# Final Exam (Solutions)

May 11, 2011

**1.** Classify the group $G = (\mathbb{Z}\times\mathbb{Z})/\langle(5,6)\rangle$ according to the *Fundamental Theorem of Theory of Finitely Generated Abelian Groups.*

*Let* $\pi:\mathbb{Z}\times\mathbb{Z}\to G$ *be the quotient map and* $\iota:\langle(1,1)\rangle\to\mathbb{Z}\times\mathbb{Z}$ *be the inclusion map. We will show that* $\phi=\pi\circ\iota$ *is an isomorphism which means that* $G$ *is infinite cyclic. Indeed,* $\langle(1,1),(5,6)\rangle$ *contains both* $(1,0)$ *and* $(0,1)$*:*

$$(1,0)=6\cdot(1,1)-(5,6)\qquad \textit{and}\qquad (0,1)=-5\cdot(1,1)+(5,6)$$

*which generate* $\mathbb{Z}\times\mathbb{Z}$*, and thus* $\langle(1,1),(5,6)\rangle=\mathbb{Z}\times\mathbb{Z}$*. It follows that* $\phi$ *is surjective. It is obviously injective, since* $\langle(1,1)\rangle\cap\langle(5,6)\rangle=\{(0,0)\}$*.*

**2.** Show that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.

*Any ring isomorphism is an isomorphism of the corresponding additive groups. Both* $2\mathbb{Z}$ *and* $3\mathbb{Z}$ *are infinite cyclic, and an isomorphism between* $2\mathbb{Z}$ *and* $3\mathbb{Z}$ *must take* $2$ *to a generator of* $3\mathbb{Z}$*, i.e. to* $3$ *or* $-3$*. In particular,* $4=2+2$ *would be sent to either* $6=3+3$*, or* $-6=-3+(-3)$ *instead of* $9=(\pm3)^2$*. Thus, none of the two isomorphisms of the additive groups is a homomorphism of multiplicative semigroups.*

**3.** Let

$$\alpha=\begin{pmatrix}1&2&3&4&5&6&7&8\\1&5&8&7&6&2&3&4\end{pmatrix}\quad\text{and}\quad\beta=\begin{pmatrix}1&2&3&4&5&6&7&8\\2&5&3&7&6&1&8&4\end{pmatrix}\tag{1}$$

Find $\sigma\in S_8$ such that $\sigma\circ\alpha=\beta\circ\sigma$.

*One has* $\alpha=(2\ 5\ 6)(3\ 8\ 4\ 7)$ *and* $\beta=(4\ 7\ 8)(1\ 2\ 5\ 6)$*. It follows that if we put*

$$\sigma=\begin{pmatrix}1&2&3&4&5&6&7&8\\3&8&1&2&4&7&5&6\end{pmatrix}$$

*then* $\sigma\circ\alpha\circ\sigma^{-1}=\beta$*.*

**4.** Let $\alpha$ be the permutation defined in formula (1) above. Find $\alpha^{2011}$.

*Permutation* $\alpha$ *is the product of disjoint cycles* $\lambda=(2\ 5\ 6)$ *and* $\mu=(3\ 8\ 4\ 7)$*. Hence*

$$\alpha^{2011}=\lambda^{2011}\mu^{2011}=\lambda\mu^{-1}=\begin{pmatrix}1&2&3&4&5&6&7&8\\1&5&7&8&6&2&4&3\end{pmatrix}$$

*in view of the fact that* $2011=1\mod 3$ *and* $2011=-1\mod 4$*.*

**5.** Find the order of the permutation $\sigma=(3\ 11\ 5)(10\ 5\ 4\ 3\ 2\ 11\ 6)(3\ 5\ 11)$ in $S_{11}$.

*Note that* $(3\ 5\ 11)=(3\ 11\ 5)^{-1}$*, so* $\sigma$ *is a conjugate of a cycle of length* $7$*, thus is a cycle iteslf and its order equals its length, i.e.* $7$*.*

**6.** Let $G$ be a group. We say that $G$-sets $X$ and $X'$ are *isomorphic* if there exists a bijection $f : X \to X'$ such that $f(g \cdot x) = g \cdot f(x)$ for any $g \in G$ and $x \in X$. Prove that the orbit $Gx$ of any element $x \in X$ is isomorphic to $G/G_x$.

*The obvious map $\phi : G \to X$, given by $g \mapsto g \cdot x$, has the desired property and is surjective. One has $g \cdot x = g' \cdot x$ precisely if $g^{-1}g' \in G_x$, i.e., $\phi(g)$ depends only on the coset $[g] = gG_x$, and the induced map $f : G/G_x \to Gx$, where $f([g]) = g \cdot x$, is a desired isomorphism.*

**7.** Let $X$ be a $G$-set and $\mathcal{O} \subseteq X$ be any orbit. Prove that $|\mathcal{O}|$ divides $|G|$ if $G$ is finite.

*The assertion of Problem 6 implies that, for any $x \in \mathcal{O}$, one has $|\mathcal{O}| = |G/G_x| = |G : G_x|$, and the index of any subgroup of $G$ divides the order of $G$ if $G$ is finite (Lagrange's Theorem).*

**8.** Prove that any subgroup $H \subset G$ of index 2 is normal.

*$G \setminus H$ is the union of left cosets distinct from $H$ and also the union of right cosets distinct from $H$. Since index of $H$ in $G$ is $2$, $G \setminus H$ consists of a single left, and of a single right coset. Thus, every left coset is a right coset and vice-versa, i.e., $H$ is normal in $G$.*

**9.** Prove that any action of a group $G$ of order 11 on a set $X$ with 2011 elements must have at least 9 fixed points.

*Set $X$ is the union of disjoint orbits. By Problem 7, $|\mathcal{O}|$ divides $|G| = 11$, so equals $1$ or $11$. Orbits of cardinality $1$ correspond to fixed points. Denote by $m$ the number of orbits having $11$ elements. It follows that $2011 = |X| = 11m + |Fix_G(X)|$ and, since $11m \leq 2011$ and the largest multiple of $11$ less or equal $2011$ is $2002$, we conclude that $|Fix_G(X)| \geq 2011 - 2002 = 9$.*

**10.** Mark each of the following **Y** (Yes, it's true) or **N** (No, it's false).

**N** **a.** *A group $G$ is abelian iff every subgroup of $G$ is normal.*

**N** **b.** *A group $G/N$ is abelian iff the commutator subgroup $[G,G]$ contains $N$.*

**N** **c.** *The quotient group $\mathbb{C}/\mathbb{Z}$ has infinitely many elements of order $10$.*

**N** **d.** *A group $G$ is* simple *iff $G$ has no nontrivial subgroups.*

**Y** **e.** *Any homomorphism $\phi: A_n \to G$ into a group of order $4$ is trivial.*

**Y** **f.** *If $H \subset S_n$ is a proper subgroup of index less than $n$, then $H = A_n$.*

**N** **g.** *A positive integer $n$ divides the order of a finite group $G$ iff $|g| = n$ for some $g \in G$.*

**Y** **h.** *Every congruence relation on a group $G$ is of the form:* $a \sim b$ iff $ab^{-1} \in N$ for some normal subgroup $N \subseteq G$.

**Y** **i.** *If $\phi: G \to H$ is a homomorphism of finite groups, then the order of the image, $\phi(G)$, divides* $\gcd(|G|, |H|)$.

**Y** **j.** *If $\phi \circ \psi$ is an isomorphism, then $\phi$ is an epimorphism and $\psi$ is a monomorphism.*

**N** **k.** *Every homomorphism $\phi$ can be expressed as $\phi = \pi \circ \psi \circ \iota$ where $\iota$ is a monomorphism, $\psi$ is an isomorphism, and $\pi$ is an epimorphism.*

**Y** **l.** *An ideal $J$ in a ring $R$ with 1 equals $R$ iff $1 \in J$.*

**N** **m.** *A subring of a noncommutative ring is noncommutative.*

**N** **n.** *A quotient ring of a noncommutative ring is noncommutative.*

**Y** **o.** *A commutative ring $R$ with 1 is a field iff the only ideals in $R$ are $0$ and $R$.*

**Y** **p.** *The quotient $R/J$ of a commutative ring with 1 is a field iff $J$ is a* maximal ideal *(i.e., $J$ and $R$ are the only ideals containing $J$).*