_1,g_2} \right)_{g\in G}, \qquad \left(\prod_{\substack{(g_1,g_2)\in G\times G \\ g_1g_2=g}} M_{g_1,g_2} \right)_{g\in G}$$

4 Derivations

4.1 The Leibniz identity

4.1.1

A central concept as well as the point of departure in Differential Calculus is the following identity:

$$\delta(ab) = \delta(a)b + a\delta(b). \tag{92}$$

A k-linear map δ satisfying (92) is called a *derivation*.

4.1.2

By its very nature, the source of a k-linear derivation must be a k-algebra, denote it A, while its target must be an (A, A)-module, denote it M.

4.1.3 $\mathbf{Der}_k(A, M)$

We shall denote the set of k-linear derivations $\delta\colon A \longrightarrow M$ by $\mathrm{Der}_k(A, M)$. It is a submodule of the k-module $\mathrm{Hom}_{k\text{-mod}}(A, M)$.

4.1.4

Given a homomorphism of (A, A)-modules $f\colon M \longrightarrow M'$, the post-composition $f \circ \delta$ with any k-linear derivation $\delta \in \mathrm{Der}_k(A, M)$ produces obviously a derivation from A to M'.

4.1.5

In particular, $\mathrm{Der}_k(A, M)$ is naturally a left module over the unital associative k-algebra $\mathrm{End}_{(A,A)\text{-mod}}(M)$.

Exercise 21 *Show that any k-linear derivation from a unital algebra to a unitary bimodule vanishes on 1_A.*

4.1.6 Inner derivations

For a given $m \in M$, let

$$\delta_m \colon A \longrightarrow M, \qquad a \longmapsto \delta_m(a) := [m, a] \qquad (a \in A). \tag{93}$$

Exercise 22 *Show that δ_m is a derivation if M is an associative A-bimodule.*

4.2 Derivations on a binary algebra

4.2.1

The special case $M = A$ deserves a special attention. We shall denote $\mathrm{Der}_k(A, A)$ by $\mathrm{Der}_k(A)$ and refer to elements of $\mathrm{Der}_k(A)$ as derivations on A.

4.2.2

If $A^2 = 0$, then every endomorphism of k-module A is a derivation. Vice-versa, if the identity endomorphism id_A satisfies the Leibniz identity, then multiplication (3) is identically zero.

4.2.3 The Lie k-algebra structure on $\mathrm{Der}_k(A)$

Given two k-linear derivations δ and δ' on A, the straighforward calculation

$$(\delta \circ \delta')(ab) \;=\; (\delta \circ \delta')(a)b \,+\, \big[\delta'(a)\delta(b) + \delta(a)\delta'(b)\big] \,+\, a(\delta \circ \delta')(b),$$

with the expression in the square brackets being symmetric in δ and δ', shows that the commutator

$$[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$$

of two k-linear derivations on A is a k-derivation. Thus, $\mathrm{Der}_k(A)$ is a *Lie subalgebra* of the Lie k-algebra $\big(\mathrm{End}_{k\text{-mod}}(A),\, [\ ,\ ]\big)$.

Exercise 23 *Show that the map that sends $a \in A$ to the inner derivation δ_a on an associative k-algebra A,*

$$a \longmapsto \delta_a, \qquad (a \in A)$$

is a homomorphism of Lie k-algebras

$$\big(A,\, [\ ,\ ]\big) \longrightarrow \mathrm{Der}_k(A).$$

4.3 The universal derivation

4.3.1 Bilinearization of a derivation

Let M be a bimodule over a unital associative k-algebra A. Given a k-linear map $\delta\colon A \longrightarrow M$, its *bilinearization*,

$$\tilde{\delta}\colon A^{\otimes 3} \longrightarrow M, \qquad \tilde{\delta}(a' \otimes a \otimes a'') = a'\delta(a)a'' \qquad (a', a', a'' \in A), \tag{94}$$

is obtained by linearizing δ in the category of *left* A-modules, and then linearizing the result in the category of *right* A-modules. It is the unique A-bimodule map that makes the following diagram commutative

$$\begin{tikzcd} & M \\ A \arrow[ur, "\delta"] \arrow[dr, "{}_A\eta_A"'] & \\ & A^{\otimes 3} \arrow[uu, dashed, "\tilde{\delta}"'] \end{tikzcd} \tag{95}$$

where

$${}_A\eta_A\colon a \longmapsto 1_A \otimes a \otimes 1_A \qquad (a \in A). \tag{96}$$

4.3.2

Note that

$$\tilde{\delta}(1 \otimes ab \otimes 1) = \delta(ab) \tag{97}$$

and

$$\tilde{\delta}(1 \otimes a \otimes b + a \otimes b \otimes 1) = \delta(a)b + a\delta(b). \tag{98}$$

The Leibniz identity for δ thus becomes equivalent to saying that $\tilde{\delta}$ annihilates tensors of the form

$$a \otimes b \otimes 1 - 1 \otimes ab \otimes 1 + 1 \otimes a \otimes b \qquad (a, b \in A).$$

The A-bimodule generated by such tensors in $A^{\otimes 3}$ coincides with the image of the A-bimodule map $A^{\otimes 4} \longrightarrow A^{\otimes 3}$ which sends

$$a_1 \otimes a_2 \otimes a_3 \otimes a_4 \qquad (a_1, a_2, a_3, a_4 \in A)$$

to

$$a_1a_2 \otimes a_3 \otimes a_4 \;-\; a_1 \otimes a_2a_3 \otimes a_4 \;+\; a_1 \otimes a_2 \otimes a_3a_4.$$

This map is denoted b'_4 and is the fourth *boundary map* of the *Bar complex* of A.

Exercise 24 *Denote the composite map*

$$A \xrightarrow{\ {}_A\eta_A\ } A^{\otimes 3} \xrightarrow{\ \pi\ } \frac{A^{\otimes 3}}{b'_4 A^{\otimes 4}} \tag{99}$$

by d_{un}. Show that d_{un} is a k-linear derivation.

4·3·3

We obtain the following diagram with commutative triangles

$$\begin{tikzcd}
A \arrow[rr, "\delta"] \arrow[dr, "{}_A\eta_A"] \arrow[ddr, "d_{\mathrm{un}}:=\pi\circ{}_A\eta_A"'] & & M \\
& A^{\otimes 3} \arrow[ur, dashed, "\tilde{\delta}"] \arrow[d, two heads, "\pi"] & \\
& \dfrac{A^{\otimes 3}}{b'_4 A^{\otimes 4}} \arrow[uur, dashed, "\bar{\delta}"'] &
\end{tikzcd}$$

Exercise 25 *Show that if δ' and δ'' are two k-linear maps*

$$\frac{A^{\otimes 3}}{b'_4 A^{\otimes 4}} \longrightarrow M$$

such that $\delta' \circ d_{\mathrm{un}} = \delta = \delta' \circ d_{\mathrm{un}}$, then $\delta' = \delta''$.

4·3·4

We conclude that any k-linear derivation δ is produced from derivation d_{un},

$$\delta = \bar{\delta} \circ d_{\mathrm{un}},$$

by a unique A-bimodule map

$$\frac{A^{\otimes 3}}{b'_4 A^{\otimes 4}} \xrightarrow{\ \bar{\delta}\ } M\ .$$

In other words, (99) is a *universal* k-linear derivation from a unital associative k-algebra A. It sends $a \in A$ to

$$d_{\mathrm{un}}a \;:=\; \text{the class of } \; 1_A \otimes a \otimes 1_A \;\; \text{modulo } b'_4 A^{\otimes 4}.$$

4.4 The Bar complex

4.4.1

Let A be a k-algebra.

Exercise 26 *Show that the map* $A^q \longrightarrow A^{\otimes(q-1)}$,

$$(a_1, \ldots, a_q) \;\longmapsto\; \sum_{i=1}^{q-1} (-1)^{i-1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q, \tag{100}$$

is k*-linear in each of its* $q+1$ *arguments.*

4.4.2

The k-module map $A^q \longrightarrow A^{\otimes(q-1)}$ induced by (100) is denoted b'_q. Note that b'_2 is the multiplication map (78) while $b'_2 \circ b'_3$ is the *associator* map

$$a_1 \otimes a_2 \otimes a_3 \;\longmapsto\; (a_1 a_2) a_3 - a_1 (a_2 a_3). \tag{101}$$

In particular, $b'_2 \circ b'_3 = 0$ if and only if A is associative.

4.4.3

Note that, for an associative algebra, b'_q are A-bimodule maps:

$$b'(a' a_1 \otimes \cdots \otimes a_q) = a' b'_q (a_1 \otimes \cdots \otimes a_q)$$

and

$$b'(a_1 \otimes \cdots \otimes a_q a'') = b'_q (a_1 \otimes \cdots \otimes a_q) a''.$$

Exercise 27 *Show that* $b'_q \circ b'_{q+1} = 0$ *for all* $q > 1$ *if* A *is associative.*

4.4.4

Thus, for an associative k-algebra, the sequence of maps

$$A \xleftarrow{b'_2} A^{\otimes 2} \xleftarrow{b'_3} A^{\otimes 3} \xleftarrow{b'_4} \cdots \tag{102}$$

forms a chain complex of A-bimodules. It is called the *Bar complex* of a k-algebra A and is denoted $B_*(A)$.

4.4.5 Bar homology

The homology groups of $B_*(A)$,

$$\mathrm{HB}_q(A) := \frac{\mathrm{Ker}\, b'_q}{b'_{q+1} A^{\otimes(q+1)}} \tag{103}$$

are called the *Bar homology groups* of A.

4.4.6 Example: group homology

The Bar homology of the augmentation ideal

$$I_k G := \mathrm{Ker} \textstyle\int_G, \tag{104}$$

cf. (43), is the group homology of G with trivial coefficients k,

$$\mathrm{HB}_q(I_k G) = H_q(G; k) \qquad (q > 0).$$

4.4.7 Example: Lie algebra homology

The *augmentation ideal* $J_k\mathfrak{g}$ of a Lie k-algebra $\mathfrak{g}$ is the kernel of the homomorphism

$$\mathcal{U}_k\mathfrak{g} \longrightarrow k$$

from the universal enveloping algebra $\mathcal{U}_k\mathfrak{g}$ to k which sends all $g \in \mathfrak{g}$ to 0. Its Bar homology is the Lie algebra homology of $\mathfrak{g}$ with trivial coefficients k,

$$\mathrm{HB}_q(I_k G) = H_q(\mathfrak{g}; k) \qquad (q > 0).$$

4.4.8

When A is unital, the Bar complex $B_*(A)$ is equipped with a sequence of maps $\eta_q := \eta_{A^{\otimes q}}$, for $q > 0$.

Exercise 28 *Show that*

$$\eta_{q-1} \circ b'_q + b'_{q+1} \circ \eta_q = \mathrm{id}_q \qquad (q > 1) \tag{105}$$

where $\mathrm{id}_q := \mathrm{id}_{A^{\otimes q}}$.

4.4.9

Note that (105) is also valid for $q \leq 1$ if we set $B_q(A) = 0$ for $q < 1$ with the corresponding η_q and b'_q being zero maps.

4.4.10

Unlike the boundary maps b'_q, which are A-bimodule maps, η_q are only right A-module maps. In fact, identity (105) expresses the fact that b'_{q+1} is a left A-linearization of

$$\mathrm{id}_q - \eta_{q-1} \circ b'_q.$$

This allows to use the sequence of identities (105) to define b'_q inductively starting from the multiplication map b'_2 which is the linearization of id_1 as we already noticed in Section 2.14.2.

4.4.11

One can show that $b'_q \circ b'_{q+1} = 0$ directly from this inductive definition. Indeed,

$$b'_2 \circ b'_3 \circ \eta_2 = b'_2 \circ (\mathrm{id}_2 - \eta_1 \circ b'_2) = b'_2 - b'_2 \circ \eta_1 \circ b'_2 = b'_2 - \mathrm{id}_1 \circ b'_2 = 0$$

which, in view of the fact that $b'_2 \circ b'_3$ is left A-linear and the image of η_2 generates $A^{\otimes 3}$ as a left A-module, implies that $b'_2 \circ b'_3 = 0$.

For $q > 2$, one has accordingly,

$$\begin{aligned} b'_q \circ b'_{q+1} \circ \eta_q &= b'_q \circ (\mathrm{id}_q - \eta_{q-1} \circ b'_q) \\ &= b'_q - (\mathrm{id}_{q-1} - \eta_{q-2} \circ b_{q-1}) \circ b'_q \\ &= \eta_{q-2} \circ b_{q-1}) \circ b'_q = 0 \end{aligned}$$

which again, since the image of η_q generates $A^{\otimes(q+1)}$ as a left A-module, implies that $b'_q \circ b'_{q+1} = 0$.

4.4.12

In the language of Homological Algebra, identities (105) mean that the sequence of right A-module maps $(\eta_q)_{q>0}$ is a *contracting homotopy* for the Bar complex $B_*(A)$. Note that it suffices that A has only a *left* identity. In particular, the Bar complex of a k-algebra with a left identity is *contractible*.

4.4.13

Vanishing of the homology groups is an immediate consequence of contractibility. Indeed, if $b'_q(\alpha) = 0$, then

$$\alpha = \mathrm{id}_q(\alpha) = (\eta_{q-1} \circ b'_q + b'_{q+1} \circ \eta_q)(\alpha) = b'_{q+1}(\eta_q(\alpha)). \tag{106}$$

4.4.14

By replacing left A-module linearizations with right A-module linearizations, one can similarly show that the Bar complex of a k-algebra with a right identity is contractible.

4.4.15

It follows that for an associative k-algebra with one-sided identity, b'_q induce isomorphisms

$$\operatorname{Ker} b'_{q-1} \xleftarrow[\simeq]{b'_q} \frac{A^{\otimes q}}{b'_{q+1} A^{\otimes(q+1)}} . \tag{107}$$

4.5 Diagonal calculus

4.5.1 The diagonal ideal

According to (107), the target of the universal k-linear derivation, cf. (99), is canonically isomorphic, via b_3', to the kernel of the multiplication map (78). Let us denote it by $I_\Delta A$ or simply by I_Δ. It can be thought of as a *left* ideal in the algebra $A \otimes A^{\mathrm{op}}$. Accordingly, we shall refer to I_Δ as the *diagonal ideal*.

4.5.2 Diagonal differential

Let

$$d_\Delta a := 1_A \otimes a^{\mathrm{op}} - a \otimes 1_{A^{\mathrm{op}}} \qquad (a \in A). \tag{108}$$

Note that

$$b_3' \circ d_{\mathrm{un}} = -d_\Delta.$$

4.5.3

The Leibniz identity for d_Δ takes the following form

$$d_\Delta(ab) = (1 \otimes b^{\mathrm{op}})d_\Delta a + (a \otimes 1)d_\Delta b \qquad (a, b \in A).$$

4.5.4

For any element $\alpha = \sum a_i \otimes b_i^{\mathrm{op}}$ of I_Δ one has

$$\alpha - \sum (a_i \otimes 1)d_\Delta b_i = \left(\sum a_i b_i\right) \otimes 1 = 0,$$

i.e., I_Δ is generated as a left A-module by $d_\Delta A$, the image of the diagonal differential.

4.5.5

Given a derivation $\delta\colon A \longrightarrow M$, let δ' denote the restriction to I_Δ of its left A-linearization. By the definition it is a map of left A-modules and

$$\delta'(d_\Delta b) = \delta\big(1 \otimes b^{\mathrm{op}} - b \otimes 1\big) = \delta(b) - b\delta(1) = \delta(b)$$

in view of Exercise (21).

4.5.6

The calculation

$$\begin{aligned}\delta'\big((1\otimes a^{\mathrm{op}})d_\Delta b\big) &= \delta'\big(1\otimes (ba)^{\mathrm{op}} - b\otimes a^{\mathrm{op}}\big)\\ &= \delta(ba) - b\delta(a) = \delta(b)a\\ &= \delta'(d_\Delta b)a\end{aligned}$$

shows that δ' is also a map of right A-modules.

4.5.7

We proved directly that any k-linear derivation δ is produced from the diagonal differential

$$\delta = \delta' \circ d_\Delta \tag{109}$$

by postcomposing it with some A-bimodule map $\delta' \colon I_\Delta \longrightarrow M$.

Exercise 29 *Show that such an A-bimodule map is unique.*

4.5.8 Diagonal commutator identities

Exercise 30 *Show that*

$$\big[d_\Delta a, d_\Delta b\big] = -d_\Delta[a,b] \qquad (a,b\in A). \tag{110}$$

In other words, $-d_\Delta$ is a homomorphism of Lie k-algebras

$$\big(A, [\ ,\ ]\big) \longrightarrow \big(I_\Delta, [\ ,\ ]\big).$$

Exercise 31 *Show that*

$$\big[(a\otimes 1), d_\Delta b\big] = -[a,b]\otimes 1 \qquad (a,b\in A). \tag{111}$$

By combining this with the fact that $I_\Delta = (A\otimes 1)d_\Delta A$, we obtain

$$I_\Delta^2 \subseteq (A\otimes 1)\big(d_\Delta A\big)^2 + \big(A[A,A]\otimes 1\big)d_\Delta A.$$

5 Superderivations

5.1 Preliminaries

5.1.1 k-supermodules

A k-supermodule is the same as a $\mathbf{Z}/2\mathbf{Z}$-graded k-module

$$V = V_0 \oplus V_1.$$

5.1.2 Parity

Elements of V_0 are said to be *even*, elements of V_1 are said to be odd. For this reason V_0 is sometimes denoted V_{ev} and V_1 is denoted V_{odd}.

Accordingly,

$$\tilde{v} := \begin{cases} 0 & \text{if } v \in V_0 \\ 1 & \text{if } v \in V_1 \end{cases} \tag{112}$$

is referred to as the *parity* of v. It is a function

$$V_0 \sqcup V_1 \longrightarrow \mathbf{Z}/2\mathbf{Z}$$

from the disjoint union of V_0 and V_1 to the abelian group $\mathbf{Z}/2\mathbf{Z}$.

5.1.3 k-superalgebras

A k-supermodule $A = A_0 \oplus A_1$ equipped with a $\mathbf{Z}/2\mathbf{Z}$-graded pairing (3) is said to be a k-superalgebra.

5.1.4 An example: the k-superalgebra of endomorphisms of a k-supermodule

One has

$$\text{End}_{k\text{-mod}}(V) = \text{End}^0_{k\text{-mod}}(V) \oplus \text{End}^1_{k\text{-mod}}(V)$$

where $\text{End}^0_{k\text{-mod}}(V)$ consists of those k-linear endomorphisms f of V which *preserve* the parity of $v \in V_0 \sqcup V_1$,

$$\widetilde{f(v)} = \tilde{v},$$

while $\text{End}^1_{k\text{-mod}}(V)$ consists of those k-linear endomorphisms f of V which *reverse* the parity

$$\widetilde{f(v)} = 1 + \tilde{v}.$$

5.1.5 Supercommutator and supercommutativity

The supercommutator is defined by

$$[a,b] \;:=\; ab - (-1)^{\tilde{a}\tilde{b}} ba \qquad (a,b \in A_0 \sqcup A_1). \tag{113}$$

In particular, we say that two elements of A *supercommute* if their supercommutator vanishes.

5.1.6 The opposite k-superalgebra functor

The concept of the opposite k-superalgebra reflects the parity of elements multiplied. The opposite multiplication μ^{op} cannot be defined as $\mu \circ \tau$ for some bijection $\tau\colon A \times A \longrightarrow A \times A$, like in Section (1.1.4) but it can be defined by extending by biadditivity the formula

$$a^{\mathrm{op}} b^{\mathrm{op}} := (-1)^{\tilde{a}\tilde{b}} (ba)^{\mathrm{op}} \qquad (a,b \in A_0 \sqcup A_1) \tag{114}$$

to arbitrary pairs of elements of $A = A_0 \oplus A_1$.

5.1.7

For any k-superalgebra A, the category of associative right A-supermodules is isomorphic to the category of associative left A^{op}-supermodules.

5.1.8

A superalgebra A is supercommutative,[1] if and only if $A = A^{\mathrm{op}}$.

5.1.9 Super Jacobi identity

When multiplication (3) is associative, the supercommutator satisfies the the following identity

$$(-1)^{\tilde{c}\tilde{a}}[[a,b],c] + (-1)^{\tilde{a}\tilde{b}}[[b,c],a] + (-1)^{\tilde{b}\tilde{c}}[[c,a],b] = 0 \qquad (a,b,c \in A_0 \sqcup A_1). \tag{115}$$

[1]It is a common practice to refer to supercommutative superalgebras as *commutative*, in order to avoid tedious repetition of the prefix *super*.

5.1.10 Lie k-superalgebras

A binary k-superalgebra with a binary operation

$$(a, b) \longmapsto [a, b]$$

is said to be a *Lie k-superalgebra* if the operation satisfies identity (115) and, additionally, the identity

$$[a, a] = 0 \qquad (a \in A).$$

5.1.11 The Leibniz identity in the super-case

In the super case the Leibniz identity reflects the parity of δ and of a when "moving" the symbol δ past the symbol a:

$$\delta(ab) = \delta(a)b + (-1)^{\tilde{\delta}\tilde{a}} a\delta(b). \tag{116}$$

Note that in expressions involving the parity of some symbols the corresponding elements are tacitly expected to be "pure", i.e., to be either *even* or *odd*.

5.1.12

A general superderivation from A to M is the sum

$$\delta = \delta^{\mathrm{ev}} + \delta^{\mathrm{odd}}$$

with both δ^{ev} and δ^{odd} satisfying identity (116). We shall use the same notation as in the non-super case

5.2 Graded derivations

5.2.1 Graded k-modules

Graded k-modules are k-modules equipped with a direct sum decomposition

$$V = \bigoplus_{q \in \mathbf{Z}} V_q.$$

Elements of V_q are said to be of *degree* q and

$$\deg\colon v \longmapsto q \qquad \text{if} \qquad v \in V_q, \tag{117}$$

is referred to as the *degree function*. It is a function

$$\bigsqcup_{q\in\mathbf{Z}} V_q \longrightarrow \mathbf{Z}$$

from the disjoint union of all V_q to the additive group of integers.

5.2.2 Morphisms of degree d

A k-linear map $f\colon V \longrightarrow W$ is of degree d if

$$f(V_q) \subseteq W_{q+d} \qquad \text{for all} \qquad q \in \mathbf{Z}.$$

Such maps form a k-module denoted $\mathrm{Hom}^q_{k\text{-mod}}(V,W)$.

5.2.3 The category of graded k-modules

Morphisms in the category of graded k-modules

$$\mathrm{Hom}_{k\text{-mod}_{\mathrm{gr}}}(V,W) := \bigoplus_{q\in\mathbf{Z}} \mathrm{Hom}^q_{k\text{-mod}}(V,W), \tag{118}$$

form a graded k-module themselves. One has

$$\mathrm{Hom}_{k\text{-mod}_{\mathrm{gr}}}(V,W) \subseteq \mathrm{Hom}_{k\text{-mod}}(V,W)$$

with equality if V_q and W_q are nonzero for only finitely many q.

5.2.4 Graded k-bilinear maps

Given graded k-modules U, V and W, a k-bilinear map

$$\alpha\colon U \times V \longrightarrow W$$

satisfying

$$\alpha(U_p, V_q) \subseteq W_{p+q} \qquad (p, q \in \mathbf{Z}) \tag{119}$$

is said to be *graded*. Strictly speaking one should be referring to such maps as graded of degree 0.

5.2.5 Graded modules over graded algebras

By using graded k-modules and graded k-bilinear pairings in the definitions of Section 1.1 we obtain the graded versions of the concept of an A-module and of an algebra.

5.2.6

A map $\delta \in \mathrm{Hom}^d_{k\text{-mod}}(A, M)$ of degree d from a graded k-algebra A to a agraded A-bimodule M is said to be a *graded derivation* if it satisfies the *graded Leibniz identity*

$$\delta(ab) = \delta(a)b + (-1)^{\deg \delta \deg a} a\delta(b). \tag{120}$$

In expressions involving the degree function the corresponding elements are tacitly expected to be of pure degree.

5.2.7 The associated super structures

One associates to any graded k-module V the corresponding k-supermodule

$$V_{\mathrm{ev}} := \bigoplus_{q \text{ even}} V_q \qquad \text{and} \qquad V_{\mathrm{odd}} := \bigoplus_{q \text{ odd}} V_q \tag{121}$$

Note that

$$\tilde{v} = \deg v \mod 2 \qquad (v \in V_q).$$

5.2.8

Graded commutator coincides then with supercommutator, graded commutativity with supercommutativity, the graded Jacobi identity becomes the super Jacobi identity and so on.

6 Differential graded algebras

6.1 The tensor algebra of a bimodule

6.1.1 A-algebras

Let A be a unital associative k-algebra. An A-bimodule $\mathcal{B}$ is said to be an A-algebra if it is equipped with an A-bilinear mapping

$$\mu\colon \mathcal{B}\times\mathcal{B} \longrightarrow \mathcal{B}.$$

In the context of noncommutative algebras '*bilinear*' involves three conditions: μ is supposed to be *left A-linear* in the *left* argument, *right A-linear* in the *right* argument, and *A-balanced*. If B is a symmetric bimodule over a commutative algebra, any pairing thats is A-linear in both arguments is automatically A-balanced.

6.1.2

The above definition supplies a proper generalization of the notion of an algebra to the case when the *ground ring* is not commutative. Unitality, associativity, commutativity remain all unaffected by this generalization.

Exercise 32 *Show that unital A-algebra structures on a k-algebra B are in natural bijective correspondence with homomorphisms of unital k-algebras*

$$A \longrightarrow B.$$

6.1.3 The universal property of the tensor algebra of a bimodule

Given an A-bimodule M, the canonical A-bilinear pairings

$$M^{\otimes_A p} \times M^{\otimes_A q} \;\longrightarrow\; M^{\otimes_A (p+q)}$$

define a structure of a unital associative A-algebra on

$$T_A M := \bigoplus_{q\geq 0} M^{\otimes_A q} \tag{122}$$

where $M^{\otimes_A 0}$ is understood to be A. Unital homomorphisms

$$T_A M \longrightarrow \mathcal{B} \tag{123}$$

in the category $A\text{-alg}_{\text{ass},1}$ of associative unital A-algebras are uniquely determined by A-bimodule maps $M \longrightarrow \mathcal{B}$. Vice-versa, any such A-bimodule map extends to a homomorphism of A-algebras (123), which means that the assignment

$$M \longmapsto T_A M$$

gives rise to a functor $A\text{-bimod} \longrightarrow A\text{-alg}_{\text{ass},1}$ that is left adjoint to the functor $A\text{-alg}_{\text{ass},1} \longrightarrow A\text{-bimod}$ that forgets the multiplication.

6.1.4 Modules over $T_A M$

Unitary left $T_A M$-module structures on a k-module N are in bijective correspondence with A-bilinear maps

$$M \times N \longrightarrow N.$$

Similarly for right $T_A M$-module structures. For bimodules the left and the right actions of M on N are expected to commute.

6.2 Derivations from $T_A M$

6.2.1

Given a $T_A M$-bimodule N and a splitting

$$\sigma\colon T_A M \longrightarrow T_A M \ltimes N, \qquad \sigma(t) = \begin{pmatrix} t \\ \delta(t) \end{pmatrix}, \tag{124}$$

of the semi-direct product extension

$$T_A M \overset{\pi}{\twoheadleftarrow} T_A M \ltimes N \overset{\iota}{\hookleftarrow} N \tag{125}$$

of A-algebras, the composition of σ with the projection onto N is a derivation $\delta\colon T_A M \longrightarrow N$.

Vice-versa, given any derivation $\delta\colon T_A M \longrightarrow N$, the map defined by (124) is a splitting.

6.2.2

Since the k-algebra $T_A M$ is generated by $A \cup M$, each derivation δ is uniquely determined by its restrictions to A and M,

$$\delta^0 \colon A \longrightarrow N \qquad \text{and} \qquad \delta^1 \colon M \longrightarrow N.$$

Since A is a k-subalgebra of $T_A M$, the restriction δ^0 of δ to A is a k-linear derivation $A \longrightarrow N$, while δ^1 satisfies the pair of identities

$$\delta^1(am) = \delta^0(a)m + a\delta^1(m) \qquad \text{and} \qquad \delta^1(ma) = \delta^1(m)a + m\delta^0(a). \quad (126)$$

Exercise 33 *Verify that the left identity in* (126) *expresses the fact that σ^1 is a map of left A-modules if one equips N with the A-module structure* ***twisted*** *by derivation δ^0:*

$$a \cdot_{\delta^0} n := (a + \delta^0)n.$$

Similarly for the right identity in (126).

6.2.3

Given any such pair (δ^0, δ^1) of k-linear mappings, the map

$$A \longrightarrow T_A M \ltimes N, \qquad a \longmapsto \sigma^0(a) := \begin{pmatrix} a \\ \delta^0(a) \end{pmatrix},$$

is a homomorphism of unital k-algebras, while

$$M \longrightarrow T_A M \ltimes N, \qquad m \longmapsto \sigma^1(m) := \begin{pmatrix} m \\ \delta^1(m) \end{pmatrix},$$

is a homomorphism of unitary A-bimodules if N is equipped with the δ^0-*twisted* A-bimodule structure.

6.2.4

In view of the universal property of $T_A M$ discussed above, σ^1 extends to a homomorphism of A-algebras (124) whose restriction to A is σ^0. Note that the endomorphism $\pi \circ \sigma$ of the algebra $T_A M$ is the identity map on A and on M. Since the k-algebra $T_A M$ is generated by $A \cup M$, $\pi \circ \sigma = \mathrm{id}_{T_A M}$, i.e., σ is a splitting of (125). In particular, the composition of σ with the projection onto N yields a k-linear derivation extending δ^0 and δ^1.

6.2.5

The k-module isomorphism between $\mathrm{Der}_k(T_A M, N)$ and the k-module formed by pairs (δ^0, δ^1) such that δ^0 is a derivation from A to N and δ^1 satisfies identities (126), yields also the description of derivations from the quotient algebras $T_A M/J$ when J is generated by a submodule of

$$A \oplus M \oplus M \otimes_A M.$$

6.2.6 Derivations from the symmetric algebra

For example, given a *symmetric* bimodule M over a commutative algebra A, derivations from $S_A M$ to an $S_A M$-bimodule N are precisely those derivations $\delta\colon T_A M \longrightarrow N$ for which

$$[\delta^1(m), m'] + [m, \delta^1(m')] = 0 \qquad (m, m' \in M). \tag{127}$$

6.2.7 Derivations from the exterior algebra

Derivations from $\Lambda_A M$ to a $\Lambda_A M$-bimodule N are precisely those derivations $\delta\colon T_A M \longrightarrow N$ for which

$$\delta^1(m)m + m\delta^1(m) = 0 \qquad (m \in M). \tag{128}$$

6.2.8 Derivations from the universal enveloping algebra

Given a Lie A-algebra, k-linear derivations from $\mathcal{U}_A\mathfrak{g}$ to a $\mathcal{U}_A\mathfrak{g}$-bimodule N are precisely those derivations $\delta\colon T_A\mathfrak{g} \longrightarrow N$ for which

$$[\delta^1(g), g'] + [g, \delta^1(g')] = \delta^1([g, g']) \qquad (g, g' \in \mathfrak{g}). \tag{129}$$

6.3 Graded derivations from $T_A M$

6.3.1

If $T_A M$ is considered as an $\mathbf{N}$-*graded* algebra, then it will be denoted $T_A^\bullet M$. Given a graded $T_A^\bullet M$-bimodule $N_\bullet$ and an integer d, the *shifted* bimodule $[p]N$ is defined as

$$([p]N)_q := N_{q-p} \qquad (q \in \mathbf{Z}),$$

with the left and right actions of $T_A^\bullet M$ given by

$$t([p]n) := (-1)^{\tilde{t}\tilde{n}}[p](tn) \qquad \text{and} \qquad ([p]n)t := [p](nt).$$

6.3.2

A k-linear map δ of degree d from $T_A^\bullet M$ to $N_\bullet$ is a graded derivation if and only if $[-d] \circ \delta$ is a derivation of degree 0. It follows that, for the δ^1-component of a derivation of degree d, the conditions (126) become

$$\delta^1(am) = \delta^0(a)m + a\delta^1(m) \qquad \text{and} \qquad \delta^1(ma) = \delta^1(m)a + (-1)^{\tilde{d}}m\delta^0(a) \tag{130}$$

since elements of M have degree 1.

6.3.3

In particular, the argument demonstrating that derivations from T_AM to N correspond to those pairs (δ^0, δ^1) yields that graded derivations

$$\delta\colon T_A^\bullet M \longrightarrow N_\bullet$$

correspond to pairs (δ^0, δ^1) where

$$\delta^0\colon A \longrightarrow N_d$$

is a k-linear derivation of A into the A-bimodule N, and

$$\delta^1\colon M \longrightarrow N_{d+1}$$

is a k-linear map satisfying identities (130).

6.3.4 Derivations from the symmetric algebra

For the δ^1-component of a derivation of degree d from $S_A^\bullet M$ identity (127) becomes

$$\big(\delta^1(m)m' - (-1)^{\tilde{d}}m'\delta^1(m)\big) - \big(\delta^1(m')m - (-1)^{\tilde{d}}m\delta^1(m')\big) = 0 \qquad (m, m' \in M). \tag{131}$$

6.3.5

Note that the symmetric k-algebra $S^\bullet_A M$ is *not* graded commutative if M is assigned degree 1. Elements of M from the graded point of view *anti-commute*. In a graded algebra *anticommutator* is denoted

$$[a,b]_+ := ab + (-1)^{\tilde{a}\tilde{b}} ba.$$

Using the anticommutator notation, identity (131) can be expressed as

$$[\delta^1(m), m']_+ + (-1)^{\tilde{d}}[m, \delta^1(m')]_+ = 0.$$

6.3.6 Graded derivations from the exterior algebra

For the δ^1-component of a derivation of degree d from $\Lambda^\bullet_A M$ identity (128) becomes

$$\delta^1(m)m + (-1)^{\tilde{d}} m\delta^1(m) = 0 \qquad (m \in M). \tag{132}$$

6.3.7

Condition (132) is automatically satisfied if the target of the derivation, $N_\bullet$, is a *graded-symmetric* bimodule over $\Lambda^\bullet_A M$, which means that the graded commutators $[\alpha, n]$ vanish for all $\alpha \in \Lambda^\bullet_A M$ and in $n \in N$.

Indeed, note that in any graded algebra $A_\bullet$, one has the formula

$$d(a^2) = \delta(a)a + (-1)^{\tilde{\delta}\tilde{a}} a\delta(a),$$

whose the right-hand-side coincides with the graded commutator

$$[\delta(a), a]$$

if a is of *odd* degree. In particular, if a commutes in the graded sense with $\delta(a)$, then $\delta(a^2)$ vanishes.

6.4 The universal cochain algebra

6.4.1 The Bar algebra

Given a k-algebra A (here not assumed to be associative or unital),

$$\bar{T}_k A = \bigoplus_{q>0} A^{\otimes q} \tag{133}$$

is equipped with the pairings

$$A^{\otimes(p+1)} \times A^{\otimes(q+1)} \longrightarrow A^{\otimes(p+q+1)}$$

given by

$$(a_0 \otimes \cdots \otimes a_p, a'_0 \otimes \cdots \otimes a'_q) \longmapsto a_0 \otimes \cdots \otimes a_p a'_0 \otimes \cdots \otimes a'_q.$$

This makes (133) into an $\mathbf{N}$-graded k-algebra if we consider $A^{\otimes(q+1)}$ as the component of degree q.

Exercise 34 *Show that the Bar boundary $b' = (b'_q)_{q \in \mathbf{Z}}$ is, for this multiplication, a derivation of degree -1.*

6.4.2

We shall now consider the *even* part of the Bar algebra which is the direct sum of *odd* tensor powers of A,

$$T_k^{\mathrm{odd}} A. \tag{134}$$

We shall consider it as an $\mathbf{N}$-graded algebra on its own, with tensors $\alpha \in A^{\otimes(2q+1)}$ assigned degree q.

6.4.3

From now on let us assume again that A is an associative unital k-algebra. In Section 4.3.1 we constructed the universal derivation from A in terms of the *bilinearization* $\eta = {}_A\eta_A$ of the identity map id_A.

Let us extend η to a map of degree 1 on (134), whose component in degree q,

$$\eta^{(q)} \colon A^{\otimes(2q+1)} \longrightarrow A^{\otimes(2q+3)},$$

sends

$$a_0 \otimes \cdots \otimes a_{2q} \tag{135}$$

to

$$\sum_{i=0}^{q} (-)^i (a_0 \otimes \cdots \otimes a_{2i-1})\, \eta a_{2i}\, (a_{2q+1} \otimes \cdots \otimes a_{2q}) \tag{136}$$

6.4.4

Employing Bar multiplication, the rank 1 tensor (135) can be rewritten as the product of elements a_{2i} of degree 0 and elements ηa_{2j+1} of degree 1,

$$a_0(\eta a_1)a_2 \cdots a_{2(q-1)}(\eta a_{2q-1})a_{2q},$$

and $\eta^{(q)}$ is the alternating sum consisting of the expressions obtained by replacing each a_{2i} by ηa_{2i}.

Exercise 35 *Show that the image of $\eta^{(q+1)} \circ \eta^{(q)}$ is contained in the ideal generated by*

$$\eta(1) = 1 \otimes 1 \otimes 1 = b_4'(1 \otimes 1 \otimes 1 \otimes 1).$$

Exercise 36 *Calculate*

$$(\eta^{(p)}\alpha)\beta - \eta^{(p+q)}(\alpha\beta) + (-1)^p \alpha \eta^{(p+q)}\beta,$$

where α has degree p and β has degree q, and show that the result belongs to the ideal generated by $b_4' A^{\otimes 4}$.

6.4.5

In particular, the quotient of graded algebra (134) by the ideal generated by the single element $b_4'(1 \otimes 1 \otimes 1 \otimes 1)$ becomes a cochain complex, whereas the quotient by the ideal generated by the submodule

$$b_4' A^{\otimes 4} \subset A^{\otimes 3}$$

becomes a cochain algebra.

Exercise 37 *Show that the quotient algebra $A^{(\bullet)}$ obtained by dividing the even part of the Bar algebra,* (134), *by the ideal generated by $b_4' A^{\otimes 4}$, is isomorphic to the bimodule tensor algebra*

$$T_A^\bullet A^{(1)} \qquad \textit{where} \qquad A^{(1)} = \frac{A^{\otimes 3}}{b_4' A^{\otimes 4}}. \tag{137}$$

In other words, you are asked to show that the multiplication maps

$$A^{(1)} \otimes_A \cdots \otimes_A A^{(1)} \longrightarrow A^{(q)} \qquad (q \textit{ times})$$

are isomorphisms.

6.4.6

In view of Exercise 36, the graded map $\eta^{(\bullet)}$ induces a differential of degree 1 and makes $A^{(\bullet)}$ into an $\mathbf{N}$-graded cochain k-algebra.

Exercise 38 *Show that, given any associative unital $\mathbf{N}$-graded cochain algebra $(\mathcal{B}, d)$, any homomorphism of unital k-algebras $f\colon A \longrightarrow \mathcal{B}^0$ has a unique extension to a homomorphism of cochain algebras*

$$A^{(\bullet)} \longrightarrow \mathcal{B}.$$

6.4.7

It follows that the assignment

$$A \longmapsto \left(A^{(\bullet)}, \eta^{(\bullet)}\right)$$

gives rise to a functor that is left adjoint to the inclusion of the category of associative unital k-algebras onto a full subcategory of the category of $\mathbf{N}$-graded cochain k-algebras.

6.4.8

Moreover, if $\mathcal{B}^0$ is a quotient algebra of A and $\mathcal{B}$ is generated as a k-algebra by $B \cup dB$, then $(\mathcal{B}, d)$ is a quotient of $\left(A^{(\bullet)}, \eta^{(\bullet)}\right)$.

6.5 The de Rham algebra

6.5.1 $d\colon \mathcal{O} \longrightarrow \Omega^1_{\mathcal{O}/k}$

Let $\mathcal{O}$ be a unital commutative k-algebra There are several models of the universal derivation from $\mathcal{O}$ to a symmetric $\mathcal{O}$-bimodule. A particularly convenient one is given by

$$\Omega^1_{\mathcal{O}/k} := \frac{O^{\otimes 2}}{b_2 \mathcal{O}^{\otimes 3}}, \qquad df := \text{the class of } 1 \otimes f. \tag{138}$$

Note that the Hochschild boundary maps $b_2\colon A^{\otimes(q+1)} \longrightarrow A^{\otimes q}$ are $\mathcal{O}$-linear in view of commutativity of $\mathcal{O}$. In particular, the cokernel of b_2 is equipped with an $\mathcal{O}$-module structure.

Exercise 39 *Show that $f \longmapsto df$ is a k-linear derivation.*

Exercise 40 *Show that a k-linear map $\delta\colon \mathcal{O} \longrightarrow E$ to an $\mathcal{O}$-module E (considered as a symmetric $\mathcal{O}$-bimodule) is a derivation if and only if*

$$\tilde{\delta} \circ b_2 = 0$$

where $b_2\colon A^{\otimes 3} \longrightarrow A^{\otimes 2}$ is the Hochschild boundary map and $\tilde{\delta}\colon A^{\otimes 2} \longrightarrow E$ is the $\mathcal{O}$-linearization of δ,

$$\tilde{\delta}(f \otimes g) := f\delta(g) \qquad (f, g \in \mathcal{O}).$$

6.5.2 $\Omega^\bullet_{\mathcal{O}/k}$

The de Rham algebra of $\mathcal{O}$ is the graded-commutative $\mathcal{O}$-algebra generated by $\Omega^1_{\mathcal{O}/k}$

$$\Omega^\bullet_{\mathcal{O}/k} := \Lambda^\bullet_{\mathcal{O}} \Omega^1_{\mathcal{O}/k}\ . \tag{139}$$

6.5.3

A k-linear derivation $\delta\colon \Omega^\bullet_{\mathcal{O}/k} \longrightarrow \Omega^\bullet_{\mathcal{O}/k}$ of degree l is uniquely determined by its components in degrees 0 and 1

$$\delta^0\colon \mathcal{O} \longrightarrow \Omega^l_{\mathcal{O}/k} \qquad \text{and} \qquad \delta^1\colon \Omega^1_{\mathcal{O}/k} \longrightarrow \Omega^{l+1}_{\mathcal{O}/k}.$$

Here δ^0 is a k-linear derivation while δ^1 is a k-linear map satisfying the identity

$$\delta^1(f\alpha) = \delta^0(f)\wedge\alpha + f\delta^1(\alpha) \qquad (f \in \mathcal{O}, \alpha \in \Omega^1_{\mathcal{O}/k}), \tag{140}$$

and any such pair of maps gives rise to a derivation of degree l.

6.5.4 The de Rham differential

The k-bilinear map

$$\mathcal{O}\times\mathcal{O} \longrightarrow \Omega^2_{\mathcal{O}/k}, \qquad (f, g) \longmapsto df\wedge dg \qquad (f, g \in \mathcal{O}),$$

induces a map $d^1\colon \mathcal{O} \otimes_k \mathcal{O} \longrightarrow \Omega^2_{\mathcal{O}/k}$.

Exercise 41 *Show that $d^1 \circ b_2 = 0$.*

The induced k-linear map

$$\Omega^1_{\mathcal{O}/k} \longrightarrow \Omega^2_{\mathcal{O}/k}$$

will be denoted d^1 as well.

Exercise 42 *Show that d^1 satisfies identity* (140).

6.5.5

The resulting graded derivation of $\Omega^\bullet_{\mathcal{O}/k}$ of degree 1 is referred to as the *de Rham differential*. Since $d^1 \circ d = 0$, the de Rham differential makes $\Omega^\bullet_{\mathcal{O}/k}$ into a cochain algebra. The cohomology of $\left(\Omega^\bullet_{\mathcal{O}/k}, d\right)$ is called the *de Rham cohomology* of the commutative unital k-algebra $\mathcal{O}$ and is denoted

$$H^\bullet_{\mathrm{dR}}(\mathcal{O}/k).$$

It is a graded commutative k-algebra itself.

Exercise 43 *Show that the inclusion of the field of algebraic numbers $\overline{\mathbf{Q}}^{\mathrm{alg}}$ into $\mathbf{C}$ induces an isomorphism*

$$\overline{\mathbf{Q}}^{\mathrm{alg}} \simeq H^0_{\mathrm{dR}}(\mathbf{C}/\mathbf{Z}).$$

6.6 The Lie superalgebra of graded derivations of $\Omega^\bullet_{\mathcal{O}/k}$

6.6.1

Exercise 44 *Show that*

$$[d, [d, \delta]] = 0$$

for any graded derivation δ of $\Omega^\bullet_{\mathcal{O}/k}$.

It follows that the graded commutator with the de Rham differential is a coboundary map on the graded k-module $\mathrm{Der}^\bullet_k\left(\Omega^\bullet_{\mathcal{O}/k}\right)$. It makes $\mathrm{Der}^\bullet_k\left(\Omega^\bullet_{\mathcal{O}/k}\right)$ into a *differential graded Lie superalgebra*.

6.6.2 Graded $\mathcal{O}$-linear derivations of $\Omega^{\bullet}_{\mathcal{O}/k}$

A derivation δ is $\mathcal{O}$-linear if and only if $\delta^{0} = 0$. This follows from unitality of $\mathcal{O}$. Identity (140) means in this case that δ^{1} is a k-module map. Given any k-linear map

$$\phi\colon \Omega^{1}_{\mathcal{O}/k} \longrightarrow \Omega^{l+1}_{\mathcal{O}/k},$$

the associated $\mathcal{O}$-linear derivation

$$f_0 df_1 \wedge \cdots \wedge df_q \ \longmapsto \ \sum_{i=1}^{q} (-1)^{(i-1)(l-1)} f_0 df_1 \wedge \cdots \wedge \phi(df_i) \wedge \cdots \wedge df_q$$

with will be denoted ι_ϕ.

Exercise 45 *Show that*

$$\iota_{\mathrm{id}_{\Omega^{1}_{\mathcal{O}/k}}}(\alpha) = \deg \alpha.$$

6.6.3 The Lie derivative

The graded commutator with the de Rham differential

$$\mathscr{L}_\phi := \left[d, \iota_\phi\right] \tag{141}$$

will be called the *Lie derivative*.

Exercise 46 *Show that $\mathscr{L}_\phi$ is $\mathcal{O}$-linear if and only if $\phi = 0$.*

In particular, $\mathscr{L}_\phi = 0$ if and only if $\phi = 0$.

Exercise 47 *Show that de Rham differential is the Lie derivative with respect to the identity endomorphism of $\Omega^{1}_{\mathcal{O}/k}$.*

$$\mathscr{L}_{\mathrm{id}_{\Omega^{1}_{\mathcal{O}/k}}} = d$$

Exercise 48 *Find formulae for ι_ϕ and $\mathscr{L}_\phi$ where ϕ is the morphism of left multiplication by a differential form $\alpha \in \Omega^{l}_{\mathcal{O}/k}$,*

$$\beta \longmapsto \alpha \wedge \beta \qquad (\beta \in \Omega^{1}_{\mathcal{O}/k}).$$

6.6.4

Given any derivation δ of degree l, its 0-component, being a derivation from the k-algebra $\mathcal{O}$ to the $\mathcal{O}$-module $\Omega^l_{\mathcal{O}/k}$, is represented as

$$d^0 = \delta \circ \psi$$

for a unique k-linear map $\psi\colon \Omega^1_{\mathcal{O}/k} \longrightarrow \Omega^l_{\mathcal{O}/k}$. Note that the 0-component of $\mathscr{L}_\psi$ is $d \circ \psi$, hence $\delta - \mathscr{L}_\psi$ is $\mathcal{O}$-linear, and thus is of the form ι_ϕ for some k-linear map $\phi\colon \Omega^1_{\mathcal{O}/k} \longrightarrow \Omega^{l+1}_{\mathcal{O}/k}$.

6.6.5

The resulting decomposition

$$\delta = \iota_\phi + \mathscr{L}_\psi \tag{142}$$

is unique. Indeed, given another decomposition

$$\delta = \iota_{\phi'} + \mathscr{L}_{\psi'}$$

we would have

$$\iota_{\phi-\phi'} = \iota_\phi - \iota_{\phi'} = \mathscr{L}_{\psi'} - \mathscr{L}_\psi = \mathscr{L}_{\psi'-\psi}.$$

In view of Exercise 46, one has $\psi' - \psi = 0$. This implies that $\iota_{\phi-\phi'} = 0$ which means that $\phi - \phi' = 0$.

We established the following important result.

Theorem 6.1 *Any derivation of degree l of the de Rham algebra is uniquely representable as the sum of an $\mathcal{O}$-linear derivation and of a Lie derivative, cf.* (142).

6.6.6

As a corollary, we deduce that the graded k-module of Lie derivatives coincides with the *centralizer* of the de Rham differential in $\mathrm{Der}^\bullet_k(\Omega^\bullet_{\mathcal{O}/k})$. In particular, it is a subalgebra of $\mathrm{Der}^\bullet_k(\Omega^\bullet_{\mathcal{O}/k})$.

6.6.7

Thus, $\operatorname{Der}^\bullet_k(\Omega^\bullet_{\mathcal{O}/k})$ decomposes into the direct sum of two of its graded Lie super-subalgebras: $\operatorname{Der}^\bullet_{\mathcal{O}}(\Omega^\bullet_{\mathcal{O}/k})$, which coincides with the centralizer of $\mathcal{O}$, and the Lie derivative subalgebra that coincides with the centralizer of d. As graded k-modules they are isomorphic to

$$\operatorname{Hom}_{\mathcal{O}\text{-mod}}(\Omega^1_{\mathcal{O}/k}, \Omega^\bullet_{\mathcal{O}/k}) \tag{143}$$

and

$$\operatorname{Hom}_{\mathcal{O}\text{-mod}}(\Omega^1_{\mathcal{O}/k}, [1]\Omega^\bullet_{\mathcal{O}/k}) = [1]\operatorname{Hom}_{\mathcal{O}\text{-mod}}(\Omega^1_{\mathcal{O}/k}, \Omega^\bullet_{\mathcal{O}/k}),$$

respectively. This endows (143) with two bracket operations: the ι-bracket $[\phi, \psi]_\iota$ is defined by

$$\iota_{[\phi,\psi]_\iota} = [\iota_\phi, \iota_\psi]$$

while the $\mathscr{L}$-bracket $[\phi, \psi]_{\mathscr{L}}$ is defined by

$$\mathscr{L}_{[\phi,\psi]_{\mathscr{L}}} = [\mathscr{L}_\phi, \mathscr{L}_\psi].$$

The former is of degree 0, the latter of degreee 1.

6.6.8

The correspondence

$$h\colon \delta \longmapsto \iota_\psi \qquad \big(\delta \in \operatorname{Der}^\bullet_k(\Omega^\bullet_{\mathcal{O}/k})\big) \tag{144}$$

defines a k-linear map of degree -1 such that

$$h \circ h = 0 \qquad \text{and} \qquad [\operatorname{ad}_d, h] = \operatorname{id}_{\operatorname{Der}^\bullet_k(\Omega^\bullet_{\mathcal{O}/k})}$$

where ad_d denotes the graded commutator with the de Rham deifferential, considered to be an inner derivation of the Lie superalgebra $\operatorname{Der}^\bullet_k(\Omega^\bullet_{\mathcal{O}/k})$,

$$\delta \longmapsto \operatorname{ad}_d(\delta) := [d, \delta].$$

In other words, (144) is an *integrable* contracting homotopy for the cochain complex

$$\Big(\operatorname{Der}^\bullet_k(\Omega^\bullet_{\mathcal{O}/k}), \operatorname{ad}_d\Big).$$

Exercise 49 *Show that the correspondence*

$$\delta \longmapsto \mathscr{L}_\xi$$

that sends a k-linear derivation δ of $\mathcal{O}$ to a derivation of $\Omega^\bullet_{\mathcal{O}/k}$ of degree 0, where ξ is the unique k-module morphism $\xi\colon \Omega^1_{\mathcal{O}/k} \longrightarrow \mathcal{O}$ such that

$$\delta = d \circ \xi,$$

is a homomorphism of Lie superalgebras

$$\mathrm{Der}_k(\mathcal{O}) \longrightarrow \mathrm{Der}^\bullet_k\big(\Omega^\bullet_{\mathcal{O}/k}\big)$$

(*with* $\mathrm{Der}_k(\mathcal{O})$ *being concentrated in degree* 0).

Exercise 50 *Show that for any*

$$\xi, \theta \in \big(\Omega^1_{\mathcal{O}/k}\big)^\vee := \mathrm{Hom}_{\mathcal{O}\text{-mod}}\big(\Omega^1_{\mathcal{O}/k}, \mathcal{O}\big),$$

one has

$$\big[\mathscr{L}_\xi, \iota_\theta\big] = \iota_{[\xi,\theta]_{\mathscr{L}}}.$$

6.7 Poincaré's Lemma

6.7.1 Description of $d\colon S_kV \longrightarrow \Omega^\bullet_{S_kV/k}$

Given a k-module V, the k-q-linear maps

$$V\times\cdots\times V \longrightarrow S_kV\otimes_k V, \qquad (v_1,\ldots,v_q) \longmapsto \sum_{i=1}^q v_1\cdots\hat{v}_i\cdots v_q\otimes v_i$$

are symmetric, hence they define k-linear maps

$$S^q_kV \longrightarrow S_kV\otimes_k V$$

that are combined into a single map

$$d_V\colon S_kV \longrightarrow S_kV\otimes_k V. \tag{145}$$

Exercise 51 *Show that* (145) *is a derivation from S_kV to the relatively free S_kV-module $S_kV\otimes_k V$.*

6.7.2

A derivation δ from the symmetric algebra S_kV to a symmetric S_kV-module E is uniquely determined by its restriction δ_1 to $V \subset S_kV$. This map $V \longrightarrow E$ possesses a unique extension to a S_kV-linear map

$$\tilde{\delta}\colon S_kV \otimes_k V \longrightarrow E.$$

Exercise 52 *Show that*

$$\delta = \tilde{\delta} \circ d_V.$$

Thus, (145) provides a model for the universal derivation from $\mathcal{O} = S_kV$ into a symmetric bimodule.

6.7.3

In particular, the following sequence

$$S_kV \otimes_k V \overset{\tilde{d}_V}{\twoheadleftarrow} \left(S_kV\right)^{\otimes 2} \overset{b_3'}{\longleftarrow} \left(S_kV\right)^{\otimes 3}$$

is exact.

Exercise 53 *Write down the formula for* $\tilde{d}_V$.

6.7.4 The exterior algebra of a relatively free module

Given an $\mathcal{O}$-algebra B and an $\mathcal{O}$-linear mapping from the induced module $\mathcal{O} \otimes_k V$ to B,

$$f\colon \mathcal{O} \otimes_k V \longrightarrow B,$$

such that

$$f(e)^2 = 0 \qquad (e \in \mathcal{O} \otimes_k V),$$

its restriction f_1 to V induces a homomorphism of k-algebras

$$f_\Lambda\colon \Lambda_kV \longrightarrow B. \tag{146}$$

Exercise 54 *Show that the $\mathcal{O}$-linearization of* (146)

$$\tilde{f}_\Lambda\colon \mathcal{O} \otimes_k \Lambda_kV \longrightarrow B$$

is a homomorphism of k-algebras.

Since the $\mathcal{O}$-algebra $\mathcal{O} \otimes_k \Lambda_k V$ is generated by $\mathcal{O} \otimes_k V$ and $e^2 = 0$ in $O \otimes_k \Lambda_k V$ for any $e \in \mathcal{O} \otimes_k V$, the canonical homomorphism

$$\mathcal{O} \otimes_k \Lambda_k V \longrightarrow \Lambda_{\mathcal{O}}\big(\mathcal{O} \otimes_k V\big)$$

must be the inverse to the canonical homomorphism

$$\Lambda_{\mathcal{O}}\big(\mathcal{O} \otimes_k V\big) \longrightarrow \mathcal{O} \otimes_k \Lambda_k V.$$

6.7.5 $\Omega^\bullet_{S_k V/k}$

It follows that the de Rham algebra $\Omega^\bullet_{\mathcal{O}/k}$ is canonically isomorphic to the tensor product of $\mathcal{O}$ and the exterior algebra $\Lambda_k V$ whenever $\Omega^1_{\mathcal{O}/k}$ is isomorphic to the relatively free module $\mathcal{O} \otimes_k V$.

6.7.6 The subalgebra of constant differential forms

The image of $\Lambda_k V$ under the map

$$\Lambda_k V \longrightarrow \Lambda_{\mathcal{O}} \Omega^1_{\mathcal{O}/k} = \Omega^\bullet_{\mathcal{O}/k}, \qquad \alpha \longmapsto 1_{\mathcal{O}} \otimes \alpha,$$

forms the subalgebra of *constant* differential forms $\Omega^{\bullet,\mathrm{const}}_{\mathcal{O}/k}$. As an $\mathcal{O}$-module, $\Omega^\bullet_{\mathcal{O}/k}$ is relatively free,

$$\Omega^\bullet_{\mathcal{O}/k} \simeq \mathcal{O} \otimes_k \Omega^{\bullet,\mathrm{const}}_{\mathcal{O}/k}.$$

6.7.7 The de Rham algebra of the symmetric algebra $S_k V$

The grading of $S_k V$ induces a $\mathbf{N}\times\mathbf{N}$-grading of $\Omega^\bullet_{S_k V/k}$ with tensors

$$u_1 \cdots u_p \otimes_k dv_q \wedge \cdots \wedge dv_q \qquad (u_1, \dots, u_p, v_1, \dots, v_q \in V)$$

having bidegree (p, q). Differential forms of bidegrees $(0, q)$ form the subalgebra of constant differential forms.

Note that the de Rham differential has bidegree $(-1, 1)$. In particular, it preserves the total degree $n = p + q$, hence $\big(\Omega^\bullet_{S_k V/k}, d\big)$ is the direct sum of cochain complices

$$\Omega^\bullet_{S_k V/k} = \bigoplus_{n \geq 0} \Omega^\bullet_{S_k V/k}(n)$$

where $\Omega^\bullet_{S_k V/k}(n)$ denotes the subcomplex of forms of total degree n.

6.7.8

Exterior multiplication induces a canonical isomorphism in the category of cochain complices (concentrated in degrees ≥ 0) of the n-th supersymmetric tensor power of $\Omega^\bullet_{S_k V/k}(1)$ with $\Omega^\bullet_{S_k V/k}(n)$,

$$S^n_k\big(\Omega^\bullet_{S_k V/k}(1)\big) \simeq \Omega^\bullet_{S_k V/k}(n).$$

The complex $\Omega^\bullet_{S_k V/k}(1)$ is concentrated in degrees 0 and 1 and is isomorphic to

$$0 \longrightarrow V \xrightarrow{id_V} V \longrightarrow 0$$

This complex is obviously contractible with id_V providing the contraction. In terms of $\Omega^\bullet_{S_k V/k}(1)$, the differential $d(0)$ is provided by the map

$$V \longrightarrow 1_k \otimes V, \qquad v \longmapsto 1 \otimes v$$

while the contracting homotopy is provided by its inverse

$$1_k \otimes V \longrightarrow V \qquad 1 \otimes v \longmapsto v.$$

We shall denote the latter map by $\int(1)$. Note that the contraction is a boundary map of degree -1 (we call such contractions *integrable*).

6.7.9

When tensoring n complices with integrable contractions, we obtain a complex with integrable contraction for the endomorphism of multiplication by n. This induces the corresponding contraction of the symmetric n-th tensor powers.

We established the following result.

Proposition 6.2 *The de Rham complex of $S_k V$ is the direct sum of complices $\big(\Omega^\bullet_{S_k V/k}(n), d(n)\big)$, each equipped with an integrable contraction of the endomorphism of multiplication by n.*

6.7.10 The integral

Assuming that n is invertible in k, the boundary map of the chain complex

$$S^n_k\big(\Omega^\bullet_{S_kV/k}(1), \textstyle\int(1)\big)$$

divided by n is a contracting homotopy for $\big(\Omega^\bullet_{S_kV/k}(n), d(n)\big)$. We shall denote it $\int(n)$. For $n = 0$ we set $\int(0) = 0$.

Corollary 6.3 (Poincaré's Lemma) *If the additive group of k is uniquely divisible, then $\Omega^\bullet_{S_kV/k}(n)$ is contractible for any $n > 0$, and the inclusion*

$$[0]k = \Omega^\bullet_{S_kV/k}(0) \hookrightarrow \Omega^\bullet_{S_kV/k}$$

is a homotopy equivalence, with the integral $\int = \big(\int(n)\big)_{n\in\mathbf{N}}$ supplying the corresponding homotopy.

6.7.11

In particular, the de Rham cohomology of S_kV vanishes in positive degrees if the additive group of $\mathcal{O}$ is a $\mathbf{Q}$-vector space.

6.8 Commutative and N-graded algebras

6.8.1

Suppose that $\mathcal{O}$ is $\mathbf{N}$-graded (note that this is in general different from *graded-commutative*),

$$\mathcal{O} = \bigoplus_{n\geq 0} \mathcal{O}(n).$$

7 Connections

7.1 The definition

7.1.1

Given an $\mathcal{O}$-module E, a k-linear map $d_\nabla\colon E \longrightarrow E\otimes_{\mathcal{O}}\Omega^1_{\mathcal{O}/k}$ is said to be a *connection on* E if it satisfies the identity

$$d_\nabla(ef) = \big(d_\nabla e\big)f + e\otimes_{\mathcal{O}} df \qquad (e \in E,\ f \in \mathcal{O}). \tag{147}$$

7.1.2

An $\mathcal{O}$-module E is a symmetric $\mathcal{O}$-bimodule, therefore ef is defined and equals fe. In the context of connections on E, the right-module-notation is preferrable to the left-module-notation, however, because it agrees with the structure of several standard chain complices like Bar and Hochschild complex.

7.1.3

Given a "vector field"

$$\xi \in \big(\Omega^1_{\mathcal{O}/k}\big)^\vee = \mathrm{Hom}_{\mathcal{O}\text{-mod}}\big(\Omega^1_{\mathcal{O}/k}, \mathcal{O}\big),$$

the composite mapping

$$\begin{tikzcd} E \arrow[r, "d_\nabla"] \arrow[d, dashed] & E\otimes_{\mathcal{O}}\Omega^1_{\mathcal{O}/k} \arrow[d, "\mathrm{id}_E \otimes_{\mathcal{O}} \xi"] \\ E & E\otimes_{\mathcal{O}}\mathcal{O} \arrow[l, "\simeq"'] \end{tikzcd} \tag{148}$$

will be denoted ∇_ξ. The calculation

$$\begin{aligned} \nabla_\xi(ef) &= \mathrm{id}_E \otimes_{\mathcal{O}} \xi\ \big(d_\nabla(e)f + e\otimes_{\mathcal{O}} df\big) \\ &= \big(\mathrm{id}_E \otimes_{\mathcal{O}} \xi\big)\big(d_\nabla(e)\big)f + e\big((\xi \circ d)\big)f \\ &= (\nabla e)f + e\big(\mathscr{L}_\xi f\big)\ . \end{aligned}$$

shows that ∇_ξ satisfies the following form of Leibniz's identity

$$\nabla_\xi(ef) = (\nabla_\xi e)f + e\big(\mathscr{L}_\xi f\big). \tag{149}$$

7.1.4

The correspondence

$$\nabla\colon \big(\Omega^1_{\mathcal{O}/k}\big)^\vee \longrightarrow \mathrm{End}_k(E), \qquad \xi \longmapsto \nabla_\xi,$$

is $\mathcal{O}$-linear,

$$\nabla_{f\xi} = f\nabla_\xi,$$

if we consider the $\mathcal{O}$-module structure on $\mathrm{End}_k(E)$ induced by the action of $\mathcal{O}$ on the *target* of the homomorphism,

$$(f\phi)(e) := f\phi(e).$$

Note that $\mathrm{End}_k(E)$ is naturally an $\mathcal{O}$-bimodule, with the other $\mathcal{O}$-module structure induced by the action of $\mathcal{O}$ on the *source* of a homomorphism,

$$(\phi f)(e) := \phi(fe).$$

7.1.5

If $\mathcal{O} = \mathcal{O}(X)$ is the algebra of functions on a *space* X, and E is the $\mathcal{O}$-module of *sections* on a vector bundle $\mathscr{E}$ on X, a traditional view is that a connection ∇ on $\mathscr{E}$ is an assignment to each vector field $\xi \in \mathscr{T}_X$ on X, of a k-linear *operator* ∇_ξ on E satisfying identity (149), the dependence of ∇_ξ on ξ supposed to be $\mathcal{O}$-linear.

7.1.6 $E\otimes_{\mathcal{O}}\Omega^\bullet_{\mathcal{O}/k}$

Exterior multiplication of differential forms equips $E\otimes_{\mathcal{O}}\Omega^\bullet_{\mathcal{O}/k}$ with the structure of a graded-symmetric bimodule over $\Omega^\bullet_{\mathcal{O}/k}$,

$$(e\otimes_{\mathcal{O}}\alpha)\wedge\beta := e\otimes_{\mathcal{O}}(\alpha\wedge\beta) \qquad \text{and} \qquad \beta\wedge(e\otimes_{\mathcal{O}}\alpha) := (-1)^{\tilde{\alpha}\tilde{\beta}}e\otimes_{\mathcal{O}}(\alpha\wedge\beta).$$

7.1.7

The de Rham differential, not being $\mathcal{O}$-linear, does not induce a degree 1 k-linear endomorphism of $E\otimes_{\mathcal{O}}\Omega^{\bullet}_{\mathcal{O}/k}$. This is what a connection on E is meant for. Let us consider the k-bilinear map

$$E\times\Omega^{q}_{\mathcal{O}/k} \longrightarrow E\otimes_{\mathcal{O}}\Omega^{q+1}_{\mathcal{O}/k}, \qquad (e,\alpha) \longmapsto d_{\nabla}e\wedge\alpha + e\otimes_{\mathcal{O}}d\alpha. \tag{150}$$

Calculating the values pairing (150) takes on (ef,α),

$$d_{\nabla}(ef)\wedge\alpha + e\otimes_{\mathcal{O}}d\alpha = (d_{\nabla}e)f\wedge\alpha + e\otimes_{\mathcal{O}}df\wedge\alpha + (ef)\otimes_{\mathcal{O}}d\alpha, \tag{151}$$

and on $(e,f\alpha)$,

$$d_{\nabla}e\wedge(f\alpha) + e\otimes_{\mathcal{O}}d(f\alpha) = d_{\nabla}e\wedge(f\alpha) + e\otimes_{\mathcal{O}}df\wedge\alpha + e\otimes_{\mathcal{O}}(fd\alpha), \tag{152}$$

we observe that the the values are equal. In other words, pairing (150) is $\mathcal{O}$-*balanced* without being $\mathcal{O}$-linear in either argument.

Exercise 55 *Show that the resulting k-linear maps*

$$d^{q}_{\nabla}\colon E\otimes_{\mathcal{O}}\Omega^{q}_{\mathcal{O}/k} \longrightarrow E\otimes_{\mathcal{O}}\Omega^{q+1}_{\mathcal{O}/k}, \qquad (q\in\mathbf{N}),$$

satisfy the identity

$$d^{p+q}_{\nabla}(\epsilon\wedge\alpha) = d^{p}_{\nabla}\epsilon\wedge\alpha + (-1)^{p}\epsilon\wedge d\alpha \qquad (\epsilon\in E\otimes_{\mathcal{O}}\Omega^{p}_{\mathcal{O}/k},\ \alpha\in\Omega^{q}_{\mathcal{O}/k}). \tag{153}$$

7.1.8 The curvature of a connection

The calculation

$$\begin{aligned}\big(d^{q+1}_{\nabla}\circ d^{q}_{\nabla}\big)(e\otimes_{\mathcal{O}}\alpha) &= d^{q+1}_{\nabla}\Big(d_{\nabla}(e)\wedge\alpha + e\otimes_{\mathcal{O}}d\alpha\Big)\\ &= \big(d^{q+1}_{\nabla}\circ d_{\nabla}\big)(e)\wedge\alpha - d_{\nabla}(e)\wedge d\alpha + d_{\nabla}(e)\wedge d\alpha + e\otimes_{\mathcal{O}}(d\circ d)\alpha\\ &= \big(d^{1}_{\nabla}\circ d_{\nabla}\big)(e)\wedge\alpha\end{aligned}$$

shows that the degree 2 k-linear endomorphism

$$d_{\nabla}\circ d_{\nabla}$$

of $E\otimes_{\mathcal{O}}\Omega^{\bullet}_{\mathcal{O}/k}$ is $\Omega^{\bullet}_{\mathcal{O}/k}$-linear. Since the $\Omega^{\bullet}_{\mathcal{O}/k}$-module $E\otimes_{\mathcal{O}}\Omega^{\bullet}_{\mathcal{O}/k}$ is generated by E, $d_{\nabla}\circ d_{\nabla}$ is uniquely determined by its degree 0 component,

$$d^{1}_{\nabla}\circ d_{\nabla}\colon E \longrightarrow E\otimes_{\mathcal{O}}\Omega^{2}_{\mathcal{O}/k}.$$

The latter is denoted R_{∇} and referred to as the *curvature* (operator) of connection ∇.

7.1.9 Integrable connections

We say that a connection is *integrable* or *flat*), if its curvature operator is zero.

Exercise 56 *Show that a connection d_∇ is integrable if the $\mathcal{O}$-module E is generated by a k-submodule E_0 consisting of vectors annihilated by d_∇.*

7.2 The category of $\mathcal{O}$-modules-with-connection

7.2.1

Modules over $\mathcal{O}$ equipped with a connection form a category with morphisms being $\mathcal{O}$-linear maps $\phi E \longrightarrow E'$ for which the diagram

$$\begin{tikzcd} E' \arrow[r, "d_{\nabla'}"] & E' \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}/k} \\ E \arrow[u, "\phi"] \arrow[r, "d_\nabla"] & E \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}/k} \arrow[u, "\phi \otimes_{\mathcal{O}} \mathrm{id}_{\Omega^1_{\mathcal{O}/k}}"'] \end{tikzcd}$$

commutes.

7.2.2 The direct sum of modules-with-connection

Given modules-with-connection $(E', d_{\nabla'})$ and $(E'', d_{\nabla''})$, their direct sum is the module $E' \oplus E''$ equipped with the connection

$$\begin{pmatrix} e' \\ e'' \end{pmatrix} \longmapsto \begin{pmatrix} d_{\nabla'}(e') \\ d_{\nabla''}(e'') \end{pmatrix}.$$

We shall denote this module-with-connection by $(E', d_{\nabla'}) \oplus (E'', d_{\nabla''})$.

Exercise 57 *Show that the k-bilinear pairing*

$$E' \times E'' \longrightarrow E' \otimes E'' \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}/k}, \qquad (e', e'') \longmapsto d_{\nabla'}(e') \otimes_{\mathcal{O}} e'' + e' \otimes_{\mathcal{O}} d_{\nabla''}(e'') \tag{154}$$

is $\mathcal{O}$-balanced.

Exercise 58 *Show that the k-linear map induced by* (154)*,*

$$d_{\nabla} \colon E' \otimes_{\mathcal{O}} E'' \longrightarrow E' \otimes E'' \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}/k} \tag{155}$$

is a connection on $E' \otimes_{\mathcal{O}} E''$*.*

7.2.3 The tensor product of modules-with-connection

Given modules-with-connection $\left(E', d_{\nabla'}\right)$ and $\left(E'', d_{\nabla''}\right)$, their tensor product is the module $E' \otimes_{\mathcal{O}} E''$ equipped with connection (155). We shall denote it by $(E', d_{\nabla'}) \otimes_{\mathcal{O}} (E'', d_{\nabla''})$.

Exercise 59 *Show that* $(E, d_{\nabla}) \otimes_{\mathcal{O}} (\mathcal{O}, d_{\mathrm{dR}})$ *is canonically isomorphic to* (E, d_{∇})

7.3 The space of connections

7.3.1

Given any connection d_{∇} on E and an $\mathcal{O}$-linear map

$$\theta \colon E \longrightarrow E \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}/k},$$

the map $d_{\nabla} + \theta$ is a conection. Vice-versa, the difference of two connections $d_{\nabla} - d_{\nabla'}$ is $\mathcal{O}$-linear in view of the fact that the commutator of d_{∇} with the multiplication by a function $f \in \mathcal{O}$,

$$[d_{\nabla}, f]$$

is the operator of multiplication by df. In particular, the deviation of d_{∇} from being $\mathcal{O}$-linear is independent of the connection.

7.3.2

It follows that the set of connections $\mathrm{Conn}(E)$ on an $\mathcal{O}$-module E is a *torsor* over $\mathrm{Hom}_{\mathcal{O}\text{-mod}}(E, E \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}/k})$ when it is nonempty.

7.3.3 Torsors

A *torsor* over a group G is a transitive free G-set, i.e., a G-set X with the property that given any two elements $x, x' \in X$, there exists a unique element $G \in G$ such that $x' = gx$.

7.3.4 Existence of a connection

Suppose that an $\mathcal{O}$-module E' is a *retract* of an $\mathcal{O}$-module E which means that there exists a pair of $\mathcal{O}$-linear maps $\iota\colon E' \longrightarrow E$ and $\pi\colon E \longrightarrow E$ such that $\pi \circ \iota = \mathrm{id}_{E'}$. Fixing a retraction pair (ι, π) allows one to produce a connection on E' from a connection d_∇ on E,

$$\begin{tikzcd} E' \arrow[r, dashed] \arrow[d, tail, "\iota"'] & E' \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}/k} \\ E \arrow[r, "d_\nabla"] & E \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}/k} \arrow[u, "\pi \otimes_{\mathcal{O}} \mathrm{id}_{\Omega^1_{\mathcal{O}/k}}"'] \end{tikzcd}$$

Denote this connection by $d_{\nabla'}$. We shall refer to it as the *retract* of d_∇ via (ι, π), and to the module-with-connection $(E', d_{\nabla'})$ as the *retract* of (E, d_∇) via (ι, π).

7.3.5

We infer that the set of connections on E' is nonempty if E' is a retract of a module E with $\mathrm{Conn}(E) \neq \emptyset$.

7.3.6

In fact, the map

$$(\iota, \pi)_\sharp\colon \mathrm{Conn}(E) \longrightarrow \mathrm{Conn}(E'), \qquad d_\nabla \longmapsto \left(\pi \otimes_{\mathcal{O}} \mathrm{id}_{\Omega^1_{\mathcal{O}/k}}\right) \circ d_\nabla \circ \iota,$$

is surjective. Indeed, given *any* connection $d_{\nabla'}$ on E' and *any* connection $d_{\nabla''}$ on the kernel $E'' = \ker \pi$ (such a connection exists since E'' is a retract of E), their direct sum defines a connection on $E' \oplus E'' \simeq E$ that is sent by $(\iota, \pi)_\sharp$ to $d_{\nabla'}$.

7.3.7 Free $\mathcal{O}$-modules

Any free $\mathcal{O}$-module F is isomorphic to a direct sum of a family of free rank 1 modules $\mathcal{O}$. Selecting a basis $B \subset F$ equips F with the unique connection that annihilates basis vectors $b \in B$. Such a connection is integrable, cf. Exercise 56.

7.3.8 Projective $\mathcal{O}$-modules

Retracts of free modules coincide with the class of *projective* modules. In particular, projective modules admit a connection.

7.4 Chern character

For any pair of $\mathcal{O}$-modules E and F, the map

$$E^\vee \otimes_{\mathcal{O}} F \longrightarrow \mathrm{Hom}_{\mathcal{O}\text{-mod}}(E, F)$$

that sends a tensor $\varepsilon \otimes_{\mathcal{O}} f$ to the $\mathcal{O}$-linear map

$$e \longmapsto \varepsilon(e) f,$$

is a map of $\mathcal{O}$-bimodules. If E is a free module of finite rank, this map is an isomorphism. As a corollary, the same holds for retracts of free modules of finite rank, i.e., for finitely projective modules.

7.4.1 The contraction $E^\vee \otimes_{\mathcal{O}} E \longrightarrow \mathcal{O}$

Note that $E^\vee \otimes_{\mathcal{O}} E$ is, for *any* module E, naturally equipped with the $\mathcal{O}$-bilinear *evaluation* pairing

$$\tau \colon E^\vee \otimes_{\mathcal{O}} E \longrightarrow \mathcal{O}, \qquad (\varepsilon, e) \longmapsto \varepsilon(e).$$

In classical theory of invariants this is referred to as *contraction of tensors*.

7.4.2 The contraction algebra $E^\vee \otimes_{\mathcal{O}} E$

The 4-linear map

$$(\varepsilon', e', \varepsilon'', e'') \longmapsto \varepsilon''(e')\varepsilon' \otimes_{\mathcal{O}} e'', \tag{156}$$

induces a multiplication on $E^\vee \otimes_{\mathcal{O}} E$,

$$\big(E^\vee \otimes_{\mathcal{O}} E\big) \otimes_{\mathcal{O}} \big(E^\vee \otimes_{\mathcal{O}} E\big) \longrightarrow E^\vee \otimes_{\mathcal{O}} E$$

Exercise 60 *Show that the contraction multiplication induced by* (156) *is associative and that the canonical mapping*

$$E^\vee \otimes_{\mathcal{O}} E \longrightarrow \mathrm{End}_{\mathcal{O}\text{-mod}}(E) \tag{157}$$

is a homomorphism of $\mathcal{O}$-algebras.

Exercise 61 *Calculate*

$$\tau\big((\varepsilon'\otimes_{\mathcal{O}}e')(\varepsilon''\otimes_{\mathcal{O}}e'')\big)$$

and use your calculation to show that

$$\tau(u\circ v)=\tau(v\circ u)\qquad\big(u,v\in E^{\vee}\otimes_{\mathcal{O}}E\big).$$

7.4.3 Trace

Contraction τ for a finitely projective module E induces a $\mathcal{O}$-bimodule map

$$\mathrm{Tr}\colon\ \mathrm{End}_{\mathcal{O}\text{-mod}}(E)\longrightarrow\mathcal{O}$$

called the *trace* map.

7.4.4 The N-graded algebra $E^{\vee}\otimes_{\mathcal{O}}E\otimes_{\mathcal{O}}\Omega^{\bullet}_{\mathcal{O}/k}$

The tensor product of the $\mathcal{O}$-algebras $E^{\vee}\otimes_{\mathcal{O}}E$ and of the **N**-graded $\mathcal{O}$-algebra $\Omega^{\bullet}_{\mathcal{O}/k}$,

$$E^{\vee}\otimes_{\mathcal{O}}E\otimes_{\mathcal{O}}\Omega^{\bullet}_{\mathcal{O}/k},$$

can be considered as an $\Omega^{\bullet}_{\mathcal{O}/k}$-alegbra. The contraction pairing on $E^{\vee}\otimes_{\mathcal{O}}E$ induces the contraction

$$\tau^{\bullet}:=\tau\otimes_{\mathcal{O}}\mathrm{id}_{\Omega^{\bullet}_{\mathcal{O}/k}}\longrightarrow\Omega^{\bullet}_{\mathcal{O}/k}.$$

7.4.5 Chern character forms

For a finitely generated projective module E, the curvature operator of any connection becomes an element of

$$E^{\vee}\otimes_{\mathcal{O}}E\otimes_{\mathcal{O}}\Omega^{2}_{\mathcal{O}/k}.$$

Proposition 7.1 *The contractions of the powers of the curvature of a connection are closed differential forms*

$$d_{\mathrm{dR}}\big(\tau^{2n}(R^{n}_{\nabla})\big)=0.$$

Proposition 7.2 *Given two connections on E, the difference*

$$\tau^{2n}(R^{n}_{\nabla})-\tau^{2n}(R^{n}_{\nabla'})$$

is an exact form.

7.4.6

Assuming that multiplication by $n!$ is an automorphism of the additive group of $\Omega^{2n}_{\mathcal{O}/k}$, one defines the *Chern character* forms as

$$\mathrm{Ch}_n(E,\nabla) := \frac{1}{n!}\tau^{2n}(R^n_\nabla).$$

The *Chern character* of a finitely generated projective module E (E, d_∇) is defined as the element of

$$\prod_{n\in\mathbf{N}} H^{2n}_{\mathrm{dR}}(\mathcal{O}/k)$$

whose component in degree $2n$ is the cohomology class of $\mathrm{Ch}_n(E,\nabla)$. It depends only on E and not on the chosen connection.

7.4.7 The K-group of a unital ring

Given a unital ring R, isomorphism classes of finitely generated projective (left) R-modules form a commutative monoid. Addition is induced by direct sum

$$[E] + [E'] := [E \oplus E'].$$

The isomorphism class of the zero module provides the neutral element. The reflection of this monoid in the category of groups is an abelian group, denoted $K(R)$, and called the *K-group* of R.

7.4.8 The K-functor

Assignment

$$R \longmapsto K(R)$$

gives rise to a functor from the category of unital rings to the category of abelian groups.

7.4.9 Multiplication on $K(R)$

When R is commutative, tensor product

$$E, E' \longmapsto E \otimes_R E',$$

induces commutative associative multiplication on $K(R)$ which is biadditive. The class of the rank 1 free module R provides the neutral element. In particular, $K(R)$ is a unital commutative ring.

Theorem 7.3 *The Chern character is a unital ring homomorphism*

$$K(\mathcal{O}) \longrightarrow \prod_{n \in \mathbf{N}} H^{2n}_{\mathrm{dR}}(\mathcal{O}/k).$$