The Conceptbook

Supplementary notes on Calculus

Mariusz Wodzicki

October 23, 2015

1 Functions

1.1 The concept of a function

1.1.1

A *function* consists of a pair of sets X and Y, and a *relation* between the elements of X and Y such that

$$\textit{for any } x \in X \textit{ there is exactly one } y \in Y \textit{ related to } x. \tag{1}$$

1.1.2 Notation

General functions are usually denoted by single letters of Latin or Greek alphabet. A very common notation for a function is f (since it is the first letter of the word *function*.)

1.1.3 Terminology

If the function is denoted f, then the set X is referred to as the *domain* of f and Y could be called the *target* of f. For this reason, we shall also refer to X as the *source* of f. Notation

$$f\colon X \longrightarrow Y \tag{2}$$

shows you in one glimpse that the function is denoted f, its source is X and its target is Y.

1.1.4 The value of f at x

If x is an element of X, then the *unique* element y of Y that is related to x is denoted $f(x)$ and called the *value* of f *on* (or *at* x).

1.1.5

The rules specifying the value of f at x can be given in many different ways.

1.1.6 Equality of functions

We say that functions

$$f\colon X \longrightarrow Y \qquad \text{and} \qquad g\colon V \longrightarrow W$$

are *equal* if

$$X = V, \qquad Y = W,$$

and

$$f(x) = g(x) \quad \text{for every} \quad x \in X.$$

In other words, two functions are equal if they have the same domain, the same target and take the same values on all all elements of the domain.

1.1.7 Example: the inclusion functions i_{XY}

If X is a subset of Y, then the function with X as its domain, Y as its target and the values given by

$$f(x) := x \qquad (x \in X) \tag{3}$$

is the associated *inclusion* function. Note that the "rule" determining the value is common to all such inclusion functions but the two inclusion functions: for a subset $X \subseteq Y$ and for a subset V of another set W, are equal if and only if

$$X = V \qquad \text{and} \qquad Y = W.$$

1.1.8 The identity functions $\mathbf{id}_X$

In the special case when $X = Y$, the inclusion function is called the *identity function* of a set X.

1.1.9 A special case: the empty domain

A special case when the domain is *empty* merits special attention. In this case *no* relation is involved since X has no elements, thus the only piece of data that one needs in order to specify a function from the empty set to a set Y is the target, i.e., set Y itself. In particular, there is no more than a single function

$$\emptyset \longrightarrow Y \qquad (\emptyset \text{ is the standard notation for the empty set}).$$

Such a function exists: it is the inclusion function of $\emptyset$ viewed as a subset of Y.

1.1.10 The image of a subset under a function

For a subset A of the domain of f, the set formed by the values $f(x)$ for all $x \in A$,

$$f(A) := \{y \in Y \mid y = f(x) \text{ for some } x \in A\}, \tag{4}$$

is called the *image of A under f*.

1.1.11 The range of a function

The image $f(X)$ of the domain under f is often called the *range* of f.

1.1.12 The preimage of a subset under a function

For a subset B of the target of f, the set formed by all $x \in X$ such that the value $f(x)$ belongs to B,

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}, \tag{5}$$

is called the *preimage of B under f*.

1.1.13

You should think of the *image under f* and the *preimage under f* as two operations on sets which are induced by the function. The notation used is traditional and logical but you must not confuse $f(A)$ with being the value of f: here A is a *subset* of the domain, not an element.

Exercise 1 *What is the image of the empty set $\emptyset$ under f?*

Exercise 2 *What is the preimage of the empty set $\emptyset$ under f?*

Exercise 3 *Let X and B be subsets of a set Y. Describe the preimage of B under the inclusion function i_{XY}.*

Exercise 4 *Is $f^{-1}\big(f(A)\big) = A$?*

1.1.14 A comment about exercises in the form of a question

You are expected not as much to provide an answer as to provide an explanation of your answer. Giving an answer without being able to provide an explanation that is *relevant* has little value.

Exercise 5 *Is $f\big(f^{-1}(B)\big) = B$?*

1.2 Composition of functions

1.2.1

Given two functions $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$, we can *compose* them. The result is the function denoted $g \circ f$. Its value at $x \in X$ is obtained by evaluating f first, then evaluating g,

$$g(f(x)).$$

Note that the domain of the composite function $g \circ f$ is the domain of f (the function applied first), while the target of $g \circ f$ is the target of g (the function applied last).

1.2.2

Note that the rule specifying the value of $g \circ f$ is applicable to all elements of the domain of f if the *range* of g is *contained* in the domain of f. This is less restrictive than asking that the target of g equals the domain of f.

You are allowed to compose such functions but you should understand that in order to do that, you are *restricting* the domain of f and the target of g to a common set containing the range of g.

1.2.3

Composition is perhaps the most important operation involving functions.

1.2.4

For a function $g\colon Y \to X$, one can form both

$$g \circ f \qquad \text{and} \qquad f \circ g.$$

The first of these composite functions is a function $X \longrightarrow X$, the second one is a function $Y \longrightarrow Y$.

Exercise 6 *Explain why*

$$\mathrm{id}_Y \circ f \;=\; f \;=\; f \circ \mathrm{id}_X\,. \tag{6}$$

1.3 The inverse function

1.3.1

A function $g\colon Y \longrightarrow X$ is said to be *an inverse to* f if

$$g \circ f = \mathrm{id}_X \tag{7}$$

and

$$f \circ g = \mathrm{id}_Y \,. \tag{8}$$

1.3.2 Uniqueness

If such a function exists, then it is unique. Indeed, suppose $h\colon Y \longrightarrow X$ is another function that is inverse to f. Then,

$$g = g \circ \mathrm{id}_Y = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_X \circ h = h. \tag{9}$$

1.3.3

In view of its uniqueness, a function that is inverse to f is referred to as *the* inverse function and is denoted f^{-1}. Functions $f\colon X \longrightarrow Y$ for which such a function exists are said to be *invertible*.

1.3.4

For a subset $B \subseteq Y$ you must not confuse its preimage $f^{-1}(B)$ with being the value of the inverse function: here B is a *subset* of Y, not an element. Moreover, the inverse function f^{-1} exists only for invertible functions and $f^{-1}(y)$ makes sense only when f is invertible. In contrast, the preimage of a subset B of Y makes sense *for any function*.

Exercise 7 *Let $f\colon X \longrightarrow Y$ be a function. Suppose that a function $g\colon Y \longrightarrow X$ exists that satisfies* (7)*. Explain why f is "one-to-one", i.e., it satisfies the following property*

$$\textit{if } f(x_1) = f(x_2)\textit{, then } x_1 = x_2. \tag{10}$$

1.3.5

Note that condition (10) is equivalently stated as

$$\textit{if } x_1 \neq x_2\textit{, then } f(x_1) \neq f(x_2). \tag{11}$$

Exercise 8 *Explain why a function f is "one-to-one" if and only if the preimage of any singleton set $B = \{y\}$ has no more than a single element.*

Exercise 9 *Let $f\colon X \longrightarrow Y$ be a function. Suppose that a function $g\colon Y \longrightarrow X$ exists that satisfies* (8)*. Explain why f is "onto", i.e., it satisfies the following property*

$$\text{for any } y \in Y, \text{ there is } x \in X \text{ such that } y = f(x). \tag{12}$$

Exercise 10 *Explain why a function f is "onto" if and only if the preimage of any nonempty subset $B \subseteq Y$ is not empty.*

1.3.6 More elegant terminology

Functions that are "onto" are also said to to be *surjective* while functions that are "one-to-one" are said to to be *injective*.

1.3.7 Bijections

It follows from Exercises 7 and 9 that an invertible function is simultaneously injective and surjective. Such functions are said to be *bijections* between elements of a set X and of a set Y.

This condition is not only necessary but is also sufficient: if f is bijective, then the correspondence

$$y \longmapsto \text{the unique } x \in X \text{ such that } y = f(x),$$

defines a function inverse to f.

1.4 Functions encountered in Calculus

1.4.1

In one-dimensional Calculus, X and Y are usually subsets of the real line $\mathbf{R}$. Calculus as taught at American universities and colleges at the freshmen level reflects the approach and habits that go back several centuries ago. It is, for example, a common practice to assume by default that the target of a real valued function is $\mathbf{R}$ if it is not explicitly indicated.

1.4.2

If the domain of f is not explicitly indicated, then it is assumed to be the largest subset of the real line for which the rule determining the value of f at x makes sense. We shall refer to it as the *natural domain* of f.

1.4.3

For example, if one encounters in Calculus a function whose value is given by the formula

$$f(x) := \sqrt{\frac{x-1}{x(x+1)}}, \tag{13}$$

and no information about the domain is given, then one should consider it as being defined on the set X of real numbers for which the right-hand-side of (13) is defined. This is the set of those real numbers x for which

$$x(x+1) \neq 0 \qquad \text{and} \qquad \frac{x-1}{x(x+1)} \geq 0.$$

One could think of f as the composition $g \circ h$ of functions

$$g(x) = \sqrt{x}$$

and

$$h(x) = \frac{x-1}{x(x+1)}$$

but the range of h is not contained in the domain of g. The natural domain of g is $[0, \infty) = \{x \in \mathbf{R} \mid x \geq 0\}$ while the natural domain of h is the set

$$\{x \in \mathbf{R} \mid x \neq -1, 0\}$$

which is the union of three open intervals

$$(-\infty, -1), \qquad (-1, 0) \qquad \text{and} \qquad (0, \infty).$$

Function h takes positive values on $(-1, 0)$ and nonnegative values on $[1, \infty)$. At other points of the domain of h it takes negative values. Therefore the natural domain of f is the union of the following two intervals

$$(-1, 0) \cup [1, \infty).$$

1.4.4

The above example illustrates the practice common in Calculus. We compose functions $g \colon Y \longrightarrow \mathbf{R}$ and $f \colon X \longrightarrow \mathbf{R}$ even when they are, strictly speaking, not composable. In order to define $g \circ f$, we determine first those points of the domain of f, for which the value $g(f(x)$ is defined. They form the natural domain of $g \circ f$.

From this point of view, one can say that any two real valued functions with the domains being subsets of the real line can be composed. It can happen that the domain of such a composite function is very small or even empty.

Exercise 11 *Let a be a real number and $f(x) = a - \sqrt{x}$. What is the domain of $f \circ f$?*

1.4.5 Operations on real valued functions

Given two real valued functions $f\colon X \to \mathbf{R}$ and $g\colon X \to \mathbf{R}$ *with the same domain* X, we define

$$f+g, \qquad fg, \qquad f-g \qquad \text{and} \qquad \frac{f}{g} \tag{14}$$

by performing the corresponding operations *on the values*. Thus, the value of the function $f+g$ on x is the sum of the values of f and g on x,

$$(f+g)(x) := f(x) + g(x)$$

and similarly for thre other operations. The domain of

$$f+g, \qquad ,fg \qquad \text{and} \qquad f-g \tag{15}$$

equals X. The domain of

$$\frac{f}{g} \tag{16}$$

is obtained by removing those $x \in X$ where the denominator function vanishes,

$$\{x \in \mathrm{X} \mid g(x) \neq 0\}.$$

1.4.6

To perform these operations on real valued functions, the common domain can be an *arbitrary* set. It does not need to be a subset of the real line. In Calculus, we apply these operations even when the domains of f and g differ. The domain of either of the three functions in (15) is understood to be the intersection of the domains of f and g,

$$\mathrm{Domain}\,(f) \cap \mathrm{Domain}\,(g)\,,$$

whereas for (16) it is understood to be the intersection of the two domains from which the points where g vanishes are removed,

$$\mathrm{Domain}\left(\frac{f}{g}\right) = \{x \in \mathrm{Domain}\,(f) \cap \mathrm{Domain}\,(g) \mid g(x) \neq 0\}.$$

1.5 Monotonic functions

1.5.1 Increasing functions

Let X be a set of real numbers. A function $f\colon X \to \mathbf{R}$ is said to be *increasing on a subset* $A \subseteq X$ if

$$f(x) < f(x') \qquad \text{whenever } a < a' \text{ and } a, a' \in A.$$

1.5.2 Nondecreasing functions

A function $f\colon X \to \mathbf{R}$ is said to be *nondereasing on a subset* $A \subseteq X$ if

$$f(x) \leq f(x') \qquad \text{whenever } a < a' \text{ and } a, a' \in A.$$

Exercise 12 *By analogy, state the definitions of functions that are decreasing (respectively, nonincreasing) on a subset* A *of the domain.*

Exercise 13 *Suppose that* $I = (a', a'')$ *is an interval contained in the domain of an increasing function* f*. What is its image under* f*?*

Exercise 14 *How does the answer change if we assume that* f *is only nondecreasing?*

Exercise 15 *How does the answer change if* f *is decreasing?*

2 Limits and neighborhoods

2.1 The intuition about the limit of a function at a point

2.1.1

A typical "definition" of the limit of a function reads like this

A point $q \in Y$ is the limit of a function $f\colon X \to Y$ at a point $p \in X$ if

$$f(x) \text{ approches } q \text{ whenever } x \text{ approaches } p. \tag{17}$$

First, one needs to point out that whatever "x approaches p" means, x is not allowed to equal p. Secondly, one needs to make precise the meaning of the phrase:

$$f(x) \text{ approches } q \text{ whenever } x \text{ approaches } p. \tag{18}$$

One way of doing that is:

$$f(x) \text{ is as close to } q \text{ as one wishes whenever } x \text{ is sufficiently close to } p. \tag{19}$$

This is still not precise but making it precise is not difficult. For this we need a concept of *neighborhoods* of a point.

2.1.2 A set equipped with a neighborhood structure

Let X be a set. Suppose that, for every element $p \in X$, somebody specified a family of subsets $Nbhds(p)$ of X whose members are referred to as *neighborhoods of p*. The precise meaning of (19) then becomes clear:

For any neighborhood N of q, there exists a neighborhood M of p, such that

$$f(x) \in N \quad \text{whenever} \quad x \in M. \tag{20}$$

2.1.3

Recalling that we are supposed to exclude p from those points "sufficiently close to p", the precise definition of "q being the limit of f at p" reads:

A point $q \in Y$ is the limit of a function $f\colon X \to Y$ at $p \in X$ if, for any neighborhood N of q, there exists a neighborhood M of p, such that

$$f(x) \in N \quad \text{whenever} \quad x \text{ belongs to } M \text{ and } x \neq p. \tag{21}$$

2.1.4 Additional comments

In the definition of the limit at a point p the point itself does not need to belong to the domain of f. More than that, it is clear that if p does belong to the domain, then we proceed by *ignoring the value of* f at p. This is the same as narrowing the domain of f by removing point p from it. We do this *before* we look at what f does with neighborhoods of p.

2.1.5

We need just one property from a point p itself in order to be able to talk about the limit of f at p. If D is a subset of X on which f is defined, then we assume that

for any neighborhood M of p, there is a point x in it that is in the domain of f and which is different from p. (22)

In other words, for any neighborhood of p,

$$M \cap \text{Domain}\,(f)$$

must contain at least one point different from p. If this is so, we say that a point p of X is a *limit point* of a subset $D \subseteq X$.

2.1.6 The definition of the limit made precise

Let f be a function from a subset D of X to a set Y. We assume that both X and Y are equipped with a neighborhood structure. Let p be a limit point of D.

A point $q \in Y$ is the limit of a function $f\colon X \to Y$ at $p \in X$ if, for any neighborhood N of q, there exists a neighborhood M of p, such that

$$f(x) \in N \quad \textit{whenever} \quad x \textit{ belongs to } M \textit{ and } x \neq p. \tag{23}$$

We denote this fact by writing

$$\lim_{x \to p} f(x) \;=\; q. \tag{24}$$

2.1.7 A simple but important observation

The following fact must be stressed: the limit of f at a point p is the limit of f *restricted* to $D' := D \setminus \{p\}$.

2.1.8

Before proceeding any further, let us record a simple observation about the limit.

Proposition 2.1 *Suppose that* (24) *holds and D' is a subset of D such that p is also a limit point of D'. Then the restriction of f to D' has the same limit at p:*

$$\lim_{x \to p} f(x) \;=\; \lim_{x \to p} f_{|D'}(x) \tag{25}$$

Indeed, if N is a neighborhood of q and M is a neighborhood of p such that

$$f\big(M \cap D \setminus \{p\}\big) \subseteq N,$$

then obviously also

$$f\big(M \cap D' \setminus \{p\}\big) \subseteq N$$

since

$$M \cap D' \setminus \{p\} \quad \text{is contained in} \quad M \cap D \setminus \{p\}$$

2.1.9 Terminology

Sometimes we are more interested in knowing whether the limit of f at a point p exists at all than in determining the actual value of the limit. So, if there exists a point $q \in Y$ such that (24) holds, then we say that f *has a limit at* p or, that the limit of f at p *exists*.

2.1.10 A few warnings

Saying "the limit of a given function at a given point exits" and saying "the concept of the limit exists" are two different things. The former makes sense and may be true or may be false. The latter, on the other hand, is nonsense.

2.1.11

Saying "q is a limit point of a subset E of Y" has very different meaning from saying "q is the limit of a function f at some point p".

Exercise 16 *Explain the difference.*

2.1.12

Observe also that we use an indefinite article "a" when talking about limit points of *subsets* and the definite article "the" when talking about the limits of functions at various points.

2.2 One-sided limits

2.2.1

For a function defined on a subset $D \subseteq \mathbf{R}$ of the real line, and a point $a \in \mathbf{R}$, let

$$D' := \{x \in D \mid x < a\} \qquad \text{and} \qquad D'' := \{x \in D \mid a < x\}.$$

The limit at a of f *restricted* to D' is called the *left* limit of f at a, and is denoted

$$\lim_{x \to a^-} f(x) \qquad \text{or} \qquad \lim_{x \nearrow a} f(x).$$

This is one of the two *one-sided* limits, it is also called the *limit-from-below* at a.

2.2.2

The *right* limit, also called the *limit-from-above* at a, is defined similarly, as the limit at at point a of f *restricted* to D''. It is denoted

$$\lim_{x \to a^+} f(x) \qquad \text{or} \qquad \lim_{x \searrow a} f(x).$$

2.2.3

Point a may be a limit point of D but not of D' or D''. In particular, $\lim_{x \to a} f(x)$ may make sense but not necessarily the left or the right limit, as the obvious example of a function defined on the interval $D = [a, b]$ demonstrates: here D' is empty.

2.2.4

When a is a limit point of both D' and D'', and

$$\lim_{x \to a} f(x) = c$$

exists, then both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and their values are equal,

$$\lim_{x\to a^-} f(x) = c = \lim_{x\to a^+} f(x).$$

This is an immediate corollary of Proposition 2.1.

2.2.5

Vice-versa, if the one-sided limits exist but are not equal, then the limit $\lim_{x\to a} f(x)$ *does not* exist.

2.3 What is expected of a family of neighborhoods?

2.3.1

We made the definition of the limit of a function possible by using the notion of the families of neighborhoods of $p \in X$ and $q \in Y$ but we did not say which families of subsets qualify to be called neighborhoods of the corresponding points. There are remarkably few things that one needs to assume about a family of subsets in order that is qualified to serve as the family of neighborhoods of a point.

2.3.2

We expect that

$$\textit{point } p \textit{ belongs to each of its neighborhoods}. \tag{26}$$

We also expect that

$$\begin{array}{l}\textit{a subset } N' \subseteq X \textit{ is a neighborhood of } p \textit{ if} \\ \textit{it already contains a neighborhood } N \textit{ of } p.\end{array} \tag{27}$$

2.3.3

One more condition is normally expected:

$$\textit{the intersection of two neighborhoods } N_1 \cap N_2 \textit{ of } p \textit{ is a neighborhood of } p. \tag{28}$$

2.4 The neighborhood structure of the real line

2.4.1 Neighborhoods of a point

Let us say first that any open interval $I = (a', a'')$ of the real line is declared to be a neighborhood of every $a \in I$. One does not need to say anything else, this alone *completely* determines the neighborhood structure of the real line.

Indeed, according to Property (27), a subset of the real line N is then a neighborhood of a if and only if it contains an open interval I whose member is a.

2.4.2

Such an interval can always to be chosen so that a is at its center. To describe I one then needs just to specify the *distance* of the endpoints of I from a. If that distance is $\rho > 0$, then

$$I = (a - \rho, a + \rho). \tag{29}$$

Exercise 17 *Explain why* (29) *coincides with the set*

$$\{x \in \mathbf{R} \mid |x - a| < \rho\}. \tag{30}$$

2.4.3

Property (28) is automatically satisfied too. This is so because the intersection of two open intervals I_1 and I_2 is either empty or an open interval itself. If a is a member of both, then their intersection is not empty, therefore it is an open interval whose member is a.

Thus, if N_1 contains an interval I_1 while N_2 contains an interval I_2, and a is a member of both, then $N_1 \cap N_2$ contains open interval $I_1 \cap I_2$ whose member is a.

2.4.4 A few observations

Any neighborhood of a point p on the real line contains other points beyond p. Indeed, such a neighborhood contains an open interval and every open interval has infinitely many points and all but one are distinct from p.

2.4.5

Given two different points $p \neq q$ of the real line, there is a neighborhood M of p and a neighborhood N of q, such that they are disjoint, i.e., they have no points in common:

$$M \cap N = \emptyset. \tag{31}$$

This property of the neighborhood structure of the real line is called the *separability* (*"distinct points can be separated by their neighborhoods"*).

Exercise 18 *Let $\rho = \frac{1}{2}|p - q|$. Show that the intervals $M = (p - \rho, p + \rho)$ and $N = (q - \rho, q + \rho)$ have no points in common.*

2.4.6

The separability of the real line has an important consequence. If a function $f \colon X \to \mathbf{R}$ has number b as its limit at a point p, then it cannot have a different number b as it limit at p.

Indeed, let N and N' be neighborhoods of b and b', respectively, such that they are disjoint. If

$$\lim_{x \to p} f(x) = b,$$

then there is a neighborhood M of p such that

$$f\big(M \setminus \{p\}\big) \subseteq N.$$

If

$$\lim_{x \to p} f(x) = b',$$

then there is a neighborhood M' of p such that

$$f\big(M' \setminus \{p\}\big) \subseteq N'.$$

It follows that $M \cap M'$ is a neighborhood of p such that

$$f\big(M \cap M' \setminus \{p\}\big)$$

is contained in both N and N'. Since they have no points in common, the image of $M \cap M' \setminus \{p\}$ is empty. This is possible only when $M \cap M' \setminus \{p\}$ is empty, i.e., when $M \cap M'$ has only one point, namely p. This contradicts the fact that every neighborhood of a point p on the real line has points other than p.

2.4.7

Another consequence of separability of the real line is the following corollary of Proposition 2.1.

Theorem 2.2 *Let $f\colon D \to \mathbf{R}$ be function with the domain $D \subseteq X$ and let D' and D'' be two subsets of D, each having p as its limit point. If f has the limit b', when restricted to D', and the limit b'', when restricted to D'', and if*

$$b' \neq b'',$$

then f has no limit at p.

Indeed, if

$$\lim_{x \to p} f(x)$$

exists, then, according to Proposition 2.1 one has

$$b' \;=\; \lim_{x \to p} f_{|D'}(x) \;=\; \lim_{x \to p} f(x) \;=\; \lim_{x \to p} f_{|D''}(x) \;=\; b''$$

and this contradicts the hypothesis that $b' \neq b''$.

2.4.8

I would like you to observe how elegant is the argument that allowed us to establish this very useful fact. An attempt to demonstrate Theorem 2.2 directly would likely lead into quite a complex reasoning.

You are seeing here how one should be doing analysis: collect various general observations first about the concepts you are studying, record them for the future use, and later use them as you use tools—to achieve various tasks.

The process of "collecting the tools" is essentially where 80% of your learning process should go.

2.4.9 An example: $f(x) = \sin \frac{1}{x}$

We shall now demonstrate how powerful is the theorem we established. The function

$$f(x) = \sin \frac{1}{x} \tag{32}$$

is defined on $D = \mathbf{R} \setminus \{0\}$. Consider the following two subsets of D:

$$D' = \left\{ \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \dots \right\} \qquad \text{and} \qquad D'' = \left\{ \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \dots \right\}.$$

The point 0 is a limit point of both. Function (32) when restricted to D' becomes the constant function

$$f(x) = 0 \qquad (x \in D').$$

When restricted to D'', it becomes the constant function

$$f(x) = 1 \qquad (x \in D'').$$

In particular,

$$\lim_{x \to 0} f_{|D'}(x) = 0 \qquad \text{and} \qquad \lim_{x \to 0} f_{|D''}(x) = 1.$$

It follows that

$$\lim_{x \to 0} \sin \frac{1}{x}$$

does not exist.

Exercise 19 *For each of the following subsets $D \subseteq \mathbf{R}$ find the set of its limit points:*

(a) $D = (a, b)$ *(here and below $a < b$),*

(b) $D = (a, b]$,

(c) $D = [a, b)$,

(d) $D = [a, b]$,

(e) $D = (0, 1) \cup (1, 2)$,

(f) $D = [0, 1] \cap [1, 2]$,

(g) $D = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \right\}$,

(h) $D = \mathbf{R} \setminus \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \right\}$.

(i) D *is the set of all integers* $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

2.4.10 An example: $\lim_{x\to a} x^2 = a^2$

Let us assume that $a > 0$. If N is a neighborhood of a^2, it contains an interval $I = (a', a'')$ such that

$$0 < a' < a^2 < a''.$$

Then

$$0 < \sqrt{a'} < a < \sqrt{a''}$$

and since the function $f(x) = x^2$ is *increasing* on $[0, \infty)$, the image of the interval

$$M = \big(\sqrt{a'}, \sqrt{a''}\big)$$

is contained in I (in fact, is equal to I). In particular, $f(M) \subseteq N$. This shows that

$$\lim_{x\to a} x^2 = a^2$$

for $a > 0$.

Exercise 20 *Modify the above argument for* $a < 0$.

Exercise 21 *Show that* $\lim_{x\to 0} x^2 = 0$.

2.4.11 Functions bounded around a point

We say that a function $f\colon D \longrightarrow \mathbf{R}$ is *bounded around* a point p if there exists a positive number $B > 0$ and a neighborhood M of p such that

$$|f(x)| < B$$

for all x belonging to $M \cap D$.

Exercise 22 *Suppose that a real valued function* f *has a limit at a point* p*. Explain why such a function is bounded around* p*.* (*You can, if you wish, assume that the domain of* f *is a subset of the real line. This will neither make this exercise easier nor will make it more difficult, however.*)

Exercise 23 *Suppose that the limit is not zero. Explain why there exists a neighborhood* M *of* p *such that*

$$f(x) \neq 0 \qquad \big(x \in M \setminus \{p\}\big).$$

3 Arithmetic operations on real valued functions and the operation of passing to the limit

3.1 A certain property of arithmetic operations on real numbers

3.1.1 Addition

Suppose that the sum of two real numbers $b_1 + b_2$ belongs to an interval

$$(c', c'').$$

The pairs y_1 and y_2 of real numbers such that

$$y_1 + y_2 \in (c', c'')$$

form the region in the plane located strictly between the lines

$$y_1 + y_2 = c' \qquad \text{and} \qquad y_1 + y_2 = c'',$$

see Figure 1 where the two lines are marked thick red.

3.1.2

As one can clearly see from Figure 1, if one takes sufficiently small open intervals I_1 and I_2 around points b_1 and b_2, the corresponding rectangle

$$I_1 \times I_2 = \{(y_1, y_2) \mid y_1 \in I_1 \text{ and } y_2 \in I_2\} \tag{33}$$

fits entirely inside this region. This means that

$$y_1 + y_2 \in (c', c'')$$

for *all* $y_1 \in I_1$ and $y_2 \in I_2$.

Exercise 24 *Let b_1 and b_2 be a pair of real numbers whose sum satisfies the double inequality*

$$1 < b_1 + b_2 < 2.$$

Find open intervals I_1 and I_2 containing b_1 and, respectively, b_2, such that

$$1 < y_1 + y_2 < 2$$

for all $y_1 \in I_1$ and $y_2 \in I_2$.

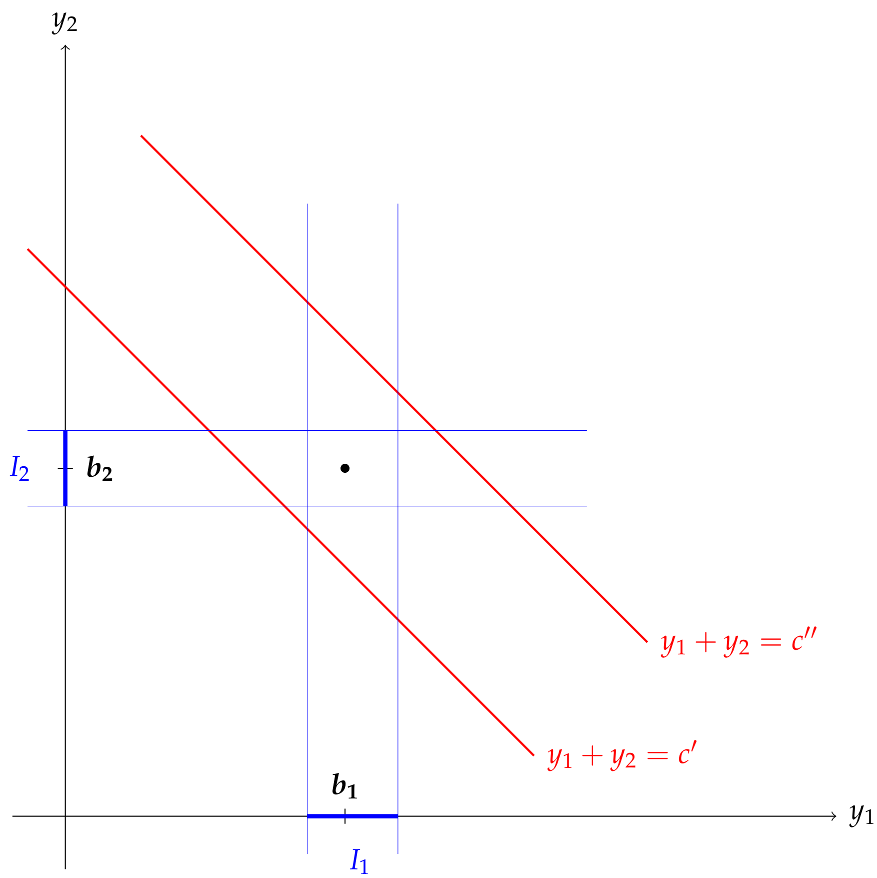

Figure 1: Points inside the region between the red lines correspond to pairs of points of the real line such that $y_1 + y_2 \in (c', c'')$. Sufficiently small intervals I_1 and I_2 around b_1 and b_2 have the property that their sum $y_1 + y_2$ belongs to (c', c'') for all $y_1 \in I_1$ and $y_2 \in I_2$.

3.1.3 Multiplication

A similar property holds for multiplication. Suppose that the product of two real numbers $b_1 b_2$ belongs to an interval

$$(c', c'').$$

The pairs y_1 and y_2 of real numbers such that

$$y_1 y_2 \in (c', c'')$$

form the region in the plane located strictly between the curves

$$y_1 y_2 = c' \qquad \text{and} \qquad y_1 y_2 = c'',$$

see Figure 2.

3.1.4

As one can clearly see from Figure 2, if one takes sufficiently small open intervals I_1 and I_2 around points b_1 and b_2, rectangle (33) fits entirely inside the mentioned region which means that

$$y_1 y_2 \in (c', c'')$$

for *all* $y_1 \in I_1$ and $y_2 \in I_2$.

Exercise 25 *Let b_1 and b_2 be a pair of real numbers whose sum satisfies the double inequality*

$$2 < b_1 b_2 < 3.$$

Find open intervals I_1 and I_2 containing b_1 and, respectively, b_2, such that

$$2 < y_1 y_2 < 3$$

for all $y_1 \in I_1$ and $y_2 \in I_2$. Consider first the case when b_1 and b_2 are both positive, then consider the case when they are both negative.

3.1.5 Division

Exactly the same property holds also for idivision. Suppose that the ratio of two real numbers b_1/b_2 belongs to an interval

$$(c', c'').$$

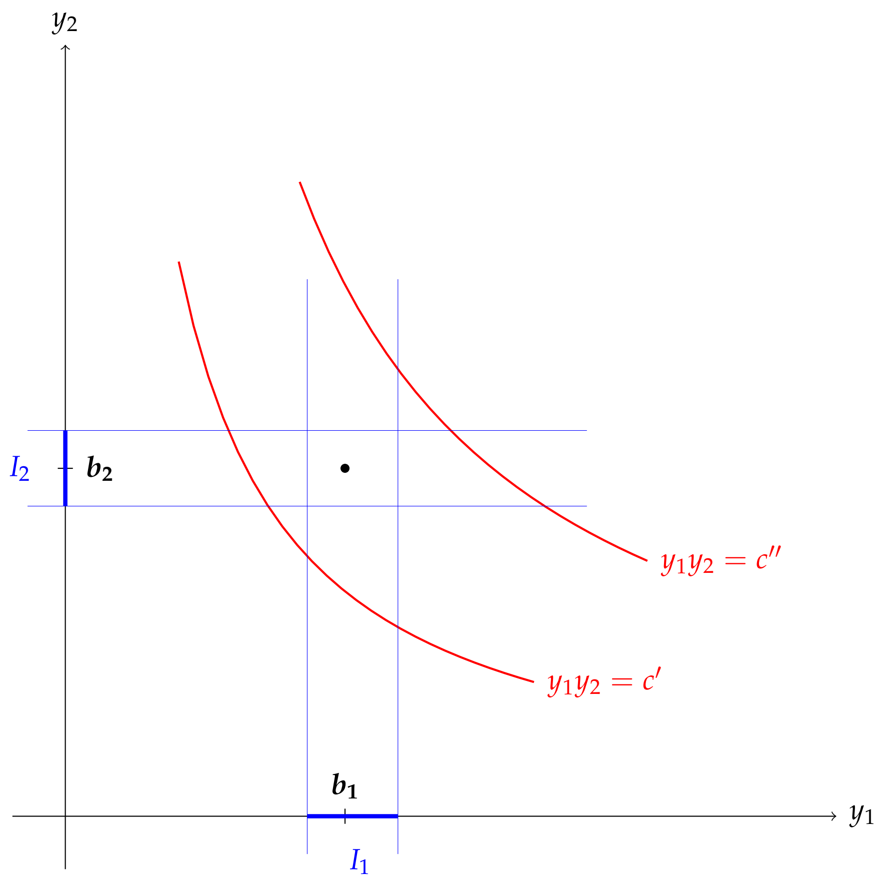

Figure 2: Points inside the region between the red curves correspond to pairs of points of the real line such that $y_1y_2 \in (c', c'')$. Sufficiently small intervals I_1 and I_2 around b_1 and b_2 have the property that their product y_1y_2 belongs to (c', c'') for all $y_1 \in I_1$ and $y_2 \in I_2$.

The pairs y_1 and y_2 of real numbers such that

$$y_1/y_2 \in (c', c'')$$

form the region in the plane located strictly between the lines

$$y_1/y_2 = c' \qquad \text{and} \qquad y_1/y_2 = c'',$$

see Figure 3.

3.1.6

As one can clearly see from Figure 3, if one takes sufficiently small open intervals I_1 and I_2 around points b_1 and b_2, rectangle (33) fits entirely inside the mentioned region which means that

$$y_1/y_2 \in (c', c'')$$

for *all* $y_1 \in I_1$ and $y_2 \in I_2$.

Exercise 26 *Let b_1 and b_2 be a pair of real numbers whose sum satisfies the double inequality*

$$2 < b_1/b_2 < 3.$$

Find open intervals I_1 and I_2 containing b_1 and, respectively, b_2, such that

$$2 < y_1/y_2 < 3$$

for all $y_1 \in I_1$ and $y_2 \in I_2$. Consider first the case when b_1 and b_2 are both positive, then consider the case when they are both negative.

3.1.7 Subtraction

Exercise 27 *State the corresponding property for subtraction of real numbers and illustrate it by drawing the appropriate picture.*

3.1.8 Exponentiation

Later we shall see that this property is also shared by the operation of exponentiation $y_1^{y_2}$.

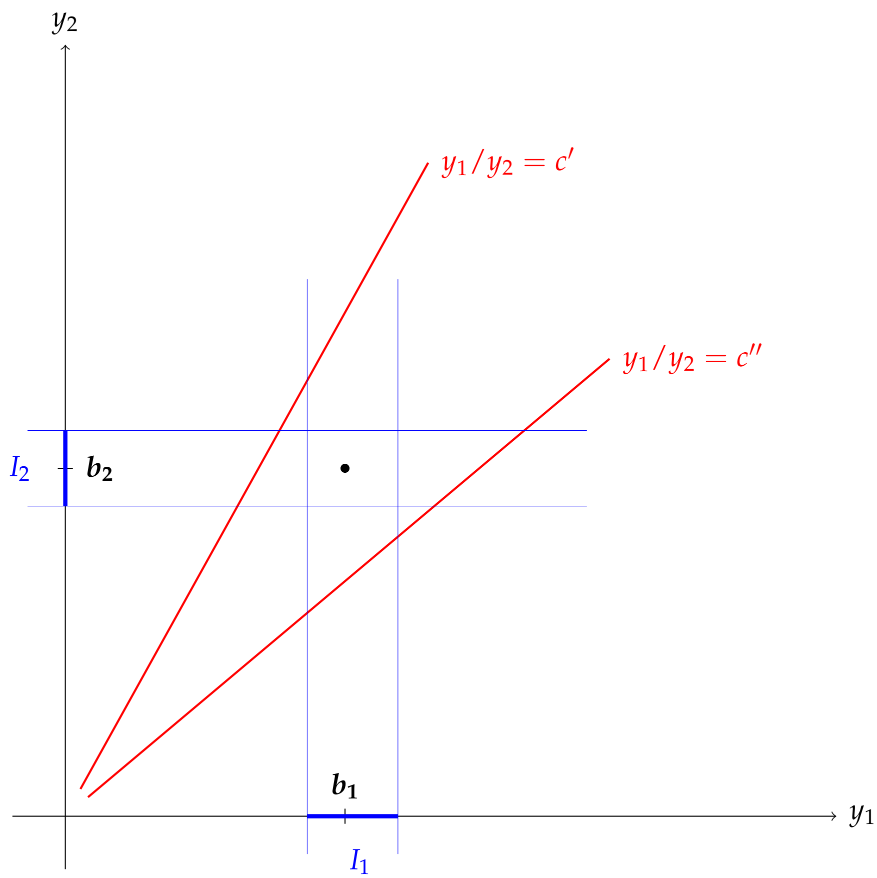

Figure 3: Points inside the region between the red lines correspond to pairs of points of the real line such that $y_1/y_2 \in (c', c'')$. Sufficiently small intervals I_1 and I_2 around b_1 and b_2 have the property that their ratio y_1/y_2 belongs to (c', c'') for all $y_1 \in I_1$ and $y_2 \in I_2$.

3.2 The Limit Laws

3.2.1

Theorem 3.1 *Suppose that f_1 and f_2 are two real valued functions with the same domain D. Suppose that*

$$\lim_{x\to p} f_1(x) = b_1 \qquad \textit{and} \qquad \lim_{x\to p} f_2(x) = b_2. \tag{34}$$

Then

$$\lim_{x\to p}\big(f_1(x) + f_2(x)\big) \;=\; b_1 + b_2 \tag{35}$$

and

$$\lim_{x\to p}\big(f_1(x) f_2(x)\big) \;=\; b_1 b_2. \tag{36}$$

If $\lim_{x\to p} f_2(x) \neq 0$, then also

$$\lim_{x\to p} \frac{f_1(x)}{f_2(x)} \;=\; \frac{b_1}{b_2}. \tag{37}$$

3.2.2

Indeed, if (c', c'') is an interval containing point $b_1 + b_2$, there exist open intervals I_1 and I_2 containing points b_1 and b_2 such that

$$y_1 + y_2 \;\in\; (c', c'')$$

holds for all $y_1 \in I_1$ and $y_2 \in I_2$, cf. Section 3.1.2.

3.2.3

Since $\lim_{x\to p} f_1(x) = b_1$, there is a neighborhood M_1 of p such that

$$f_1(x) \;\in\; I_1 \tag{38}$$

for all x belonging to $M_1 \cap D \setminus \{p\}$.

3.2.4

Similarly, since $\lim_{x\to p} f_2(x) = b_2$, there is a neighborhood M_2 of p such that

$$f_2(x) \;\in\; I_2 \tag{39}$$

for all x belonging to $M_2 \cap D \setminus \{p\}$.

3.2.5

It follows that $M_1 \cap M_2$ is the desired neighborhood: for all x belonging to $M_1 \cap M_2 \cap D \setminus \{p\}$, both (38) and (39) hold, and therefore

$$f_1(x) + f_2(x) \in (c', c'').$$

3.2.6

This completes the demonstration of the Limit Law for Addition. The Limit Law for Multiplication is demonstrated exactly same way, by replacing everywhere the sums by the corresponding products.

3.2.7

In the case of division, we first notice that that f_2 does not vanish on some neighborhood of p, cf. Exercise 23. The argument is otherwise exactly the same as in the cases of addition and multiplication, with the sums being everywhere replaced by the corresponding ratios.

3.2.8

Note the unifying principle that allowed us to derive the limit laws for the arithmetic operations involving real numbers: it is the common property that they share, cf. Sections 3.1.2, 3.1.4 and 3.1.6.

4 Improper limits

4.1 The extended real line

4.1.1 Infinite intervals

You have already encountered symbols ∞ and $-\infty$ in the notation employed for infinite *open* intervals

$$(a,\infty) := \{x \in \mathbf{R} \mid a < x\}, \qquad (-\infty,a) := \{x \in \mathbf{R} \mid x < a\}, \tag{40}$$

and for infinite *closed* intervals

$$[a,\infty) := \{x \in \mathbf{R} \mid a \leq x\}, \qquad (-\infty,a] := \{x \in \mathbf{R} \mid x \leq a\}, \tag{41}$$

4.1.2

What if we treat ∞ and $-\infty$ as actual points of the *extended* real line? They are *ideal* points in the sense that they are not *real numbers*, they represent the fact that every neighborhood of every point p of the real line has *two* sides—one consisting the points *to the right* of p, the other one consisting the points *to the left* of p.

4.1.3

In the extended real line real numbers form the infinite open interval

$$(-\infty,\infty). \tag{42}$$

4.1.4 The neighborhood systems of the ideal points

We declare the infinite intervals

$$(a,\infty] := (a,\infty) \cup \{\infty\} \tag{43}$$

to be neighborhoods of ∞. Likewise, we declare the infinite intervals

$$[-\infty,a) := \{-\infty\} \cup [-\infty,a) \tag{44}$$

to be neighborhoods of $-\infty$. This completely determines which subsets of the extended real line are considered neighborhoods of ∞ or $-\infty$. Thus, a subset E of the extended real line is a neighborhood of ∞ if there exists such a real number a such that E contains *all points greater* than a.

Exercise 28 *Which subsets E of the extended real line are neighborhoods of $-\infty$?*

Exercise 29 *Which subsets $D \subseteq \mathbf{R}$ of the real line have ∞ as their limit point? Answer the same question also for $-\infty$.*

4.2 Extended arithmetic

4.2.1 Extended addition and subtraction

We shall extend the operation of addition of real numbers as follows as follows;

$$\textit{adding or subtracting a real number } a \textit{ to } \pm\infty \textit{ yields } \pm\infty. \tag{45}$$

We also declare

$$\infty + \infty = \infty - (-\infty) = \infty \quad \text{and} \quad (-\infty) + (-\infty) = -\infty - \infty = -\infty \tag{46}$$

but we *do not* declare the values of

$$\infty - \infty, \qquad (-\infty) + \infty \qquad \text{or} \qquad \infty + (-\infty)\,. \tag{47}$$

4.2.2 Extended multiplication

We shall extend the operation of multiplication of real numbers as follows

$$\textit{multiplying a } \boldsymbol{positive} \textit{ real number } a \textit{ by } \pm\infty \textit{ yields } \pm\infty \tag{48}$$

while

$$\textit{multiplying a } \boldsymbol{negative} \textit{ real number } a \textit{ by } \pm\infty \textit{ yields } \mp\infty. \tag{49}$$

We also declare

$$\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty \quad \text{and} \quad (-\infty) \cdot \infty = \infty \cdot (-\infty) = -\infty \tag{50}$$

but we *do not* declare the values of

$$0 \cdot \pm\infty \qquad \text{or} \qquad \pm\,\infty \cdot 0\,. \tag{51}$$

4.2.3 Extended division

We shall extend the operation of division of real numbers as follows

$$\textit{dividing } \pm\infty \textit{ by a } \textbf{\textit{positive}} \textit{ real number yields } \pm\infty \tag{52}$$

and

$$\textit{dividing } \pm\infty \textit{ by a } \textbf{\textit{negative}} \textit{ real number yields } \pm\infty \tag{53}$$

but we *do not* declare the values of

$$\frac{\pm\infty}{\pm\infty} \qquad \text{or} \qquad \frac{e}{0} \tag{54}$$

where e can be any point of the extended real line.

4.3

4.3.1

We shall be viewing real valued functions as functions with the target being the extended real line $[-\infty, \infty]$, whose *range* is contained in $(-\infty, \infty)$.

4.3.2

For such functions,

$$\lim_{x \to p} f(x) = \infty$$

has the obvious meaning:

for any real number c', there exists a neighborhood M of p, such that

$$c' < f(x) \quad \textit{whenever} \quad x \textit{ belongs to } M \textit{ and } x \neq p. \tag{55}$$

4.3.3

Similarly obvious is the meaning of

$$\lim_{x \to p} f(x) = \infty\,,$$

namely:

for any real number c'', there exists a neighborhood M of p, such that

$$f(x) < c'' \quad \textit{whenever} \quad x \textit{ belongs to } M \textit{ and } x \neq p. \tag{56}$$

4.3.4 The limits at $\pm\infty$.

For functions defined on subsets of the real line $f\colon D \to Y$, we can also talk about the limits at ∞ or $-\infty$. For example,

$$\lim_{x\to\infty} f(x) = q \tag{57}$$

has the obvious meaning:

for any neighborhood N of point q, there exists a real number a', such that

$$f(x) \in N \quad \textit{whenever} \quad x \textit{ belongs to } (a', \infty). \tag{58}$$

4.3.5

One has, of course, to remember that (57) makes sense only when ∞ is a limit point of D, the domain of function f.

Exercise 30 *State the appropriate definition of*

$$\lim_{x\to-\infty} f(x) = q.$$

4.3.6

For real valued functions $f\colon D \to \mathbf{R}$ defined on subsets $D \subseteq \mathbf{R}$ of the real line we can think of 4 more possibilities involving the limits at ∞ or $-\infty$ and the values of the limits being ∞ or $-\infty$.

Exercise 31 *State the appropriate definition of*

$$\lim_{x\to\infty} f(x) = -\infty.$$

5 Continuity

5.1 Continuity at a point

5.1.1

Let X and Y be sets equipped with a neighborhood structure and D be a subset of X. A function $f\colon D \to Y$ is said to be *continuous at a point* $p \in D$ if

> *For any neighborhood N of $f(p)$, there exists a neighborhood M of p, such that*
> $$f(x) \in N \quad \textit{whenever} \quad x \in M. \tag{59}$$

5.1.2 Continuity at a limit point

By comparing this with the definition of the limit at p, we observe that, if p is a limit point of D, then f is continuous at p if and only if

$$\lim_{x\to p} f(x) = f(p). \tag{60}$$

5.1.3 Isolated points

What if $p \in D$ is not limit point? This happens precisely when

$$\textit{there exists a neighborhood } M \textit{ of } p\textit{, such that } M \cap D = \{p\}, \tag{61}$$

i.e., in that neighborhood p is *the only* point from D. Such points in D are said to be *isolated*.

5.1.4

The image under f of such a neighborhood is the set with just one element, $f(p)$, thus it is contained in *every* neighborhood N of $f(p)$. In particular, every function is continuous at every isolated point of its domain. This shows that failure of being continuous can occur only at points of the domain which are its limit points.

5.1.5 Removable discontinuities

There are two basic types of disconuities. The first, when $\lim_{x \to p} f(x)$ does not exist and the second, when $\lim_{x \to p} f(x)$ exists but is not equal to $f(p)$. The second type is called a *removable discontinuity* because if we modify the function by changing its value at p, then f becomes continuous at p.

5.1.6

Discontinuities of the first type are said to be *nonremovable*: no matter how we define the value of f at p, the resulting function will not be continuous.

5.2 Continuous functions

5.2.1

We say that a function is *continuous* if it is continuous at every point of its domain.

5.2.2 The algebra of continuous real-valued functions

It is an immediate consequence of the Limit Laws that the sum and the product of any two continuous functions $D \to \mathbf{R}$ is continuous. The set of continuous functions from D to $\mathbf{R}$, equipped with the operations of addition and multiplication is denoted $C(D)$ and is called the *algebra of continuous functions* on D.

5.2.3

The algebra of continuous functions is an object of fundamental importance in Mathematics. To be precise, we just encountered the algebra of *real-valued* continuous functions. Frequently, one considers the algebra of *complex-valued* continuous functions.

5.3 The Composition Limit Law

5.3.1

You saw how the concept of the limit interacts with arithmetic operations on real-valued functions. Now you will see how it behaves with respect to the operation of composition of functions.

5.3.2

Suppose $g\colon D \to Y$ is a function defined on a subset D of X with

$$\lim_{x \to p} g(x) = q.$$

Suppose $f\colon E \to Z$ is a function defined on a subset E of Y which contains point q.

5.3.3

If p is a limit point of the domain of $f \circ g$, which is equal to

$$g^{-1}(E) = \{x \in D \mid g(x) \in E\},$$

then we can talk about the limit of $f \circ g$ at p.

5.3.4

Suppose that f is *continuous* at q. Thus, given any neighborhood N of $f(q)$, there exists a neighborhood M' of q, such that

$$f(y) \in N \qquad \text{whenever} \qquad y \in M'.$$

Since q is the limit of g at p, there exists a neighborhood M of p such that

$$g(x) \in M' \qquad \text{whenever} \qquad x \in M \setminus \{p\}.$$

In particular,

$$f\big(g(x)\big) \in N \qquad \text{whenever} \qquad x \in M \setminus \{p\}.$$

5.3.5

We established our last limit law.

Theorem 5.1 *If*

$$\lim_{x \to p} = q$$

and f is continuous at q, then

$$\lim_{x \to p} f\big(g(x)\big) = f(q)$$

provided the limit of $f \circ g$ at p makes sense, i.e., provided p is a limit point of the domain of $f \circ g$.

6 Differentiability

6.1 Negligible functions

6.1.1

A function $\nu \colon D \to \mathbf{R}$ is said to be *negligible* at a point $p \in D$ if

$$\nu(x) \longrightarrow 0$$

faster than the distance from x to p when $x \longrightarrow p$. This property, when expressed in rigorous terms becomes

For any number $C > 0$, there is a neighborhood M of p such that

$$|\nu(x)| \leq C \operatorname{dist}(x, p) \qquad \textit{whenever} \qquad x \in M \cap D. \tag{62}$$

6.1.2 Distance functions

The above definition makes sense if the domain D is a subset of a set X equipped with a *distance* function.

What is a "distance function"? It is a function whose argument is a pair of points in X. If X is the real line $\mathbf{R}$,

$$\operatorname{dist}_{\mathbf{R}}(x, y) = |x - y|$$

is a standard distance function. If X is the n-dimensional Euclidean space $\mathbf{R}^n$,

$$\operatorname{dist}_{\mathbf{R}^n}(x, y) = \|x - y\|$$

where $\| \ \|$ is the function on $\mathbf{R}^n$ that plays the role of the absolute value function:

$$\big\|(a_1, \dots, a_n)\big\| = \sqrt{a_1^2 + \cdots + a_n^2}. \tag{63}$$

Function (63) is referred to as the *norm*, or more precisely, the *Euclidean norm* on the n-dimensional coordinate space $\mathbf{R}^n$.

6.1.3 Metric spaces

A distance function defined on pairs of points of a set X is supposed to possess the following properties: the distance from x to y equals the distance from y to x, i.e.,

$$\operatorname{dist}(x, y) = \operatorname{dist}(y, x)\,;$$

the distance from x to y is a nonnegative real number and it vanishes precisely when $x = y$. Finally, given any three points x, y and z, the distance from x to z is not greater than the sum of the distances from x to y and from y to z,

$$\operatorname{dist}(x,z) \leq \operatorname{dist}(x,y) + \operatorname{dist}(y,z) \qquad (x,y,z \in X). \tag{64}$$

Inequality (64) is called the *Triangle Inequality* because for the Euclidean distance it expresses the fact that the length of one side in a triangle in the Euclidean space does not exceed the sum of lengths of the other two sides.

6.1.4

Any function satisfying above properties is called a *distance function* on a set X. A set equipped with a distance function is called a *metric* space.

6.1.5 Balls

The set consisting of points x in a metric space whose distance to a given point p does not exceed number $r > 0$,

$$\bar{B}_r(p) = \{x \in X \mid \operatorname{dist}(x,p) \leq r\} \tag{65}$$

is called the *ball of radius r with center at p*. More precisely, this is a *closed* ball of radius r. The open ball is obtained by taking the points x whose distance is *less* than r,

$$B_r(p) = \{x \in X \mid \operatorname{dist}(x,p) < r\}. \tag{66}$$

6.1.6 The neighborhood structure of a metric space

Every metric space has a very natural neighborhood structure if we declare the balls with center at p to be neighborhoods of p. In particular, a subset N of a metric space is a neighbohood of a point p if it contains a ball with center at p. Whether the ball is open or closed does not matter: an open ball of radius r is contained in the open ball of radius r and at the same time contains closed balls of radius *less* than r.

Exercise 32 *Show that the sum of functions negligible at a point p is negligible at p.*

Exercise 33 *Let f be a function bounded at a point p and v be negligible at p. Show that fv is negligible at p.*

6.1.7 Lipschitz functions

We say that a function $f\colon X \to Y$ drom a metric space X to a metric space Y has the Lipschitz property at a point p if there exists a constant $K > 0$ and a neighborhood M of p such that

$$\mathrm{dist}_Y(f(x), f(p)) \leq K \,\mathrm{dist}_X(x, p) \tag{67}$$

for all x in M.

Exercise 34 *Show that the composition $\nu \circ g$ is negligible at p if g has the Lipschitz property at p and ν is negligible at $q = g(p)$.*

6.2 Differentiablity at a point

6.2.1

Let $f\colon D \to \mathbf{R}$ be a function defined on a subset of the real line. We say that it is *differentiable at a* if D is a neighborhood of a and there exists a number m such that

$$\nu(x) := f(a) + m(x - a) \tag{68}$$

is *negligible* at a.

Exercise 35 *Show the function*

$$f(x) = l(x - a)$$

is negligible at a if and only if $l = 0$. Deduce from it that the number m for which function (68) *is neglibible is unique.*

6.2.2 The derivative at a

That unique number is called the *derivative* of f at a and is denoted $f'(a)$.

Exercise 36 *Show that a function differentiable at a point a of the real line has the Lipschitz Property for any $K > |f'(a)|$. Derive from this that $\nu \circ g$ is negligible at a if g is diferentiable at a and ν is negligible at $b = g(a)$.*