Solving Cubic Equations

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A general cubic equation:

$$ax^3 + bx^2 + cx + d = 0 \qquad (a \neq 0) \tag{1}$$

reduces, after one divides both sides by a, to the equivalent equation

$$x^3 + b'x^2 + c'x + d' = 0 \tag{2}$$

where

$$b' := \frac{b}{a}, \qquad c' := \frac{c}{a}, \qquad \text{and} \qquad d' := \frac{d}{a}. \tag{3}$$

Note that $x^3 + b'x^2$ is equal to the cube of $x + \frac{b'}{3}$ up to a linear term in x:

$$x^3 + b'x^2 = \left(x + \frac{b'}{3}\right)^3 - 3\left(\frac{b'}{3}\right)^2 x - \left(\frac{b'}{3}\right)^3. \tag{4}$$

Thus we obtain

$$\begin{aligned} x^3 + b'x^2 + c'x + d' &= \left(x + \frac{b'}{3}\right)^3 + \left(c' - \frac{(b')^2}{3}\right) + d' - \left(\frac{b'}{3}\right)^3 \\ &= \left(x + \frac{b'}{3}\right)^3 + \left(c' - \frac{(b')^2}{3}\right)\left(x + \frac{b'}{3}\right) + \left(d' - \frac{b'c'}{3} + \frac{2(b')^3}{27}\right) \\ &= \xi^3 + c''\xi + d'' \end{aligned}$$

after substituting

$$\xi := x + \frac{b'}{3} \tag{5}$$

and setting

$$c'' := c' - \frac{(b')^2}{3} \quad \text{and} \quad d'' := d' - \frac{b'c'}{3} + \frac{2(b')^3}{27}. \tag{6}$$

The reduction of the general cubic equation, (1), to the equation without a quadratic term:

$$\xi^3 + c''\xi + d'' = 0 \tag{7}$$

was an easy part. In order to solve equation (7), we represent ξ as the sum of two new unknowns

$$\xi = u + v. \tag{8}$$

Note that

$$\begin{aligned} \xi^3 + c''\xi &= u^3 + v^3 + 3uv(u+v) + c''(u+v) \\ &= u^3 + v^3 + (3uv + c'')\xi \\ &= u^3 + v^3 \end{aligned}$$

precisely when u and v satisfy equality

$$uv = -\frac{c''}{3}. \tag{9}$$

Then equation (7) becomes

$$u^3 + v^3 + d'' = 0. \tag{10}$$

Let us multiply both sides of equation (10) by u^3 and use equality (9). What we get is the following particularly simple equation in u of degree 6:

$$u^6 - d''u^3 - \left(\frac{c''}{3}\right)^3 = 0 \tag{11}$$

which is a quadratic equation in u^3. Due to symmetry between u and v we can select u^3 to be

$$u^3 = -\frac{d''}{2} + \sqrt{\left(\frac{d''}{2}\right)^2 + \left(\frac{c''}{3}\right)^3} = -\frac{d''}{2} + \sqrt{\Delta} \tag{12}$$

and v^3 to be

$$v^3 = -\frac{d''}{2} - \sqrt{\left(\frac{d''}{2}\right)^2 + \left(\frac{c''}{3}\right)^3} = -\frac{d''}{2} - \sqrt{\Delta} \tag{13}$$

where we put

$$\Delta := \left(\frac{d''}{2}\right)^2 + \left(\frac{c''}{3}\right)^3. \tag{14}$$

Then, at least formally,

$$u = \sqrt[3]{-\frac{d''}{2} + \sqrt{\Delta}}, \tag{15}$$

and

$$v = \sqrt[3]{-\frac{d''}{2} - \sqrt{\Delta}}, \tag{16}$$

and finally

$$x = \sqrt[3]{-\frac{d''}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{d''}{2} - \sqrt{\Delta}}. \tag{17}$$

Now, one has to use the above formulae *very cautiously*, especially the last one.

If we exclude the case $c'' = 0$ which renders equation (7) trivial:

$$\xi^3 + d'' = 0, \tag{18}$$

then any u and v satisfying condition (9) are nonzero. In particular, u^3 and v^3 are nonzero. A nonzero (complex) number, or more generally an element of any field in which any cubic equation has a solution, has exactly three different roots.

Let u_0 be *any* cubic root of

$$-\frac{d''}{2} + \sqrt{\Delta}. \tag{19}$$

The remaining two roots are

$$u_1 = \zeta u_o \qquad \text{and} \qquad u_2 = \zeta^2 u_0 = \zeta^{-1} u_0 \tag{20}$$

where ζ is a nontrivial, i.e., not equal 1, cubic root of 1. There are two completely symmetric choices for ζ:

$$\frac{-1+\sqrt{-3}}{2} \qquad \text{and} \qquad \frac{-1-\sqrt{-3}}{2}. \tag{21}$$

They correspond to the two possible choices for the square root of -3. It does not depend which choice we make. If we choose one to be ζ then the other is ζ^{-1}.

Having discussed all three choices for the cubic root of (19), the corresponding three choices for the cubic root of

$$-\frac{d''}{2} - \sqrt{\Delta} \tag{22}$$

are

$$v_0 = -\frac{c''}{3u_0}, \qquad v_1 = -\frac{c''}{3u_1} = \zeta^{-1} v_0, \qquad \text{and} \qquad v_2 = -\frac{c''}{3u_2} \zeta v_0. \tag{23}$$

The final result is this complete set of solutions of the original equation (1):

$$x_0 = u_0 + v_0 - \frac{b'}{3} = u_0 - \frac{c''}{3u_0} - \frac{b'}{3}, \tag{24}$$

where u_0 is an *arbitrarily chosen* cubic root of (19),

$$x_1 = u_0\zeta + v_0\zeta^{-1} - \frac{b'}{3} = u_0\zeta - \frac{c''\zeta^{-1}}{3u_0} - \frac{b'}{3}, \tag{25}$$

and

$$x_2 = u_0\zeta-1 + v_0\zeta^{-}\frac{b'}{3} = u_0\zeta^{-1} - \frac{c''\zeta}{3u_0} - \frac{b'}{3}. \tag{26}$$

Note that the number of possible choices for u_0 is 6, not 3, in general: *two* choices for the square root of Δ, assuming Δ is nonzero, and *three* choices for the cubic root of (19) when the c'' is nonzero.

Now, it is important to understand that *any* of these six choices will produce all three roots of equation (1). As we have seen, any particular choice assigns labels 0, 1, and 2 to these roots, but when we move through all 6 choices for u_0, the labels 0, 1, and 2, are assigned to the roots differently each time!

You are seeing here the *Galois group* of a general cubic equation in action!

It would take us too far to attempt here an introduction to Galois Theory. Let me just say that the Galois group of a general cubic polynomial acts as the group of *all* permutations of the set of roots $\{x_0, x_1, x_2\}$, and is therefore icomorphic to the symmetric group S_3 that you have encountered in your *Introduction to Abstract Algebra*.