Riemann integration from Measure Theory's point of view
Homework 9

due April 6, 2012

1. Let f_c be the function defined in Homework 8. Show that $f_c > 0$ if $f\colon X \to (0,\infty)$ is *lower semicontinuous*[1] and $c > 0$.

The lower- and upper-semicontinuous envelopes of a function For any function $f\colon X \to [-\infty,\infty]$ on an arbitrary topological space X, define

$$\underline{f}(x) := \sup_{N\in\mathscr{N}_x}\ \inf_{\xi\in N} f(\xi) \qquad \text{and} \qquad \bar{f}(x) := \inf_{N\in\mathscr{N}_x}\ \sup_{\xi\in N} f(\xi)$$

where $\mathscr{N}_x$ denotes the family of neighborhoods of a point $x \in X$.

2. Show that

$$\underline{f}(x) = \lim_{n\to\infty}\left(\inf_{\xi\in I_n} f(\xi)\right) \qquad \text{and} \qquad \bar{f}(x) = \lim_{n\to\infty}\left(\sup_{\xi\in I_n} f(\xi)\right) \tag{1}$$

for any nonincresing sequence of intervals

$$I_0 \supseteq I_1 \supseteq \ldots$$

whose lengths decreases to zero, $l(I_n) \searrow 0$, and $x \in \mathring{I}_n$ for each n.

3. Show that $\underline{f} \leq f \leq \bar{f}$, and

$$\underline{f} \leq \underline{g} \qquad \text{and} \qquad \bar{f} \leq \bar{g}$$

whenever $f \leq g$.

4. Show that $\underline{f}$ is lower-semicontinuous and $\bar{f}$ is upper-semicontinuous.

5. Show that f is lower-semicontinuous (resp., upper-semicontinuous) if and only if $f = \underline{f}$ (resp., $f = \bar{f}$).

6. Show that

$$\underline{f} = \sup\{g\colon X \to [-\infty,\infty] \mid g \leq f \text{ and } g \text{ is lower-semicontinuous}\}$$

and

$$\bar{f} = \inf\{h\colon X \to [-\infty,\infty] \mid f \leq h \text{ and } h \text{ is upper-semicontinuous}\}.$$

7. Show that f is continuous at a point $p \in X$ if and only if $\underline{f}(p) = \bar{f}(p)$, i.e., the set of *discontinuity* of f,

$$\mathrm{Disc}(f) := \{p \in X \mid f \text{ is not continuous at } p\},$$

coincides with the set

$$\{p \in X \mid \mathrm{osc}_f(p) > 0\}$$

[1] This hypothesis was by mistake missing in the formulation of Exercise 7 in the last Homework.

where the function $\mathrm{osc}_f := \bar{f} - \underline{f}$ is called the *oscillation* of f.

8. Show that the set of *discontinuity* of f is an F_σ-set.

Riemann integration With every function $f\colon [a,b] \to [-\infty, \infty]$ we associate two nets of simple functions, $\{\underline{s}_{\mathscr{P}}\}$ and $\{\bar{s}_{\mathscr{P}}\}$,

$$\underline{s}_{\mathscr{P}} := \sum_{I \in \mathscr{P}} \inf_{\xi \in I} f(\xi)\, \chi_{\mathring{I}} \qquad \text{and} \qquad \bar{s}_{\mathscr{P}} := \sum_{I \in \mathscr{P}} \sup_{\xi \in I} f(\xi)\, \chi_I,$$

labelled by partitions

$$\mathscr{P} = (I_1, \dots, I_r), \qquad I_j = [t_{j-1}, t_j] \qquad (a = t_0 < \cdots < t_r = b),$$

of interval $[a,b]$ (here $\mathring{I}$ denotes the interior of I). Functions $\underline{s}_{\mathscr{P}}$ are lower-semicontinuous and $\bar{s}_{\mathscr{P}}$ are upper-semicontinuous.

For any partition $\mathscr{P}$, let $E_{\mathscr{P}} = \{t_0, \dots, t_r\}$ be the set of the end-points of the subintervals $I \in \mathscr{P}$. Recall, that the set of partitions of interval $[a,b]$ is partially ordered by the *refinement* relation

$$\mathscr{P} \preccurlyeq \mathscr{P}' \qquad \text{if} \qquad E_p \subseteq E_{\mathscr{P}'},$$

or, equivalently, if each $I \in \mathscr{P}$ is contained in some $I' \in \mathscr{P}'$. Note, that the set of partitions of $[a,b]$ is directed by $\preccurlyeq$.

Let

$$\mathscr{P}_0 \preccurlyeq \mathscr{P}_1 \preccurlyeq \cdots \tag{2}$$

be a *chain* (i.e., a linearly ordered subset in a partially ordered set) of partitions of $[a,b]$. Denote by

$$E = \bigcup_{n \in \mathbf{N}} E_{\mathscr{P}_n}.$$

9. Show that, for any $x \in [a,b] \setminus E$,

$$\underline{s}_0(x) \le \underline{s}_1(x) \le \cdots \le \underline{f}(x) \le \bar{f}(x) \le \cdots \le \bar{s}_1(x) \le \bar{s}_0(x)$$

where $\underline{s}_n := \underline{s}_{\mathscr{P}_n}$ and $\bar{s}_n := \bar{s}_{\mathscr{P}_n}$. Moreover, show that

$$\lim_{n\to\infty} \underline{s}_n(x) = \underline{f}(x) \qquad \text{and} \qquad \lim_{n\to\infty} \bar{s}_n(x) = \bar{f}(x) \qquad (x \in [a,b] \setminus E)$$

if the the *mesh*[2] of $\mathscr{P}_n$ tends to zero.

Darboux sums and Darboux integrals The integrals of simple functions $\{\underline{s}_{\mathscr{P}}\}$,

$$\underline{S}_{\mathscr{P}}(f) = \sum_{I \in \mathscr{P}} \left(\inf_{\xi \in I} f(\xi) \right) l(I), \tag{3}$$

form the net of *lower Darboux sums* which is *nondecreasing* in view of the inequality

$$s_{\mathscr{P}}(x) \le s_{\mathscr{P}'}(x) \tag{4}$$

which holds for all x in $[a,b]$ except for those that belong to the finite set $\setminus(E_{\mathscr{P}'} \setminus E_{\mathscr{P}}) \subset [a,b]$.

[2]Recall that $\mathrm{mesh}(\mathscr{P}) := \max\{l(I) \mid I \in \mathscr{P}\}$.

The supremum of net (3) is called the *lower Darboux integral*

$$\underline{\int_a^b} f(x)\,dx := \sup_{\mathscr{P}} \underline{S}_{\mathscr{P}}(f).$$

Similarly, the integrals of simple functions $\{\bar{s}_{\mathscr{P}}\}$,

$$\bar{S}_{\mathscr{P}}(f) = \sum_{I \in \mathscr{P}} \left(\sup_{\xi \in I} f(\xi) \right) l(I), \tag{5}$$

form the net of *upper Darboux sums* which is *nonincreasing* in view of the same inequality (4)

The infimum of net (5) is called the *upper Darboux integral*

$$\overline{\int_a^b} f(x)\,dx dx := \inf_{\mathscr{P}} \bar{S}_{\mathscr{P}}(f).$$

10. Show that

$$\underline{\int_a^b} f(x)\,dx = -\infty$$

if and only if f is not bounded below. Formulate the corresponding assertion for the upper Darboux integral.

10. Show that a function $f\colon [a,b] \to (-\infty,\infty)$ is Riemann integrale if and only if it is bounded and

$$\overline{\int_a^b} f(x)\,dx dx = \underline{\int_a^b} f(x)\,dx.$$

11. Show that, for any *bounded* function $f\colon [a,b] \to (-\infty,\infty)$ and for any sequence of partitions (2) whose mesh decreases to zero, one has

$$\sup_n \underline{S}_{\mathscr{P}_n}(f) = \int_{[a,b]} \underline{f} \qquad \text{and} \qquad \inf_n \bar{S}_{\mathscr{P}_n}(f) = \int_{[a,b]} \bar{f}$$

and deduce from this that

$$\underline{\int_a^b} f(x)\,dx = \int_{[a,b]} \underline{f} \qquad \text{and} \qquad \overline{\int_a^b} f(x)\,dx = \int_{[a,b]} \bar{f}.$$

Here $\int_{[a,b]}$ is the *Lebesgue* integral, i.e., the integral associated with a certain Borel measure on $[a,b]$ which is called *Lebesgue measure*, such that the integral of any *continuous* function coincides with Riemann integral. In particular, for any function $g\colon [a,b] \to [-\infty,\infty]$ which is a pointwise limit of a nondecreasing sequence of *continuous* functions $\phi_i \nearrow g$,

$$\int g = \sup_i \int_a^b \phi_i(x)\,dx$$

and for a function which is a pointwise limit of a nonincreasing sequence of *continuous* functions $\phi_i \searrow g$,

$$\int g = \inf_i \int_a^b \phi_i(x)\,dx.$$

It follows that a bounded function $f\colon [a,b] \to (-\infty,\infty)$ is Riemann integrable if and only if

$$\int_{[a,b]} \operatorname{osc}_f = \int_{[a,b]} (\bar{f} - \underline{f}) = 0.$$

12. Show that $f\colon [a,b] \to (-\infty,\infty)$ is Riemann integrable if and only if it is bounded and the set of discontinuity of f has Lebesgue measure zero.